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(C.I M E)

GENERAL-RELATIVISTIC KINETIC THEORY OF GASES (1)

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Introduction

The relativistic kinetic theory of gases, which will be presented in the following lectures, is of interest for a number of reasons: It offers a simple, microscopic model for matter in bulk which is sufficiently general to provide a basis for hydrodynamics and thermodynamics of simple and multi-component systems. Definite conservation laws, balance equations, equations of state, transport and reactions can be derived from it, and if cross from a microscopic scattering theory are fed in, kinetic theory gives transport and reaction coefficients. As in the non-relativistic theory, the arbitrariness of the constitutive equations and the indefiniteness of the transport coefficients inherent in the phenomenological continuum approach are overcome by the kinetic theory.

Moreaver, kinetic theory provides a description of gases under conditions where fluid dynamics does not apply, e.g., when collisions are rare and the mean free path is large.

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Applied to macroscopic particles like stars or galaxies, kinetic theory offers a method of treating systems or the sustem of galaxies, the "gas" of cosmology.

Another asset of relativistic kinetic theory is its uniform treatment of gases consisting of particles with positive mass and those having zero mass particles; its application to photons gives the cosmologically and astrophysically important theory of the transport of radiation.

Specific applications of relativistic kinetic theory to astrophysical problems which illustrate the usefulness of this theory will be mentioned later.

Although the domains of applicability of fluid dynamics and kinetic theory overlap, neither of them contains the other one. Nevertheless, kinetic theory may be considered as the more fundamental of the two theories, since within it one can derive from simple microscopic laws and plausible statistical assumptions and approximation methods the general forms of all the laws which are postulated in fluid dynamics; only the numerical values of (e.g.) transport coefficients have to be changed on leaving the domain of validity of kinetic theory.

Ideally, one would like to derive both kinetic theory and fluid dynamics from statistical mechanics; at the relativistic level, this has not yet been achieved. Therefore, we have to introduce the basic concepts and laws of kinetic theory on the basis of plausibility considerations as did Boltzmann.

There are many unsolved problems in relativistic kinetic theory, questions concerning the foundations, the mathematical structure, and specific physical applications. We shall refer to some of them in the following lectures.

Several systematic expositions of relativistic kinetic theory exist which naturally have much in common with the following lectures, in particular those by N. A. Chernikov (1963, 1964), C. Marle (1969), J. Ehlers and R. K. Sachs (1968), and J. Ehlers (1969). The elementary aspects of the special-relativistic theory which precede the Boltzmann equation (or sidestep it) are contained in the well-known book by J. L. Synge (1957) whose geometrical spirit has strongly influended the present lectures. (More specific references will be given at appropriate places in the lectures.)

In order to free equations of inessential factors, we shall use the following convention regarding physival dimensions and units: We put

$$c = 8\pi G = \hbar = 1,$$

where c is the speed of light in vacuo, G Newton's constant of gravitation, $\hbar$ the quantum of angular momentum, and k Boltzmann's constant. All physical quantities are then measured by pure numbers.

1. Assumptions on spacetime. Notation

Let X denote spacetime which we assume to be a real, four-dimensional, connected, differentiable Hausdorff manifold. In addition, we assume X to be oriented, and take always oriented local coordinate systems (x^a), $a = 1, \ldots, 4$.

The tangent space to X at p is denoted as $T_p(x)$; its dual, $T_p^*(X)$. Natural, dual bases in T_p and T_p^* are $(\frac{\partial}{\partial x^a})$ and (dx^a), respectively.

X carries a normal hyperbolic metric whose signature we take as + 2. The metric tensor or gravitational potential is written g_{ab}, the Riemannian connection is Γ^a_{bc}, and the Riemanni curvature tensor is R^a_{bcd}. The Ricci tensor is given by $R_{ab} := R^c{}_{acb}$, and the Einstein tensor by $G_{ab} := R_{ab} - \frac{1}{2} R g_{ab}$, where $R := R^a{}_a$. The sign of the curvature tensors is fixed by the Ricci identity.

$$2 v_{a;[bc]} = v_d R^d{}_{abc}.$$

We assume that X is time-oriented with respect to g_{ab}, so that it is meaningful to distinguish between future directed and past directed timelike and lightlike vectors, respectivbly [1].

An orthonorlaal basis (e_j) of T_p is always chosen to be oriented and such that e_4 is future-directed.

[1] An example of a pair (X, g_{ab}) which is not time-orientable is given in appendix I of Ehlers (1969).

A coordinate-system (x^a) is said to be inertial at p, p $\in$ X, if $\Gamma^a_{bc}\big|_p = 0$ and $\left(\frac{\partial}{\partial x^a}\Big|_p\right)$ is orthonorlaal.

The physical interpretation of general relativity theory is largely based on the correspondence principle that physical laws in the presence of gravitation retain their special relativistic form at p if expressed with respect to coordinates which are inertial at p. This guiding principle is not unambigious, however.

The assumption that spacetime is oriented is not necessary for kinetic theory; it is made here only for convenience. Without this assumption several quantities appearing in kinetic theory would have to be defined with respect to oriented domains of X, and it would have to be shown that a change of orientation preserves all relevant equations. This can be done.

The assumption that spacetime is time-oriented is also not necessary for those parts of kinetic theory which are independent of the Boltzmann equation. Without it, some quantities would have to be defined relative to time oriented domains of X, and the relevant equations would have to be shown to be insensitive to changes of the time orientation; that can easily be done. The Boltzmann equation, however, can only be formulated in a time-oriented spacetime, and its form is not preserved under a change of that orientation. The reason is that the occupation numbers of initial and final states enter the collision integral in a non-symmetrical manner, as will be seen later and as is known from ordinary kinetic theory. The arrow of time built into the Boltzmann equation shows up particularly clearly in the H-theorem, to be derived later.

2. Some facts about differential forms and integration [1]

Kinetic theory deals with various kinds of averages which are expressed as integrals. The domain of integration is sometimes a hypersurface in X, sometimes a specetime region, sometimes a hypersurface or a region in phase space (to be defined below). The appropriate tools for forming such integrals-volume elements, hypersurface elements etc. - are differential forms. We assume that the elements of the theory of differential forms on manifolds are known, and collect here a number of facts which we need later.

On an n-dimensional manifold N, the differential form fields can be expressed, with respect to local coordinates, as sums of homogeneous forms like $\psi = \frac{1}{r!}\psi_{a_1 \ldots a_r} dx^{a_1 \ldots a_r}$; where the components $\psi_{a_1 \ldots a_r}$ are real functions and

$$dx^{a_1 \ldots a_r} := dx^{a_1} \wedge dx^{a_2} \ldots \wedge dx^{a_r}$$

are exterior products of the coordinate differentials. With respect to the operations of addition, multiplication with real numbers and exterior multiplication the form fields from an associative algebra. The exterior differentiation operator d maps this algebra into itself.

An r-form ψ can be contracted, like any covariant tensor, with a vector A; the result is a (r - 1)-form $\varphi = A \cdot \psi$ with components $\varphi_{a_2 \ldots a_r} = A^{a_1}\psi_{a_1 a_2 \ldots a_r}$. A coordinate-independent definition of this operation

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(1) See the "references"about mathematical tools" in the bibliography at the end of these lecture notes.

is contained in the following assertion:
For any system of r-1 vectors $A_2, \dots, A_r$, we have

$$\varphi(A_2, \dots, A_r) = \psi(A_1, A_2, \dots, A_r).$$

For a fixed vector A the mapping $\psi \to A \cdot \psi$ of the algebra of forms into itself in an antiderivation, i. e., it is linear and satisfies the product rule

$$A \cdot (\varphi \wedge \psi) = (A \cdot \varphi) \wedge \psi + (-1)^{\deg \varphi} \varphi \wedge (A \cdot \psi).$$

Moreover,

$$A \cdot (A \cdot \psi) = 0.$$

A trivial, but useful consequence of the definitions is the

Lemma 1. If Ω is a nonzero n-form at some point p of N, then the map $L \to L \cdot \Omega =: \omega$ is a vector space isomorphism of $T_p(N)$ onto the space of (n-1) -forms at p.

This lemma immediately leads to

Lemma 2. If Ω is an n-form at p, $\Omega \neq 0$, and $L \in T_p(N)$, $L \neq 0$, then the most general (n-1)-form ω at p such that $\omega(A_1, \dots, A_{n-1}) \neq 0$ whenever $(L, A_1, \dots, A_{n-1})$ is linearly independent, is given by $\omega = a\, L \cdot \Omega$ where $a \neq 0$.

Corollary. If ω has the property stated in Lemma 2, then $L \cdot \omega = 0$, and $\omega(A_1, \ldots, A_{n-1}) = 0$ whenever $(L, A_1, \ldots, A_{n-1})$ is linearly dependent.

Lemma 2 and its corollary should be visualized by considering Ω and ω as volume-functions for n-dimensional and (n-1)-dimensional parallelotopes, respectively.

Another useful fact needed later the proof of which is left as an exercise is

Lemma 3. If Ω is an n-form field on N, L a vector field and f a function, then

$$df \wedge (L \cdot \Omega) = L(f)\,\Omega. \tag{1}$$

We here recall that a vector is (identified with) a linear differential operator acting on functions: $L(f) = L^a f_{,a}$.

Finally we recall the fundamental theorem (of Stokes):
If M is an oriented, compact, m-dimensional submanifold-with-boundary ∂M of an n-manifold N, and φ is a (smooth) (m-1)-form field of N defined on M, then

$$\int_M d\varphi = \int_{\partial M} \varphi \tag{2}$$

The assumption that M is compact can be omitted provided φ decreases to zero sufficiently strongly at infinity of M; also, ∂M may be allowed to have "corners".

3. Volume elements in spacetime

Under the assumptions about spacetime stated in section 1 the expression

$$\eta = \sqrt{-g}\,dx^{1234}, \tag{3}$$

where $g := \det(g_{ab})$ and $\sqrt{-g} > 0$, is a nonvanishing 4-form such that $\eta(e_1, e_2, e_3, e_4) = 1$ for any orthonormal basis (e_j); it is the volume element of spacetime.

Let A be a vector field on X and D a 4-dimensional, oriented, compact submanifold-with-boundary of X - henceforth called a region. Then, according to (2),

$$\int_D d(A\cdot\eta) = \int_{\partial D} A\cdot\eta \tag{4}$$

The integrand on the left can be rewritten as

$$d(A\cdot\eta) = (\sqrt{-g}\,A^a)_{,a}\,dx^{1234} = A^a{}_{;a}\,\eta$$

and that on the right as

$$A\cdot\eta = A^a\,\sigma_a, \quad \sigma_a := \frac{1}{6}\,\eta_{abcd}\,dx^{bcd}, \tag{5}$$

where

$$\eta_{abcd} = \eta_{[abcd]}, \quad \eta_{1234} = \sqrt{-g} > 0 \tag{6}$$

are the components of η.

σ_a are the components of the (vectorial) hypersurface element in X; the latter is a vector-valued 3-form.

With this notation, (4) goes over into

$$\int_D A^a{}_{;a}\,\eta = \int_{\partial D} A^a\,\sigma_a, \tag{7}$$

the familiar metric-dependent version of Gauss's theorem in Riemannian space.

We shall heneeforth use the term hypersurface for "oriented hypersurface".

Since each tangent space T_q of X is itself a (flat, oriented) pseudoriemannian space, it has its own volume element

$$\pi = \sqrt{-g}\, dp^{1234}. \tag{8}$$

g is to be evaluated at q with respect to coordinates (x^a), and the p^a from $p = p^a \frac{\partial}{\partial x^a}$ define an oriented coordinate-system on T_q.

Physically important hypersurfaces of T_q are the mass-shells for masses $m \geqslant 0$. The mass-shell $P_m(q)$ con sists of all future directed (4 momentum) vectors p at q which belong to (proper) mass m; $p^2 = - m^2$. An oriented coordinate-system on $P_m(q)$ is defined as follows. Take coordinates on X around q such that $\frac{\partial}{\partial x^\nu}$, $\nu = 1, 2, 3,$ are spacelike and $\frac{\partial}{\partial x^4}$ is future-directed and timelike at q. Then the restrictions of the natural coordinates p^ν to $P_m(q)$ form an oriented coordinate-system on $P_m(q)$, and p^4 (>0) is determined by

$$g_{ab}(x^c)p^a p^b = -m^2. \tag{9}$$

$P_0(q)$ is the future light cone of q.

