Effective Hamiltonian constraint from group field theory

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Abstract
Spinfoam models provide a covariant formulation of the dynamics of loop
quantum gravity. They are non-perturbatively defined in the group field theory
(GFT) framework; the GFT partition function defines the sum of spinfoam
transition amplitudes over all possible (discretized) geometries and topologies.
The issue remains, however, of explicitly relating the specific form of the GFT
action and the canonical Hamiltonian constraint. Here, we suggest an avenue
for addressing this issue. Our strategy is to expand GFTs around non-trivial
classical solutions and to interpret the induced quadratic kinematical term as
defining a Hamiltonian constraint on the group field and thus on spin-network
wavefunctions. We apply our procedure to Boulatov GFT for 3D Riemannian
gravity. Finally, we discuss the relevance of understanding the spectrum of this
Hamiltonian operator for the renormalization of GFTs.

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(Some figures may appear in colour only in the online journal)

Introduction

The spinfoam program [1] was originally developed in order to implement the dynamics
of loop quantum gravity [2, 3] in a covariant, sum-over-histories fashion and, moreover, to
compute transition amplitudes between spin-network states of quantum geometry. However,
the spinfoam formalism was later found to have a much greater applicability, being intimately
tied to the quantization of topological field theories (of background field (BF) type) and sitting
very naturally even in a lattice gauge theory context. The basic setting is that the spacetime
structure is described by an abstract cellular 2-complex dressed with algebraic data. These
data are given by representations and intertwiners (invariant tensors) of the gauge group
(usually $SU(2)$ or the Lorentz group spin(3, 1) for quantum gravity). A spinfoam model

defines a probability amplitude for each such discrete spacetime structure. In most models, the cellular complexes are (topologically) dual to spacetime triangulations. Thus, spinfoams can be interpreted as quantized simplicial geometries. From this point of view, spinfoam amplitudes are closely related to discretized general relativity and Regge calculus [1]. In fact, the relation between spinfoam amplitudes and simplicial gravity path integrals has been known from early on [1], and was recently clarified further [9–12]. Finally, the full spinfoam quantum dynamics is defined as the sum over all possible 2-complexes. A non-perturbative definition of this sum is provided by the group field theory (GFT) formulation [4–6].

A GFT is a field theory on a group manifold (the product of a number of copies of the relevant gauge group) and with a peculiar type of non-local interaction. It is a rather rich theory but we shall list here some of its main characteristics. To begin, its Feynman diagrams can be mapped onto 2-complexes and the associated Feynman amplitudes define the spinfoam amplitude of the corresponding 2-complex. Furthermore, the perturbative expansion of the GFT partition function defines the sum over all (admissible) 2-complexes weighted with the spinfoam amplitude in question. From this perspective, GFT can be considered as a generalized matrix/tensor model [7, 8], which generates triangulations of spacetime as Feynman diagrams (see the cited literature for further details). Nevertheless, the particularity of GFT is that it is a field theory in its own right, and we can use standard field theory techniques to investigate and analyze its properties and its quantization (in particular, tools from perturbative renormalization [26, 14] of quantum field theory, QFT). In fact, whatever its historic origin, one may take GFT as an independent arena for research in its own right. It is in this vein that a large portion of research on the subject is done nowadays [14]. Of course, the original motivations still stand and, with this in mind, we shall attempt here to refocus on one of the initial goals: to provide a consistent quantum dynamics for canonical loop quantum gravity states.

Indeed, we propose to interpret the kinetic term of the GFT as a Hamiltonian constraint acting on the group field and more generally on spin-network states. Of course, this is not the kinetic term related to the trivial solution of the GFT but rather that related to the effective dynamics around a non-trivial background solution. The rationale for this proposal lies in the fact that GFTs can be interpreted as a second quantized approach to spin-network dynamics, and therefore, in the spirit of 'third quantizations' of gravity [5, 6, 15]. As such, one expects the dynamics of the first quantized theory, here canonical loop quantum gravity, to be encoded in the classical action of the second quantized theory, here GFT. Moreover, in such a framework, one expects that both the geometry *and* topology of spacetime to be fully dynamical.

GFTs fulfil these expectations. In their perturbative expansion, they incorporate both a sum over all 2-complexes/triangulations of given topology and a sum over all topologies. However, they do so in a peculiar way. In the GFT action, a trivial kinetic term is usually chosen, so the whole quantum dynamics of both geometry (Hamiltonian constraint) and topology are encoded in the interaction term of the theory. Treating this interaction term as an operator on spin-network states, and allowing it to act repeatedly on them, generates both a graph-changing evolution of geometry and a change in the underlying topology of space.

We would like to separate geometrical evolution from topology change. More precisely, we wish to reformulate the action so that it contains a non-trivial kinetic term that could be held responsible for the dynamics of geometry, leaving the change of topology confined to the GFT interaction. Recall also that, from the point of view of spin-network states, the geometrical dynamics can be defined either in terms of graph-changing or non-graph-changing Hamiltonian operators [2]. Obviously, one should not neglect the possibility that the interaction term provides additional contributions to the dynamics of geometry (see [15] for a discussion of this issue). We achieve this proposal here for the effective GFT dynamics. Moreover, we see that the effective Hamiltonian constraint is of a non-graph-changing type.

The effective GFT vacuum is a non-perturbative configuration, from the point of view of standard GFT. It encodes infinitely many fundamental excitations around the $\phi = 0$ configuration. Thus, in principle, it is a manifestation of a continuum space comprised of an infinity of fundamental simplices. This is in contrast to the $\phi = 0$ vacuum, which is a state corresponding to 'no space'. It is only after we possess this continuum picture that we expect the dynamics of fluctuations to reproduce the quantum dynamics of continuum geometry.

Let us also mention that the exploration of the non-perturbative sector of GFT models, and of their effective dynamics around background configurations, has already proceeded along different directions, recently [13, 18–22]. In some works [13, 18–21] the idea being investigated was that some simplified GFT perturbations around classical solutions could be interpreted as emergent (non-commutative) matter fields. Another possibility being explored [22] was to obtain effective equations for geometry from the GFT equations of motion, as conditions for a given background configuration to define a solution of the same, in the spirit of mean field theory in Bose condensates. Here we explore the other logical possibility that the dynamics of geometry should be looked for in the effective dynamics of generic perturbations around background solutions.

There are two main ingredients to our proposal.

