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The spacetime of a Dirac fermion

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ABSTRACT

We present an approximate solution to the minimally coupled Einstein–Dirac equations. We interpret the solution as describing a massive fermion coexisting with its own gravitational field. The solution is axisymmetric but is time dependent. The metric approaches that of a flat spacetime at the spatial infinity. We have calculated a variety of conserved quantities in the system.

1. Introduction

Explicit solutions in General Relativity (coupled to matter) and in supergravity theories play a significant role in the study of quantum gravity. Several important progresses (for example, [1–4]) have been achieved with the help of some concrete solutions. Due to the weakness of the gravitational interaction, most of the phenomenologically interesting solutions describe physics at the astronomical scale, such as black holes, p-branes, black rings [5] and so on.

From the theoretical point of view, however, it is also interesting to look for exact solutions at the microscopic scale. Quantum field theory (QFT) in flat spacetime tells us that if the zero-point energy gravitates, then extreme fine tuning is needed to render the cosmological constant to its currently observed value. There have been a lot of effort towards solving the problem, but without too much success [6]. The zero-point energy is a byproduct of the quantization of fields in flat spacetime. When it comes to curved spacetime, the situation is much more complicated (see, e.g. [7]). As implied by the Unruh effect [2], even the notion of *particle* is no longer fundamental, but depends on the observer. It is our hope that an exact solution to gravity coupled to quantum matter may help us better understand the nature of QFT in curved spacetime, and offer some clue to the persisting problem of the cosmological constant. When a spacetime is flat at the spatial infinity, the usual notion of *particle* is still useful for an observer far away from the distribution of matter. One can imagine a scenario where the observer sees nothing but a single particle at the center. It is then interesting to ask how the wave function of the particle (such as a neutrino or electron) behaves under its own gravity. To find this, we will also need an exact solution to classical gravity coupled with quantum matter. More recently, the development in non-Fermi liquid and holographic superconductors [8–10] has also made it interesting to study gravitational solutions with back reaction from spinor fields [11]. In this work, we will present an approximate solution to the Einstein–Dirac system. It describes how a fermion exists, under its own gravity, in a flat spacetime background.

Compared with the long list of literature on other systems, there have been relatively fewer work on solutions to the Einstein–Dirac system (for examples, see [12–26]). Most of the existing effort came after Brill and Wheeler's 1957 paper [12]. An early review of works focused on neutrinos can be found in [13]. Some of the solutions describe ghost spinors (see, e.g., [14]). Solutions in dimensions other than four can be found in [15,16]. Particle-like solutions have been previously studied in [25].

Apart from the complexity of the calculation, the reason discouraging people from dealing with the spinor field might be its anti-commutative nature upon quantization. Here we will treat the spinor field as a quantum mechanical wave function, but not as a quantized field operator. The reason is the following. For a bosonic field, the wave-function treatment can be justified in the low energy limit, when there is no particle production. On the other hand, a spinor field is often believed to be intrinsically quantum, and one should always quantize it first before using it. However, the problem with gravity is that there is NO known ways to quantize the spinor field that is completely satisfactory. There is always a negative energy associated with the un-quantized spinor field. In the usual QFT treatment, this negativeness is shifted to hide in

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the zero-point energy, rendering it physically irrelevant [27]. But in theories with gravity, the problem reappears in the form of a too large contribution to the cosmological constant. In this sense, the cosmological constant problem is essentially the negative energy problem. So when there is gravity, the usual way of quantizing the spinor field is no longer a good fix of all the problems. Our hope is that, an exact particle-like solution to the Einstein–Dirac system (but with an un-quantized spinor field) may offer some hint on how to correctly quantize the spinor field with gravity.

Due to the complexity of the coupled Einstein–Dirac equations, we start by looking for an approximate solution to the system. We expand the solution in terms of the radial coordinate, and our solution is approximately valid at the spatial infinity. At the present stage, it is very difficult to discuss the stability of the solution.[1] In fact, it is not even clear if our approximate solution comes from a *well behaved* full solution or not. On the other hand, it is already non-trivial to find an ansatz (such as (12) and (13)) which allows for an approximate solution to exist.[2] So as a first step towards finding an exact particle-like solution to the Einstein–Dirac system, we will be content with the approximate solution for now, but will try to solve all the remaining problems in future works.

