

Cyclic, ekpyrotic and little rip universe in modified gravity

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We propose the reconstruction of $F(R)$ gravity in such a way that corresponding theory admits cyclic and ekpyrotic universe solutions. The number of explicit examples of such $F(R)$ model is found. The comparison with the reconstructed scalar-tensor theory is made. We also present $F(R)$ gravity which provides the little rip evolution and gives the realistic gravitational alternative for Λ CDM cosmology. The time for little rip dissolution of bound structures in such theory is estimated. We demonstrate that transformed little rip $F(R)$ solution becomes qualitatively different cosmological solution with Big Bang type singularity in Einstein frame.

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I. INTRODUCTION

It is widely accepted nowadays that universe evolution has passed through two (super)accelerating epochs: early-time inflation and late-time cosmic acceleration. Despite the close similarity between these two accelerating epochs there is a number of differences. For instance, inflationary era is characterized by large value of curvature while dark era occurs at small curvature. Moreover, the effective equation of state parameter w is qualitatively different for inflation or late-time cosmic acceleration. Inflation occurs right after Big Bang/Big Crunch singularity while number of quintessence/phantom dark energy models end up at finite time future singularity. This suggests the very natural conjecture: Big Bang should be identified with finite time singularity. In other words, the universe realizes cyclic evolution which may be produced by multi-fluid [1] or by the theory with oscillating equation of state parameter [2]. It is natural to construct the cyclic universe within the theory which describes the inflation and dark energy in unified way.

Modified gravity is realistic alternative for unified description of inflation with dark energy (for review of such unification in modified gravity, see [3]). Moreover, its weak-field limit may lead to newtonian regime (for review, see [4]). Hence, modified gravity represents very natural candidate where cyclic cosmology may be realized. The purpose of this work is precisely this one: reconstruction of modified $F(R)$ gravity which leads to cyclic evolution. In the next section the reconstruction technique for scalar-tensor theory and $F(R)$ gravity is developed. We find the explicit examples of scalar-tensor theory and $F(R)$ gravity which have the solutions corresponding to cyclic universe. In third section $F(R)$ action is presented as General Relativity plus ideal fluid. The reconstruction is developed for such formulation of modified gravity. Again, the example of $F(R)$ gravity which admits cyclic universe solution is found. In fourth section using above methods we show that ekpyrotic scenario may be also realized in frames of modified gravity. In fifth section we reconstruct $F(R)$ gravity which induces little rip cosmology. Such non-singular dark energy model was recently proposed in Ref. [5] as an alternative to Λ CDM model. Furthermore, we demonstrate that reconstruction of $F(R)$ theory which leads to non-singular phantom evolution is possible. Some summary is given in the last section.

II. CONFORMALLY CYCLIC UNIVERSE

Let us show how to construct the cyclic universe solution in modified gravity. The starting point is the spatially flat FRW metric

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2. \quad (1)$$

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where $a(t)$ is the monotonically expanding function of the cosmological time t . Let $a(t)$ has the following form

$$\ln a(t) = H_0 t + h(t). \quad (2)$$

Here H_0 is a positive constant and $h(t)$ is a function with period T :

$$h(t + T) = h(t). \quad (3)$$

Then

$$\begin{aligned} ds^2 &= -dt^2 + a(t + T)^2 \sum_{i=1,2,3} (dx^i)^2 \\ &= -dt^2 + a(t)^2 e^{2H_0 T} \sum_{i=1,2,3} (dx^i)^2 \\ &= -dt^2 + a(t)^2 \sum_{i=1,2,3} (d\tilde{x}^i)^2. \end{aligned} \quad (4)$$

Here $\tilde{x}^i \equiv e^{H_0 T} x^i$. Eqs. (1) and (4) show that the metric at t is physically identical with that at $t + T$. In this sense, the universe is cyclic although the universe is monotonically expanding. Hence, it called as conformally cyclic universe. A simplest example is

$$h(t) = H_1 \cos\left(\frac{2\pi t}{T}\right). \quad (5)$$

Here H_1 is a positive constant less than H_0 , which guarantees that the universe is monotonically expanding.

We now consider the models which reproduce the scale factor $a(t)$ given by (2) with (5). First, let us investigate the scalar-tensor model, whose action is given by

$$S = \int d^4 x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R - \frac{1}{2} \omega(\phi) \partial_\mu \phi \partial^\mu \phi - V(\phi) + L_{\text{matter}} \right\}. \quad (6)$$

Here, $\omega(\phi)$ and $V(\phi)$ are functions of the scalar ϕ . For the model where $\omega(\phi)$ and $V(\phi)$ are given by a single function $f(\phi)$, as follows,

$$\omega(\phi) = -\frac{2}{\kappa^2} f''(\phi), \quad V(\phi) = \frac{1}{\kappa^2} (3f'(\phi)^2 + f''(\phi)), \quad (7)$$

the exact solution of the FRW equations has the following form:

$$\phi = t, \quad H = f'(t). \quad (8)$$

In case of (2) with (5), one finds

$$f(t) = \ln a(t), \quad f'(t) = H_0 - \frac{2\pi H_1}{T} \sin\left(\frac{2\pi t}{T}\right). \quad (9)$$