In order to obtain a scalar volume element on $P_m(q)$, consider the T_q-analogue of (5),

$$\tau_a := \frac{1}{6}\eta_{abcd}\,dp^{bcd}. \tag{10}$$

Its restriction to $P_m(q)$ has values proportional to the normal of $P_m(q)$, hence there exists a 3-form π_m such that

$$\tau_a \Big|_{P_m} = p_a \pi_m, \tag{11}$$

since p_a is a normal of P_m. Setting a = 4 in (10) and (11) gives explicitely

$$\pi_m = \frac{\sqrt{-g}}{|p_4|}\, dp^{123}. \tag{12}$$

The same volume element is formally obtained from

$$\pi_m = 2\ H(p)\ \delta(p^2 + m^2)\,\pi, \tag{13}$$

in which H is the Heaviside function of p^4 and δ is the Dirac distribution.

For $m > 0$, $m\ \pi_m$ is the induced Riemannian volume element of $P_m(q)$ as a hypersurface of T_q.

In inertial coordinates at q, we have the familiar expression

$$\pi_m = \frac{dp^{123}}{E} \tag{14}$$

where $E = p^4$ is the energy. Taking polar coordinates in $\vec{p}$-space we obtain

$$\pi_m = \frac{\vec{p}^{\,2}}{\sqrt{m^2 + p^2}}\ d|\vec{p}| \wedge (\sin\vartheta\ d\vartheta \wedge d\varphi) \tag{15}$$

or also

$$\pi_m = \sqrt{E^2 - m^2}\ \ dE \wedge (\sin\vartheta\ d\vartheta \wedge d\varphi) \tag{16}$$

The consideration which led to the volume element π_0 on the tangent null cone $P_o(q)$ can be generalized to the actual null cone of q in X. We leave it as an exercise to the reader to verify

Lemma 4. Let $N_q^{\downarrow}$ be the past null cone of q, and let u_q be a future-directed timelike unit vector at q. A normal k_a of $N_q^{\downarrow}$ is obtained by drawing null geodesics through q, choosing tangent vectors k to them such that, at q, $k \cdot u_q = 1$, and parallely propagating these k's along the null geodesics. Also, put $v = 0$ at q, and put $k^a = \frac{dx^a}{dv}$, obtaining a field of affine distances v on $N_q^{\downarrow}$. Denote as $d\Omega_q$ the solid angle obtained by projecting a small bundle of null rays through q into that 3-space through q which is orthogonal to u_q, and call D the distance from apparent size of an arbitrary point $r \in N_q^{\downarrow}$ from q, as measured by an observer at q with 4-velocity u_q. Then

$$\sigma_a \Big|_{N_q} = k_a D^2 d\Omega_q \wedge dv \ , \qquad (17)$$

so that $D^2 d\,\Omega_q \wedge dv$ is a natural scalar volume element on N_q.

4. Basic assumptions about a relativistic gas. Geometry of phase space.[1]

The history of a system of many (classical) particles of negligible size is represented in relativity theory as a complex of timelike or lightlike wordlines. The particles may be thought of as being macroscopic (stars, galaxies) or microscopic (molecules, atoms, ions, nuclei, photons, . . .), and they may be interacting through long-range and/or short range forces.

Without attempting to give a detailed description of the dynamics of such a general system, we lay down a special, simple model for some systems which we call gases. In these systems, the particles are assumed to move like test particles in a mean gravitational field g_{ab} and elec-

(1) The geometric treatment given in this section follows essentially that of Bichteler (1965). See also Chernikov (1963), Lindquist (1966) (Appendix), and Marle (1969).

tromagnetic field F_{ab}, except during encounters due to short range interactions which are idealised as point collisions. (I. e., the range of these interactions must be much smaller than the mean free path.) The mean fields may be external fields - we then speak of a test gas - or may be collectively generated by the gas particles themselves, in which case we have a selfgravitating gas (or a Vlasov plasma).

We proceed to formalize this qualitative picture of a gas.

A particle of mass m ($\geqslant 0$) and charge e has a worldline $x^a(v)$ which obeys the Lorentz-Einstein equations of motion

$$\frac{dx^a}{dv} = p^a, \qquad \frac{Dp^a}{dv} = e\, F^a{}_b p^b, \tag{18}$$

if radiation reaction is neglected. The parameter v is so chosen that the tangent vector p^a is the (future-directed) 4-momentum. If $m > 0$, mv is proper time. $\frac{D}{dv}$ denotes, here and in the sequel, the absolute derivative along the world line,

$$\frac{Dp^a}{ds} = \frac{dp^a}{ds} + \Gamma^a{}_{bc} p^b p^c. \tag{19}$$

If a particle participates in a collision at $x \in X$, its world line may have a corner at x, or the world line may end or begin at x, if the particle is annihilated or created in the collision.

In the case of many particles the spacetime figure of a gas is a complicated network of curves, since several trajectories with different directions can pass through the same event, and the trajectories through nearby events can have quite different directions.

A simplification of the geometrical representation is achieved, as in nonrelativistic kinetic theory, by introducing a phase space. Since in relativity no preferred space sections $t = \text{const.}$ exist, the relativistic phase space cannot be defined in strict analogy to the ordinary $(\vec{x}, \vec{p})$ phase space (of one particle), but will correspond to the $(\vec{x}, t, \vec{p}, E)$ - -space. We define the (relativistic) one particle phase space for particles of arbitrary mass m to be the manifold

$$M: = \left\{ (x,p): x \in X, \quad p \in T_x(X), \; p^2 \lessgtr 0, \; p \text{ future directed.} \right\} \tag{20}$$

This set is indeed a 8-dimensional manifold, if we agree to take as local coordinates (x^a, p^a), where (x^a) is a coordinate-system on X and p^a are the corresponding natural vector components.
M is, in fact, a manifold with boundary, the boundary ∂M being the set of states (x, p) having mass zero, $p^2 = 0$.

M is a fiber bundle with base X. The fiber at x is the set of non-spacelike, future-directed vectors at x, i.e., the 4-momentum space at x. (If all vectors had been admitted, M would be the tangent bundle $T(X)$ over spacetime.)

M is obviously oriented, the (x^a, p^a)-systems being oriented coordinate-systems.

The equations of motion (18), (19) define on M a vector field

$$L = p^a \frac{\partial}{\partial x^a} + (e\, F^a{}_b p^a - \Gamma^a{}_{bc} p^b p^c) \frac{\partial}{\partial p^a} \tag{21}$$

called the Liouville vector (or operator). The oriented integral curves $(x^a(v), p^a(v))$ form a congruence in M, the phase flow generated by L. Physically, the phase flow represents the set of all test particle motions which are possible in the combined gravitational and electromagnetic fields occuring in L.

The rest mass m as given by equation (9) is a scalar function on M. It is constant on each phase orbit,

$$L(m) = 0. \tag{22}$$

Hence the restriction L_m of L to the hypersurface M_m of M defined by m = const. is tangent to M_m. We note that

$$M_m = \bigcup_{x \in X} P_m(x). \tag{23}$$

M_m, with its Liouville vector L_m and its phase flow, is the phase space for particles of fixed mass m; it is seven-dimensional and corresponds to the Newtonian $(\vec{x}, t, \vec{p})$ -space.(1) It is also a fiber

(1) In classical mechanics, this space is sometimes called "augmented phase space". See e.g. Liboff (1969). p. 16.

bundle with base X, the fiber over x now being $P_m(x)$, the mass-shell at x.

M_m, being the boundary of the oriented submanifold of M given by $p^2 \leqslant - m^2$, is also orientable. We orient it by choosing a coordinate system (x^a, p^a) on M such that $p_4 p^4 < 0$ whenever $p \in P_m(x)$, and then take (x^a, p^ν) as an oriented coordinate-system on M_m. We then have

$$L_m = p^a \frac{\partial}{\partial x^a} + (e F^\nu{}_b p^b - \Gamma^\nu_{bc} p^b p^c) \frac{\partial}{\partial p^\nu} \tag{24}$$

We know from ordinary statistical mechanics the usefulness of a measure on phase space which is invariant under canonical transformations and, in particular, under the phase flow.

Let us consider, therefore, the coordinate-independent 8 form

$$\Omega := \eta \wedge \pi = - g \, dx^{1234} \wedge dp^{1234} \tag{25}$$

on M (formed by means of (3) and (8)) and the 7-form

$$\Omega_m := \eta \wedge \pi_m = \frac{-g}{|p_4|} dx^{1234} \wedge dp^{123} \tag{26}$$

on M_m. Obviously, at each point.

$$\Omega \neq 0, \ \Omega_m \neq 0. \tag{27}$$

Ω and Ω_m are related as follows (exercise):

$$\Omega = m \; dm \wedge \Omega_m . \tag{28}$$

To see whether Ω is invariant with respect to the phase flow we compute $\mathcal{L}_L \Omega$, the Lie derivative of Ω with respect to L. Because of the identity [1].

$$\mathcal{L}_L \Omega = d\,(L \cdot \Omega) + L \cdot d\Omega$$

and $d\Omega = 0$, we get $\mathcal{L}_L \Omega = d\omega$, if we put

$$\omega := L \cdot \Omega = p^a \, \sigma_a \wedge \pi + \frac{1}{6} \, \eta_{abcd} (eF^a{}_d p^d - \Gamma^a{}_{de} p^d p^e) dp^{bcd} \wedge \eta . \tag{29}$$

The differential of this 7-form vanishes. This is really verified by using inertial coordinates at some (arbitrary) event x.

Hence,

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(1) See, e.g. Hicks (1965), p. 94.

$$\mathcal{L}_L \Omega = d\omega = 0; \tag{30}$$

i.e., <u>Ω is invariant under the phase flow</u> (Liouville's theorem).

The 7-form ω which arose here rather naturally as a tool will be seen in the next section to be important in itself; let us note some of its properties. From its definition (29) and from $\mathcal{L}_L\omega = d(L\cdot\omega) + L(d\omega)$
we infer:

$$L\cdot\omega = 0, \quad \mathcal{L}_L\omega = 0. \tag{31}$$

These properties express that ω induces a nonzero 7 form on the quotient manifold M/L; <u>ω can be consedered as a measure on the 7-manifold of phase orbits.</u> Indeed, if we introduce on M comoving local coordinates ξ^A with respect to L, i.e., such that $L = \frac{\partial}{\partial\xi^8}$, then (31) means that $\omega = \sum_{A=1,\dots,7}\omega_A(\xi^1,\dots\xi^7)\,d\xi^A$, which "is" a form on M/L. If $\mathfrak{T}$ is a tube of phase orbits and Σ a cross section of $\mathfrak{T}$, $\int_\Sigma\omega$ measures, loosely speaking, the "number" of orbits contained in $\mathfrak{T}$; it is independent of the cross section.

The preceding considerations can be carried over straightforwardly from $M, L,$ to M_m, L_m (exercise); one obtains

$$\omega_m := L_m \cdot \Omega_m = p^a\, \sigma_a \wedge \pi_m + \frac{1}{2|p_4|}\, \eta_{\lambda\mu\nu 4}\,(F^\lambda{}_b p^b - \Gamma^\lambda_{bc} p^b p^c)\, dp^\mu \wedge \eta \tag{32}$$

$$\mathcal{L}_{L_m}(\Omega_m) = d\,\omega_m = \mathcal{L}_{L_m}(\omega_m) = L_m \cdot \omega_m = 0 \qquad (33)$$

5. Distribution function, collision density, Liouville's equation

An individual gas-history - a particular complex of world-lines - is too complicated to be useful; we are interested only in the typical, average properties of gases. Therefore, we imagine a large collection of microscopically different, but macroscopically indistinguishable gas histories, a Gibbs-ensemble of gases. The average properties of such an ensemble are the subject of kinetic theory. (The averaging may have the additional merit that it disposes of certain all-too-classical features of our gas model like sharply defined worldlines and collision events; the average properties may well provide an approximate macroscopic description of a gas whose particles obey quantum laws. (1)

Consider, then, a gas consisting of particles of different species. Concentrate on one component the particles of which have mass m and charge e. A definite microstate, or history, of the gas can be represented, as far as the specified component is concerned, as a collection of

(1) Nonrelativistically the Boltzmann equation, e.g., can be "derived" from classical as well as from quantum mechanics; see Kadanoff and Baym (1962): Lowry (1970).

segments of phase orbits in M_m, the states occupied by particles between collisions. (We do not assign phase-orbits to particles during collisions; hence there are no particle orbits in M_m transverse to the phase flow.)