I will present the solution in next section. Then I will discuss some of the (approximately) conserved quantities. A short summary is at the end.

2. The solution

The action of the Einstein–Dirac system is given by

$$S = \int d^n x \sqrt{|g|}\left\{\frac{R-2\Lambda}{16\pi G_{(n)}} - \frac{i}{2}\bar{\psi}\gamma^\mu D_\mu \psi + \frac{i}{2}(D_\mu\bar{\psi})\gamma^\mu\psi - i\mu\bar{\psi}\psi\right\}, \tag{1}$$

where $G_{(n)}$ is Newton's constant in n spacetime dimensions, Λ is the cosmological constant, and μ is the mass of the spinor field. The signature of the metric is mostly positive. It is often convenient to use the Planck mass $M_p^2 = \frac{1}{8\pi G_{(n)}}$ in place of Newton's constant. We will let $16\pi G_{(n)} = 1$ from now on. The notations related to the spinor field are

$$\begin{aligned}
&D_\mu\psi = \left(\partial_\mu + \frac{i}{2}w_{ab\mu}\gamma^{ab}\right)\psi, \quad \bar{\psi} = \psi^\dagger\gamma^0,\\
&D_\mu\bar{\psi} = \partial_\mu\bar{\psi} - \frac{i}{2}w_{ab\mu}\bar{\psi}\gamma^{ab}, \quad \gamma^{ab} = -\frac{i}{4}[\gamma^a,\gamma^b]\\
&\{\gamma^a,\gamma^b\} = 2\eta^{ab}, \quad g_{\mu\nu} = \eta^{ab}e_{a\mu}e_{b\nu}, \quad \gamma^\mu = \gamma^a e_a{}^\mu,\\
&w_{ab\mu} = (\Gamma^\rho_{\mu\nu}e_{a\rho} - \partial_\mu e_{a\nu})e_b{}^\nu = -(\nabla_\mu e_{a\nu})e_b{}^\nu.
\end{aligned} \tag{2}$$

Note γ^0 in $\bar{\psi}$ is defined in the vielbein basis. The equations from (1) are

$$0 = \gamma^\mu D_\mu\psi + \mu\psi = -(D_\mu\bar{\psi})\gamma^\mu + \mu\bar{\psi}, \tag{3}$$

$$R_{\mu\nu} = \frac{i}{8}\bar{\psi}(\gamma_\mu D_\nu + \gamma_\nu D_\mu)\psi - \frac{i}{8}(D_\mu\bar{\psi}\gamma_\nu + D_\nu\bar{\psi}\gamma_\mu)\psi + \frac{2}{n-2}\left(\Lambda + \frac{i}{4}\mu\bar{\psi}\psi\right)g_{\mu\nu}. \tag{4}$$

We will work in four spacetime dimensions ($n=4$) from now on and will take $\Lambda = 0$. As a matter of choice, the gamma matrices in the vielbein basis are taken to be

$$\gamma^0 = i\begin{pmatrix} & \mathbf{1}_2\\ \mathbf{1}_2 & \end{pmatrix}, \qquad \gamma^{1,2,3} = i\begin{pmatrix} & \sigma^{3,1,2}\\ -\sigma^{3,1,2} & \end{pmatrix}, \tag{5}$$

where $\mathbf{1}_2$ is the two-dimensional unit matrix and $\sigma^{1,2,3}$ are the usual Pauli matrices. The corresponding charge conjugating operator is $\mathcal{C} = \gamma^3$.

We are interested in an axisymmetric spacetime. The ansatz for the metric is given by

$$ds^2 = -f_t^2(dt + f_z\,d\phi)^2 + f_x^2\,dx^2 + f_y^2\,d\theta^2 + (f_p\,d\phi - \tilde{f}_z\,dt)^2, \tag{6}$$

where $|f_t| > |\tilde{f}_z|$ in general. We expect x to be the asymptotic radial coordinate, θ the latitudinal angle, t the time and ϕ the azimuthal angle. It is convenient to write the vierbeins as

$$\begin{aligned}
&e^0 = f_t(dt + f_z\,d\phi), \qquad e^1 = f_x\,dx,\\
&e^2 = f_y\,d\theta, \qquad e^3 = f_p\,d\phi - \tilde{f}_z\,dt.
\end{aligned} \tag{7}$$