Therefore if we consider the model

$$\begin{aligned} \omega(\phi) &= \frac{2H_1}{\kappa^2} \left(\frac{2\pi}{T}\right)^2 \cos\left(\frac{2\pi\phi}{T}\right), \\ V(\phi) &= \frac{1}{\kappa^2} \left\{ 3 \left(H_0 - \frac{2\pi H_1}{T} \sin\left(\frac{2\pi\phi}{T}\right) \right)^2 - \left(\frac{2\pi}{T}\right)^2 \cos\left(\frac{2\pi\phi}{T}\right) \right\}, \end{aligned} \quad (10)$$

the conformal cyclic universe (2) with (5) can be reproduced.

We now consider $F(R)$ theory

$$S_{F(R)} = \int d^4 x \sqrt{-g} \left(\frac{F(R)}{2\kappa^2} + \mathcal{L}_{\text{matter}} \right). \quad (11)$$

Here $F(R)$ is an appropriate function of the scalar curvature R . The action (11) is equivalently rewritten as

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} (P(\phi)R + Q(\phi)) + \mathcal{L}_{\text{matter}} \right\}. \quad (12)$$

Here, P and Q are proper functions of the auxiliary scalar ϕ . By the variation over ϕ , it follows that $0 = P'(\phi)R + Q'(\phi)$, which may be solved with respect to ϕ as $\phi = \phi(R)$. By substituting the obtained expression of $\phi(R)$ into (12), one arrives again at the $F(R)$ gravity action (for reconstruction of modified gravities, see Refs. [3, 6, 7]):

$$S = \int d^4x \sqrt{-g} \left\{ \frac{F(R)}{2\kappa^2} + \mathcal{L}_{\text{matter}} \right\}, \quad F(R) \equiv P(\phi(R))R + Q(\phi(R)). \quad (13)$$

If the scale factor a is given by a proper function $g(t)$ as $a = a_0 e^{g(t)}$ with a constant a_0 , by solving the following second-rank differential equation,

$$0 = 2 \frac{d^2 P(\phi)}{d\phi^2} - 2g'(\phi) \frac{dP(\phi)}{d\phi} + 4g''(\phi)P(\phi) + \sum_i (1 + w_i) \rho_{i0} a_0^{-3(1+w_i)} e^{-3(1+w_i)g(\phi)}, \quad (14)$$

we can find the form of $P(\phi)$ which gives a solution $a = a_0 e^{g(t)}$. One also finds the form of $Q(\phi)$ as follows:

$$Q(\phi) = -6(g'(\phi))^2 P(\phi) - 6g'(\phi) \frac{dP(\phi)}{d\phi} + \sum_i \rho_{i0} a_0^{-3(1+w_i)} e^{-3(1+w_i)g(\phi)}.$$

In case of (2) with (5), when we neglect the contribution from matter, Eq. (14) has the following form:

$$0 = \frac{d^2 P(\phi)}{d\phi^2} - \left(H_0 - \frac{2\pi H_1}{T} \sin\left(\frac{2\pi\phi}{T}\right) \right) \frac{dP(\phi)}{d\phi} - \frac{2(2\pi)^2}{T^2} \cos\left(\frac{2\pi\phi}{T}\right) P(\phi). \quad (15)$$

Since one cannot solve (15) explicitly, we assume $P(\phi)$ can be expanded by the Fourier series:

$$P(\phi) = \sum_{n=0}^{\infty} p_n \cos\left(\frac{2\pi\phi}{T}\right) + \sum_{n=1}^{\infty} q_n \sin\left(\frac{2\pi\phi}{T}\right). \quad (16)$$

Then one gets $q_1 = p_1 = 0$ and for $n \geq 2$,

$$\begin{aligned} p_n &= -\frac{2(n-1)^2}{H_1(n+2)} p_{n-1} - \frac{4\pi H_0(n-1)}{T(n+2)} q_{n-1} + \frac{n-4}{n+2} p_{n-2}, \\ q_n &= \frac{2(n-1)^2}{H_1(n+2)} q_{n-1} - \frac{4\pi H_0(n-1)}{T(n+2)} p_{n-1} - \frac{n-4}{n+2} q_{n-2}. \end{aligned} \quad (17)$$

In (17), we put $q_0 = 0$. Eq. (17) shows the existence of the solution of (15) and the corresponding $F(R)$ gravity model.