The distribution of occupied states in M_m can be fully characterized by the functional $\Sigma \rightarrow N_m[\Sigma]$ which assigns to any compact hypersurface Σ the number of occupied orbit segments intersecting it. By a hypersurface in M_m we mean here and henceforth an oriented, 6-dimensional submanifold with boundary of M_m. The intersection of an orbit k with Σ is counted positively (negatively) if, at the event of intersection, the vector basis $(L_m, A_1, \ldots, A_6)$ has the same (opposite) orientation as the basis of an oriented coordinate system of M_m, L_m being the tangent to k and $(A_1, \ldots, A_6)$ an oriented basis tangent to Σ.

If D is any region in M_m, then $N_m[\partial D]$ is the number of collisions in D, if creations are counted positively, and annihilations negatively.

For a macrostate, let $\bar{N}_m[\Sigma]$ be the ensemble average of N_m. Since $\bar{N}_m[\Sigma]$ is a kind of flux through Σ of a fictitious fluid streaming in M_m with velocity L_m, we expect it to be expressible as an integral. We thus need a volume element for hypersurfaces in M_m.

It is natural to ask whether there exists a 6-form on M_m which could serve as such a volume element. From the meaning of $\bar{N}_m$ it is clear that this form would have to assign a nonzero volume to any hypersurface-element not tangent to L_m, since there could be a flux through it. Using the fact that Ω_m is a non-vanishing 7-form on M_m (see eq. (27)) and remembering lemma 2, we infer that such a 6-form must coincide with ω_m as defined in (32), except for a non-vanishing factor. Because of Liouville's theorem, eq. (33), it is advisable to choose this factor to be constant on phase orbits in order that the 6-form is L_m-invariant $(0 = d(f\omega_m) = df \wedge \omega_m = df \wedge (L_m \cdot \Omega_m) = L_m(f)\Omega_m \Longrightarrow L_m(f) = 0$; we have used (1)). Hence, ω_m reoommends itself as an almost unique candidate for the required measure.

A hypersurface Σ in M_m whose projection into X is a space-like hypersurface corresponds to a region of an "inttantaneous" ordinary $(\vec{x},\vec{p})$-phase space. On such a Σ (and, more generally, on any Σ whose projection into X is a hypersurface), ω_m from eq. (32) reduces to its first part,

$$\omega_m = p^a \sigma_a \wedge \pi_m \,. \tag{34}$$

If we choose at some point x with $(x,p) \in \Sigma$ an inertial coordinate system such that $\frac{\partial}{\partial x^4}$ is, at x, normal to Σ, (34) gives, at x,

$$\omega_m = - dx^{123} \wedge dp^{123} \tag{35}$$

which is, except for the (conventional) sign, the ordinary phase volume element of an observer at x with 4 velocity $\frac{\partial}{\partial x^4}$. Therefore, ω_m

is the appropriate 6-form we have been looking for..

We return to our study of M_m. According to its physical meaning, we make the following <u>smoothness assumptions</u> about N_m, i. e. about a macrostate of a gas:

D_1) On any fixed hypersurface $\Sigma \subset M_m$ there exists a continuous, nonnegative density function f_Σ such that for all compact parts $\Sigma' \subset \Sigma$

$$\bar{N}_m [\Sigma'] = \int_{\Sigma'} f_\Sigma \, \omega_m . \tag{36}$$

D_2) Every point $(x, p) \in M_m$ has a neighbourhood U such that for every region $D \subset U \subset M_m$

$$\left| \bar{N}_m [\partial D] \right| \leqslant A \int_D \Omega_m \tag{37}$$

for seme constant A dppending on U.

D_1 asserts that on any fixed hypersurface Σ the measure defined by the expectation value of the number of occupied states contained in parts Σ' of Σ has a continuous derivative, or density function f_Σ with respect to the geometrical measure ω_m.

Equation (35) shows that $f_\Sigma(x, p)$ equals, for any observer whose worldline intersects the projection of Σ into X orthogonally at x, the ordinary density of states in his infinitesimal, ordinary $(\vec{x}, \vec{p})$ phase space.

D_2 asserts that the expectation value of the number of collisions in

D is at most of the order of the Ω_m volume of D; this assumption excludes, e. g., the possibility of having all (or particularly many) collisions occuring on one hypersurface of X.

The two assumptions D_1 and D_2 imply the existence of an invariant, i. e., hypersurface or observer-independent (one particle) distribution function (1) f_m on M_m such that for any hypersurface $\Sigma \subset M_m$

$$\bar{N}_m [\Sigma] = \int_{\Sigma} f_m \, \omega_m; \tag{38}$$

To prove (38), we have to show that if a point $\xi \in M_m$ is contained in two hypersurfaces Σ_1 and Σ_2, then $f_{\Sigma_1}(\xi) = f_{\Sigma_2}(\xi)$. For that, consider a tube $\mathfrak{T}$ of phase-orbits having ξ on its boundary. Then $\Sigma_1 \cap \mathfrak{T}$, $\Sigma_2 \cap \mathfrak{T}$ are two cross sections of $\mathfrak{T}$. Without loss of generality we assume that these two cross sections together with the part Λ of the cylindrical boundary $\partial\mathfrak{T}$ which lies between $\Sigma_1 \cap \partial\mathfrak{T}$ and $\Sigma_2 \cap \partial\mathfrak{T}$ form the boundary ∂D of an orientable, compact region D of M_m. Since no phase orbits can intersect Λ we have $\bar{N}_m [\partial D] = \bar{N}_m [\Sigma_1 \cap \mathfrak{T}] - \bar{N}_m [\Sigma_2 \cap \mathfrak{T}]$. Because of

(1) The preceding introduction of f_m is a generalised and "rigorised" version of that given in Synge (1957), p. 12-14.

D_1 and with the mean value theorem for integrals this can be re-written as $\bar{N}_m[\partial D] = f_{\Sigma_1}(\xi_1)\int_{\Sigma_1\cap\mathfrak{Z}}\omega_m - f_{\Sigma_2}(\xi_2)\int_{\Sigma_2\cap\mathfrak{Z}}\omega_m$, where $\xi_i \in \Sigma_i \cap \mathfrak{Z}$. But we know from Liouville's theorem that the two integrals on the right-hand side are equal. Hence, using also D_2, $\left| f_{\Sigma_1}(\xi_1) - f_{\Sigma_2}(\xi_2) \right| \leqslant A \int_D \Omega_m \left(\int_{\Sigma_1\cap\mathfrak{Z}}\omega_m\right)^{-1}$. If one now lets $\mathfrak{Z}$ shrink towards the orbit passing through ξ, the right-hand side tends to zero since the numerator is "one order smaller" than the denominator. Also, $\xi_i \rightarrow \xi$. Consequently, $f_{\Sigma_1}(\xi) = f_{\Sigma_2}(\xi)$. We call the common value $f_m(\xi)$, in order to emphasize that f_m is defined on M_m.

It is easy to verify that our orientation and sign conventions imply

$$f_m \geqslant 0. \qquad (39)$$

It is techhically desirable and physically not harmful to require also

D_3) f_m is continuously differentiable on M_m.

Having obtained a phase space density f_m which measures the average density of occupied states, we obtain straightforwardly a collision density in M_m. The average number of collisions in the region $D \subset M_m$, i.e., the difference between creations and annihilations of particles of the specified kind in D, is given by $\bar{N}_m[\partial D] =$
$= \int_{\partial D} f_m \,\omega_m = \int_D d(f_m\,\omega_m) = \int_D df_m \wedge \omega_m = \int_D df_m \wedge (L_m \cdot \omega_m) =$

$$= \int_D L_m(f_m)\, \Omega_m .$$

We have used equations (38), (2), (33), (32), and (1). Hence

$$L_m(f_m) = p^a \frac{\partial f_m}{\partial x^a} + (e\, F^\lambda{}_b p^b - \Gamma^\lambda{}_{bc} p^b p^c) \frac{\partial f_m}{\partial p^\lambda} \qquad (40)$$

is the collision density in M_m with respect to Ω_m (in the sense defined above).

Note that if $(x^a(v), p^a(v))$ is the phase orbit passing through (x, p) for $v = 0$, then the expression (40), evaluated at $(x, p,)$, equals $\left(\frac{d}{dv} f_m(x(v), p(v))\right)_{v=0}$, a fact that is often useful.

The preceding considerations prove the following theorem. The distribution function f_m of a component of a (possibly heterogeneous) gas satisfies Liouville's equation

$$L_m(f_m) = 0 \qquad (41)$$

in a region $D \subset M_m$ if and only if there is detailed balancing everywhere in D, i.e., if the average number of creations of particles of that component equals everywhere in D the average number of annihilations (1).

(1) Note that, in our terminology, even an elastic collision involves two annihilations and two creations.

Corollary 1.[1] If the particles of a particular species do not participate in any collisions in D, then the corresponding distribution function satisfied, in D, equation (41).

Corollary 2. If the assumptions of the theorem hold, then f_m is, in D, an integral of the motion defined by (18).

As an application of the invariance (observer-independence) of the distribution function, let us consider a radiation field as a photon gas with distribution function f_γ. Relative to an observer with 4-velocity u^a, it is customary to define a specific intensity I_ν of the radiation field, as the limit of the radio "(energy of photons with frequency in $d\nu$ and direction in solid angle $d\Omega$ passing in time dt normally through an area dA/ $(d\nu\, d\Omega dt dA)$". It is related (exercise) to f_γ by

$$I_\nu = 2\pi \left| u_a p^a \right|^3 f_\gamma (x, p). \tag{42}$$

Since $\nu = (2\pi)^{-1} \left| u_a p^a \right|$, the observer-independence of f_γ im-

(1) For geodesic motion (e = o), this assertion has first been stated by Walker (1936).

plies that of I_ν/ν^3, a fact that is important, e.g., in cosmology; its direct, kinematical proof is somewhat cumbersome.

If the photons are emitted by a source S (galaxy, e.g.) and do not interact with matter on their journey to the observer O, Liouville's equation (41) for f_γ and (42) give the important relation

$$I_{\nu_0} = \frac{I_{\nu_S}}{(1+z)^3} \qquad (43)$$

between I_{ν_S}, "measured" near the source by a fictitious comoving observer, and I_{ν_0}, the intensity actually measured by 0. z is the usual redshift of S relative to 0. (43) is basic for the derivation of observable relations in cosmology. Notice that the derivation just sketched holds in any spacetime, not only in the standard Robertson-Walker universes.

If one assumes that the famous 3° K "fireball" radiation was emitted thermally from the recombination hypersurface ($T \approx 3500^\circ$) in the early universe, one obtains from (43) the predicted intensity distribution in each direction in an arbitrary model universe, provided one can compute z from the null geodesics.(1)
This idea was used by R. K. Sachs and A. M. Wolfe (1967) to estimate the influence of material "lumps" on the radiation, and similar applications have been made more recently. The same method has been employed by W. L. Ames and K. S. Thorne (1968) to determine the optical appearance

(1) It is also assumed that no scattering occurred between emission and absorption.

of a collapsing star to a distant observer. Several other applications of (41) have been made, particularly in cosmology and stellar dynamics.

6. Macroscopic fluid variables, balance equations, conservation laws.

Let us rewrite (38) for a hypersurface Σ whose projection into X is a hypersurface G. We obtain, using (34),

$$\bar{N}_m[\Sigma] = \int_G \sigma_a \left\{ \int_{K_x} f_m \, p^a \pi_m \right\}. \qquad (44)$$

K_x is that part of the mass shell $P_m(x)$ which is contained in Σ.

In particular, the integral $\int_G \sigma_a \left\{ \int f_m p^a \pi_m \right\}$ gives the average total number of particles of the species considered whose world lines intersect G. Here we have used the convention, to be maintained throughout the remainder, that $\int \pi_m \ldots$ denotes an integral over the whole mass-shell $P_m(x)$. Therefore, the spacetime vector field

$$N_m{}^a(x) := \int f_m p^a \pi_m \qquad (45)$$

is the particle 4-current density of the respective species. It is always timelike and future-directed under our assumptions. (If we would permit f_m to be a distribution, $N_m{}^a$ could be lightlike in one particular case: $m = 0$, and there is no 4-momentum dispersion at any event).

Similarly,

$$J^a := e\, N_m{}^a \tag{46}$$

is the electric 4-current density of the species considered.