First, we focus on the free GFT defined by the quadratic part of the GFT action. The equation of motion of this free GFT is a linear equation of the type $\hat{H}\phi = 0$, where ϕ is the field and $\hat{H}$ can be interpreted as a Hamiltonian constraint. However, the standard formulation of GFTs uses a trivial kinetic term and is of the type

$$S_{\text{GFT}}[\phi] = \frac{1}{2}\int \phi^2 - \lambda \int \mathcal{V}[\phi], \tag{1}$$

where $\mathcal{V}[\phi]$ defines the interaction term. Obviously, the free GFT defined as such is trivial. Our strategy here is to follow the procedure first used in [13]. We can expand the GFT around $\phi = \phi_0$, a non-trivial classical solution to its full equation of motion:

$$\phi_0 = \lambda \frac{\delta \mathcal{V}}{\delta \phi}[\phi_0]. \tag{2}$$

This classical solution defines a background structure for the GFT[3], and we can define an (effective) action describing the field variations around ϕ_0 (instead of describing its variations around the 'no-space' state $\phi = 0$):

$$S_{\phi_0}[\phi] \equiv S_{\text{GFT}}[\phi_0 + \phi] - S_{\text{GFT}}[\phi_0] = \frac{1}{2}\int \phi \hat{H}_{\phi_0}\phi + \cdots. \tag{3}$$

The kinetic term is then non-trivial and provides us with a tentative Hamiltonian constraint for our proposal. This operator $\hat{H}_{\phi_0}$ depends on the background structure defined by the field ϕ_0, which encodes some non-trivial dynamical information, since its definition involves the fundamental interaction term $\mathcal{V}$.

The second ingredient is to view spin-network functionals ψ as multi-particle states of the GFT. They are indeed constructed as the group-averaged tensor product of group fields, as we will explain in more detail:

$$\psi \sim \phi \otimes \cdots \otimes \phi.$$

From this point of view, the Hilbert space of spin networks can be seen as a Fock space of the quantized GFT. Then the linear operator $\hat{H}_{\phi_0}$ acts on states ψ and we can investigate its spectrum on spin-network states.

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[3] Note that generically these solutions are purely non-perturbative configurations as is testified by their dependence on the GFT coupling constant.

Finally, we have underlined the interpretation of the free GFT and the role of the kinetic term $\hat{H}_{\phi_0}$ as defining a constraint operator acting on the group field ϕ and spin-network functionals ψ. However, more generally, $\hat{H}_{\phi_0}$ defines the (inverse of the) propagator for the point of view of QFT and it is crucial to understand its properties and spectrum for the computation of the GFT correlations (which define the spinfoam transition amplitudes) and the renormalization of the GFT (which reflects the coarse-graining of spinfoam models). In fact, as natural in field theory context, and in third quantized gravity [15], the presence of interactions will necessarily involve excitations of quantum geometry outside the space of solutions of the Hamiltonian constraint ('virtual', 'off-shell' geometries akin to virtual particles in ordinary field theory), and their understanding requires then a control over the full spectrum of the Hamiltonian constraint.

The present paper consists of two parts. The first section will review the basics of the GFT formalism. We shall introduce its expansion around non-trivial classical solutions and discuss how to define spin-network functionals as multi-particle states. In the second section, we shall explicitly apply our program to Boulatov's GFT for the Ponzano–Regge spinfoam model of 3D quantum gravity. We expand it around the flat solutions introduced in [13] and analyze the spectrum of the induced Hamiltonian constraint. We compare it to the known Hamiltonian constraint of topological BF theory, and we will see that it can be interpreted as a Klein–Gordon-like operator with a spectrum of the type '$p^2 - m^2$'. Finally, we shall discuss the relevance of our procedure to the study of GFTs and their interpretation as quantum gravity models.

1. Group field theory for spinfoams

1.1. Generating the spinfoam partition function

A GFT is defined by the choice of a gauge group $\mathcal{G}$ and an action of the form

$$S_\lambda[\phi] = \frac{1}{2} \int [\mathrm{d}g_a][\mathrm{d}\tilde{g}_a] \, \phi^\sigma(g_a) \, \mathcal{K}(g_a, \tilde{g}_a) \, \phi^{\tilde{\sigma}}(\tilde{g}_a) - \lambda \int \Big(\prod_{a=1}^{m} [\mathrm{d}g_{ab}] \, \phi^{\sigma_a}(g_{ab}) \Big) \mathcal{V}(\{g_{ab}\}),$$

(4)

where ϕ is a real (or complex)-valued function acting on n copies of the group manifold $\mathcal{G}$:

$$\phi : \mathcal{G}^{\otimes n} \to \mathbb{R} \text{ (or } \mathbb{C}); \qquad (g_1, \ldots, g_n) \to \phi(g_1, \ldots, g_n) =: \phi(g_a). \qquad (5)$$

The label $\sigma \in S_n$ denotes the action of the permutation group on the arguments of the field:

$$\phi^\sigma(g_a) := \phi(g_{\sigma(a)}). \qquad (6)$$

Also, let us remark that there is no summation over permutations here or elsewhere in the paper. All permutations, when stated, are fixed. Thus, there is no permutation symmetry present. Moreover, $[\mathrm{d}g_a]$ is shorthand for the normalized measure on $\mathcal{G}^{\otimes n}$, while $\mathcal{K}$ and $\mathcal{V}$ are the kinetic and potential operators, respectively. We further require the invariance of the field under the following action of the gauge group:

$$\phi(g_a) \to \phi(g_a g) = \phi(g_a), \qquad \forall g \in \mathcal{G}. \qquad (7)$$

One can realize this symmetry explicitly in a neat fashion by a simple group averaging:

$$\phi(g_a) := \int_{\mathcal{G}} \mathrm{d}g \, \varphi(g_a g), \qquad (8)$$

where φ is an auxiliary non-invariant field.

We define the partition function based on this action in the straightforward way[4]:

$$\mathcal{Z} = \int \mathcal{D}\phi \, e^{-\mathcal{S}_\lambda[\phi]} = \sum_N \sum_{\Delta_N} \frac{\lambda^N}{\text{sym}[\Delta_N]} \mathcal{Z}[\Delta_N], \qquad (9)$$

where in the second equality we have performed a perturbative expansion in λ. In that case, N is the order in λ, Δ_N denotes the Feynman diagrams with N vertices, while sym$[\Delta_N]$ is the symmetry factor associated with Δ_N.

The Feynman diagrams are identified as 2-complexes defining spacetime triangulations (or more generally cellular decompositions). As in matrix models, the interaction term generates the fundamental building blocks of the discrete manifold, while the propagator glues them together along their boundary. Then, for a given Feynman diagram Δ, its evaluation $\mathcal{Z}[\Delta]$ defines the spinfoam amplitude associated with the corresponding triangulation.

It is illuminating to illustrate these general concepts in terms of a specific model: the GFT for topological BF theory in *n*-dimensions. This is highly relevant to the spinfoam program because topological BF theory is the starting point of the whole construction of spinfoam models. Indeed, the spinfoam models for BF theory are the only ones which have been shown to provide a consistent and correct quantization of the theory. Moreover, 3D gravity is exactly a topological BF theory, while 4D general relativity can be formulated as a BF theory with non-trivial potential.