When matter can be neglected, Eq. (14) can be rewritten as

$$\frac{d}{d\phi} \left(g'(\phi) P(\phi)^{-1/2} \right) = -\frac{1}{2} P(\phi)^{-3/2} \frac{d^2 P(\phi)}{d\phi^2}, \quad (18)$$

which gives

$$g'(\phi) = -\frac{1}{2} P(\phi)^{1/2} \int d\phi P(\phi)^{-3/2} \frac{d^2 P(\phi)}{d\phi^2} = -\frac{1}{2P(\phi)} \frac{dP(\phi)}{d\phi} - \frac{3}{4} P(\phi)^{1/2} \int d\phi P(\phi)^{-5/2} \left(\frac{dP(\phi)}{d\phi} \right)^2. \quad (19)$$

In the second equality, we have used the partial integration. Furthermore by writing $P(\phi)$ as

$$P(\phi) = U(\phi)^{-4}, \quad (20)$$

(19) is rewritten as follows:

$$g'(\phi) = \frac{2}{U(\phi)} \frac{dU(\phi)}{d\phi} - \frac{12}{U(\phi)^2} \int d\phi \left(\frac{dU(\phi)}{d\phi} \right)^2. \quad (21)$$

As an example, we may consider

$$U(\phi) = U_1 + U_2 \cos \omega \phi. \quad (22)$$

Here U_1 , U_2 , and ω are constants and it is assumed $U_1 > U_2 > 0$, which show $U(\phi)$ does not vanish. Hence

$$g'(\phi) = -\frac{2U_2\omega \sin \omega \phi}{U_1 + U_2 \cos \omega \phi} - \frac{3U_2^2\omega^2 (2\phi - \frac{1}{\omega} \sin(2\omega\phi) + C)}{(U_1 + U_2 \cos \omega \phi)^2}. \quad (23)$$

Here C is a constant of the integration. Note that the obtained $g'(\phi)$ is not always positive. We also note that the $g'(\phi)$ is not periodic due to the term proportional to ϕ . This kind of term always appears for the periodic $U(\phi)$ since the integrand of the last term in (21) is positive definite and therefore the integration cannot be periodic.

Let us now consider an example where the function $P(\phi)$ is given by,

$$P(\phi) = U(\phi)^{-4} = P_0(\cos \omega \phi)^4, \quad (24)$$

where P_0 and ω are constants. Then, by the equation (18), the solution is,

$$g'(\phi) = g_0(\cos \omega \phi)^2 + 2\omega(\sin 2\omega \phi - \tan \omega \phi), \quad (25)$$

where g_0 is an integration constant. Note that here the solution contains divergences due to the term of the tangent in (24). These divergences correspond to points where the scale factor becomes null $a(t_0) = 0$, which can be identified with a Big Bang/Crunch singularity. This class of singularities is very common in cyclic Universes, where the ekpyrotic scenario is reproduced. In the same way other cyclic universes may be used for explicit reconstruction of modified gravity.

III. RECONSTRUCTED $F(R)$ AS AN EFFECTIVE PERFECT FLUID

Let us start with the modified gravity FRW equations written in the following form:

$$\begin{aligned} 3H^2 &= \frac{1}{F'(R)} \left(\frac{1}{2}F(R) + 3H\partial_t F'(R) \right) - 3\dot{H}, \\ -3H^2 - 2\dot{H} &= -\frac{1}{F'(R)} \left(\frac{1}{2}F(R) + 2H\partial_t F'(R) + \partial_t^2 F'(R) \right) - \dot{H}, \end{aligned} \quad (26)$$

where we have neglected the contributions of any other kind of matter. If we compare Eqs. (26) with the standard FRW equations ($3H^2 = \kappa^2\rho$ and $-3H^2 - 2\dot{H} = \kappa^2 p$), we may identify both right sides of Eqs. (26) with the energy-density and pressure of a perfect fluid, in such a way that they are:

$$\begin{aligned} \rho &= \frac{1}{\kappa^2} \left[\frac{1}{F'(R)} \left(\frac{1}{2}F(R) + 3H\partial_t F'(R) \right) - 3\dot{H} \right] \\ p &= -\frac{1}{\kappa^2} \left[\frac{1}{F'(R)} \left(\frac{1}{2}F(R) + 2H\partial_t F'(R) + \partial_t^2 F'(R) \right) + \dot{H} \right]. \end{aligned} \quad (27)$$

Then, Eqs. (26) take the form of the usual FRW equations, where the EoS parameter for this dark fluid is defined by:

$$w = \frac{p}{\rho} = -\frac{\frac{1}{F'(R)} \left(\frac{1}{2}F(R) + 2H\partial_t F'(R) + \partial_t^2 F'(R) \right) + \dot{H}}{\frac{1}{F'(R)} \left(\frac{1}{2}F(R) + 3H\partial_t F'(R) \right) - 3\dot{H}}. \quad (28)$$

The corresponding EoS may be written as follows:

$$p = -\rho - \frac{1}{\kappa^2} \left(4\dot{H} + \frac{1}{F'(R)} \partial_t^2 F'(R) - \frac{H}{F'(R)} \partial_t F'(R) \right). \quad (29)$$