One can always set $\tilde{f}_z = 0$ by using the following local Lorentz transformation

$$(\Lambda^a{}_b) = \frac{1}{\sqrt{1-(\tilde{f}_z/f_t)^2}}\begin{pmatrix} 1 & & & \tilde{f}_z/f_t\\ & 0 & & \\ & & 0 & \\ \tilde{f}_z/f_t & & & 1\end{pmatrix}. \tag{8}$$

So (6) and (7) are equivalent to the following,

$$ds^2 = -f_t^2(dt + f_z\,d\phi)^2 + f_x^2\,dx^2 + f_y^2\,d\theta^2 + f_p^2\,d\phi^2, \tag{9}$$

$$\begin{aligned}
&e^0 = f_t(dt + f_z\,d\phi), \qquad e^1 = f_x\,dx,\\
&e^2 = f_y\,d\theta, \qquad e^3 = f_p\,d\phi.
\end{aligned} \tag{10}$$

We take the spinor field to be

$$\Psi = \begin{pmatrix} \psi_{1a} + i\psi_{1b}\\ \psi_{2a} + i\psi_{2b}\\ \psi_{3a} + i\psi_{3b}\\ \psi_{4a} + i\psi_{4b}\end{pmatrix}, \tag{11}$$

where all the functions ψ_{ia} and ψ_{ib} (throughout the Letter, the index $i = 1,\ldots,4$) are real.

Given (9) and (11), it is difficult to solve (3) and (4) directly. In this work, we will focus on a spacetime that is flat at the spatial infinity ($x \to +\infty$). Our strategy is to expand the functions in terms of the radial coordinates x and obtain an approximate solution to the equations when $x \to +\infty$. We will assume that all the unknown functions depend on x, θ and t only. Our ansatz for the functions are

$$\begin{aligned}
&f_t = 1 + \frac{t_1}{x} + \frac{t_2}{x^2} + \mathcal{O}\left(\frac{1}{x^3}\right),\\
&f_z = \frac{z_1}{x} + \frac{z_2}{x^2} + \mathcal{O}\left(\frac{1}{x^3}\right),\\
&f_x = 1 + \frac{x_1}{x} + \frac{x_2}{x^2} + \mathcal{O}\left(\frac{1}{x^3}\right),\\
&f_y = x + y_0 + \frac{y_1}{x} + \frac{y_2}{x^2} + \mathcal{O}\left(\frac{1}{x^3}\right),\\
&f_p = \sin\theta\left[x + p_0 + \frac{p_1}{x} + \frac{p_2}{x^2} + \mathcal{O}\left(\frac{1}{x^3}\right)\right],
\end{aligned} \tag{12}$$

$$\begin{aligned}
&\psi_{ia} = \frac{a_{i1}}{x} + \frac{a_{i2}}{x^2} + \mathcal{O}\left(\frac{1}{x^3}\right),\\
&\psi_{ib} = \frac{b_{i1}}{x} + \frac{b_{i2}}{x^2} + \mathcal{O}\left(\frac{1}{x^3}\right), \quad i = 1,\ldots,4,
\end{aligned} \tag{13}$$

[1] I thank the referee for raising up this point.

[2] For example, the counterpart of (12) and (13) allow for no solution when there is an arbitrary cosmological constant.

where $x_{1,2,\ldots}$, $t_{1,2,\ldots}$, $z_{1,2,\ldots}$, $y_{0,1,2,\ldots}$, $p_{0,1,2,\ldots}$, $a_{i1,i2,\ldots}$ and $b_{i1,i2,\ldots}$ are polynomials of trigonometric functions of x, θ and t. The expansion in (12) guarantees that (9) approaches the metric of a flat spacetime as $x \to +\infty$.