Thus, for BF theory, the interaction term generates *n*-simplices, while the propagator glues them together along the shared $(n-1)$-simplices. The fundamental operators are

$$\mathcal{K}(g_a, \tilde{g}_a) = \int dg \prod_{a=1}^n \delta(\tilde{g}_{\tilde{i}(a)}^{-1} g_a g),$$

$$\mathcal{V}(\{g_{ab}\}) = \int [dg_a] \prod_{a=1}^n \prod_{b:b>a} \delta(g_{ba}^{-1} g_b g_a^{-1} g_{ab}). \qquad (10)$$

These choices for $\mathcal{K}$ and $\mathcal{V}$ are usually referred as 'trivial' in the GFT framework. For instance, $\mathcal{K}$ is simply the projector on gauge-invariant fields, that is, the identity on the space of gauge-invariant fields (7). This choice is appropriate for topological models (of BF type), and it is a widely used one also in non-topological models based on constraining/deforming topological ones. The interaction term $\mathcal{V}$ here simply identifies the group elements g_{ab} and g_{ba} up to gauge transformations. In this sense, we can call it trivial. The standard prescription for GFTs is to keep a trivial propagator $\mathcal{K}$, while encoding all the non-trivial information and dynamics in the interaction vertex $\mathcal{V}$.

With these definitions, the propagator is $\mathcal{P} := \mathcal{K}^{-1} = \mathcal{K}$ and the amplitude for a specific Feynman graph is

$$\mathcal{Z}[\Delta] = \int [dg] \prod_e \mathcal{P} \prod_v \mathcal{V} = \int \prod_e dg_e \prod_l \delta(G_l). \qquad (11)$$

In the final expression, l denotes the loops in the Feynman graph, which are dual in the topological sense to the $(n-2)$-simplices of the discrete manifold. Moreover, $G_l \equiv \prod_{e \in l} g_e^{\epsilon(e,l)}$, where $\epsilon(e, l) = \pm 1$ depends on the relative orientation of e and l. In the end, we recognize the discretized quantum amplitude for BF theory, in its lattice gauge theory formulation, as expected.

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[4] Here we have taken e^{-S} in the path integral as in statistical physics, but we can also define the partition function with e^{iS} which would truly quantize the GFT and which would be more natural from a 'third quantization' point of view. Using one or the other depends on the purpose of the partition function. Although this is an important question for the interpretation of GFT in general, this issue is not relevant to the discussion in this paper.

So, in this BF case, the free theory has trivial dynamics and it does not contain any interesting information on the behavior of the full theory. In particular, the only classical solution is $\phi = 0$, i.e. the 'no-spacetime' configuration. As noted earlier, the entire non-trivial dynamics of the theory lies in the GFT interaction term, that is, it imposes the (Hamiltonian) flatness constraint (as a graph-changing operator) on geometry and at the same time governs topology changing processes. For this reason, we shall study the variations of the GFT field around non-trivial classical solutions of the full GFT. We shall see that the effective kinetic term defining the new free theory actually carries non-trivial information about the GFT dynamics; moreover, it defines a graph-preserving, and thus topology-preserving, quantum dynamics for geometry, leaving graph-changing and topology-changing processes to be generated by the (effective) interaction term.

1.2. Spin network observables

For field theories in general, physical observables are deemed to be functions of the fields that are invariant under the (gauge) symmetries of the theory. In the class of GFT models dealt with above, the fundamental field is a scalar ϕ, which is invariant under the action of the symmetry (rather then covariantly transforming with respect to it). Thus rather arbitrary functions of the field suffice to encode acceptable physical observables.

Consider an arbitrary product of V fields:

$$\psi_{\{\sigma_v\}}^{\{\phi_v\}}(g_{vw}) = \prod_v^V \phi_v^{\sigma_v}(g_{vw}). \tag{12}$$

This tensor product $\psi^{\{\phi_v\}} = \bigotimes_v \phi_v$ represents an arbitrary multi-particle state for the GFT. There is a neat graphical interpretation of such an entity: a vertex v describes the field, while the edges vw incident at that vertex denote the arguments of this field. Thus, one views each $\phi_v^\sigma(g_1, \ldots, g_n)$ as an n-valent vertex. The element $\sigma \in S_n$ defines an ordering of the edges around the vertex (as when projected onto a plane). Moreover, there is no coupling among the fields; thus, there is no sense in which these vertices are connected to each other in any manner. The index w merely denotes the open end of the edge vw.

Then, from the quantum gravity viewpoint, a particularly interesting subclass of observables are those that can be labeled by connected graphs Γ:

$$\psi_{\Gamma,\{\sigma_v\}}^{\{\phi_v\}}(G_e) = \int [\mathrm{d}g_{vw}] \prod_v \phi_v^{\sigma_v}(g_{vw}) \prod_e \delta(g_{t(e)i(e)}^{-1} g_{s(e)i(e)} G_e^{-1}). \tag{13}$$

We note that the first product is exactly that occurring in the observable $\psi^{\{\phi_v\}}$ above. The second product serves to couple these fields, in effect by gluing pairs of edges at their *a priori* free endpoints. More precisely, $s(e)$ and $t(e)$ denote the source and target vertices of the edge e in the graph Γ. The index $i(e)$ is the intermediate index of the group elements at the vertices $s(e)$ and $t(e)$ which allows us to glue the group elements along the edge e. It can be thought of as an intermediate vertex along the edge e.[5] This can be seen in figure 1. Ultimately, in the language of quantum gravity, the coupling term imposes that the holonomies along the two segments of the edge compose (under group multiplication) to form a holonomy G_e for the whole edge.

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[5] One can also define a slightly different but equivalent gluing procedure, giving the same spin-network functionals from products of GFT fields. Instead of inserting a delta function per edge constraining the arguments of the GFT fields, one can impose, by projection, an extra 'gluing symmetry' by considering only those products of fields whose arguments referring to the (would be) same edge e of the closed spin-network graph are invariant under translations by the same group element $h_{i(e)}$.
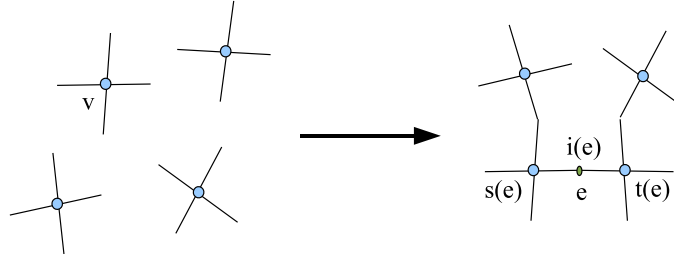
Figure 1. The functional $\psi^{\{\phi_v\}}$ consisting of group fields φ_v at unrelated vertices v and the functional $\psi_\Gamma^{\{\phi_v\}}$ constructed by gluing these vertices along the edges of a graph Γ. For a given edge e, the two group field living at the source vertex $s(e)$ and target vertex $t(e)$ are glued by an intermediate (fiducial) vertex $i(e)$.

From the form of the observable, the symmetry of the field ϕ under the diagonal action of the group ensures the following symmetry for $\psi_{\Gamma,\phi}$:

$$\psi_\Gamma^{\{\phi_v\}}(G_e) = \psi_\Gamma^{\{\phi_v\}}\big(h_{s(e)}G_e h_{t(e)}^{-1}\big), \tag{14}$$

where we have removed the subscripts $\{\sigma_v\}$ to lighten the notations. Reversely, any function satisfying this gauge invariance can be written as a multi-particle states (13) of the GFT.

