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Theory of Neutrino Oscillations in the framework of Quantum Mechanics

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Zusammenfassung

Die Theorie von Neutrinooszillationen, erstmals eingeführt zur Interpretation des Sonnenneutrinoproblems, hat sich bald als nützlich zur Deutung von weiteren Neutrinoexperimenten erwiesen sowie berechtigte Zweifel an der Endgültigkeit des heutigen Standardmodells aufgeworfen. In dieser Arbeit wird die Theorie von Neutrinooszillationen im Vakuum im Rahmen der Quantenmechanik behandelt, wobei zunächst die Wahrscheinlichkeit für Flavoroszillationen auf einfache Weise hergeleitet wird. Die resultierende Formel wird auf Abhängigkeiten untersucht und für den Zwei-Neutrino-Fall gemittelt, was Rückschlüsse auf den Mischungswinkel erlaubt. Die Fragwürdigkeit der bei der einfachen Herleitung gemachten Annahmen wird erläutert und im Anschluss daran der konsistentere Wellenpaketansatz vorgestellt, von welchem zwei zusätzliche Kohärenzbedingungen in Form eines Lokalisierungs– und Kohärenzterms abgeleitet werden. Zwei unterschiedliche Herleitungen der Oszillationsformel mithilfe einer gaußschen und einer allemeineren Wellenpaketform werden gegenübergestellt und die Äquivalenz der resultierenden Kohärenzbedingungen aufgezeigt. Die Grenzen des Wellenpaketansatzes bezüglich der Beschreibung von Produktions– und Detektionsprozess werden beschrieben und zugehörige quantenmechanische Unschärfen diskutiert. Schließlich werden Materie-Effekte für den Zwei-Neutrino-Fall erläutert und die Möglichkeit des Auftretens von CP-Verletzung im Drei-Neutrino-Fall gezeigt.

Abstract

Being first applied to solar neutrinos related to the solar neutrino problem, the theory of neutrino oscillations has shown to provide an expedient explanation of successive neutrino measurements as well as given rise to reasonable doubts about the definitiveness of the present Standard Model. In this thesis, the theory of vacuum neutrino oscillations is reviewed in the framework of quantum mechanics, beginning with the standard derivation of the oscillation probability which is analyzed and, for two neutrino mixing, also averaged, allowing for constraints on the mixing angle. The lack of justification of the underlying assumptions of the standard approach is pointed out and consequently the more consistent wave packet approach is introduced, followed by the analysis of the therein implied (de)coherence effects, embodied in a localization and coherence condition. A Gaussian and a more general approach are contrasted, showing to yield equivalent coherence conditions. The limits of the wave packet approach regarding the account for neutrino production/detection and the corresponding uncertainties are disclosed. Finally, matter effects are addressed in the case of two neutrino mixing and the possibility of occurrence of CP violation for three neutrino mixing is shown.

Contents

1 Introduction

In a way, the discovery of the neutrino is representative for physics in the 20th century until now, where the postulation of particles or effects precedes the experimental discovery and represents an approved method to promote progress in science in contrast to the former custom of solely “inductive” science in a strict sense [1].

In order to remedy the contradictions in energy conservation as well as spin-statistics in the β decay, Wolfgang Pauli postulated in 1930 the existence of a neutral weakly interacting particle, a so-called “neutron”. In particular, the energy spectrum of the β decay was continuous in comparison to discrete α or γ radiation, which could not been understood by considering β decay as a pure two-body decay. Furthermore, in a $\beta^{\pm}$ decay both parent and daughter nucleus have either integer or half-integer spin, so a single $e^{\pm}$ with spin $\frac{1}{2}$ is not able to comply with angular momentum conservation. In 1932, the present-day name “neutrino” was introduced by Enrico Fermi. Finally, in 1956, the (electron) neutrino was discovered by F. Reines and C.L. Cowan via the inverse β decay $\bar{\nu}_e + p \ \rightarrow \ e^+ + n$.

Following the discovery of muon and the lepton, the muon and tau neutrino were discovered.

In the Standard Model (SM), neutrinos – being described by left-handed (chiral) Weyl spinors, implying that neutrinos are massless in the SM – interact only through weak interactions due to their charge neutrality and zero mass. Though neutrino oscillations, which are discussed below, indicate that neutrinos have non-zero, albeit low mass, the effect of gravitational interaction is negligible for most purposes and shall not be taken into account.