The Ricci scalar is $R = 6(2H^2 + \dot{H})$, then $F(R)$ is a function of the Hubble parameter H and its derivative \dot{H} . The form of the EoS is written as: $p = -\rho + g(H, \dot{H}, \ddot{H}, \dots)$, where

$$g(H, \dot{H}, \ddot{H}, \dots) = -\frac{1}{\kappa^2} \left(4\dot{H} + \partial_t^2 (\ln F'(R)) + (\partial_t \ln F'(R))^2 - H\partial_t \ln F'(R) \right). \quad (30)$$

Then, by combining the FRW equations, it yields the following differential equation:

$$\dot{H} + \frac{\kappa^2}{2}g(H, \dot{H}, \ddot{H}, \dots) = 0. \quad (31)$$

Hence, for a given cosmological model, the function g given in (30) may be seen as a function of cosmic time t , and then by the time-dependence of the Ricci scalar, the function g is rewritten in terms of R . Finally, the function $F(R)$ is recovered by the expression (30). In this sense, Eq. (31) combining with the expression (30) results in [8]:

$$\frac{dx(t)}{dt} + x(t)^2 - H(t)x(t) = \dot{H}(t), \quad (32)$$

where $x(t) = \frac{d(\ln F'(R(t)))}{dt}$. At the next step we can explore several oscillating/cyclic solutions for the Hubble parameter.

Let us consider the oscillating Universe described by the Hubble parameter,

$$H(t) = H_1 \cos \omega t. \quad (33)$$

Then, by introducing this expression in the equation (32), it yields,

$$x(t) = \frac{d(\ln F'(R(t)))}{dt} = H_1 \cos \omega t. \quad (34)$$

Inverting the expression for the Ricci scalar $R = 6(2H^2 + \dot{H})$, and the expression (34), the corresponding $F(R)$ is obtained,

$$F(R) = \frac{24\omega^2(\omega - \sigma(R))\zeta(R) + (3\omega^2 - R + 24H_1^2)\sigma(R) + \sqrt{3}(4R - 3\omega^2) - 48\sqrt{3}H_1^2\omega}{48H_1^2\omega\zeta^3(R)}(\sqrt{3}\omega - \sigma(R))e^{\zeta(R)}, \quad (35)$$

where,

$$\begin{aligned} \sigma(R) &= \sqrt{48H_1^2 + 3\omega^2 - 4R}, \\ \zeta(R) &= \frac{\sqrt{24H_1^2 + 3\omega^2 - \sqrt{3}\omega\sigma(R) - 2R}}{2\sqrt{6}\omega}. \end{aligned} \quad (36)$$

Hence, we have reconstructed the action for $F(R)$ that is able to reproduce a cyclic evolution (33). Note that the action (35) might be expanded in power series of the curvature, so that we have the Einstein-Hilbert action plus corrections.

One can consider now an oscillating Universe described by the Hubble parameter,

$$H(t) = H_0 + H_1 \sin \omega t, \quad (37)$$

where H_0 and H_1 are positive constants and Eq. (37) corresponds to (2) with (5) by identifying $\omega = 2\pi/T$. The solution of the equation (32) gives,

$$x(t) = \frac{d(\ln f'(R(t)))}{dt} = H_0 + H_1 \sin \omega t. \quad (38)$$

Then, the $F(R)$ action in terms of the time coordinate is given by,

$$F(R(t)) = \int e^{H_0 t + \frac{H_1 \sin \omega t}{\omega}} [6H_1\omega^2 \sin \omega t - 24H_1\omega \cos \omega t (H_0 + H_1 \sin \omega t)] dt. \quad (39)$$

In order to get the explicit expression of $F(R)$ for this case, one has to invert the expression of the Ricci scalar, $R = 6(2H^2 + \dot{H})$, which is not analytically invertible for the example (37), and solve the integral in (39), which is neither possible to solve analytically. However, note that given explicit values of the constants $\{H_0, H_1\}$, and of the frequency ω , some information may be obtained for different times, since each term in the integral (39) becomes important for different values of the time coordinate, i.e. at different epochs of the Universe evolution. In the same way, one can reconstruct $F(R)$ gravity for any other oscillating cosmology.

IV. THE EKPYROTIC SCENARIO IN $F(R)$ GRAVITY

The inflationary epoch (early-time acceleration) just after the initial singularity that was the origin of the Universe, is able to solve some of intrinsic problems of the Big Bang model. These problems, known as the *flatness* problem, the question on the homogeneity at large scales, the absence of monopoles or the origin of the inhomogeneities that lead to the formation of large scale structure, can be solved quite well by the inflationary paradigm. However, a decade ago a new proposal, alternative to inflation, was suggested, the so-called Ekpyrotic/cyclic Universe (see Ref. [9]). This new scenario solves additionally the puzzles of the standard cosmological model as well as it provides perhaps a more complete picture of the Universe evolution. While in the inflationary scenario, the initial conditions are required to complete the cosmological picture, as the time begins at the Big Bang singularity, in the ekpyrotic Universe, this is not a requirement due to its cyclic nature. The evolution presented by this class of cyclic Universe contains in general four stages per cycle: a first initial hot state similar to the standard Big Bang model, then a phase of accelerated expansion, after which a phase where the Universe starts to contract occurs and finally ends in a Big Bang/Crunch transition, when the cycle starts again. The corresponding period where the main problems enumerated above are solved occurs during the contracting phase. In the usual ekpyrotic models, a scalar field is included to reproduce a cyclic Universe. However, it is clear that modified gravity, and precisely $F(R)$ gravity, can perfectly reproduce the ekpyrotic scenario. In order to show that the above models, reproduced by $F(R)$ gravity, can solve the initial problems by means of a cyclic Universe containing a contracting phase, let us consider a FRW Universe described by,