This symmetry is very familiar from the spin-network observables arising in the spinfoam approach. In that context, one has gauge invariant functions of the connection with support on graphs, that is, functions of the form $\psi_\Gamma(G_e)$ with the same symmetry (14). Here, one has gauge invariant *functionals* of the connection with support on graphs. What is even more appealing is that in the quantum gravity setting, these spin-network functions form a basis for the kinematical state space. Indeed, the GFT functionals can be expanded in terms of these spin-network functions. In other words, the GFT observables may be viewed as functionals of these states, i.e. functionals of the same wavefunctions defining quantum states of geometry in canonical loop quantum gravity. This is in agreement with the interpretation of GFTs as second quantizations of canonical loop quantum gravity [5, 6, 15].

Following this, we can introduce a natural set of observables for the GFT:

$$\psi_\Gamma[\phi](G_e) = \int [\mathrm{d}g_{vw}] \prod_v \phi^{\sigma_v}(g_{vw}) \prod_e \delta\big(g_{t(e)i(e)}^{-1} g_{s(e)i(e)} G_e^{-1}\big), \tag{15}$$

which is a polynomial function of the group field ϕ.

These observables have a natural field theoretic interpretation, namely that

$$\langle \psi_\Gamma[\phi](G_e) \rangle = \frac{1}{Z} \int \mathcal{D}\phi \; \psi_\Gamma[\phi](G_e) \; \mathrm{e}^{-S_\lambda[\phi]},$$

$$\big\langle \psi_{\Gamma^1}[\phi]\big(G_e^1\big) \; \psi_{\Gamma^2}[\phi]\big(G_e^2\big) \big\rangle = \frac{1}{Z} \int \mathcal{D}\phi \; \psi_{\Gamma^1}[\phi]\big(G_e^1\big) \; \psi_{\Gamma^2}[\phi]\big(G_e^2\big) \; \mathrm{e}^{-S_\lambda[\phi]}, \qquad \text{etc.} \tag{16}$$

define the probability amplitude for the boundary state $\psi_\Gamma(G_e)$ and the transition amplitude between two spin-network states $\psi_{\Gamma^1}(G_e^1)$ and $\psi_{\Gamma^2}(G_e^2)$, respectively. Indeed, if we expand these correlations perturbatively in the coupling λ, we recover the standard sum over all spinfoam structures compatible with the boundary graph(s) (see e.g. [4, 16] for more details). Here, we parameterize the boundary data with a graph Γ and group elements G_e (up to gauge transformations) on the graph edges. If we want to go to the standard spin-network basis, we

just have to carry out harmonic analysis on the group $\mathcal{G}$ and go to boundary data labeled by representations and intertwiner states[6].

Finally, we note that all the terms in the GFT action are given as spin-network observables of the above type. For the GFT formulation of BF theory, the kinetic term corresponds to the Θ-graph (made up of two vertices), while the potential corresponds to an n-vertex graph (corresponding to an n-simplex). More precisely, the kinetic and potential terms are given by the evaluation of the corresponding spin-network functionals at the identity $G_e = \mathbb{I}$. Under renormalization, we may expect effective terms of the type $\psi_\Gamma[\phi](G_e)$ for other graphs Γ to enter the effective GFT action as quantum corrections (or counter-terms). Of course, we expect terms given by the evaluation at the identity $G_e = \mathbb{I}$ as before, but effective terms with evaluations on more general group elements or derivative terms would be probably a sign of non-trivial curvature corrections.

1.3. Non-trivial backgrounds and effective action

As mentioned earlier, the standard GFT action is usually prescribed with a trivial kinetic operator and hence propagator. It does not contain any derivative terms and is a simple mass term. The induced equation of motion is

$$\frac{\delta \mathcal{S}_\lambda[\phi]}{\delta \phi(g_a)} = \phi(g_a) - \lambda \int [\mathrm{d}g_{ab}] \frac{\delta}{\delta \phi(g_a)} \Big(\prod_{a=1}^{m} \phi^{\sigma_a}(g_{ab}) \Big) \mathcal{V}(\{g_{ab}\}) = 0. \tag{17}$$

The trivial classical solution is obviously $\phi = 0$. This is to be compared to the equation of motion of the *free* theory defined by solely considering the kinetic term (and discarding the interaction term)

$$\frac{\delta \mathcal{S}^{\mathrm{free}}[\phi]}{\delta \phi(g_a)} = \phi(g_a) = 0. \tag{18}$$

Thus, for the free theory, the *only* classical solution is $\phi = 0$.

To go further, it is rather convenient to rescale the field so that λ disappears from the equations of motion: $\phi \to \lambda^{-\frac{1}{m-2}} \phi$. The action, under this redefinition, transforms as

$$\mathcal{S}_\lambda[\phi] \to \mathcal{S}_\lambda[\lambda^{-\frac{1}{m-2}} \phi] = \lambda^{-\frac{2}{m-2}} \mathcal{S}[\phi]. \tag{19}$$

To maintain a certain level of generality, let us assume that we have some non-trivial solution: $\phi = \phi_0$ such that $\frac{\delta \mathcal{S}}{\delta \phi}\big|_{\phi_0} = 0$. Then one may rewrite any field configuration as $\phi = \phi_0 + \varphi$. Now we can substitute this decomposition into the action to obtain (schematically)

$$\lambda^{-\frac{2}{m-2}} \mathcal{S}[\phi] = \lambda^{-\frac{2}{m-2}} \left[\mathcal{S}[\phi_0] + \frac{\delta \mathcal{S}}{\delta \phi}\bigg|_{\phi_0} \varphi + \frac{1}{2} \frac{\delta^2 \mathcal{S}}{\delta \phi^2}\bigg|_{\phi_0} \varphi^2 + \sum_{a=3}^{m} \frac{1}{m!} \frac{\delta^a \mathcal{S}}{\delta \phi^a}\bigg|_{\phi_0} \varphi^a \right]. \tag{20}$$

Naturally, the $\mathcal{S}[\phi_0]$ term may be dropped since it is constant and does not affect the classical dynamics. What is more, in the quantum theory, it cancels the evaluation of normalized correlation functions. The second term is zero since ϕ_0 is a classical solution. So, it is the third term and onward that contain the effective dynamics of the field φ:

$$\mathcal{S}_{\phi_0}[\varphi] := \lambda^{-\frac{2}{m-2}} \left[\frac{1}{2} \frac{\delta^2 \mathcal{S}}{\delta \phi^2}\bigg|_{\phi_0} \varphi^2 + \sum_{a=3}^{m} \frac{1}{m!} \frac{\delta^a \mathcal{S}}{\delta \phi^a}\bigg|_{\phi_0} \varphi^a \right]. \tag{21}$$

This is just a simple recasting of the theory in terms of different variables, i.e. we are not changing the non-perturbative dynamics of the model. However, the power and motivation of

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[6] One can also use a spinorial representation of the same functions [28], or go to a triad (flux) representation [29], using the non-commutative group Fourier transform [23, 24]

the approach, it amounts to considering, in perturbation theory, the dynamics around a new, non-trivial phase of the theory, and one that can be reached only non-perturbatively, from the point of view of the 'no-space' vacuum $\phi = 0$. This simply follows from the fact that the field φ is a perturbation around the classical background solution ϕ_0.