In analogy with (45) we define

$$T_m{}^{ab}(x) := \int p^a p^b f_m\, \pi_m \tag{47}$$

as the kinetic stress energy momentum or matter tensor of the species. (If is possible to define a 4-momentum flux through a hypersurface $G \subset X$ and to show that (47) is the corresponding 4-momentum flux density, but this has no further use and is therefore not treated in detail here.)

We have assumed here, and will do so throughout these lectures, that f_m vanishes at infinity on $P_m(x)$ so that integrals like (45), (47) exist. (Sufficient for this is exponential boundedness on $P_m(x)$, as defined at the end of section 8.)

Excluding the trivial case where f_m vanishes on $P_m(x)$ (and the singular distribution mentioned below (45)) we infer from (47)

Lemma 5. If v^a is not spacelike and $v^a \neq 0$, then

$$\underset{m}{T}_{ab} v^a v^b > 0. \tag{48}$$

This lemma and a theorem due to J. L. Synge [1] imply

Lemma 6. Any kinetic stress energy momentum tensor is normal [2], i. e., admits a decomposition

$$\underset{m}{T}^{ab} = \mu u^a u^b + p^{ab} \tag{49}$$

with

$$u_a u^a = -1, \quad p_{ab} u^b = 0. \tag{50}$$

u^a can and will be chosen future-directed, and then (49) is unique.

The physical meaning of $\underset{m}{N}^a$, $\underset{m}{T}^{ab}$ for a local observer in terms of "3-dimensional" quantities is obtained by evaluating (45) and

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(1) See Synge (1956), p. 292.

(2) See Lichnerowicz (1955).

(47) in an inertial coordinate system at x. We obtain, for an arbitrary observer at x:

$$
\begin{aligned}
&\overset{4}{N}_m \text{ is the } \underline{\text{number density}}\\
&\vec{N}_m := N_m^\lambda \frac{\partial}{\partial x^\lambda} = \overset{4}{N}_m \langle \vec{v} \rangle_x^{(m)} \text{ is the } \underline{\text{particle flux density,}}\\
&\overset{44}{T}_m = \overset{4}{N}_m \langle E \rangle_x^{(m)} \text{ is the } \underline{\text{energy density,}} \qquad (51)\\
&\vec{T}_m := T_m^{\lambda 4} \frac{\partial}{\partial x^\lambda} = \overset{4}{N}_m \langle \vec{p} \rangle_x^{(m)} \text{ is the } \underline{\text{momentum density,}}\\
&\overleftrightarrow{T}_m := T_m^{\lambda\mu} \frac{\partial}{\partial x^\lambda} \otimes \frac{\partial}{\partial x^\mu} = \overset{4}{N}_m \langle \vec{p} \otimes \vec{v} \rangle_x^{(m)} \text{ is the } \underline{\text{kinetic}}
\end{aligned}
$$

<u>pressure tensor</u>.

Here $\vec{v}$, $E\ (= \overset{4}{p})$, and $\vec{p}$ are the 3-velocity, energy, and 3-momentum, and $\langle\ \rangle_x^{(m)}$ denotes the conditional expectation value at x, evaluated by means of the probability distribution defined by f_m with respect to the chosen inertial system.

We also define <u>mean kinetic pressure</u> p by (tr := trace)

$$
p := \frac{1}{3} \operatorname{tr} \overleftrightarrow{T}_m = \frac{1}{3} \overset{4}{N}_m \langle \vec{p} \cdot \vec{v} \rangle_x^{(m)} \qquad (52)
$$

and recognize the classical Bernoulli formula.

The rest mass density is $\rho := m \overset{4}{N}_m$. Writing μ for the energy density in $(51)_3$, we formulate

<u>Lemma 7.</u> For any observer and any distribution f_m, the inequa-

lities

$$0 \leq 3p \leq \frac{3}{2}\,p + \sqrt{(\frac{3}{2}\,p)^2 + \rho^2} \leq \mu \leq \mu + 3p \qquad (53)$$

hold.

Most of these inequalities are obvious from (51), (52), and $E = \frac{m}{(1-v^2)^{1/2}}$, $\vec{p} = E\,\vec{v}$; the only nontrivial inequality is the third one, due to A. H. Taub (1948). It follows by considering $(1-v^2)^{-\frac{1}{4}}$ and $(1-v^2)^{\frac{1}{4}}$ as elements of the Hilbert space $\mathcal{L}^2(P_m, f_m dp^{123})$, and applying Schwartz's inequality to them.

Equations (51) and (52) imply the well-known relations:

$$\begin{aligned}
&\text{If } p \ll \rho, \text{ then } \mu \approx \rho + \frac{3}{2}\,p \text{ (nonrel. monatomic gas)},\\
&\text{If } p \gg \rho, \text{ then } \mu \approx 3p \quad \text{(ultrarel. gas)}, \qquad (54)\\
&\text{If } m = 0, \text{ then } \mu = 3\,p \quad \text{(photon or neutrino gas)}.
\end{aligned}$$

In order to obtain balance equations for various macroscopic fluid variables we observe that these latter quantities are <u>moments</u> of the distribution function in 4-momentum space, given by $\int p^{a_1} p^{a_2} \ldots p^{a_r} f_m \pi_m$. The 0th moment is, at least for $m > 0$, essentially the trace of the matter tensor. Indeed. (47), (51), and (52) give

$$0 \leq m^2 \int f_m \pi_m = -\,T_{m\;a}^{\;a} = \mu - 3p. \qquad (55)$$

The 1st moment is the particle current density $N_m{}^a$, and the 2nd moment is the matter tensor. We would like to evaluate the divergence of the r-th order moment. We first establish

<u>Lemma 8.</u> If g is a C^1 -function on M_m, then

$$\left(\int p^a \, g \, \pi_m \right)_{;a} = \int L_m(g) \, \pi_m . \tag{56}$$

To prove this, take an arbitrary region $D \subset X$, and let $\hat{D} := \{ (x,p) : x \in D,\ p \in P_m(x) \}$ be the cylindrical region of M_m lying over D. Then, as in the derivation of the collision density above eq. (40), $\int_{\partial \hat{D}} g \, \omega_m = \int_{\hat{D}} L_m(g) \, \Omega_m$. We transform both these integrals into integrals over D:

$$\int_{\partial \hat{D}} g \, \omega_m = \int_{\partial D} \sigma_a \left\{ \int p^a g \, \pi_m \right\} = \int_D \eta \left(\int p^a \, g \, \pi_m \right)_{;a},$$

(Use (34) and (7))

$$\int_{\hat{D}} L_m \, (g) \Omega_m = \int_D \eta \left\{ \int L_m(g) \, \pi_m \right\} , \quad \text{from (26).}$$

Since D is arbitrary, (56) follows.

Next, we generalize this to obtain an "equation of transfer".

Lemma 9. [1] If f_m is an arbitrary distribution function on M_m, then (for $r \geqslant 2$)

$$\left(\int p^{a_1} p^{a_2} \dots p^{a_r} f_m \pi_m \right)_{;a} = \int p^{a_2} \dots p^{a_r} L_m (f_m) \pi_m +$$

$$+ \sum_{\lambda=2}^{r} e F^{a_\lambda}{}_b \int p^{a_2} \dots p^b \dots p^{a_r} f_m \pi_m . \qquad (57)$$

(The integrand in the sum is to be understood such that b replaces a_λ in the sequence $a_2 \dots a_r$.)

Proof: Take an arbitrary tensor field $v_{a_2 \dots a_r}$ which satisfies $v_{a_2 \dots a_r ; b} = 0$ at some arbitrary event x_0. Put $p^{a_2} \dots p^{a_r} v_{a_2 \dots a_r} f_m = g$, and apply lemma 8, to obtain at x_0:

$$v_{a_2 \dots a_r} \left(\int p^{a_1} \dots p^{a_r} f_m \dot{\pi}_m \right)_{; a_1} = \int L_m (v_{a_2 \dots a_r} p^{a_2} \dots p^{a_r} f_m) \pi_m . \quad (58)$$

Evaluate $L_m(\dots)$ at x_0 by taking the phase-orbit through x_0 and

---

[1] Tauber - Weinberg (1961)

differentiate $(\ldots)$ with respect to the parameter at x_0, getting

$$L_m(\ldots) = \frac{D}{dv}(\ldots) = v_{a_1 \ldots a_r} \frac{D}{dv}(p^{a_1} \ldots p^{a_r} f_m) =$$

$$= v_{a_1 \ldots a_r}(p^{a_1} \ldots p^{a_r} L_m(f_m) + e\, F^{a_1}{}_b p^b \ldots p^{a_r} f_m + \ldots),$$

where we have used (18) . Insertion into (58) and "dividing" by $v\ldots$ gives (57).

(Generalizations of Lemma 9 have been given by Ph. M. Quan (1966) and C. Marle (1969), but they do not seem to have found applications yet.)

Applying (56) (with $g = f_m$) and (57) (with $r = 2$) and using the definitions (45), (46), and (47) we obtain

$$N_m{}^a{}_{;a} = \int L_m(f_m)\, \pi_m \tag{59}$$

and

$$T_m{}^{ab}{}_{;b} = F^a{}_b J^b + \int p^a L_m(f_m)\, \pi_m. \tag{60}$$

Equation (59) is the balance equation for particles of the species considered; since $L_m(f_m)$ has been shown to be the collision density in phase space, the right-hand side of (59) is the (spacetime) production density of these particles.

Equation (60) is the 4-momentum balance equation for the given species; the vectors on the right-hand side represent the electromagnetic and the collisional 4-force densities acting on the component of the gas with distribution function f_m. An example for the latter is the force exerted on an electron gas by photons due to Compton scattering.

So far, we always concentrated on one component of a gas which may contain other kinds of particles as well; all our equations are valid for any component of a mixture.

Let us now first specialize to the case of a monocomponent, or simple gas consisting of particles all having (proper) mass m and charge e. Then, assuming conservation of particles in collisions (59) gives the conservation law

$$N_m{}^a{}_{;a} = \int L_m(f_m)\,\pi_m = 0 \tag{61}$$

which, of course, implies also charge conservation, $J^a{}_{;a} = 0$. Assuming also 4-momentum conservation during collisions, (60) results in

$$T_m{}^{ab}{}_{;b} = F^a{}_b J^b, \quad \int p^a L_m(f_m)\,\pi_m = 0. \tag{62}$$

These equations give on the one hand the macroscopic conservation laws basic to fluid mechanics, and they impose restrictions on the evolution of f_m, required by the microscopic conservation laws.

For a simple gas, there are two sensible ways to define the mean 4-velocity. One can either use the fact that the particle current vector $N_m{}^a$ is timelike and put

$$N_m{}^a = n\, u_k{}^a, \quad u_k{}^a u_{ka} = -1, \quad u_k{}^a \text{ future-directed,} \tag{63}$$

or one can use the normality of the matter tensor $T_m{}^{ab}$ and use the u^a of Lemma 6, i.e., require

$$u_D{}^{[a} T_m{}^{b]}{}_c u_D^c = 0, \quad u_D^a u_{Da} = -1, \quad u_D{}^a \text{ future-directed} \tag{64}$$

$u_k{}^a$ is called the kinematic mean velocity.[1] An observer travelling with $u_k{}^a$ is characterized by the property that in his local inertial frames there is no particle flux density (see $(51)_2$); n from (63) is the proper particle number density.

u_D^a is called the dynamic mean 4-velocity.[1] An observer travell-

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(1) The distinction and terminology is due to J. L. Synge; see Synge (1960)

ing with it will measure no momentum density, and this characterizes $u_D{}^a$ (see $(51)_4$). The energy density μ in (49) is the minimum of the energy densities measured by all possible observers; this property also characterizes $u_D{}^a$ (exercise).

The two mean velocities $u_k{}^a$ and $a_D{}^a$ are in general distinct, their equality characterizes (by definition) adiabatic processes. They are physically characterized by the existence of an observer u^a who finds neither a particle flux nor an energy flux in his local inertial systems. The necessary and sufficient condition for that is that $N_m{}^{[a}T_m{}^{b]}{}_c N_m{}^c = 0$, a very complicated restriction on the distribution function.