$$\frac{3}{\kappa^2}H^2 = \frac{\rho_{m0}}{a^3} + \frac{\rho_{r0}}{a^4} + \frac{\rho_{\sigma 0}}{a^6} - \frac{k}{a^2} + \rho_{F(R)}. \quad (40)$$

Here ρ_{m0} is the energy-density for pressureless matter, ρ_{r0} for radiation, $\rho_{\sigma 0}$ for the anisotropies, k is the spatial curvature of the Universe and $\rho_{F(R)}$ is the effective energy-density defined in the previous section for the extra geometrical terms. Then, in order to get a homogeneous and isotropic spatially flat Universe in a contracting phase, the effective EoS parameter $w_{F(R)} > 1$, so that when the scale factor tends to zero, the $F(R)$ terms in the equations dominate over the rest, and the result is the same as in the inflationary scenario.

Looking at the above examples, we have that for the model given in (33) and (35), the effective EoS parameter for the perfect fluid defined with the extra terms of $F(R)$ is given by,

$$w_{F(R)} = \frac{p_{F(R)}}{\rho_{F(R)}} = -1 + \frac{2 \sin \omega t}{3\omega H_1 \cos^2 \omega t}. \quad (41)$$

Hence, for $t \sim \frac{\pi}{2\omega}$, the effective EoS $w_{F(R)} \gg 1$, so that the ekpyrotic scenario takes place, and the result is the observable Universe. Hence, we have shown that cyclic Universe can be well reconstructed in frames of modified gravity, and the initial problems, as the *flatness* or *horizon* problems, can be solved. However, generally the cyclic Universes contain singularities of the type of Big Bang/Crunch, where the scale factor goes to zero, and the energy densities for matter/radiation grow to infinity, as, for example, occurs for the model (24). In the original ekpyrotic model, based on a scalar field, the divergences are controlled by a function introduced in the action as a strong coupling between the scalar field and the other components. In the kind of ekpyrotic scenario proposed here, a smooth transition along the Big Bang/Crunch can be reproduced by means of the reconstruction of models that avoid the singularity (as it is realized for several examples given above) or by introducing a coupling in the matter action,

$$S_M = \int dx^4 \sqrt{-g} \beta(R) \mathcal{L}_m. \quad (42)$$

Hence, by an appropriate choice of the function $\beta(R)$, the matter divergences can be avoided, and the transition across the Big Bang/Crunch can be realized smoothly.

Therefore, the ekpyrotic scenario can be realized with no need to introduce an additional field but only in terms of modified gravity.

V. LITTLE RIP COSMOLOGY IN MODIFIED GRAVITY

In the previous sections, several oscillating Universes have been reconstructed via $F(R)$ gravity. Cyclic Universes seem to be well reproduced in these theories and the ekpyrotic scenario can be realized. Here we are interesting in the study of cosmological models which are able to reproduce super-accelerated phase in the context of modified gravity. In this case, the corresponding effective EoS parameter (28) crosses the phantom barrier, that is $w_{F(R)} < -1$. It is well-known that cosmological phantom models usually contain the so-called big rip singularity [10]. In the case of

the big rip singularity, the phantom energy-density and the scale factor diverges in a finite future. One of the most surprising consequences of the big rip is the dissolution of bounded systems, as the Solar System and atoms before the singularity (see Ref. [10, 11]).

In this section, we are interesting to reconstruct $F(R)$ gravity which is able to reproduce a phantom behavior, being free of future singularity. Nevertheless, such singularity free or little rip cosmology (see Refs. [5, 12]) also leads to dissolution of bound structures. Let us recall the first FRW equation in $F(R)$,

$$\frac{3}{\kappa^2}H^2(t) = \rho_{F(R)}(t), \quad (43)$$

where the energy-density is defined in (27). By definition, a big rip singularity occurs in a finite time, usually denoted by t_s , when the scale factor $a(t)$ and the energy-density diverges. So in order to avoid a big rip singularity, $\rho_{F(R)}(t)$ must remain positive and finite for all t . Looking at the expression for $\rho_{F(R)}(t)$ in (27), it seems complicated to obtain a general condition on the form of $F(R)$, but a natural condition (not sufficient) seems $F'(R) > 0$ for all R in order to avoid divergences. On the other hand, in the absence of matter, the EoS parameter (28) can be rewritten as,