Given (12) and (13), the leading contribution to (3) is of the order $\mathcal{O}(\frac{1}{x})$. The equations are given by

$$\begin{aligned}
&\mu b_{31} - (\partial_x - \partial_t) a_{11} = 0, && \mu a_{31} + (\partial_x - \partial_t) b_{11} = 0, \\
&\mu b_{41} + (\partial_x + \partial_t) a_{21} = 0, && \mu a_{41} - (\partial_x + \partial_t) b_{21} = 0, \\
&\mu b_{11} + (\partial_x + \partial_t) a_{31} = 0, && \mu a_{11} - (\partial_x + \partial_t) b_{31} = 0, \\
&\mu b_{21} - (\partial_x - \partial_t) a_{41} = 0, && \mu a_{21} + (\partial_x - \partial_t) b_{41} = 0.
\end{aligned} \tag{14}$$

These equations can be completely solved by

$$\begin{aligned}
a_{i1} = {} & a_{i1a}(\theta)\cos(wt)\cos(kx) + a_{i1b}(\theta)\cos(wt)\sin(kx) \\
& + a_{i1c}(\theta)\sin(wt)\cos(kx) + a_{i1d}(\theta)\sin(wt)\sin(kx),
\end{aligned} \tag{15}$$

where $i = 1, \ldots, 4$ and $w^2 = k^2 + \mu^2$. b_{i1}'s are solved in terms of a_{i1}'s in an obvious way. The leading contribution to (4) is of the order $\mathcal{O}(x)$. The equations are given by

$$\left(\partial_x^2 - \partial_t^2\right) y_0 = \left(\partial_x^2 - \partial_t^2\right) p_0 = 0. \tag{16}$$

They do not depend on the spinor field. The solutions are

$$\begin{aligned}
y_0 &= y_{0y}(\theta) + y_{01}(\theta, x - t) + y_{02}(\theta, x + t), \\
p_0 &= p_{0y}(\theta) + p_{01}(\theta, x - t) + p_{02}(\theta, x + t).
\end{aligned} \tag{17}$$

It is hard to see how can functions of $wt \pm kx$ and $x \pm t$ coexist with each other in the case $w \neq k$. So we will assume $y_{01} = y_{02} = p_{01} = p_{02} = 0$ when $\mu \neq 0$ (in fact, our solution exists only when $\mu \neq 0$). Note the leading order equations (14) and (16) are independent of each other.

To the order $\mathcal{O}(\frac{1}{x})$ of (4), one has equations that determine z_1, t_1, x_1, y_1 and p_1, in terms of a_{i1} and b_{i1}. The assumption about the functions z_1, t_1, x_1, y_1 and p_1 impose extra constraints on the structure of a_{i1} and b_{i1}. For example, two of the equations are

$$\begin{aligned}
\partial_x \partial_y t_1 = {} & -\frac{1}{4\sqrt{1-y^2}} \left[a_{11} b_{21} \partial_x \ln\left(\frac{a_{11}}{b_{21}}\right) + a_{21} b_{11} \partial_x \ln\left(\frac{a_{21}}{b_{11}}\right) \right. \\
& \left. - a_{31} b_{41} \partial_x \ln\left(\frac{a_{31}}{b_{41}}\right) - a_{41} b_{31} \partial_x \ln\left(\frac{a_{41}}{b_{31}}\right) \right],
\end{aligned} \tag{18}$$

$$\begin{aligned}
\partial_y \partial_t x_1 = {} & -\frac{1}{4\sqrt{1-y^2}} \left[a_{11} b_{21} \partial_t \ln\left(\frac{a_{11}}{b_{21}}\right) + a_{21} b_{11} \partial_t \ln\left(\frac{a_{21}}{b_{11}}\right) \right. \\
& \left. - a_{31} b_{41} \partial_t \ln\left(\frac{a_{31}}{b_{41}}\right) - a_{41} b_{31} \partial_t \ln\left(\frac{a_{41}}{b_{31}}\right) \right].
\end{aligned} \tag{19}$$

Using (15), one can find that the right-hand side of (18) does not depend on x, while that of (19) does not depend on t. To be consistent with the assumption that t_1 and x_1 only depend on x or t in the form of trigonometric functions, the right-hand sides of both (18) and (19) must vanish. This condition constrains both t_1, x_1 and a_{i1}, b_{i1}. As a special case that satisfies this condition, we let

$$b_{11} \sim a_{21}, \qquad b_{21} \sim a_{11}, \qquad b_{31} \sim a_{41}, \qquad b_{41} \sim a_{31}, \tag{20}$$

up to some constant factors. This will reduce (15) to a much simpler form.