If one chooses any mean 4-velocity u^a, one can decompose the matter tensor uniquely according to the scheme (Eckart 1940):

$$T_m{}^{ab} = \mu u^a u^b + 2 u^{(a} q^{b)} + p h^{ab} + \pi^{ab}, \tag{65}$$

where

$$h^a{}_b = \delta^a{}_b + u^a u_b \tag{66}$$

projects $T_x(X)$ onto the 3-space orthogonal to u^a, and where

$$u_a q^a = u_a \pi^{ab} = \pi^a{}_a = 0. \tag{67}$$

μ is the mean energy density, q^a the mean energy flux density, p the mean kinetic pressure and π^{ab} the shear pressure tensor relative to u^a, These quantities change with u^a. If $u^a = u_D{}^a$, then $q^a = 0$. Adiabatic processes are characterized by the property that $q^a = 0$ for $u^a = u_k{}^a$. If, in addition, $\pi^{ab} = 0$, the gas behaves, in the process considered, as an ideal gas. We shall extend these mechanical considerations later on to the regime of thermodynamics.

Consider next a multicomponent gas. We distinguish the particle species by indices $A, B, \dots$; particles of species A have mass m_A, charge e_A, and (if we have microscopic particles) further characteristics like baryon number b_A etc. Each species has its phase space which we denote by M_A (instead of M_{m_A}), and its Liouville operator L_A; its distribution function f_A, current densities $N_A{}^a$, $J_A{}^a$, and its matter tensor $T_A{}^{ab}$, and the quantities which we have defined in terms of these, like $u^a_{D,A}$.

Reqiiring again 4-momentum conservation, we have instead of (62)

$$T_k{}^{ab}{}_{;b} = F^a{}_b J^b, \tag{68}$$

where

$$T_k^{\ ab} := \sum_A T_A^{ab} \tag{69}$$

is the total kinetic stress energy momentum tensor of the mixture, and

$$J^a := \sum_A J_A^{\ a} \tag{70}$$

is the total electric 4-current density. Moreover,

$$\sum_A \int p^a L_A(f_A)\, \pi_A = 0. \tag{71}$$

The individual particles will in general not be conserved during collisions, but certain combinations of the $N_A^{\ a}$ will have vanishing divergence. For example, if we define the baryon current density

$$B^a := \sum_A b_A N_A^{\ a} \tag{72}$$

and assume conservation of baryon number during collisions, we obtain

$$B^a{}_{;a} = 0 \tag{73}$$

and

$$\sum_A b_a \int L_A(f_A)\, \pi_A = 0. \tag{74}$$

Similarly, we will have

$$J^a{}_{;a} = 0 \tag{75}$$

and

$$\sum_A e_A \int L_A(f_A)\, \pi_A = 0. \tag{76}$$

Thus, we obtain macroscopic conservation laws for a mixture and corresponding integral conditions for the distribution functions.

Resonable mean 4-velocities for a mixture are the dynamic mean 4-velocity u^a_D defined as in (64), with T^{ab}_m replaced by T^{ab}_k,

the barycentric mean velocity, defined by

$$\sum_A m_A N_A{}^a = \rho u^a_{CM} \quad (\rho > 0), \tag{77}$$

and the baryonic mean 4-velocity, defined by

$$B^a = b\, u^a_B \tag{78}$$

provided B^a is timelike, as it is for "ordinary" matter.

With any choice of mean 4-velocity, one can decompose T_k^{ab} according to (65), obtaining μ, p etc. for a mixture. Which 4-velocity is the most useful one depends on the circumstances; a careful investigation is not known to the author.

7. The selfconsistent Einstein-Maxwell-Liouville equations [1]

Consider a collisionless mixture of particles, so that (41) holds for each component, and consequently (71), (74), (76) are trivially satisfied. Then, we have the macroscopic conservation laws (73), (75) and the (generalized) Poynting equation (68). It is, therefore, permissible to assume that g_{ab}, F_{ab} are the mean fields produced by the gas, i.e., to require that they satisfy the Einstein-Maxwell field equations:

$$G^{ab} + \Lambda g^{ab} = T_K^{ab} + T_M^{ab}, \tag{79}$$

$$F_{[ab,c]} = 0, \qquad F^{ab}{}_{;b} = J^a,$$

---

[1] Compare with Tauber-Weinberg (1961) who apparently first advocated these equations.

where

$$T_M^{\ ab} = F^{ac}F^b_{\ c} - \frac{1}{4} g^{ab}F_{cd}F^{cd} \tag{80}$$

is the Maxwell stress energy momentum tensor. Indeed, (80) and $(79)_2$ imply $T_M^{\ ab}{}_{;b} = - F^a_{\ b}J^b$, and if this is combined with (68), there results $(T_k^{\ ab} + T_M^{\ ab})_{;b} = 0$, as required by (79).

Hence, the equations (79) together with the Liouville equations

$$L_A(f_A) = 0 \tag{81}$$

seem to provide a closed, consistent system of dynamical equations for a gravitating plasma (in the Vlasov approximation).

For neutral particles, $(79)_1$ (with $T_M^{\ ab} = 0$) and (81) give a relativistic version of the equations of stellar dynamics (for collisionless systems).

It is natural to pose the Cauchy initial value problem for the system (79), (81). Formally, there seems to be no obstacle to solving it in the usual way by separating initial constraints from evolutions equations, the former being propagated off the initial hypersurface in consequence of the evolution equations, which in turn can be solved for the highest derivatives off the initial hypersurface, provided that is not characteristic. A careful elaboration for the present system (79), (81) does not seem to have been performed, however. (²)

Examples of sulutions to the equations $(79)_1$ (with $T_M{}^{ab} = 0$), (81) are known, see E. D. Fackerell (1966), 1968), J. Ehlers, P. Geren and R. K. Sachs (1868), R. Hakim (1968), R. Berezdivin and R. K. Sachs (1970); see also Misner (1968), Stewart (1969), (¹) Matzner (1969). Solutions with electromagnetic fields do not seem to be known at present (Problem).

8. The Boltzmann equation

Consider again a multicomponent gas with particle species A, B, ... If collisions occur, then the phase space density of all collisions in which particles of type A participate, $L_A(f_A)$, will be a sum (or integral) of various contributions due to different kinds of collisions, e.g., elastic and inelastic binary collisions, absorptions and emissions.

(¹) For a series of papers on the stability theory of static, spherically symmetric solutions of (79), (81) (for Fab = 0), see J. R. Ipser and K. S. Thorne, Ap. J. 154, 251 (1968), and subsequent papers by Ipser in the same Journal.

(²) (Note added in proof) Meanwhile, the problem has been solved by Y. Choquet-Bruhat; see Journ. Math. Phys., 1970, and another forthcoming paper.

Let the symbol

$$(x; p_A, p_B, \ldots \quad p_C, \ldots) \tag{82}$$

stand for a collisions in which particles of types A, B, ... with respective 4-momenta $p_A, p_B, \ldots$ collide at $x \in X$ and produce particles C, ... with $p_C, \ldots$; the numbers of incoming and outgoing particles may be arbitrary. (If, e.g., A = B, one has to write p_A, p'_A instead of p_A, p_A; this is tacitly assumed here and in the sequel.)

The set of all collisions (82) of a particular type, with $x \in X$, $p_A \in p_A(x), \ldots$ is again a bundle over X, the collision bundle. It carries a measure, viz., $\eta \wedge \pi_A \wedge \pi_B \wedge \ldots \wedge \pi_C \wedge \ldots$.
Augmenting our former smoothness assumption D_2 concerning the probability distribution of collisions we make the hypothesis:
C_1) In any macrostate of a gas, the average number of collisions (82) in a compact region U of the collision bundle is

$$\int_U V(x; p_A, p_B, \ldots \rightarrow p_C, \ldots;)\ \delta(\Delta p)\, \eta \wedge \pi_A \wedge \pi_B \ldots \wedge \pi_C \ldots \tag{83}$$

where V is a nonnegative (ordinary, measurable) function.(1) (In order to avoid ambiguities in the definition of V, U must be such that 4-momentum ranges $K_A(x), K_B(x), \ldots$ of indistinguishable incoming or outgoing particles

(1) Because of the S-factor in (83) the physically important domain of definition of V is $(X; p_A, p_B, \ldots \rightarrow p_C, \ldots) : \Delta p = 0$.

($A = B = \ldots$) do not overlap.

In (83), δ is the Dirac distribution (on R^4), and

$$\Delta p := p_A + p_B + \ldots - p_C - \ldots \tag{84}$$

is the 4-momentum difference between "in" and "out" states. The δ-factor in (83) expresses that collisions (82) occur only if they conserve 4-momentum.

It then follows that the distribution functions $f_A, f_B, \ldots$ of a gas satisfy equations of the form

$$L_A(f_A)(x, p_A) = \sum \Gamma_A^V \int V(x; p_A, p_B, \ldots \rightarrow p_C, \ldots)\, \delta(\Delta p)\, \pi_B \ldots \wedge \pi_C \ldots \tag{85}$$

In this "collision balance" the sum is to be taken over all kinds of collisions in which A-particles participate, either as incoming or as outgoing collision partners. The integral goes over the mass - shells of all colliding particles except the one whose state occurs on the left-hand side of (85). Γ_A^V is a numerical factor depending on the type of collision and on whether the state p_A on the left-hand side of (85) is an "in" or an "out" state; it is defined thus: $\Gamma_A^V > 0$ (< 0) if A is an "out" ("in") state, and

$\left|\Gamma^V_A\right| = n_A (n_A ! \, n_B ! \ldots n_C !)^{-1}$, where $n_A, n_B, \ldots$ are the numbers of (indistinguishable) particles of types $A, B, \ldots$ entering or leaving the V-collision, and n_A refer to the number of particles to which the left-hand state in (85) belongs.
(If we have a collision $(p_A, p'_A \rightarrow p''_A, p_B, p'_B)$ with $A \neq B$ and p_A is the state occuring on the left-hand side of (85), then $\Gamma^V_A = -2 \, (2! \; 1! \; 2!)^{-1} = \frac{-1}{2}$.) This factor is necessary in order that the various collisions involving identical in (or out) particles are not counted several times in the balance (85).

The equations (85) are useless as long as the dependences of the functions V on the state of the gas are not specified.[1] It is clear from nonrelativistic statistical mechanics that in a rigorous many-particle theory V will depend, not only on the one-particle distribution functions $f_A, f_B, \ldots,$ but also (at least) on pair

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[1] See, e.g., Liboff (1969).

correlations $g_{AB}(x, p_A; x', p'_B)$. No attempt will be made here to cope with these difficulties which pose important and interesting problems. Rather, I shall write down a "reasonable Ansatz" (as people say); then I shall make some remarks about the "philosophy" which is used to motivate that Ansatz; then modify it so as to account for the non-classical symmetry character of Bosons and Fermions; and then simply proceed on the basis of the resulting (generalized) Boltzmann equation.

Consider the hypothesis

$$C_2)\ V(x;p_A, p_B, \ldots \rightarrow p_C, \ldots) = f_A(x, p_A)\, f_B(x, p_B) \ldots R(p_A, p_B, \ldots \rightarrow p_C, \ldots) \tag{85}$$

in which the factors $f_A, \ldots$ refer to the "in" states only.

(C_2) is suggested by the assumptions that

(a) particles which are about to collide have uncorrelated momenta,

(b) the ranges of the collisional interactions are small in comparison with the scale on which the f_A's change appreciably with x,

(c) collisions take place in spacetime regions so small that the mean differential gravitational field $R^a{}_{bcd}$ (geodesic deviation etc. !) and the mean electromagnetic field $\mathbf{F}_{ab}$ do not affect their frequency appreciably.

(d) the presence of particles not participating directly in a collision does not affect the probability of occurence of that collision.

These assumptions, which essentially express that the gas is dilute ((d) and, for a gravitating gas, (c)), not too inhomogeneous in spacetime (b)), and in a state of high randomness ((a)), indicate the range of validity of the "Boltzmann collision hypothesis" C_2; each of them poses a problem of justification and indicates desirable generalizations. If, e.g., (c) were not true, then R might be expected (1) to depend on the principal directions and eigenvalues of $R^a{}_{bcd}$ and $F^a{}_b$.

In order to support the assumption C_2 further and to relate it to scattering theory, let us consider a collision $(p_A, p_B \rightarrow p_C, \ldots)$ with two incident and q emerging particles, and let us consider

(1) Compare Marle (1969), p. 88.

those collisions for which the momenta are contained in small ranges $K_A \subset P_A(x)$ etc.