$$w_{F(R)} = -1 - \frac{2\dot{H}}{3H^2}. \quad (44)$$

Hence, for a super-accelerated expansion, the Hubble parameter is required to be an increasing function of time. Looking at some of the models reconstructed in the previous sections, we can see that the Hubble parameter (33) is a periodic function that reproduces a cyclic Universe. From the effective EoS parameter (41), it is clear that this model exhibits periods of super-accelerating expansion and it does not contain any future singularity. To find out if the model (33) may lead to a little rip, one has to determine the duration of each cycle and the strength of the effective repulsive force reproduced by the $F(R)$ terms and compare with the binding forces of coupled systems (as for example the Solar System). However, to ensure that a little rip is reproduced, one has to study models of eternal acceleration but free of future singularities, which may have a stronger growth in time than those models containing big rip singularities. Let us use the reconstruction technique using Eqs. (12-21). As an example, we consider the function,

$$U(\phi) = e^{-\beta e^{\alpha\phi}}, \quad (45)$$

where α and β are constants. Then, by the expression (21), the Hubble parameter and the scale factor yield,

$$H(t) = h_0 e^{\alpha t} + h_1, \quad \rightarrow \quad a(t) = a_0 e^{4\beta e^{\alpha t} + 6\alpha t}. \quad (46)$$

where $h_0 = 4\alpha\beta$ and $h_1 = 6\alpha$. It is straightforward to see that the function (46) describes a Universe, where for small times $t \ll \alpha$, the Hubble parameter can be approximated as a constant, reproducing a de Sitter solution, as in the case of Λ CDM model. For large times, the Universe ends in an eternal phantom phase, where the EoS parameter $w_{F(R)} < -1$, but without big rip singularity. Nevertheless, a little rip (dissolution of bound structures) might occur in a finite time, similarly to the model presented in Ref. [5], as it is pointed out below. The functions $P(\phi)$ and $Q(\phi)$ can be easily reconstructed by the expressions (15) and (20),

$$P(\phi) = e^{4\beta e^{\alpha\phi}}, \quad Q(\phi) = -6\alpha^2(3 + 4\beta e^{\alpha\phi})(3 + 8\beta e^{\alpha\phi})e^{4\beta e^{\alpha\phi}}. \quad (47)$$

The $F(R)$ action that reproduces the solution (46) can be calculated by inverting the expression of the Ricci scalar $R = 6(2H^2 + \dot{H})$ and by using the expression (13), which yields,

$$F(R) = \left[C_1 + C_2 \sqrt{4\frac{R}{R_0} + 75} \right] e^{\sqrt{\frac{R}{12R_0} + \frac{25}{16}}}. \quad (48)$$

where $R_0 = \alpha^2$, $C_1 = -24e^{-39/12}R_0$, and $C_2 = 2\sqrt{3}R_0$. Hence, we have reconstructed $F(R)$ model which is able to reproduce a super-accelerated (eternal) phase, which does not lead to a future singularity. Note that the action (48) turns out to be the Einstein-Hilbert action plus some corrections for small values of the Ricci curvature R , where the exponential functions are expanded in power series,

$$F(R) \sim \kappa_1 R + \kappa_2 \frac{R^2}{R_0} + \kappa_3 \frac{R^3}{R_0^2} + \dots \quad (49)$$

Here the couplings κ_i are constants depending on $C_{1,2}$. Hence, for small values of the Ricci scalar the action reduces to the action for General Relativity plus power-law curvature corrections, as R^2 , which is known recipe to cure the

singularities and which has a viable behavior ([13]). Hence, the action (48) represents a viable model where GR can be recovered while the curvature scalar corrections remain small. It is important that such additional corrections become relevant close to the little rip evolution. In order to estimate the time for the little rip induced dissolution of bound structures in a naive way, one might compare the energy-density of a bound system as the Solar System with the density $\rho_{F(R)}$. For the model (46), such density can be approximated for large times by

$$\rho_{F(R)} = \rho_0 e^{2\alpha t}, \quad (50)$$

where ρ_0 is a constant that can be set by imposing that the current value of the energy-density is $\rho_{F(R)}(t_0) = \frac{3}{\kappa^2} H_0^2 \sim 10^{-47} \text{ GeV}^4$, where the age of the Universe is taken to be $t_0 \sim 13.73 \text{ Gyrs}$, according to Ref. [14]. One can set the time of the little rip dissolution occurrence when the gravitational coupling of the Sun-Earth system is broken due to the cosmological expansion. By assuming a mean density of the Sun-Earth system given by $\rho_{\odot-\oplus} = 0.594 \times 10^{-3} \text{ kg/m}^3 \sim 10^{-21} \text{ GeV}^4$, the time for the little rip dissolution of bound structures is,

$$t_{\text{LR}} = 13.73 \text{ Gyrs} + \frac{29.93}{\alpha}. \quad (51)$$

Hence, depending on the parameter α , the appearance of the little rip may last shorter or longer. For example, when $\alpha = 10^{-1} \text{ Gyrs}^{-1}$, the little rip occurs at the Universe age of $t_{\text{LR}} = 313.03 \text{ Gyrs}$, while for $\alpha \geq 1 \text{ Gyrs}^{-1}$, the time for the decoupling will be much shorter.