The equations from higher orders of the expansion are more complicated and we will only summarize some of the main features here. There are eight equations at each order $\mathcal{O}(1/x^n)$ of (3). These equations are just enough to determine a_{in} and b_{in} in terms of $a_{i,n-1}$, $b_{i,n-1}$, x_{n-1}, y_{n-1}, p_{n-1}, t_{n-1}, z_{n-1} and lower order functions. On the other hand, the equations at the order $\mathcal{O}(1/x^n)$ of (4) determine x_{n-1}, y_{n-1}, p_{n-1}, t_{n-1} and z_{n-1} in terms of $a_{i,n-1}$, $b_{i,n-1}$, x_{n-2}, y_{n-2}, p_{n-2}, t_{n-2}, z_{n-2} and lower order functions. What's more, there are always two equations that involve z_n, t_n, x_n, y_n, p_n, a_{in} and b_{in} at each order $\mathcal{O}(1/x^n)$ of (4), while the rest only involve lower order functions. These equations can always be solved together with those from the order $\mathcal{O}(1/x^{n+1})$ of (4). Because of the constraints from equations like (18) and (19), it is not guaranteed that one can find a self-consistent solution to all orders in $\frac{1}{x}$. We have done the calculation up to the order $\mathcal{O}(1/x^2)$ for both (3) and (4), and still we do not see any true obstacle (other than the tediousness) to push the calculation to higher orders. This is an encouraging sign that (12) and (13) might be the right ansatz to give us a consistent solution.

At the moment, we will be content with the solution approximate up to the order $\mathcal{O}(1/x^2)$. Even at this stage, the full result is already very unwieldy and contains many free parameters and functions. Presumably these free functions and parameters should be determined by equations from higher orders of the expansion. But for the purpose of giving an accessible example, we have (not so rigorously) chosen the parameters and functions in such a way that the solution has a simpler structure. One cannot expect to get the correct approximation to some exaction solution (if it exists) in this way. But the solution so obtained is still approximately valid in its own right, in the sense that it solves (3) and (4) up to the order $\mathcal{O}(1/x^2)$.

For the metric, we find that

$$\begin{aligned}
f_t &= 1 - \frac{N^2 w \cos\theta}{4k\mu x} + \mathcal{O}\left(1/x^3\right), \quad f_p = \sin\theta f_y, \\
f_x &= 1 - \frac{N^2 w^3 \cos\theta}{4k^3 \mu x} + \frac{N^2 k \sin(2wt - 2\zeta)\sin\theta}{4w^2 \mu x^2} + \mathcal{O}\left(1/x^3\right), \\
f_y &= x + \frac{N^2 w(2k^2 + w^2)\cos\theta}{4k^3 \mu} - \frac{N^2 k \sin(2wt - 2\zeta)\sin\theta}{4w^2 \mu x} \\
&\quad + \frac{N^4 (2k^2 - w^2)\sin(2wt - 2\zeta)\sin 2\theta}{16k^2 w \mu^2 x^2} + \mathcal{O}\left(1/x^3\right), \\
f_z &= + \frac{N^2 [6w\cos(2kx)\sin^2\theta + k\sin(2kx)\sin(2wt - 2\zeta)\sin(2\theta)]}{8k^2 \mu^2 x^2} \\
&\quad - \frac{N^2 w \sin(2kx)\sin^2\theta}{2k\mu^2 x} + \mathcal{O}\left(1/x^3\right).
\end{aligned} \tag{21}$$

For the spinor field, we find

$$\begin{aligned}
\psi_{1a} = {} & \frac{N\sqrt{\sin\theta}}{x\sqrt{w-k}} \left[\cos(wt + kx - \zeta) \right. \\
& - \frac{N^2 (4k - w) w \cos\theta \cos(wt + kx - \zeta)}{8k^2 \mu x} \\
& \left. + \frac{(2k + w)\cot\theta \sin(wt - kx - \zeta)}{4k(w + k)x} \right] + \mathcal{O}\left(1/x^3\right), \\
\psi_{1b} = {} & \frac{N\sqrt{\sin\theta}}{x\sqrt{w+k}} \left[\cos(wt - kx - \zeta) \right. \\
& - \frac{N^2 (4k + w) w \cos\theta \cos(wt - kx - \zeta)}{8k^2 \mu x} \\
& \left. + \frac{(2k - w)\cot\theta \sin(wt + kx - \zeta)}{4k(w - k)x} \right] + \mathcal{O}\left(1/x^3\right), \\
\psi_{2a} = {} & \frac{N\sqrt{\sin\theta}}{x\sqrt{w-k}} \left[\cos(wt - kx - \zeta) \right. \\
& - \frac{N^2 (4k - w) w \cos\theta \cos(wt - kx - \zeta)}{8k^2 \mu x}
\end{aligned}$$