According to (83) and (86), the number of those collisions per unit spacetime volume is

$$f_A(x, p_A)\, \pi_A(K_A)\, f_B(x, p_B)\, \pi_B(K_B)\, \delta(\Delta p)\, R\,(p_A, p_B \rightarrow p_C, \ldots)\, \pi_C(K_C) \ldots \tag{87}$$

Regarding the K_A-particles as a beam which hits the K_B-particles forming the target, we recognize that the number densities of projectiles and target particles, relative to any inertial frame with 4-velocity u, are given by (see (44))

$$n_A = f_A(x, p_A) \left| u \cdot p_A \right| \pi_A(K_A),$$

$$n_A = f_B(x, p_B) \left| u \cdot p_B \right| \pi_B(K_B),$$

whereas the relative velocity of these particles is

$$\left| \vec{v}_B - \vec{v}_A \right| = \frac{\left| (u \cdot p_A)\, p_B - (u \cdot p_B)\, p_A \right|}{(u \cdot p_A)\,(u \cdot p_B)} . \tag{88}$$

(Take $u^a = \delta^a_4$.)

Hence, we can rewrite (87) as

$$n_A n_B \left| \vec{V}_B - \vec{V}_A \right| \; dQ^u, \tag{89}$$

where

$$dQ^u = \frac{R(p_A, p_B \rightarrow p_C, \ldots .)}{\left|(u \cdot p_A)\, p_B - (u \cdot p_B)\, p_A\right|} \; \delta(\Delta p)\; \pi_c(K_c) \ldots . \tag{90}$$

Equation (89) is recognized as the standard definition of the <u>differential scattering cross section</u> dQ^u for scattering of p_A, p_B-particles into the ranges $K_c, \ldots,$ relative to the u-frame, and equation (90) is indeed the correct expression for that cross section which can be derived

α) in the nonrelativistic limit either from classical or from quantum mechanics, and

β) in the relativistic domain from quantum scattering theory.[1]

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[1] See, e.g., Brenig and Haag (1959)

(In this case R is simply related to the S operator [1])

(In the relativistic case, a classical derivation is not available, since there is no well developed theory of interacting particles.)

In a certain sense, we have now justified (85), since under the assumtions stated above the A- and B-particles in the gas should behave as if they were members of beams in a collision experiment.

One correction, or generalization, of (85) shall now be made. If the particles are atomic or sub-atomic, then assumption (d) is definitely wrong. In the case of Fermions, the presence of particles in the final states decreases, because of the Pauli principle, the collision probability, whereas for Bosons it enhances that probability (stimulated emission and scattering). This is incorporated by writing, instead of (86),

$$C_2') \quad V(x;p_A, p_B, .. \rightarrow p_C, ..) = f_A(x, p_A)\, f_B(x, p_B) \ldots \times (1 \pm s_C f_C(x, p_C)) \ldots R(p_A, p_B, .. \rightarrow p_C, ..) . \tag{91}$$

(1) See, e.g., Brenig and Haag (1959)

Here and in the sequel the upper sign refers to Bosons, the lower one to Fermions.

$$s_C := \frac{(2\pi)^3}{r_C} = \frac{h^3}{r_C}$$

is the volume of a phase-cell which corresponds asymptotically to a non-degenerate p-eigenstate of a free (quantum) particle[1] of spin degeneracy r_C. Hence, $s_C f_C(x, p_C)$ equals approximately the average occupation number of simple one-particle p_C-eigenstates localised near x. (In the "classical limit" $f_C \ll s_C^{-1}$, (91) reduces, of course, to (86).

A "pseudoproof" of (91) can be given within the Fock-space formalism, but that will not be reproduced here[2].

One simplification is possible and useful in (90).
If $u = \lambda p_A + \mu p_B$ - and these frames include the center - of - mass frame of the collision as well as the rest frames of the incoming particles-

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(1) Weyl (1911), Peierls (1936)

(2) See Bichteler (1965), Ehlers and Sachs (1968), Ehlers (1969).

then $\left|(u\cdot p_A)p_B - (u\cdot p_B)p_A\right| = \sqrt{(p_A\cdot p_B)^2 - m_A^2 m_B^2}$, independently of u, whence the corresponding expression

$$dQ = \frac{R\,(p_A, p_B \rightarrow p_C, \dots)}{\sqrt{(p_A\cdot p_B)^2 - m_A^2 m_B^2}}\ \delta(\Delta p)\ \pi_C \wedge \dots \tag{92}$$

is often called "the" (relativistic) cross section.
We have now suppressed the arguments $K_C, \dots,$ considering dQ henceforth as a (3q - 4) - form, where q is the number of final states and we imagine that δ(...) has been "absorbed" into four of the differentials $\pi_C \wedge \dots$;(If q = 1, dQ is a δ-"function":
$dQ = \sigma_a(-p_A\cdot p_B)\,\delta\!\left(\frac{1}{2}\left[m_A^2 + m_B^2 - m_C^2\right] - (p_A\cdot p_B)\right)$; σ_a is the absorption cross section.)

Inserting (91) into (85) we obtain the <u>generalized Boltzmann equation</u>

$$L_A(f_A) = \sum \Gamma_A^R \int f_A f_B \dots f^C \dots \delta(\Delta p)\, R_{AB\dots}^{C\dots}\ \pi_B \wedge \dots \wedge \pi_C \dots, \tag{93}$$

where we have simplified the notation in an obvious way. In particular, in (93) and henceforth,

$$f^C := s_C^{-1} \pm f_C; \tag{94}$$

the factors s_C have been absorbed into $R_{..}$. (Γ^R_A equals the former Γ^V_A.)

The equation (93) has first been formulated in special relativity for a classical (i.e., Boltzmannian) gas with elastic binary interactions by Lichnerowicz and Marrot (1940); for other treatments and generalisations see Tauber-Weinberg (1961), Israel (1963), Bichteler (1965) and the papers mentioned in the introduction.

Henceforth we shall require the Boltzmann equation (93) to hold for the distribution functions of any gas. (Other "reasonable" alternatives for V which lead to different kinetic equations are possible, but will not be discussed in these lectures.)

One important symmetry needs to be mentioned. If the microscopic collision law (S-matrix) is invariant with respect to the total reflection, PT, then the collision "matrix" R :: is invariant with respect to an interchange of incoming and outgoing states(1):

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(1) See Brenig and Haag (1959).

$$R^{C\ldots}_{AB\ldots} = R^{AB\ldots}_{C\ldots} . \tag{95}$$

We also add that, for Fermions, it is necessary that

$$s_A f_A \leqslant 1, \tag{96}$$

due to the exclusion principle.

The conservation law (71) is satisfied by (93), because of the $\delta(\Delta p)$ - factor. Other conservation laws like (74) can and have to be incorporated by similar restrictions on the R-functions; this will be assumed in the sequel.

It is now clear that we can generalize the selfconsistent field equations of section 7 so as to take into account collisions; we just have to replace equation (81) by (93). The remarks about the Cauchy problem made in section 7 still hold; a rigorous analysis for the system (79), (93) has not been performed, however.

In a given spacetime X (with a metric of class C^2), i.e., for a test gas, Bichteler (1967) has solved the (local) Cauchy problem for (93). Besides existence and uniqueness of exponentially bounded[1],

(1) on the mass-shell, i.e., $|f_A(x,p)| \leqslant b(x)\, e^{\beta_a(x)p^a}$, with b and β_a depending continuously on x.

nonnegative, continuous and a.e. differentiable solutions for given initial distributions of the same type, Bichteler has established the continuous dependence of the solution on the initial distributions, the metric, and the cross section (i.e., $R^{\cdot\cdot}_{\cdot\cdot}$). He assumes throughout that the total cross sections $\int_{\text{(all final states)}} dQ^{\cdot}$ are bounded. (This last assumption, though perhaps valid for strong interactions.[1], does not seem to hold, e.g., in the case of weak interactions.[2] Bichteler obtained his results by applying Banach's fixed point theorem to an operator given naturally by means of (93), defined on a suitably chosen complete metric space of exponentially bounded distribution functions. As Bichteler pointed out, his results lend some credibility to the (formal) Chapman-Enskog approximation which will briefly be discussed later.

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(1) See Eden (1966)

(2) See Bahcall (1964)

9. The second law of thermodynamics (H-theorem).

We define the entropy current density of a gas to be the 4-vector field

$$S^a := -\sum_A \int \left[f_A \log (s_A f_A) \mp f^A \log (s_A f^A) \right] p^a \pi_A \tag{97}$$

with s_A defined as before (below (91)).

The expression (97) can, in a sense, be derived from an information-theoretic point of view as indicated in Ehlers (1969). In the classical limit $s_A f_A \longrightarrow 0$ it reduces to

$$S^a = -\sum_A \left\{ \int p^a f_A \log (s_A f_A) \; \pi_A - N^a_A \right\}, \tag{98}$$

and one recognizes in the first term of S^4 the Boltzmann entropy density. Generally, $-S^a u_a$ is to be interpreted as the entropy density relative to an observer with 4-velocity u^a.

Using Lemma 8 one obtains

$$S^a{}_{;a} = \sum_A \int L_A(f_A) \log\left((s_A f_A)^{-1} \pm 1 \right) \pi_A . \tag{99}$$

inserting $L_A(f_A)$ from the Boltzmann equation (93) and assuming the

PT - symmetry (95) for all collisions involvee one gets a sum of terms; one from each kind of collision and its inverse, of the form

$$\log\left(\frac{f_A f_B \dots f^C}{f^A f^B \dots f_C \dots}\right) (f_A f_B \dots f^C \dots - f^A f^B \dots f_C \dots)\ \delta(\Delta p) \times$$

$$\times R^{C\dots}_{AB\dots}\ \pi_A \wedge \pi_B \dots \wedge \pi_C \dots, \tag{100}$$

where we have again used the notation (94). Each such integral is nonnegative, since its integrand has the form $(\log \frac{a}{b})(a-b)$. Hence,

$$S^a{}_{;a} \geqslant 0. \tag{101}$$

This is the relativistic form of Boltzmann's H-theorem (Tauber-Weinberg (1961), Ehlers (1961), Chernikov (1963), which expresses locally the content of the second law of thermodynamics in the framework of kinetic theory.

(101) implies that the flux of S^a through any closed hyper-surface in X is nonnegative. Hence, for an adiabatically enclosed

or isolated gaseous body the total entropy

$$S[\Sigma] := \int_{\Sigma} S^a \sigma_a, \tag{102}$$

evaluated on a spacelike cross section of the world tube of the body, never decreases towards the future. (Notice that in the classical limit (98), $S[\Sigma]$ consists of the total number of particles and the Boltzmann contribution. If the total particle number is not constant, the Boltzmann S-term alone does not necessarily increase.)

Notice that (101) does not follow from (93) if the collisions are due to PT-violating interactions.

If (95) does hold, and if collisions occur frequently in a gas, then the competition between collisions of a certain kind and their inverses suggests the tendency of the gas to evolve in such a way that the difference in the integrand of (100) tends towards zero, so that ultimately the entropy production density $S^a{}_{;a}$ vanishes and the Liouville equations (41) holds, provided there are no disturbing external influences. Unfortunately, precise theorems supporting this physical expectation are so far missing in relativistic kinetic theory; even at the nonrelativistic level little is known. (For a brief discussion see, e.g., Uhlenbeek and Ford (1963), p. 81.) Any result in this direction would be of interest. It would also be of some interest to know whether in situations of gravitational collapse S may increase towards infinity,

10) Stationary states, equilibrium, and thermostatics

A gas given by g_{ab}, F_{ab}, f_A is said to be in a stationary state in a region $D \subset X$ if there exists, in D, a one-dimensional local group G of fixed-point free local isometries with timelike orbits which leaves F_{ab} and the f_A invariant. In terms of the generating vector field ξ^a of G the last two conditions can be expressed as

$$\mathcal{L}_\xi F_{ab} = 0, \tag{103}$$

$$\left(\xi^a \frac{\partial}{\partial x^a} + \xi^\lambda{}_{,a} p^a \frac{\partial}{\partial p^\lambda} \right) f_A = 0; \tag{104}$$

moreover , we then have Killing's equation

$$\xi_{(a;b)} = 0. \tag{105}$$

The last two equations imply

$$\mathcal{L}_\xi S^a = 0 \tag{106}$$

and similar statements for N^a_A, T^{ab}_A etc. Because of (105) it follows further that

$$(\mathcal{L}_{\xi} S^{a})_{;a} = \mathcal{L}_{\xi}(S^{a}{}_{;a}) = 0, \tag{107}$$

i.e., the entropy production is constant on the G-orbits.