Let us start with analyzing at the free theory. Looking at just the quadratic term, one notes immediately that it now contains a non-trivial kinetic operator. Thus, now even in the free theory, one has non-trivial propagation. Thus, in analogy with other field theories it is tempting to consider it as the Hamiltonian operator for the corresponding single particle theory or, in the context of gravity, as the Hamiltonian constraint operator:

$$\mathcal{S}^{\text{free}}_{\phi_0}[\varphi] = \int \varphi \, \hat{\mathcal{H}} \, \varphi, \qquad \text{where} \qquad \hat{\mathcal{H}} = \left.\frac{\delta^2 \mathcal{S}}{\delta \phi^2}\right|_{\phi_0}. \tag{22}$$

This operator $\hat{\mathcal{H}}$ not only contains the trivial contribution coming from the original kinetic term, but more importantly it also contains a non-trivial contribution coming from the original interaction term of the GFT action. Thus it carries non-trivial information about the *full* dynamics of the theory. The equation of motion of this free theory is now simply $\hat{\mathcal{H}} \varphi = 0$ and, thus, at this level, we are interested in the zero modes in the spectrum of $\hat{\mathcal{H}}$.

One can also go further and let $\hat{\mathcal{H}}$ act on the multi-particle states of the theory, that is, on the spin-network states:

$$\hat{\mathcal{H}} \rhd \psi^{\{\phi_v\}}_{\Gamma,\{\sigma_v\}}(G_e) = \int [\mathrm{d}g_{vw}] \sum_a \left[\hat{\mathcal{H}} \rhd \varphi^{\sigma_a}_a(g_{aw}) \prod_{v:v\neq a} \varphi^{\sigma_v}_v(g_{vw}) \right]$$
$$\times \prod_e \delta\big(g^{-1}_{t(e)i(e)} g_{s(e)i(e)} G^{-1}_e\big). \tag{23}$$

Still looking for the zero modes of this operator, this means that we have defined a constraint operator acting on spin-network states. This is our proposal to define a tentative Hamiltonian constraint for LQG's spin networks from GFT. In order to assert the physical relevance of our procedure, we would have to test it on some specific spinfoam model(s). This is what we will do in the following section, where we will apply it to the GFT for 3D quantum gravity.

Let us stress that this procedure for extracting an effective Hamiltonian constraint for spin-network states has one obvious advantage and one possible limitation. The advantage is that it allows us to separate clearly and rather easily the contribution to the quantum dynamics coming from topology change (encoded in the GFT interaction) from the purely geometric ones encoded in the GFT kinetic term. The limitation that is intrinsic to our procedure is that it can only result in non-graph-changing Hamiltonian constraints, if these are to be read out of the GFT kinetic term. In the canonical quantum gravity literature, the question of whether the quantum Hamiltonian constraint acting on spin-network states should be graph changing or not does not have a unique and definite answer [2, 3], and different definitions have been proposed and explored. Therefore, it remains unclear, at present, whether the formal limitation of our procedure corresponds to a physical one or not. It remains true, as we stressed already, that further contributions to the dynamics of quantum geometry are also encoded in the effective GFT interaction, so these will also have to be studied.

Next, we need to investigate the full theory, beyond its kinetic term. From this more general perspective, $\hat{\mathcal{H}}^{-1}$ defines the propagator of the GFT, which enters the evaluation of the Feynman diagrams for the perturbative expansion of the path integral of the theory. Thus, we are interested in the interacting theory in the complete spectrum of $\hat{\mathcal{H}}$, as we expect states to be allowed to go 'off-shell' with respect to the constraint. Moreover, as we know from standard QFT, it is essential to know the full spectrum of the propagator in order to study the properties of the field theory and its renormalization.

2. The 3D GFT formulation of topological BF theory

2.1. The Boulatov model and flat Solutions

Now, it is time to specialize to an explicit example to see how our strategy plays out. We shall apply it to Boulatov's GFT for quantum BF theory in three dimensions with the gauge group $SU(2)$ [17] (equivalently, 3D Riemannian quantum gravity), whose Feynman amplitudes give the spinfoam amplitudes of the Ponzano–Regge model.

We choose a compact semi-simple Lie group $\mathcal{G}$ and consider invariant fields on $\mathcal{G}^{\times 3}$:

$$\phi(g_1, g_2, g_3) = \phi(g_1 g, g_2 g, g_3 g), \qquad \forall g \in \mathcal{G}.$$

Explicitly, the action for Boulatov's GFT is

$$\mathcal{S}_\lambda[\phi] = \frac{1}{2} \int [\mathrm{d}g] \, \phi(g_1, g_2, g_3)\phi(g_3, g_2, g_1) - \frac{\lambda}{4!} \int [\mathrm{d}g] \, \phi(g_1, g_2, g_3)\phi(g_3, g_5, g_4)$$
$$\times \phi(g_4, g_2, g_6)\phi(g_6, g_5, g_1). \tag{24}$$

The equations of motion $\frac{\delta \mathcal{S}}{\delta \phi} = 0$ take the form

$$\phi(g_3, g_2, g_1) - \frac{\lambda}{3!} \int [\mathrm{d}g] \phi(g_3, g_5, g_4)\phi(g_4, g_2, g_6)\phi(g_6, g_5, g_1) = 0. \tag{25}$$

To this equation, there exists a family of classical solutions labeled by functions $f : \mathcal{G} \rightarrow \mathbb{R}$ identified in [13], namely

$$\phi_f(g_1, g_2, g_3) = \sqrt{\frac{3!}{\lambda}} \int \mathrm{d}h \, \delta(g_1 h) f(g_2 h) \delta(g_3 h), \qquad \text{provided} \qquad \int \mathrm{d}g \, f(g)^2 = 1. \tag{26}$$

There exists of course other classical solutions (see e.g. [25]) as well as approximate solutions [22], but we will focus on the family of solutions defined above and referred to as 'flat solutions'.

Let us point out that these flat solutions are not normalizable in the sense that the evaluation of the action on them goes as $\delta(\mathbb{I})$ and is thus infinite. From this point of view, defining the effective action $\mathcal{S}_{\phi_f}$ involves subtracting the infinite constant term $\mathcal{S}[\phi_f]$ [13]. This trivial divergence is due to the translational symmetry satisfied by the flat solutions ϕ_f [25] and it will not play any role in the procedure and results presented here.