Weak interaction can be classified into two types, mediated by different gauge bosons. The first one, in which the electric charge varies by one, $\Delta Q = \pm 1$, is called charged-current (CC) interaction and is mediated by charged $W^{\pm}$ bosons. The second one is called neutral-current (NC) interaction, mediated by neutral Z^0 bosons and does not cause any variation in charge.

Interacting only weakly, the discovery of parity violation in weak interaction had considerable consequences for neutrino physics. Parity violation in weak interaction was indicated by the θ-τ puzzle in 1956 and confirmed in 1957 by C.S. Wu who observed the β decay of polarized ^{60}Co. Likewise, it was observed in the $\pi^+ \rightarrow \mu^+ + \nu_\mu$ decay, indicating that the muon neutrino ν_μ has helicity $h = -1$, i.e. the spin of the neutrino is oriented antiparallel to the direction of its momentum. Indeed, the helicity of neutrinos was found to be always $h = -1$ while for antineutrinos $h = +1$.

It is very important to realize, however, that because it is a neutral particle with a very low mass which has not yet been experimentally determined, the flavor or kind of a neutrino is determined solely by means of its leptonic interaction. That is, for instance, the neutral particle released in the β^- decay concomitant with the electron is called antineutrino $\bar{\nu}_e$ while it is called neutrino ν_e when released concomitant with a positron in the β^+ decay. This feature is crucial for flavor detection and in a way is what allows neutrino oscillations in the first place.

Initially, neutrino oscillations were proposed by B. Pontecorvo in 1957 in analogy with kaon oscillations K^0-$\bar{K}^0$, based on a quantum mechanical description of neutrino states. Though this analogy did not prove successful, neutrino oscillations in terms of flavor oscillations have de facto shown to be of great importance for neutrino physics which became apparent in the late 1960s when the Homestake experiment first measured the solar electron neutrino flux. The measurement data confirmed the predicted Solar Neutrino Problem (SNP) which is at present believed to be due to electron neutrino oscillations in the Sun, pursuant to the MSW effect.

In this thesis, the quantum mechanical description of neutrino oscillations is being discussed, beginning with the Standard derivation of the vacuum neutrino oscillation formula in Sec. 2. Subsequently, the oscillation formula is analyzed, using two neutrino mixing as an example. In Sec. 3, the vacuum oscillation probability is derived using the more proper wave packet approach and therein appearing (coherence) conditions for the occurrence of observable neutrino oscillations are studied. In addition, the impact of quantum mechanical uncertainties in the production/detection process is addressed. A generalization of the phenomenon of neutrino oscillations is performed in Sec. 4, regarding matter effects as well as the consequences of three neutrino mixing. Two (historically) prominent types of neutrino oscillation experiments are discussed briefly in Sec. 5, followed by a concluding discussion in Sec. 6. The calculations only being shortly mentioned in the main part are shown in more detail in Appendix A.

The following discussion is based mainly on the book by C. Giunti and C. W. Kim [2]; Refs. [3, 4] have also been used when so indicated. Moreover, Ref. [2] contains a more detailed discussion of neutrino physics' history which has been mostly followed here.

In order to ensure better readability, some conventions have been adopted in the text. First of all, natural units are used, i.e.

$$c = \hbar = 1. \tag{1.1}$$

Moreover, in the notation of quantum mechanical neutrino states, Greek letters as indices imply flavor states while Roman letters imply mass states.

2 Vacuum oscillation probability

2.1 The Standard Derivation

Considering the properties of flavor neutrinos mentioned above (i.e. smallness of mass, discriminable detection only via CC-interactions), it suggests itself – from a quantum mechanical point of view and in analogy to the mixing of quark flavors participating in weak interaction (cf. Sec. 4.2) – to describe the neutrino flavour state α as a superposition of massive neutrino states

$$|\nu_\alpha\rangle = \sum_j U^*_{\alpha j} |\nu_j\rangle \quad (\alpha = e, \mu, \tau), \tag{2.1}$$

with U being the lepton mixing matrix which specifies the composition of each neutrino flavor state[1]. Conversely, by inverting the relation (2.1) one obtains the massive neutrino state

$$|\nu_j\rangle = \sum_\alpha U_{\alpha j} |\nu_\alpha\rangle. \tag{2.2}$$

Furthermore, the massive neutrino states (or mass eigenstates) are chosen in such a way that they are orthonormal:

$$\langle \nu_j | \nu_k \rangle = \delta_{jk}. \tag{2.3}$$

For physical reasons (i.e. the number of particles must not change[2]), U has to be a unitary transformation, which means

$$U^\dagger U = U U^\dagger = \mathbb{1} \quad \Leftrightarrow \quad \sum_j U^*_{\alpha j} U_{\beta j} = \delta_{\alpha\beta}, \quad \sum_\alpha U^*_{\alpha j} U_{\alpha k} = \delta_{jk}, \tag{2.4}$$

and implies that the flavor states are orthonormal as well, as was expected. In the following discussion, U is assumed to be non-diagonal, or else the whole reasoning would become meaningless.