Let us assume now that an adiabatically isolated gas is in a stationary state in D, and that the boundary of the world tube $\mathfrak{J}$ of the gas is G-invariant; $\mathfrak{J} \subset$ D. Let Σ be a spacelike cross section of $\mathfrak{J}$ and a $\in$ G. Then a(Σ) is again such a cross section, and because of the assumed stationarity $S[a(\Sigma)] = S[\Sigma]$. Applying Gauss's theorem to the part of $\mathfrak{J}$ between Σ and a(Σ), using the adiabatic condition along the wall $\partial\mathfrak{J}$, and taking account of (101) we obtain in $\mathfrak{J}$

$$S^{a}{}_{;a} = 0. \tag{108}$$

This conclusion, combined with the expectation described at the end of the previous section, leads us to define:

A gas is in <u>local equilibrium at</u> x $\in$ X if, at x, $S^{a}{}_{;a} = 0$.

The formula (100) for a summand of $S^{a}{}_{;a}$ shows the validity of the first part of the

<u>theoreme</u>. If the collision functions $R^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}$ of a gas are all strictly positive almost everywhere (w.r.t. the measure $\delta(\Delta p)\ \pi_{A} \wedge \ldots$) and continuous, then the gas is in local equilibrium at x if and

only if at x

$$f_A f_B \dots f^C \dots = f^A f^B \dots f_C \dots \tag{109}$$

whenever $\Delta p = 0$, for all types of collisions which occur; or, equivalently, if and only if for each particle species on $P_A(x)$ there holds

$$L_A\,(f_A) = 0. \tag{110}$$

The second part of this theorem follows from the first part by means of equations (93) and (99).

The restriction $R \because > 0$ is not unsatisfactory from the physical point of view, since the R-functions are usually analytic functions of the momentum variables on the "collision fiber" $\Delta p = 0$, and hence they vanish only on sets of measure zero.

The problem of finding the general continuous solutions $(f_A, \dots)$ of (109) has been solved for binary elastic collisions between Boltzmann particles, where (109) reduces to

$$f_A f_B = f'_A f'_B \qquad \text{whenever} \qquad p_A + p_B = p'_A + p'_B. \tag{111}$$

In this case, the general solution is given by

$$f_A(x,p) = a_A(x)\ e^{\beta_a(x)p^a} \tag{112}$$

and a similar formula for f_B and with β_a the same for both species. (Chernikov (1964), Marle (1969) and, in the case where the f_A's are assumed C^1, Bichteler (1965), Boyer (1965). The nicest proof is that of Marle, the shortest that of Bichteler.)

If we consider elastic binary collisions between Bosons or Fermions (or a mixture) and assume that all factors in

$$f_A f_B f'^A f'^B = f'_A f'_B f^A f^B \tag{113}$$

are positive on their mass-shells, we may divide by $f_A f_B f'_A f'_B$ and obtain for $\dfrac{1}{s_A f_A} \pm 1$ etc. the same relation as for Boltzmann particles, so that we obtain

$$f_A(x,p) = \frac{r_A}{(2\pi)^3}\left[e^{-\alpha_A(x) - \beta_a(x)p^a} \mp 1 \right]^{-1} \tag{114}$$

Whereas it is easy to deduce from (111) that $f_A f_B \neq 0$ <u>everywhere</u> provided that holds for <u>some</u> pair $p_A \neq p_B$, this does not seem so obvious in the case (113). Nevertheless I shall accept (114) as the general form of an equilibrium distribution at an event x for particles participating in some kind of binary elastic collision.

If particles in a gas undergo not only binary elastic collisions, but in addition other kinds of reactions, then (114) and (109) show that the α_A must obey

$$\alpha_A + \alpha_B + \ldots = \alpha_C + \ldots \tag{115}$$

for all permissible collisions $A + B + \ldots \rightleftarrows C +$

With (114) and (115) we have obtained the general local equilibrium distributions $(f_A, f_B, \ldots)$.

Since the f_A's have to vanish at infinity on the mass-shells, $\beta^a(x)$ must be a future-directed timelike vector. We put

$$\beta^a = \beta u^a, \quad u_a u^a = -1, \quad \beta > 0. \tag{116}$$

It is a straightforward matter to obtain from (114) the quantities N_A^a, T_A^{ab}, S_A^a, n_A, μ_A, p_A, u_K^a, u_D^a defined in eqs. (45), (47), (97), (63), (64), (49), (52), respectively. Working in the rest frame of u^a one gets

$$u_K^a = u_D^a = u^a, \tag{115}$$

$$S^a_A = s_A u^a, \tag{116}$$

$$T^{ab}_A = (\mu_A + p_A)\, u^a u^b + p_A\, g^{ab}, \tag{117}$$

with the scalars (we omit temporarily the index A) n, μ, p, s given in terms of α, β and the constants m, r by

$$n = \frac{r}{2\pi^2} \int_m^\infty \frac{\sqrt{E^2 - m^2}\, E\, dE}{e^{-\alpha + \beta E} \mp 1} \tag{118}$$

$$\mu = \frac{r}{2\pi^2} \int_m^\infty \frac{\sqrt{E^2 - m^2}\, E^2\, dE}{e^{-\alpha + \beta E} \mp 1} \tag{119}$$

$$p = \frac{r}{6\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{\frac{3}{2}}\, dE}{e^{-\alpha + \beta E} \mp 1} \tag{120}$$

$$s = \frac{r}{2\pi^2} \int_m^\infty \left\{ \frac{-\alpha + \beta E}{e^{-\alpha + \beta E} \mp 1} \mp \log\left(1 \mp e^{\alpha - \beta E}\right) \right\} \sqrt{E^2 - m^2}\; E\, dE. \tag{121}$$

These functions and further thermostatic relations obtained from them have been studied extensively; see, e;g., Landsberg and Dunning-Davies (1965) and the references given there.

The thermostatic meaning of the two parameters α, β is recognized thus: observe that

$$s = -\alpha n + \beta \mu \mp \frac{r}{2\pi^2} \int_m^\infty \log\left(1 \mp e^{\alpha - \beta E}\right) \sqrt{E^2 - m^2}\; E\, dE.$$

Transform the last term by partial integration and get, with (120),

$$s = -\alpha n + \beta\mu + \beta p. \tag{122}$$

Use (120) and compute, again integrating by parts,

$$dp = \frac{n}{\beta} d\alpha - \frac{\mu + p}{\beta} d\beta. \tag{123}$$

(122) and (123) give

$$d\mu = \beta^{-1} ds + \alpha\beta^{-1} dn \tag{124}$$

Now, $\mu(s, n)$ is a thermostatic potential, and $d\mu = T\,ds + \tilde{\mu}dn$, where T is the temperature and $\tilde{\mu}$ is the chemical potential (per particle). Hence we conclude

$$\beta = T^{-1}, \quad \alpha = \frac{\tilde{\mu}}{T}. \tag{125}$$

(125) can now be rewritten in terms of the $\tilde{\mu}_A$'s and reveals itself as the law of mass action.

For the thermodynamics of mixtures see Ehlers (1969), and for applications of the preceding theory to cosmology see Ehlers and Sachs (1968).

Let us now investigate which restrictions are imposed on the parameters α, β and on the mean velocity u^a by the requirement that there is <u>global equilibrium</u>, i.e., that there is local equilibrium at each event of a region $D \subset X$. According to the theorem above, the functions (114) must then obey Lionville's equation; i.e. $L_A(\alpha_A + \beta_a p^a) = 0$ in D. This equation is easily evaluated (see, e.g., Ehlers (1969) and leads to the <u>theorem</u>. Global equilibrium requires that

(a) β^a is a conformal Killing vector and, if at least one component of the gas consists of particles with positive rest masses, a Killing vector, and

(b) the electric field strength $E_a := F_{ab} u^b$ is related to T and α by

$$T\, d\alpha = e\, E. \tag{126}$$

For a gas containing (also) ordinary particles ($m > 0$), equilibrium requires a <u>stationary</u> spacetime. Defining in such a spacetime a scalar gravitational potential U in terms of the Killingvector $\xi^a = T_o \beta^a$ by $e^{2U} = -\xi^2$ we obtain <u>Tolman's law</u>

$$e^U = \frac{T_o}{T}, \tag{127}$$

and if $E = 0$, then α = const., so that $\tilde{\mu}$ depends on the potential like the temperature. (For the general evaluation of (126) see Ehlers (1959).)

It is possible to characterize the global equilibrium solutions in a given, stationary spacetime by means of a variational principle in which S is maximised under certain constraints, see Marle (1969) pp. 107. For examples of equilibrium solutions, see Chernikov (1964).

By means of (42) and (114) it can be verified that Planck's distribution law results for $r_\gamma = 2$, $\alpha_\gamma = 0$, as it should be; $\alpha_\gamma = 0$ results from the relations (115), since there are always some processes which change the photon number but not the numbers of the other particles involved (ex.: e-e collisions).

A gas is said to be nondegenerate if the ∓ 1-term can be neglected without serious error, so that (112) holds. Otherwise, it is called degenerate.

One consequence of the last theorem is that a gas with $m > 0$ cannot maintain an equilibrium distribution if it expands isotropically, in contrast to an $(m = 0)$-gas (photons, neutrinos). A physical reason for this deviation from the nonrelativistic behaviour of a (m 0) gas will be given in the last section.

Since the thermostatic functions of a relativistic gas are explicitely known (cf. eqs. (118)-(121)) one can compute, e.g., the <u>velocity of sound</u> in such a gas, and one can check the validity of <u>Weyl's condition</u> for shock waves. For a Boltzmann gas with m 0 this has been done in detail by Synge (1957), with the result that the sound velocity increases monotonically with the temperature and approaches the limit $\frac{c}{\sqrt{3}}$ as $T \longrightarrow \infty$ (the value for a photon gas); shock speeds are always less than c. Shock waves in a gas of Fermions or Bosons have been investigate by israel (1960).

11. <u>Irreversible processes in small deviations from equilibrium; hydrodynamics</u>.

Whereas the equilibrium solutions of the Boltzmann equation can be written down exactly, there is not much hope to find rigorous solutions describing irreversible ($S^{a}{}_{;a} > 0$) processes-in fact no such (relativistic) solution is known at present. In physics, however, one is mostly interest in non-equilibrium situations. Therefore, in order to proceed one has to resort to <u>approximations.</u> We shall briefly describe such approximation methods in this section, and refer to research papers for details. Our main goal here will be to indicate how one may obtain from kinetic theory a <u>complete system of equations for thermo-hydrodynamics</u> which is sufficiently general to include heterogeneous systems in which transport processes and reactions take place, by applying suitable approximations to the Boltzmann equation. Partly our exposition will be a <u>program</u> rather than the exposition of a completed theory. For simplicity I shall consider here only <u>neutral fluids</u>, thus in the sequel "$e_A = J^a = F^a{}_b = 0$". Also, we shall only consider processes <u>close to equilibrium</u>, which will (for most of the sequel) mean states which are infinitesimal perturbations (first order variations) away from local equilibrium.

Two distributions f_A, $\bar{f}_A$ will describe nearly the same macrostate of a gas if their moments in p-space are everywhere nearly equal. This will be the case if $\bar{f}_A = f_A\,(1 + \varepsilon\,\Phi_A)$ provided Φ_A is a.e. bounded on M_A and the numerical "perturbation" parameter ε is small. With this motivation, we shall now consider a one-parameter family $f_A\,(\varepsilon)$ of states which is, for $\varepsilon = 0$, in local equilibrium, i.e., is such that for $\varepsilon = 0$ the f_A's have the form (114), with unspecified spacetime fields α_A, β_a, and we shall denote by f'_A the variations $\left.\frac{df_A}{d\varepsilon}\right|_{\varepsilon=0}$. Notice that the "local equilibrium functions" α_A, $\beta^a = \frac{U^a}{T}$ are independent of ε. For "small" ε, the moments computed by means of the "perturbed distribution functions" $f_A(0) + \varepsilon\, f'_A$ will be considered to be the macroscopic variables describing a "state close to equilibrium".