With these solutions at our disposal, we may perturb around them as specified earlier using $\phi = \phi_f + \varphi$. The effective action for the field φ is

$$\mathcal{S}_{\phi_f}[\varphi] = \frac{1}{2} \int [\mathrm{d}g]^3 \varphi(g_1, g_2, g_3)\varphi(g_3, g_2, g_1)$$
$$- \int [\mathrm{d}g]^4 f(g_2 g_1^{-1}) f(g_5 g_1^{-1}) \varphi(g_1, g_2, g_6)\varphi(g_6, g_5, g_1)$$
$$- \frac{1}{2} \int [\mathrm{d}g]^4 f(g_2 g_1^{-1}) \varphi(g_1, g_5, g_4) f(g_2 g_4^{-1}) \varphi(g_6, g_5, g_1)$$
$$- \frac{\sqrt{\lambda}}{\sqrt{3!}} \int [\mathrm{d}g]^5 f(g_2 g_3^{-1}) \varphi(g_3, g_5, g_4)\varphi(g_4, g_2, g_6)\varphi(g_6, g_5, g_3)$$
$$- \frac{\lambda}{4!} \int [\mathrm{d}g]^6 \varphi(g_1, g_2, g_3)\varphi(g_3, g_5, g_4)\varphi(g_4, g_2, g_6)\varphi(g_6, g_5, g_1), \tag{27}$$

where the corrections to the kinetic term and the new cubic interaction come from the original interaction vertex of the GFT. Focusing on the kinetic term, we see that the free theory does not depend on the coupling λ at all and is given by the quadratic action

$$\mathcal{S}_{\phi_f}^{\text{free}}[\varphi] = \frac{1}{2} \int [\mathrm{d}g] \, \varphi(g_1, g_2, g_3)\mathcal{H}(g_1, g_2, g_3; \tilde{g}_1, \tilde{g}_2, \tilde{g}_3)\varphi(\tilde{g}_1, \tilde{g}_2, \tilde{g}_3), \tag{28}$$

with the kinetic operator

$$
\mathcal{H}(g_a; \tilde{g}_b) = \delta\big(g_3 \tilde{g}_1^{-1}\big) \left[\delta\big(g_2 \tilde{g}_2^{-1}\big) - \delta\big(g_2 \tilde{g}_2^{-1}\big) \int \mathrm{d}h f\big(hg_1^{-1}\big) f\big(hg_3^{-1}\big) - 2 f\big(g_2 g_1^{-1}\big) f\big(\tilde{g}_2 g_1^{-1}\big) \right]
$$
$$
\times \, \delta\big(g_1 \tilde{g}_3^{-1}\big). \tag{29}
$$

2.2. The spectrum of the propagator

We define the operator $\hat{\mathcal{H}}$ acting on invariant fields following the formula for the free theory given above:

$$
\hat{\mathcal{H}}\varphi(g_1, g_2, g_3) = \varphi(g_1, g_2, g_3) \left(1 - \int \mathrm{d}h f\big(hg_1^{-1}\big) f\big(hg_3^{-1}\big) \right) \tag{30}
$$
$$
- 2 f\big(g_2 g_3^{-1}\big) \int \mathrm{d}\tilde{g}_2 \varphi(g_1, \tilde{g}_2, g_3) f\big(\tilde{g}_2 g_3^{-1}\big).
$$

With the standard scalar product, $\langle \varphi | \tilde{\varphi} \rangle = \int [\mathrm{d}g]^3 \overline{\varphi}(g_a) \, \widetilde{\varphi}(g_a)$, it is straightforward to check that this operator is Hermitian (since f is a real function). Moreover, for a field satisfying the reality condition $\overline{\varphi}(g_1, g_2, g_3) = \varphi(g_3, g_2, g_1)$, the kinetic term (28) of the free theory is exactly given by the scalar product $\langle \varphi | \hat{\mathcal{H}} | \varphi \rangle$. Thus the equation of motion of our free theory defined by the quadratic term of the effective action S_{ϕ_f} is simply $\hat{\mathcal{H}}\varphi = 0$. We will now look for the eigenstates satisfying $\hat{\mathcal{H}}\varphi = \mu\varphi$ and fully diagonalize the operator $\hat{\mathcal{H}}$.

   We consider the action of the operator $\hat{\mathcal{H}}$ on the space on invariant fields, $\varphi(g_1, g_2, g_3) = \varphi(g_1 g, g_2 g, g_3 g)$ for all $g \in \mathcal{G}$. Such invariant functions can be looked at as functions of the gauge invariant combinations $g_1 g_3^{-1}$ and $g_2 g_3^{-1}$. Thus, we are acting with $\hat{\mathcal{H}}$ on the Hilbert space $L^2(\mathcal{G}^{\times 3}/\mathcal{G}) \sim L^2(\mathcal{G}^2)$. A basis of functions on $L^2(\mathcal{G}^2)$ is given by tensor product states such $\varphi(g_1, g_2, g_3) = \rho(g_1 g_3^{-1}) A(g_2 g_3^{-1})$ or in short $\varphi = \rho \otimes A$. It turns out that such simple states already diagonalize $\hat{\mathcal{H}}$. We distinguish two cases as follows.

- $A \perp f$. if $\langle f | A \rangle = \int \mathrm{d}h f(h) A(h) = 0$, the action of $\hat{\mathcal{H}}$ on $\varphi(g_1, g_2, g_3) = \rho\big(g_1 g_3^{-1}\big) A\big(g_2 g_3^{-1}\big)$ simplifies to

$$
\hat{\mathcal{H}}\rho\big(g_1 g_3^{-1}\big) A\big(g_2 g_3^{-1}\big) = \rho\big(g_1 g_3^{-1}\big) A\big(g_2 g_3^{-1}\big) \left(1 - \int \mathrm{d}h f(h) f\big(hg_1 g_3^{-1}\big) \right). \tag{31}
$$

  From this, it is clear that taking $\rho(g) = \delta_G(g) = \delta(gG^{-1})$ for a fixed group element $G \in \mathcal{G}$ will diagonalize this action. Then the tensor product states $\varphi = \delta_G \otimes A$ with $A \perp f$ being eigenstates of $\hat{\mathcal{H}}$:

$$
\hat{\mathcal{H}}\delta\big(g_1 g_3^{-1} G^{-1}\big) A\big(g_2 g_3^{-1}\big) = \left(1 - \int \mathrm{d}h f(h) f(hG) \right) \delta\big(g_1 g_3^{-1} G^{-1}\big) A\big(g_2 g_3^{-1}\big). \tag{32}
$$

  The corresponding eigenvalues are $\mu = (1 - \int \mathrm{d}h f(h) f(hG))$ and do not depend on the choice of the function A. Since $\int f^2 = 1$ is normalized, the Cauchy–Schwarz inequality ensures that $\int \mathrm{d}h f(h) f(hG)$ is bounded by 1 in the absolute value; thus we have $\mu \in [0, +2]$.

     The lowest eigenvalue $\mu = 0$ is reached when saturating the Cauchy–Schwarz inequality, that is, for $G = \mathbb{I}$. In that case, the eigenvector $\varphi(g_1, g_2, g_3) = \delta(g_1 g_3^{-1}) A(g_2 g_3^{-1})$ is just the flat classical solution ϕ_A (up to a proportionality factor).