$$+\frac{(2k+w)\cot\theta\sin(wt+kx-\zeta)}{4k(w+k)x}\bigg]+\mathcal{O}(1/x^3),$$

$$\psi_{2b}=\frac{N\sqrt{\sin\theta}}{x\sqrt{w+k}}\bigg[\cos(wt+kx-\zeta)$$
$$-\frac{N^2(4k+w)w\cos\theta\cos(wt+kx-\zeta)}{8k^2\mu x}$$
$$+\frac{(2k-w)\cot\theta\sin(wt-kx-\zeta)}{4k(w-k)x}\bigg]+\mathcal{O}(1/x^3),$$

$$\psi_{3a}=-\frac{N\sqrt{\sin\theta}}{x\sqrt{w-k}}\bigg[\sin(wt-kx-\zeta)$$
$$-\frac{N^2(4k-w)w\cos\theta\sin(wt-kx-\zeta)}{8k^2\mu x}$$
$$+\frac{(2k+w)\cot\theta\cos(wt+kx-\zeta)}{4k(w+k)x}\bigg]+\mathcal{O}(1/x^3),$$

$$\psi_{3b}=\frac{N\sqrt{\sin\theta}}{x\sqrt{w+k}}\bigg[\sin(wt+kx-\zeta)$$
$$-\frac{N^2(4k+w)w\cos\theta\sin(wt+kx-\zeta)}{8k^2\mu x}$$
$$+\frac{(2k-w)\cot\theta\cos(wt-kx-\zeta)}{4k(w-k)x}\bigg]+\mathcal{O}(1/x^3),$$

$$\psi_{4a}=-\frac{N\sqrt{\sin\theta}}{x\sqrt{w-k}}\bigg[\sin(wt+kx-\zeta)$$
$$-\frac{N^2(4k-w)w\cos\theta\sin(wt+kx-\zeta)}{8k^2\mu x}$$
$$+\frac{(2k+w)\cot\theta\cos(wt-kx-\zeta)}{4k(w+k)x}\bigg]+\mathcal{O}(1/x^3),$$

$$\psi_{4b}=\frac{N\sqrt{\sin\theta}}{x\sqrt{w+k}}\bigg[\sin(wt-kx-\zeta)$$
$$-\frac{N^2(4k+w)w\cos\theta\sin(wt-kx-\zeta)}{8k^2\mu x}$$
$$+\frac{(2k-w)\cot\theta\cos(wt+kx-\zeta)}{4k(w-k)x}\bigg]+\mathcal{O}(1/x^3), \tag{22}$$

where $w^2=k^2+\mu^2$, N is a dimensionless normalization constant and ζ is an arbitrary phase. For the results given above, it is possible to set $\zeta=0$ by a shift in t. The function $\cot\theta$ appearing in (22) indicates that the solution is divergent at $\sin\theta=0$. This could be a big problem. But such divergence could also be an artifact of the expansion in (12) and (13). For example, something like (a and b are constants)

$$\frac{a}{b+x\sin\theta}$$

is obviously regular at $\sin\theta=0$, but it diverges at $\sin\theta=0$ if firstly expanded around $x\to+\infty$. For (22), it is possible to get rid of the divergence in a similar fashion.

