Supporting Information for "Piecewise Evolutionary Spectra: A practical approach to understanding projected changes in spectral relationships between circulation modes and regional climate under global warming"

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Contents of this file

Introduction

This supporting information provides:

A) Brief summary of the theory of ES proposed by Priestley.

B) Figure S1: Changes in squared coherence between NAO and surface temperature averaged over the Sahel region

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A: Theory of evolutionary spectra (ES)

We provide here, without going too much into the mathematical details, a brief summary of the theory of ES proposed by Priestley (1965, 1967, 1981, 1988). We consider our piecewise evolutionary spectra (PES) as a special case of ES. In A.1, we discuss the filter function, $g(u)$, which is used by Priestley to determine his ES. This filter is replaced for our PES by a simple box window having unit weights inside and zero weights outside the window. In A.2, we describe a weighting function, $w(u)$, which is used by Priestley to reduce sample fluctuations. This weighting function is replaced in our study by an ensemble averaging, taking the advantage that we are in possession of a large ensemble. ES are only defined for processes, which can be represented by a family of oscillatory functions (defined below). Priestley developed the theory of ES for continuous processes. In the following t refers to time as a continuous parameter.

A.1: *Definition of ES*

The theory of ES is based on two key ideas. The first one concerns the fact that the family of complex exponential functions (or sine and cosine functions) cannot be used to describe non-stationary processes. Instead, one has to replace the family of complex exponential functions by a family of *oscillatory functions* having the form

$$\phi_t(\omega) = A_t(\omega)e^{i\omega t}, \tag{S-1}$$

where the modulating amplitude $A_t(\omega)$ has the property that the modulus of its Fourier transform has an absolute maximum at the origin, i.e. $A_t(\omega)$ varies extremely slowly with t. Generally, the modulating amplitude $A_t(\omega)$ and the ES are functions of both time t and frequency ω. The choice of considering families of oscillatory functions in form of Eq.(S-1) is made to meet the desire that in Eq.(S-1), frequency ω is not just a mathematical parameter, but has the usual physical meaning. A process $X(t)$ is termed as an oscillatory process, if it can be represented in terms of oscillatory functions $\phi_t(\omega)$ defined in Eq.(S-1) as

$$X(t) = \int_{-\infty}^{\infty} \phi_t(\omega) dZ(\omega), \tag{S-2}$$

where $Z(\omega)$ is an orthogonal process with $E(|dZ(\omega)|^2) = d\mu(\omega)$. Priestley defines the evolutionary spectral density function at time t, $h_t(\omega)$, as

$$h_t(\omega)d\omega = |A_t(\omega)|^2 d\mu(\omega) \tag{S-4}$$

which satisfies for each t

$$Var(X(t)) = \int_{-\infty}^{\infty} h_t(\omega)d\omega. \tag{S-5}$$

Thus, different from the spectrum of a stationary process that describes the power-frequency distribution for the whole process (i.e. over all time), an ES describes the local power-frequency distribution at each instance of time. The condition imposed on $A_t(\omega)$ implies that $h_t(\omega)$ is only defined for the class of processes whose non-stationary second moment statistics change smoothly over time.

Given a realisation of an oscillatory process $X(t)$, $x(t)$, of length T, its ES $h_t(\omega)$ is determined using

$$y(t) = \int_{t-T}^{t} g(u)x(t-u)e^{-i\omega_o(t-u)}du \tag{S-6}$$

which is a realisation of a general linear transformation $Y(t)$ of $X(t)$

$$Y(t) = \int_{-\infty}^{\infty} g(u)X(t-u)e^{-i\omega_o(t-u)}du. \tag{S-6}$$

In the above equations, ω_o is a constant frequency, and $g(u)$ is a linear filter. The discrete representation of Eq.(S-6) is given in Eq. (1)in the main text. Assume that the width of $g(u)$ is much smaller than the characteristic width of $X(t)$, defined via the bandwidth of the family F of oscillatory functions used to represent $X(t)$. Assume further that $h_t(\omega)$ is much smoother than the transfer function of the filter $g(u)$. A meticulous consideration of the width of $g(u)$ and the bandwidth of the family F shows that $h_t(\omega)$ at $\omega = \omega_o$ is given approximately by $E(|y(t)|^2)$.