- $A \propto f$. In the case that $A = f$ (since the operator is linear, the proportionality factor is irrelevant here), the action also simplifies:

$$
\hat{\mathcal{H}}\rho\big(g_1 g_3^{-1}\big) f\big(g_2 g_3^{-1}\big) = \rho\big(g_1 g_3^{-1}\big) f\big(g_2 g_3^{-1}\big) \left(1 - \int \mathrm{d}h f(h) f\big(hg_1 g_3^{-1}\big) - 2 \int \mathrm{d}h f(h)^2 \right). \tag{33}
$$

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Using the normalization $\int f^2 = 1$ and once again taking the ansatz $\rho(g) = \delta_G(g)$, we obtain the remaining eigenstates:

$$\hat{\mathcal{H}}\delta(g_1 g_3^{-1} G^{-1}) f(g_2 g_3^{-1}) = -\left(1 + \int dh f(h) f(hG)\right) \delta(g_1 g_3^{-1} G^{-1}) f(g_2 g_3^{-1}), \qquad (34)$$

which gives eigenvalues $\mu = -1 - \int dh\, f(h) f(hG) \in [-2, 0]$.

All the eigenvalues in this sector are lower than in the previous section $A \perp f$. And once again, the lowest eigenvalue is reached for $G = \mathbb{I}$ and $\rho = \delta$, whose corresponding eigenstate of our Hamiltonian constraint $\hat{\mathcal{H}}$ is the state $\varphi(g_1, g_2, g_3) = \delta(g_1 g_3^{-1}) f(g_2 g_3^{-1})$, which is (up to an irrelevant factor) the initial flat solution ϕ_f around which we have expanded the GFT.

To summarize, we have checked that the spectrum of our (Hermitian) constraint operator $\mathcal{H}$ is bounded both from the above and below; the eigenvalues are $\pm 1 - \int dh\, f(h) f(hG)$ and are thus parameterized, as the corresponding eigenfunctions, by an arbitrary group element G; the spectrum is therefore generically continuous and it depends explicitly on the background classical solution ϕ_f defined by the function f.

It is also interesting to compare the constraint operator we have obtained, and its spectrum, to the standard $p^2 - m^2$ of a scalar field theory. Indeed, the quantity $-\int dh\, f(h) f(hG)$ can be roughly identified as the momentum squared p^2. This was already noted in [13] where the kinetic operator of the effective non-commutative field theory for a scalar field coupled to 3D quantum gravity (as derived in [23]) can be in such a fashion.

More precisely, let us take the gauge group $\mathcal{G} = SU(2)$ and assume that f is a central function, $f(hgh^{-1}) = f(g)$. Then f can be expanded over the characters χ_j of the irreducible representations of $SU(2)$ labeled by the spin $j \in \mathbb{N}/2$:

$$f(g) = \sum_j f_j \chi_j(g), \qquad \sum_j f_j^2 = 1, \qquad \int dh f(h) f(hG) = \sum_j f_j^2 \frac{\chi_j(G)}{d_j}, \qquad (35)$$

where the coefficients f_j are real and the factors $d_j = \chi_j(\mathbb{I}) = (2j + 1)$ are the dimension of the $SU(2)$ representation of spin j. A simple manipulation allows one to write

$$-\int dh f(h) f(hG) = -1 + \sum_j f_j^2 \left(1 - \frac{\chi_j(G)}{d_j}\right),$$

due to the normalization condition $\sum_j f_j^2 = 1$. Now, since the characters reach their absolute maximum in the identity, $\chi_j(\mathbb{I}) = d_j$ by definition, then the series in the equation above is always positive and vanishes at the identity $G = \mathbb{I}$. Thus we can write

$$-\int dh f(h) f(hG) = P^2(G) - 1 \qquad \text{with} \quad P^2(G) \equiv \sum_j f_j^2 \left(1 - \frac{\chi_j(G)}{d_j}\right), \qquad (36)$$

where the shift -1 can be interpreted as a mass term. This kinetic term can truly be written through a (group) Fourier transform as a Laplacian operator (corresponding to an actual 3-momentum) acting on the non-commutative $\mathbb{R}^3$ dual to the group manifold $\mathcal{G} = SU(2)$ [23, 24].

Thus, through our procedure we have recovered a non-trivial kinetic term which can be interpreted and used in analogy with the usual $(p^2 - m^2)$ of standard QFT. This is also interesting when one recalls that the Hamiltonian constraint of geometrodynamics takes the form of a (functional) Klein–Gordon-type quadratic operator on superspace. However, this analogy must be taken with care because the situation is different in three spacetime dimensions. (This is actually the case we are dealing with here.) With this point being understood, the main interest in the above result is that it confirms that the kinetic operator we have defined is non-trivial
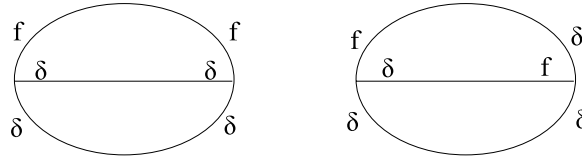
Figure 2. The Θ-graph with two vertices and three edges linking them: the two possibilities for defining the spin network functional made from ϕ_f group field insertions at the two vertices but with different choices of permutations at each vertex.

but manageable, and that the corresponding propagator has a non-trivial spectrum and it is thus directly amenable to more standard constructive techniques in the context of GFT renormalization [26].

2.3. The effective Hamiltonian constraint

We are ready to realize the action of this effective Hamiltonian constraint on a generic spin-network state for the graph Γ, given, as we have seen, by the tensor product of the group field φ_v associated with each of the vertices v of the graph:

$$\psi_\Gamma(G_e) = \int [\mathrm{d}g_{vw}] \prod_v \varphi^{\sigma_v}(g_{vw}) \prod_e \delta\big(g_{t(e)i(e)}^{-1} g_{s(e)i(e)} G_e^{-1}\big), \qquad (37)$$

so that a spin-network state is interpreted as a multi-particle state (in the Fock space) of the GFT. Then we let the linear operator $\hat{\mathcal{H}}$ act as expected on the tensor product $\psi_\Gamma = \bigotimes_v \varphi_v$ as earlier in (23):

$$\hat{\mathcal{H}} \triangleright \psi_\Gamma(G_e) = \int [\mathrm{d}g_{vw}] \sum_a \left[\hat{\mathcal{H}} \triangleright \varphi^{\sigma_a}(g_{aw}) \prod_{v:v\neq a} \varphi^{\sigma_v}(g_{vw}) \right] \prod_e \delta\big(g_{t(e)i(e)}^{-1} g_{s(e)i(e)} G_e^{-1}\big);$$
$$(38)$$

ψ_Γ is therefore an eigenstate of $\hat{\mathcal{H}}$ if (and only if) the fields φ_v are all eigenvectors of the Hamiltonian operator.

For any choices of φ_v, it is straightforward to compute the associated spin-network functional by explicitly performing the integrals over the gauge group $\mathcal{G}$. Here we focus on identifying and interpreting the state corresponding to the lowest eigenvalue of the effective Hamiltonian constraint operator, which we call for simplicity the 'ground state'.

For instance, the ground state on a graph Γ will be given by taking the ground state of the group field everywhere, i.e. $\varphi_v = \phi_f$ for all vertices v as we have derived in the previous section. A minor point is that the eigenvalue of $\hat{\mathcal{H}}$ associated with $\varphi = \phi_f$ is -2, so that it might be a better convention to shift $\hat{\mathcal{H}}$ by 2 in order to define the ground state as having vanishing eigenvalue (then, solutions of the Hamiltonian constraint equation would have eigenvalue 2).