We have found (21) and (22) by assuming that a_{in} and b_{in} are polynomials of $\{\cos(kx),\sin(kx)\}$ and $\{\cos(wt),\sin(wt)\}$ up to the $(2n-1)$'th power, and z_n, t_n, x_n, y_n and p_n are polynomials of $\{\cos(kx),\sin(kx)\}$ and $\{\cos(wt),\sin(wt)\}$ up to the $2n$'th power. The dependence on θ appears as coefficient functions of the polynomials. This assignment is inspired by the structure of (15), (17) and that of the equations from different orders of (3) and (4). After solving equations up to the order $\mathcal{O}(1/x^2)$, we find that many coefficient functions are still undetermined. What's more, we have also found a lot integration constants in the process. Although these functions and constants are arbitrary at the order $\mathcal{O}(1/x^2)$, one should expect many further constraints to arise from higher order equations in the expansion. So presumably these functions and constants should be determined by pushing the calculation to higher orders, which is currently a daunting task. In reaching (21) and (22), we have thrown away most of the functions that do not depend on the strength of the spinor field (i.e. N) in an obvious way. These functions may be important when one wants to go to higher orders. As a result, (21) and (22) may not be the correct approximation to an exact solution of (3) and (4). On the other hand, (21) and (22) is still a valid approximate solution, good to the order $\mathcal{O}(1/x^2)$.

3. Conserved quantities

From (3), one can derive a conserved current of the spinor field,

$$\mathcal{J}^\mu=\bar\psi\gamma^\mu\psi \quad\Longrightarrow\quad D_\mu\mathcal{J}^\mu=0. \tag{23}$$

If there is a time-like Killing vector ξ, then one can write down the probability density of the spinor field as

$$\rho=\mathcal{J}\cdot\xi. \tag{24}$$

For the energy density of the spinor field, there are two possibilities. One is by using the energy–momentum tensor,

$$\rho_1=\xi^\mu\xi^\nu T_{\mu\nu},\qquad T_{\mu\nu}=R_{\mu\nu}-\frac{R}{2}g_{\mu\nu}. \tag{25}$$

The other is by using the energy–momentum four-vector,

$$\rho_2=P\cdot\xi,\qquad P_\mu=-\frac{1}{2}\bar\psi D_\mu\psi+\frac{1}{2}D_\mu\bar\psi\psi. \tag{26}$$

It can be checked that $\nabla_\mu P^\mu=0$. We will see that ρ_1 and ρ_2 are quantitatively vastly different.

From (21), one can find an approximate time-like Killing vector $\xi=\partial_t-\frac{\partial_t y_1}{x}\partial_x$ at the spatial infinity $x\to+\infty$,

$$\mathcal{L}_\xi g_{\mu\nu}=\begin{pmatrix} \cdot & -\frac{\partial_\theta\partial_t y_1}{x} & \cdot & -\frac{\partial_t^2 y_1}{x}\\ -\frac{\partial_\theta\partial_t y_1}{x} & \frac{2\partial_t y_2}{x} & \cdot & \cdot\\ \cdot & \cdot & \frac{2\sin^2\theta\,\partial_t y_2}{x} & \cdot\\ -\frac{\partial_t^2 y_1}{x} & \cdot & \cdot & \cdot \end{pmatrix}+\mathcal{O}\left(\frac{1}{x^2}\right), \tag{27}$$

where $\mu,\nu=\{x,\theta,\phi,t\}$, and y_1,y_2 can be read of (12) and (21). Now the probability density of the spinor field (24) can be calculated as

$$\rho=\frac{M_p^2}{2}\cdot\frac{4N^2w\sin\theta}{\mu^2x^2}+\mathcal{O}\left(\frac{1}{x^3}\right). \tag{28}$$

Here we have restored the Planck mass to make the dimension of the density manifestly correct. If we interpret the solution as describing a particle with its center of mass located at $x=0$, and if we suppose that the chance of finding the particle far away from $x=0$ is ϱ, then

$$\varrho\approx\oint dV\,\rho\approx\frac{2\pi^2M_p^2N^2wL}{\mu^2}\quad\Longrightarrow\quad N^2\approx\frac{\mu^2\varrho}{2\pi^2M_p^2wL}, \tag{29}$$

where

$$\oint dV=\int_0^L dx\int_0^\pi d\theta\int_0^{2\pi}d\phi\,x^2\sin\theta. \tag{30}$$

The integral (29) is divergent over the whole space, so we have introduced a cutoff L to regularize the divergence. Now if the full wave function is finite near the center, then we must have $\varrho = 1$. This is because the distribution probability is then dominated by the divergent integral in (29). We will keep ϱ explicit to cover the possibility that the wave function may actually diverge at $x = 0$. In this case, a significant portion of the probability may be distributed near the center.