In contrast to the flavor neutrino states in (2.1), the massive neutrino states hold the property that they are eigenstates of the Hamiltonian,

$$\mathscr{H}_0 |\nu_j\rangle = E_j |\nu_j\rangle, \tag{2.5}$$

with E_j being the energy eigenvalue of state j that can be Taylor expanded for ultra-relativistic neutrinos (i.e. $p = |\vec{p}| \gg m$), yielding

$$E_j = \sqrt{\vec{p}^2 + m_j^2} \simeq E + \frac{m_j^2}{2E}, \tag{2.6}$$

at first order in m_j^2, neglecting the mass contribution to the energy, i.e. $E = |\vec{p}|$.

This means, however, that it is the massive neutrino states which propagate through space with a definite momentum $\vec{p}$ and a definite energy E, not the flavor states. It is obviously a specific feature of neutrinos in contrast to charged leptons that flavor

[1] For antineutrinos, the complex conjugation of U has to be dropped. In the following, only neutrinos are being considered; however, the generalization is straightforward.

[2] A possible deviation due to neutrino decay or absorption is not taken into account since the corresponding probabilities are negligible [5].

states and mass states do not coincide[3].

Knowing that the time evolution of a quantum mechanical state is given by the Schrödinger equation

$$i \frac{\mathrm{d}}{\mathrm{d}t} |\nu_j(t)\rangle = \mathscr{H}_0 |\nu_j(t)\rangle \,, \tag{2.7}$$

it is easily seen that the time evolution of the massive neutrino states is given by the plane wave solution

$$|\nu_j(t)\rangle = e^{-iE_j t} |\nu_j\rangle \,, \tag{2.8}$$

provided that the Hamiltonian is time-independent.

With these results in mind, we can proceed in the derivation of the flavor transition probability, recalling that in quantum mechanics the amplitude of a transition is given by the projection of the final state on the initial state,

$$A_{\psi_\alpha \to \psi_\beta} \equiv \langle \psi_\beta | \psi_\alpha \rangle, \tag{2.9}$$

whereas the transition probability is given by the absolute square of the amplitude,

$$P_{\psi_\alpha \to \psi_\beta} = |A_{\psi_\alpha \to \psi_\beta}|^2. \tag{2.10}$$

Consequently, the transition amplitude $A_{\nu_\alpha \to \nu_\beta}$ for the states specified above is

$$A_{\nu_\alpha \to \nu_\beta}(t) \equiv \langle \nu_\beta | \nu_\alpha(t) \rangle = \sum_{k,j} U^*_{\alpha j} U_{\beta k} e^{-iE_j t} \langle \nu_k | \nu_j \rangle = \sum_j U^*_{\alpha j} U_{\beta j} e^{-iE_j t}, \tag{2.11}$$

which leads to

$$P_{\nu_\alpha \to \nu_\beta}(t) = \sum_{j,k} U^*_{\alpha j} U_{\beta j} U_{\alpha k} U^*_{\beta k} e^{-i(E_j - E_k)t}. \tag{2.12}$$

Assuming that the different mass eigenstates have same momentum (though distinct mass) and – being ultrarelativistic – propagate almost at the speed of light, we can approximate the energy according to Eqn. (2.6) as well as the time between production and detection by the distance L between source and detector, i.e. $L \simeq t$, which allows for the final expression

$$P_{\nu_\alpha \to \nu_\beta}(L, E) = \sum_{j,k} U^*_{\alpha j} U_{\beta j} U_{\alpha k} U^*_{\beta k} \exp\left(-i \frac{\Delta m^2_{jk} L}{2E} \right), \tag{2.13}$$

with Δm^2_{jk} being the squared-mass difference $m_j^2 - m_k^2$. In the following, for convenience, the massive neutrino states are assumed to be ordered in such a way that the heavier the massive neutrino state, the larger the number of the index j, leading e.g. in the two-neutrino mixing case to the positive squared-mass difference Δm^2_{21}.