The second key idea can be expressed a form of an uncertainty principle that "in determining ES, one cannot obtain simultaneously a high degree of resolution in both the time domain and frequency domain" (Priestley, 1981). This principle is most effectively illustrated by considering $Y(t)$, a function identical to $y(t)$ when $T \to \infty$. Priestley derives $E(|Y(t)|^2)$ for two extreme forms of $g(u)$. One is a filter with infinite width, defined by constant weights in a time interval of length T in the limit $T \to \infty$. With this filter, $E(|Y(t)|^2)$ is independent of t, and reduces to the classical definition of the spectrum of $X(t)$ at frequency ω_o, as if $X(t)$ were stationary. Since $g(u)$ is a filter with constant weights over T in the limit $T \to \infty$, we sacrifice the accuracy in time to get the spectrum at frequency ω_o. The other filter is the delta function $g(u) = \delta(0)$, which has a zero width. With this filter, $Y(t) = X(t)$, and $E(|Y(t)|^2)$ is independent of ω. We sacrifice the accuracy in frequency to get the spectrum at exactly one time instance t. Generally, the more accurate we try to determine $h_t(\omega)$ as a function of time t, the less accurately we determine it as a function of frequency ω, and vice versa.

Thus, defining an ES of an oscillatory process requires a trade-off between identifying as accurate as possible the time dependence of spectral properties on the one hand and resolving as many frequencies as possible for a given time on the other hand. The trade-off boils down to the choice of the linear filter $g(u)$. For the problem considered in the present paper, we choose $g(u)$ to be a simple box window with unit weights inside and zero outside the window.

A.2: ***Estimation of ES***

Consider a given realisation of $X(t)$, $x(t)$, of length T and its linear transformation given in Eq.(S-6), where $g(u)$ has a non-zero width, which is small with respect to the bandwidth of the family F of oscillatory functions used to represent $X(t)$. Priestley shows that for suitable $g(u)$ and F, $E(|y(t)|^2)$ is an approximately unbiased estimate of $h_t(\omega_o)$. However, $E(|y(t)|^2)$ has a variance independent of T, and is hence not a consistent estimator of $h_t(\omega_o)$, just like a raw periodogram is not a consistent estimator of a stationary spectrum. To reduce the sample fluctuations, Priestley considers

$$z(t) = \int_{\infty}^{-\infty} w_{T'}(u)|y(t-u)|^2 du, \tag{S-8}$$

where $w_{T'}(t)$ is a weight-function depending on the time window parameter T' and decays sufficiently fast so that the above integral can be evaluated from a finite length of $|y(t)|^2$. $z(t)$ can be shown to be a consistent estimator in the sense that the variance of $z(t)$ is proportional to $1/T'$.

Priestley uses $z(t)$ as an approximately unbiased estimate of the (weighted) average of $h_t(\omega_o)$ in the neighborhood of the time instance t. Since we are in possession of an ensemble, we replace the average obtained with the weighting function $w_{T'}(u)$ by a simple ensemble average.

B: Figure S1

In contrast to the general decrease in piecewise evolutionary coherence (PEC) seen between NAO and UK surface temperature (Fig. 3a), especially on longer timescales like decadal and bidecadal, PEC between NAO and surface temperature over the Sahel region [10-25E, 12-20N] shows a gradual increase in PEC for all timescales except for the longest timescale (bidecadal). This is another example showing not only the non-stationary coherence changes with respect to the direction (gradual increase vs gradual decrease), but also the timescale-dependence of these changes.

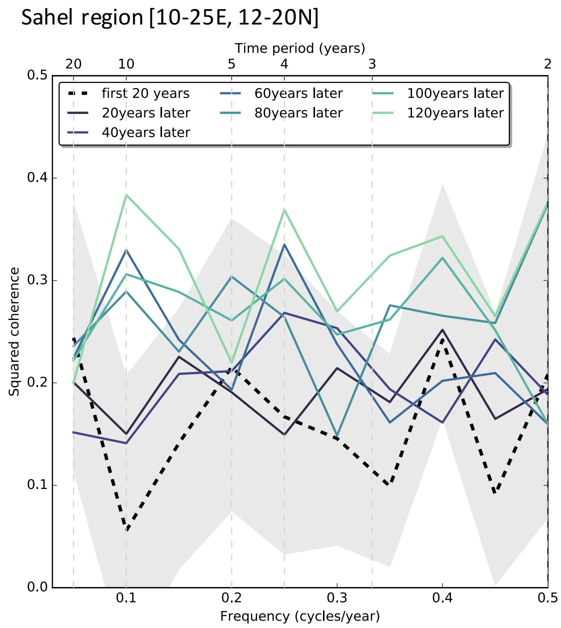

Figure S1. Piecewise evolutionary coherence spectra (PEC) between NAO and the Sahel region surface temperature (averaged over [10-25E, 12-20N]) for the seven consecutive 20-year intervals. Grey shaded region marks the 2.5% and 97.5% confidence bounds for changes in PEC from the first to another 20-year interval. Outside these bounds, change in PEC is statistically significant at 5% level.

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