Computing the integrals of the product $\prod_v \varphi_v$, we will get a certain combination of δ functions and convolutions of f of every loop of the graph Γ. Let us compute this explicitly for the Θ-graph, made up of two vertices and three edges linking them. The group field $\varphi = \phi_f = \int \delta f \delta$ has three legs with which are associated twice the δ-distribution and once the function f. Thus, we have two possible configurations on the Θ-graph depending on the choice of permutations σ_v: either the two f-insertions are on the same edge (let us say 1) or they are on different edges (let us say 1 and 2). These two possibilities are illustrated in figure 2.
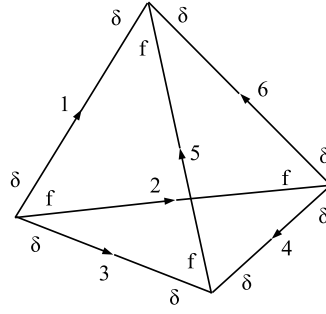
13

Figure 3. The oriented tetrahedron graph with its four vertices and six edges and a particular choice of permutations at the four vertices in order to define the spin-network functional.
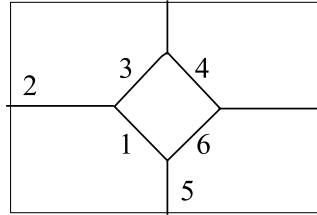

Figure 4. The tetrahedron graph (faithfully) embedded on the 2-torus with the edges 2 and 5 wrapped around the two cycles.

The corresponding spin-network states are easy to compute:

$$\psi_\Theta^{(1)}(G_a) = \delta\big(G_2 G_3^{-1}\big) \int \mathrm{d}h f(h) f\big(h G_1 G_2^{-1}\big) = \delta\big(G_2 G_3^{-1}\big) f \circ f\big(G_1 G_2^{-1}\big),$$

$$\psi_\Theta^{(2)}(G_a) = f\big(G_1 G_3^{-1}\big) f\big(G_2 G_3^{-1}\big),$$

where we have assumed that f is central[7] for simplicity's sake.

As expected, these are gauge-invariant functionals invariant under the action of $\mathcal{G}$ at each vertex of the graph. They assign a certain convoluted power δ or f or $f \circ f$ or more generally $f^{\circ n}$ to each loop depending on the number of f-insertions along that loop. When we have the δ-distribution, we are imposing that the holonomy along that loop is trivial and thus that the connection is flat. When we have a non-trivial power $f^{\circ n}$ on a loop, it can be interpreted on the other hand as a topological defect or a non-trivial cycle of the space topology.

To assert this interpretation, let us have a try at the tetrahedron graph and consider the choice of permutations as depicted in figure 3. The corresponding spin-network state is straightforward to compute

$$\psi_T(G_a) = \delta\big(G_4 G_3^{-1} G_1 G_6^{-1}\big) f \circ f\big(G_4 G_3^{-1} G_2\big) f \circ f\big(G_6 G_5 G_4^{-1}\big), \tag{39}$$

which is interpreted as a flat state on the 2-torus (figure 4) or the 2-sphere with two punctures (topological defects).

Finally, we can also play with the choice of the classical solution ϕ_f. Indeed, the ground state and, more generally, all the eigenstates depend on the choice of the function f. For

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[7] For the gauge group $\mathcal{G} = SU(2)$, if the function f is invariant under conjugation, $f(g) = f(hgh^{-1})$, then it is automatically invariant under inversion $f(g) = f(g^{-1})$. Then $\circ$ is simply the conventional convolution product between functions over $SU(2)$.

instance, if we take the (ill-defined) limit $f \rightarrow \delta/\sqrt{\delta(\mathbb{I})}$, then the ground state becomes the completely flat state imposing that the holonomies are trivial along all the loops of the graph. This correctly corresponds to the physical state of topological BF theory for a trivial topology of space. On the other hand, as soon as f is different from the δ-distribution, holonomies become non-trivial and are interpreted as a non-trivial space topology (or topological defects). This describes the ground state of our Hamiltonian constraint. Then we can see that excited states will introduce more and more topological defects and curvature.

3. Conclusion

Within the framework of spinfoam models for quantum gravity, we proposed to focus on the free theory defined by the quadratic term of the group field theory (GFT) and to interpret the kinetic operator as a Hamiltonian constraint. This Hamiltonian constraint $\hat{\mathcal{H}}$ defines the classical equation of motion of the free theory and acts on the GFT field φ, which represents a single intertwiner of a spin-network state for loop quantum gravity. Nevertheless, we have also shown to interpret general spin-network states as multi-particle states of the GFT, as tensor products of the initial field φ. Then we have a natural action of the Hamiltonian operator $\hat{\mathcal{H}}$ on spin networks.

A standard prescription for GFTs is to start with a trivial propagator (but see the last reference of [26]) and to encode all the dynamics (of both geometry, thus the Hamiltonian constraint, and topology) in the interaction term. Nevertheless, even in this case, we follow the proposal from [13] to expand the GFT around a non-trivial solution of its classical equations of motion. This background classical solution contains dynamical information from the full theory, since it depends on the original interaction term. Then we have shown that the effective GFT describing the field variations around that background acquires a non-trivial kinetic term, which we can then interpret as defining an effective Hamiltonian constraint for loop quantum gravity.

We have applied this program explicitly to Boulatov's GFT for 3SD quantum gravity [17]. We expanded it around the flat classical solutions introduced in [13] and analyzed in detail the spectrum of the induced Hamiltonian constraint operator. We have seen that it can be interpreted as a kinetic operator of the $(p^2 - m^2)$ type. This not only supports the physical relevance of our procedure but also supports the idea that GFT can be understood (at least in certain phases, or for certain perturbation fields) as the momentum representation of a field theory on an actual spacetime manifold, which would be obtained through a Fourier transform [11, 20, 25].

More work is certainly needed. In particular, one should investigate different choices of background GFT configurations around which to expand, since this determines a big deal of the effective Hamiltonian constraint.

Having a non-trivial propagator of the GFT of the $(p^2 - m^2)$ type, thus with a more standard scale dependence, opens the door to an easier use of standard QFT tools to study GFT renormalization; in particular, it would be interesting to see how our procedure can be used within the tentative framework that has been recently developed [26]. We should also apply our program to spinfoams for 4D quantum gravity and see if we can extract some interesting and physically relevant effective Hamiltonian constraint from the EPRL-FK spinfoam models for instance [30–33].

Finally, we believe that it would also be interesting to investigate the group Fourier transform [23, 24] of all our procedure. This can be given two possible interpretations: it may mean going from the momentum representation given by our GFT on a group manifold to a (non-commutative) field theory on a spacetime manifold, if the non-commutative dual

variables are interpreted directly as coordinates on it [25, 20], or it could mean simply re-writing the same field theory of geometry from connection to triad/flux variables [11, 12, 29]. In both readings, it would certainly help to understand the physical and geometrical meaning of the induced propagator and effective Hamiltonian constraint.

Acknowledgments

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