It is clear that the perturbed macroscip variables will satisfy the conservation laws

$$T^{ab}{}_{;b} = 0, \quad B^{a}{}_{;a} = 0 \tag{128}$$

and similar ones, if we impose additional "scalar" conservation laws like b-conservation. Also, we shall have the "Clausius inequality"

$$\xi := S^{a}{}_{;a} \geq 0. \tag{129}$$

Again we can write the decomposition (65) for the total, perturbed tensor T^{ab}, with $\mu = \mu(0) + \varepsilon\mu'$, $p = p(0) + \varepsilon p'$, $q^{a} = \varepsilon (q^{a})'$, $\pi^{ab} =$ $= \varepsilon (\pi^{ab})'$, because of (117) for $\varepsilon = 0$.
Similarly,

$$N^{a}_{A} = n_{A} u^{a} + i_{A}{}^{a}, \quad u_{a} i_{A}{}^{a} = 0, \tag{130}$$

and

$$S^{a} = s\, u^{a} + s^{a}, \tag{131}$$

with $n_A = n_A(0) + \varepsilon n'_A$, $i_A{}^a = \varepsilon (i_A{}^a)'$, $s = s(0) + \varepsilon s'$, $s^a = \varepsilon (s^a)'$, from (115) and (116) for $\varepsilon = 0$.

It is a straightforward matter to derive from (128) the <u>energy balance equation</u>

$$\dot{\mu} + (\mu + p)\vartheta + \pi_{ab}\sigma^{ab} + q^a{}_{;a} + q^a \dot{u}_a = 0, \tag{132}$$

where the kinematical quantities ϑ, σ_{ab}, $\dot{u}_a$, ω_{ab} are defined by

$$\begin{gathered} u_{a;b} = \omega_{ab} + \sigma_{ab} + \frac{1}{3}\vartheta h_{ab} - \dot{u}_a u_b, \\ \omega_{ab} u^b = \sigma_{ab} u^b = \omega_{(ab)} = \sigma_{[ab]} = \sigma^a{}_a = 0 \end{gathered} \tag{133}$$

and are interpreted as the rate of rotation (ω_{ab}), rate of shear (σ_{ab}), rate of expansion (ϑ), and 4-acceleration ($\dot{u}_a$) of the flow given by u^a (see, e.g., Ehlers (1961), Synge (1960)). Here and henceforth we write

$$(\ )^{\bullet} := (\ )_{;a} u^a . \tag{134}$$

Also, one obtains the <u>momentum balance equation</u>

$$(\mu+p)\,\dot{u}_a + h^b_a\,(\dot{q}_b + p_{,b} + \pi^c_{b;c}) + (\omega_{ab} + \sigma_{ab})\,q^b + \frac{4}{3}\,\vartheta q_a = 0 \qquad (135)$$

(h_{ab} has been defined in (66).)

Let us now assume that there are Q conserved scalar quantities, like b, which we call c_{qA}, where $1 \leqslant q \leqslant Q \leqslant N$ and where N is the number of species A of particles; the c_{qA} are given, constant "gharges". Then the reactions in the system are restricted by

$$\Big(\sum_A c_{qA}\ N^a_A\Big)_{;a} = 0, \quad 1 \leqslant q \leqslant Q. \qquad (136)$$

We assume the Q "vectors" $(c_{q1}, \ldots, c_{qN})$ to be linearly independent, and denote by $(r_{p1}, \ldots, r_{pN})$, $1 \leqslant p \leqslant R := N - Q$ a basis in the orthogonal space. (1) The vectors $(r_1, \ldots r_N)$ of the latter can be interpreted as (chemical or nuclear, e.g.) reaction coefficients, as is seen from the equations

$$N^a_{A;a} = \sum_p v_p\, r_{pA} \qquad (137)$$

which express the general solution of (136) in terms of the constants r_{pA} and the reaction rates v_p, giving the spacetime densities of reactions of type p.

From (130), (137) we obtain the particle balance equations

(1) I;e., $\sum_A c_{qA}\, r_{pA} = 0$ for $1 \leqslant q \leqslant Q$, $1 \leqslant p \leqslant R$.

$$\dot{n}_A + n_A \vartheta + i_A{}^a{}_{;a} = \sum_p v_p r_{pA} \tag{138}$$

Similarly, we rewrite (129), using (131), as

$$\dot{s} + s\vartheta + s^a{}_{;a} = \xi \geqslant 0. \tag{139}$$

To proceed further we vary the expression (97) for the entropy current density S^a; because of

$$\left[f \log(s\, f) \mp (s^{-1} \pm f) \log (1 \pm sf) \right]' = f' \log(\frac{1}{sf} \pm 1)^{-1}$$

and (114) we obtain

$$T s' = (\mu' - \sum_A \tilde{\mu}_A n'_A) \tag{140}$$

and

$$T s^{a\prime} = (q^{a\prime} - \sum_A \tilde{\mu}_A i_A^{a\prime}), \tag{141}$$

where the $\tilde{\mu}_A$ are the chemical potentials defined in (125).

If we combine (140) with the thermostatic <u>Gibbs relation</u>

$$d\mu = T\, d s + \sum_A \tilde{\mu}_A\, dn_A \tag{142}$$

which results from (124) by summing over the species A, and which holds for the unperturbed equilibrium functions (on the manifold $\{(s, n_1, \ldots, n_A)\}$ of equilibrium states), we get the rather remarkable

Lemma 10. The perturbed thermodynamic variables μ, s, n_A satisfy

$$\mu = F(s, n_1, \ldots, n_N) + 0(\varepsilon^2), \tag{143}$$

where F is the thermostatic potential of the system (as determined from the exact equilibrium relations of section 10).

It is, therefore, "reasonable" to use, for near-equilibrium processes, the ordinary Gibbs equation of state for the perturbed variables, neglecting the error term in (143), as we shall henceforth.

Also, we rewrite (141) for the perturbed variables:

$$T \, s^a = q^a - \sum_A \tilde{\mu}_A \, i_A{}^a. \tag{144}$$

We also recall that, from (122), the thermostatic pressure p_o associated with the perturbed state is

$$p_o = T s - \mu + \sum_A \tilde{\mu}_A \, n_A; \tag{145}$$

there is no reason why p_o should equal the *total kinetic pressure* p in (65).

We are now ready to derive an explicit expression for the entropy production density, ξ, in terms of appropriate thermodynamic and hydrodynamic quantities. Compute $\dot{s}$ from (142) *for the perturbed state*, which is permissible because of Lemma 10; insert $\dot{\mu}$ from (132), $\dot{n}_A$ from (138), and rearrange terms, using (145), (144) and the definition

$$\pi := p - p_o \tag{146}$$

for the *volume viscosity* π, to obtain the *entropy inequality*,

$$- T\xi = \pi_{ab}\,\sigma^{ab} + \pi\vartheta + \Big[q^a - \sum_A \tilde{\mu}_A\, i_A^a\Big]\Big[(\log T)_{,a} + \dot{u}_a\Big] + \sum_A i_A^a\,(\tilde{\mu}_{A,a} + \tilde{\mu}_A \dot{u}_a) + \sum_p \Big\{v_p \sum_A \tilde{\mu}_A r_{Ap}\Big\} \leqq 0. \tag{147}$$

This expression has the usual form known from ordinary irreversible thermodynamics (see, e.g., de Groot and Mazur (1962)); in relativity, it has also been worked out on the basis of phenomenological assumptions by several authors (see, e.g. Stückelberg and Wanders (1953), Kluitenberg, de Groot and Mazur (1953), Kluitenberg and de Groot (1954), (1955).

We wanted to show that (147) and the previous formulae follow, in the sense we have specified, from kinetic theory, just as in the non-relativistic case; this does not seem to have been pointed out before with the generality we have retained here. The crucial fact is that equations (143) and (144) follow from the kinetic expression (97) for the entropy current.

The expression $-T\xi$ as given by (147) is bilinear in "fluxes" $\pi_{ab}, \pi, \dots$ and "forces" $\sigma^{ab}, \vartheta, \dots$. We have shown earlier that the "fluxes" vanish at an event x if there is local equilibrium at x, and that the "forces" vanish in a region if there is (global) equilibrium in that region. Hence, one is driven to conjecture that, in a near-equilibrium process, the fluxes (which are "caused" by the forces) depend linearly and homogeneously on the forces, with coefficients depending on the thermostatic variables s, n_A. This assertion is indeed used as an assumption in phenomenological approaches, and leads to (more or less) well-known relativistic linear transport and reaction equations for the shear viscosity π_{ab}, the volume viscosity π, the heat flow $w^a := q^a - \sum_A \tilde{\mu}_A i_A^a$, the diffusion currents i_A^a, and the reaction rates v_p. The corresponding matrix which transforms $(\sigma_{ab}, \vartheta, \dots)$ into $(\pi_{ab}, \pi, \dots)$ must be positive-semi-definite because of (147) . If one requires, as is natural for a fluid, that this matrix is invariant with respect to rotations (in the 3-

tangent-space orthogonal to u^a), the matrix reduces; and one obtains a further simplification by assuming Onsager-Casimir symmetry. All this follows strictly the standard theory.

However, we should not make these assumptions, but derive them from kinetic theory. This has not yet been done in the generality maintained here, but it will undoubtedly be done soon ([1]). Such a derivation will supposedly) give not only the form of the transport and reaction equations, but will also provide formulae for the transport and reaction coefficients in terms of thermostatic variables and cross sections.

Two classical methods for doing this offer themselves; the Chapman Enskog method, and the Grad method of moments. Both these methods have, in fact, been adapted to relativity; the former by Israel (1963) and, in a mathematically more complete form, by Marle (1969). (Israel, however, gives more detailed results, particularly for a special type of "Maxwellian" gas.) The method of moments has been taken over into relativity by Chernikov (1964) and in a more geometrical (and also analytically more powerful) manner by Marle (see Marle (1966), (1969)) and, independently, by Anderson and Stewart (see Stewart (1969), Anderson (1970). Mathematically, Marle's treatment is the most complete one as regards the discussion of the "relativistic Hermite-Grad polynomials", whereas Anderson and Stewart have

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([1]) (Note added in proof) See a forthcoming paper by J. M. Stewart, to appear in Lecture Notes in Physics, Springer-Verlag.

gone further towards physical applications (transport coefficients from cross sections.) In all of this work, the gas is a simple Boltzmannian one; in that case, both methods give the transport equations for π_{ab}, π and q_a expected on the basis of (147). In particular, Israel (1963) and Anderson and Stewart (1969, 1970) both emphasized that a relativistic gas has (in general) a positive bulk viscosity, in contrast to a non-relativistic gas. The bulk viscosity vanishes both in the nonrelativistic ($T \to 0$) and in the ultrarelativistic ($T \to \infty$) limit. This result "explains" the difference between $m = 0$)-gases and ($m > 0$)-gases with respect to property a) of the theorem in section 10: A gas of the latter type behaves irreversibly if expanding isotropically, because of the term $\pi \vartheta$ in (147); a photon gas, however, behaves reversibly, since $T^a{}_a = 0$ implies that $\pi = 0$ always.

For more details concerning the transition from kinetic theory to thermo-hydrodynamics within the framework of relativity we refer to Chernikov (1964), and to the papers cited above.

The roles of temperature T, entropy S^a (or s) and of the main theorems of thermodynamics are completely clear within the framework of relativistic kinetic theory; there is no room for assumptions. (Of course, this changes if one wants to leave the domain of applicability which we have delineated above.) In particular, integration of (129) over a section of a world tube of streamlines, bounded by two spacelike cross sections Σ_1 and Σ_2, gives with the help of (131) and $s^a = \frac{w^a}{T}$ ($\Longleftrightarrow$(144)):

$$S[\Sigma_2] - S[\Sigma_1] \geqslant \int_{\Lambda} T^{-1} w^a \sigma_a \qquad (148)$$

where Λ is that part of the boundary of the world tube which lies between Σ_1, and Σ_2, and where Σ_2 is assumed to be later than Σ_1. This inequality is a precise version of the somewhat vague assertion $\delta S \geqslant \frac{\delta Q}{T}$ which has first been postulated in general relativistic thermodynamics by Tolman (1934). In a similar fashion one can derive other "global" thermodynamical laws for moving, finite systems enclosed in containers (timelike cylinders in X) from the basic differential relations discussed here; again, there is no ambiguity. (For another example of such a derivation, see Staruszkiewicz (1966).)

Last - but not least - I would like to mention that the long-discussed paradox concerning the acausal nature of temperature propagation (mathematically: the parabolic character of the corresponding system of equations) has been resolved by the observation that the general, "anormal" solutions resulting from the method of moments obey hyperbolic equations with non-spacelike characteristics (Stewart 1969), and only the special, so-called "normal" solutions give rise to the paradox, which is, therefore, due to an inadequate approximation.

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