A lot of models can be reconstructed in frames of $F(R)$ gravity which are able to reproduce a super-accelerating phase free of singularities. Let us consider, for example, a Hubble parameter that reproduces a phantom Universe without big rip [7],

$$H^2 = \frac{H_0^2}{4}(t - t_0)^2 = H_0 N + H_1. \quad (52)$$

Here we have introduced the number of e-foldings $N = \ln \frac{a}{a_0}$ instead of the cosmological time t . For the Universe described by the function (52), the effective EoS parameter is less than -1 , so it also describes a phantom evolution but free of future singularities. By using the reconstruction technique from Ref. [7], the action that reproduces the above solution is given by,

$$F(R) = K \left(-2, -\frac{1}{2}; \frac{R - 3H_0}{12H_0} \right), \quad (53)$$

where $K(a, b, x)$ is the Kummer's serie. In this model, the energy-density is given by,

$$\rho_{F(R)} = \rho_0 (t - t_0)^2, \quad (54)$$

where $t_0 = 0$ as the origin of the Universe evolution. Then, by adjusting the value for ρ_0 with the present value of the energy-density as in the above model, the little rip occurs when the Universe age is $t = 137.3 \times 10^{12} \text{ Gyrs}$, which clearly grows much slower than for above model.

Hence, we have shown here that $F(R)$ gravity is able to reproduce successfully a phantom scenario free of future singularity, where the little rip occurs at the time which depends completely on the expansion growth rate of the model.

We are now interested to explore the corresponding picture in the Einstein frame for those solutions describing a little rip in $F(R)$ gravity. In order to reconstruct the action in the Einstein frame, one has to use a conformal transformation that removes the strong coupling in the action (12),

$$g_{E\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \text{where} \quad \Omega^2 = P(\phi), \quad (55)$$

where the subscript E stands for Einstein frame. A quintessence-like action results in the Einstein frame

$$S_E = \int d^4x \sqrt{-g_E} \left[R_E - \frac{1}{2} \omega(\phi) \partial_\mu \phi \partial^\mu \phi - U(\phi) \right], \quad (56)$$

where

$$\omega(\phi) = \frac{12}{P(\phi)} \left(\frac{d\sqrt{P(\phi)}}{d\phi} \right)^2, \quad U(\phi) = \frac{Q(\phi)}{P^2(\phi)}, \quad (57)$$

while the cosmological solutions in the Jordan frame are transformed through out the conformal transformation as,

$$a_E(t_E) = P(\phi(t))^{1/2} a(t) \quad \text{where} \quad dt_E = P(\phi(t))^{1/2} dt. \quad (58)$$

As an example, one can consider the action (47) and (48), which was found to reproduce the solution (46). By applying the conformal transformation (55), the action in the Einstein frame (57) is described by the kinetic term and the scalar potential,

$$\omega(\phi) = 4\alpha^2 \beta^2 e^{2\alpha\phi}, \quad \text{and} \quad U(\phi) = -6\alpha^2 (3 + 4\beta e^{\alpha\phi})(3 + 8\beta e^{\alpha\phi}) e^{-4\beta e^{\alpha\phi}}. \quad (59)$$

The solution is transformed as,

$$a_E(t) = a_0 e^{6(\beta e^{\alpha t} + \alpha t)}, \quad t_E = \int_{-\infty}^{2\beta e^{\alpha t}} \frac{e^z}{z}. \quad (60)$$

Here the second expression is given by an exponential integral. In order to calculate an exact expression for $a_E(t_E)$, let us approximate the exponential $e^{\alpha t} \sim 1 + \alpha t + O(t^2)$ for small times, so that the solution can be expressed as,

$$a_E(t_E) = a_{E0} t_E^6 e^{6\beta\gamma t_E}, \quad H_E(t_E) = 6\beta\gamma + \frac{6}{t_E}, \quad (61)$$

where $\gamma = \alpha e^{-2\beta}$. Then, it is straightforward to see that the Universe described in the Einstein frame not only does not describe a super-accelerating expansion evolution but it also contains a singularity, a type of initial or Big Bang singularity. On the same time the solution in the Jordan frame was free of singularities. The singularity of the Einstein frame universe is probably the manifestation of little rip (dissolution of bound structures).