Let's now turn to the energy of the spinor field. The energy densities (25) and (26) are found to be

$$\begin{aligned}\rho_1 &\approx \frac{M_p^2}{2} \cdot \frac{N^2 w \cos\theta [2\mu + 3N^2 k \sin(2wt - 2\zeta) \sin\theta]}{2k\mu^2 x^3} \\ &\quad + \mathcal{O}\big(1/x^4\big), \\ \rho_2 &\approx \frac{M_p^2}{2} \cdot \frac{4N^2 w \sin\theta}{\mu x^2} + \mathcal{O}\big(1/x^3\big). \end{aligned} \tag{31}$$

Here we have again restored the Planck mass to make the dimensions manifestly correct. It is obvious that the two energy densities are very different. While the nature of ρ_1 is not very clear, we will take ρ_2 to be the true energy density of the spinor field at places far away from the center. In fact, it is easy to see that

$$\rho_2 = \rho\mu + \mathcal{O}\big(1/x^3\big). \tag{32}$$

The contribution of ρ_2 to the total energy is

$$E = \oint dV\, \rho_2 = \varrho\mu. \tag{33}$$

In the case $\varrho = 1$, ρ_2 dominates the contribution to the spinor energy. We see that the total amount is exactly μ. This is as expected: the solution describes a particle without an apparent kinetic energy, and so the total energy is nothing but the mass of the particle. However, there is a puzzle. In the solution the wave function fluctuates in both space and time, and the frequency w is apparently larger than the mass. We must have a non-vanishing wave number ($k \neq 0$) for the solution to exist. But it is still unclear how k is related to the particle mass μ. One may need to know the full solution to answer this question.

On the geometry side, the Ricci scalar of (21) is

$$R = \frac{2N^2 k \sin(2wt - 2\zeta) \sin\theta}{\mu x^2} + \mathcal{O}\big(1/x^3\big). \tag{34}$$

It is obvious that the curvature can be both positive and negative, and oscillates with a frequency of $2w$. Since the spacetime is flat at the spatial infinity, one can use the Komar formula to calculate the energy stored in the geometry [28,29],

$$\begin{aligned} M &= M_p^2 \int\limits_{x=L} *d\xi = -\frac{M_p^2 \pi^2 k N^2 L}{\mu} \sin(2wt - 2\zeta) \\ &= -\frac{\varrho}{2} \frac{k\mu}{w} \sin(2wt - 2\zeta). \end{aligned} \tag{35}$$

This energy fluctuates in time as well and it also goes negative for half of the time. We do not have a good explanation to this result at the moment. For an answer, one may have to better understand how gravity is coupled to quantum matter.

Similar to (35), one can try to calculate the angular velocity of the geometry,

$$J = -\frac{M_p^2}{2} \int\limits_{x=L} *d(\partial_\phi) = \frac{4M_p^2 \pi w N^2 L}{3\mu^2} \cos(2kL) \approx 0. \tag{36}$$

It will be interesting to compare this result with the angular momentum of the spinor field. By analogy with the result in a flat spacetime, we look at the quantity

$$S^\mu = \frac{i}{2} \epsilon^\mu{}_{\nu\rho\sigma} \xi^\nu \bar\psi \gamma^{\rho\sigma} \psi, \quad \text{with } |\epsilon_{\mu\nu\rho\sigma}| = \sqrt{|g|}. \tag{37}$$

We find that all the components vanishes as in (36). If this result is also true for the full solution, then it means that the particle described by the solution does not have a fixed axis of spin.

4. Summary

We have presented an approximate solution to the Einstein–Dirac system. It solves the coupled Einstein–Dirac equations up to the order $\mathcal{O}(1/x^2)$, with x being the radial coordinate. The solution can be interpreted as describing a single Dirac fermion coexisting with its own gravitational field. The metric approaches that of a flat spacetime at the spatial infinity. If one assumes that the full wave function is everywhere regular in the whole space, then the total energy in the spinor field is just the mass of the particle. The energy in the geometry fluctuates in time, and it is negative for half of the time. For the solution to exist, we also need a non-vanishing wave number k in the radial direction. The value of the wave number is undetermined and we still know very little about its significance.

A natural generalization of the present work is to include a cosmological constant in the spacetime background. We will leave this to future works.

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