Let us consider now the inverse reconstruction by studying first the little rip scenario with quintessence/phantom fields described by the action (6), and then reconstruct the $F(R)$ action by means of a conformal transformation. It is well-known that any cosmological solution can be reconstructed with minimally coupled quintessence/phantom fields scalar fields (see Ref. [15]). As an example, let us consider the kinetic term and the scalar potential

$$\omega(\phi) = -\frac{2}{\kappa^2} h_0 \alpha e^{\alpha\phi}, \quad V(\phi) = \frac{1}{\kappa^2} (3h_0^2 e^{2\alpha\phi} + (6h_0 h_1 + h_0 \alpha) e^{\alpha\phi} + 3h_1^2). \quad (62)$$

Using the FRW equations and Eqs.(7), the solution for the Hubble parameter and the scalar field are

$$H_E(t) = h_0 e^{\alpha t_E} + h_1, \quad \phi(t_E) = t_E, \quad (63)$$

That coincides with the solution found in (46) reproducing the same little rip evolution as in $F(R)$ gravity. In this case the equation of state parameter is,

$$w_\phi = \frac{\frac{1}{2}\omega(\phi)\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\omega(\phi)\dot{\phi}^2 + V(\phi)} = -1 - \frac{2h_0\alpha}{3(h_0 e^{\frac{\alpha}{2}t_E} + h_1 e^{-\alpha t_E})^2}. \quad (64)$$

Clearly this EoS parameter describes a scalar field which has a phantom behavior, where its EoS parameter $w_\phi < -1$ and only tends to -1 for $t_E \rightarrow \infty$. However, as a novelty compared with the usual phantom scalar fields (Ref. [15]), here there is no finite time future singularity, but little rip evolution occurs. For this example, the time for the dissolution of the Sun-Earth system is given by (51).

Let us now attempt to reconstruct the corresponding $F(R)$ by applying a conformal transformation to the action (6). It is well-known that $F(R)$ gravity can be described by a non-propagating scalar field (12), so that the conformal transformation must remove the scalar kinetic term in the action (6):

$$g_{\mu\nu E} = \Omega^2 g_{\mu\nu}, \quad \text{where} \quad \Omega^2 = \exp \left[\pm \sqrt{\frac{2}{3}} \kappa \int d\phi \sqrt{\omega(\phi)} \right], \quad (65)$$

The corresponding Jordan frame action yields,

$$S = \int d^4x \sqrt{-g} \left[\frac{e^{\left[\pm \sqrt{\frac{2}{3}} \kappa \int d\phi \sqrt{\omega(\phi)} \right]}}{2\kappa^2} R - e^{\left[\pm 2\sqrt{\frac{2}{3}} \kappa \int d\phi \sqrt{\omega(\phi)} \right]} V(\phi) \right], \quad (66)$$

This action is basically the action given in (12) in vacuum, which is equivalent to $F(R)$ gravity. However, for the case of a phantom scalar that reproduces the little rip, it is straightforward to see that the corresponding action (66) turns out to be complex, as the conformal transformation (65) is also complex due to fact that kinetic term in (64) is negative. This is the known result for the case of a phantom scalar cosmology containing future singularity (see Ref. [16]). As in the case of phantom singular Universes, for the case of little rip, a way to reconstruct a consistent action in the Jordan frame can be achieved by adding an extra phantom fluid (see Ref. [17]). However, in general a little rip Universe described in the Einstein frame owns an inconsistent action in the Jordan frame. Thus, it is explicitly demonstrated that little rip evolution may be reconstructed in frames of modified gravity.

VI. DISCUSSION

In summary, we developed the reconstructed method for $F(R)$ theory which admits cyclic and ekpyrotic universe solution. As a rule, the corresponding reconstructed $F(R)$ action is given implicitly. Nevertheless, its expansion around General Relativity action with curvature corrections is always possible as is shown explicitly. The comparison with scalar-tensor theory which also leads to cyclic evolution is made. Our results indicate that geometrical actions usually considered as inflation source may lead to more complicated cyclic universe.

We also presented the reconstruction of $F(R)$ theory leading to little rip universe. Little rip dark energy represents non-singular phantom cosmology. It was proposed in Ref. [5] as viable alternative to Λ CDM model because it is consistent with observational data. Being effectively non-singular one, little rip cosmology shows the dissolution of bound structures similarly to Big Rip cosmology. It is demonstrated that same effect occurs for $F(R)$ gravity admitting little rip solution: the corresponding time left for such dissolution is estimated. It is interesting that transforming little rip $F(R)$ gravity to scalar-tensor form (the Einstein frame), the little rip universe is transformed to qualitatively different (Big Bang type) singular universe. The appearance of initial singularity in Einstein frame indicates the manifestation of dissolution time in Einstein frame. Finally, it is shown that other types of non-singular super-accelerating universe may be also reconstructed in $F(R)$ gravity.

Note that modified gravity theories under discussion for cyclic universe are relevant mainly for early universe and/or may be expanded as General Relativity action plus corrections. Hence, such theories easily pass the local tests. From another side, it remains to understand how natural and realistic the cyclic/ekpyrotic/little rip universes are. The natural recipe for this purpose is the study of perturbations. However, such investigation should be done for different epochs showing subsequent transition from the earlier epoch to the following one and also with account of corresponding matter/radiation. This is quite complicated technical problem which lies outside of the scopes of current work.

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