By the way, the same result (2.13) is obtained by assuming the mass eigenstates to have same energy. Approximating

$$p_j = \sqrt{E^2 - m_j^2} \simeq E - \frac{m_j^2}{2E}, \tag{2.14}$$

[3] To be more exact, there is the possibility of a coherent superposition of charged lepton states, as can be seen from considering a W-boson decay. Though, in contrast to neutrinos which share the common property of having very low mass irrespective of their flavor, charged leptons vary considerably in mass, so no coherent propagation of flavor states is possible (cf. Sec. 6, Ref. [6]).

and using the phase factor for the spatial propagation of the mass states, $e^{ip_j x}$, one obtains the same oscillation probability as in Eqn. (2.13) [4].

The flavor oscillation probability (2.13) is usually called *transition probability* when $\alpha \neq \beta$ while *survival probability* when $\alpha = \beta$.

Though the so obtained oscillation probability in (2.13) is correct, its derivation does not stand on firm ground since the underlying assumptions are not consistent. Let us therefore at this point summarize the assumptions which are made in the derivation of the standard oscillation probability for further discussion:

(i) Flavor neutrino states can be described by their mass states as in Eqn. (2.1), independent of the production or detection process.

(ii) The massive neutrino states have same momentum (or same energy), but different mass.

(iii) The propagation time t between production and detection can be approximated by the distance L between source and detector (*light ray approximation*).

Assumption (i) is justified only in the case that the neutrino experiments are not sensitive to mass-difference induced variations of the production/detection amplitudes. This property is somewhat discussed in Section 3 and in general valid for present-day neutrino oscillation experiments. A satisfying discussion of this topic is not possible, however, without a proper quantum field theoretic treatment (cf. Sec. 6).

Assumption (ii) is in fact in general not tenable as discussed in Section 3 and requires a justification of the derived oscillation probability from behind, which can be done by using wave packet respectively quantum field theoretic treatment.

Assumption (iii) is utilized explicitly only by the same momentum assumption, yet not justified in plane-wave treatment at all since plane-wave treatment is incompatible with neutrino propagation (see Section 3 as well).

The implied assumption that neutrinos are ultrarelativistic is certainly appropriate in general, yet a proper justification in this regard is given in Section 3.1.

Anyhow, before discussing the validity of this simple derivation in more detail, let us start by drawing some conclusions from the oscillation probability (2.13).

2.2 Two neutrino mixing

It is clear from Eqn. (2.13) that in order to obtain a concrete oscillation formula that allows for a more detailed analysis especially in regard to measurement data, the mixing matrix U has to be specified. In many cases, it is adequate to consider only two neutrino mixing, i.e. the mixing of two flavor neutrinos which may be either two of the three known flavor neutrinos (e.g. ν_e, ν_μ) or linear combinations of flavor neutrinos (e.g. $\nu_e, c_\mu \nu_\mu + c_\tau \nu_\tau$), satisfying the normalization condition $c_\mu^2 + c_\tau^2 = 1$. Such an approximation is often possible since many experiments are not sensitive to three neutrino mixing, e.g. in the case when only a certain flavor neutrino can be detected.

Furthermore, considering only two neutrino mixing allows for simple calculations with fewer parameters involved, which makes the interpretation of measurement data even more straightforward.

For two neutrino mixing, the mixing matrix can be parametrized by a simple rotation matrix because the three additional phases which a general unitary matrix would possess can be eliminated by a rephasing of the neutrino fields as discussed in Section 4.2.

Thus, specifying the mixing matrix in (2.13) by

$$U = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}, \tag{2.15}$$

one obtains after a straightforward calculation shown in Appendix A.1:

$$P_{\nu_\alpha \to \nu_\beta} = \frac{1}{2}\sin^2 2\vartheta \left[1 - \cos\left(\frac{\Delta m^2 L}{2E}\right)\right] \quad (\alpha \neq \beta). \tag{2.16}$$

It is worth noting that in Eqn. (2.16) the oscillation probability is symmetric under the exchange $\vartheta \leftrightarrow \pi/2 - \vartheta$ which implies that the oscillation probability is degenerate with respect to the mixing angle. This is the case, however, only for vacuum oscillations and is resolved by matter effects which play an important role for many of the observed neutrino oscillations (cf. Section 4.1). Though resulting in the same oscillation probability, the two mixing angles ϑ, $\pi/2 - \vartheta$ are by no means physically equivalent: the mixing angle determines the composition of the flavor neutrinos, differing substantially for distinct mixing angles $0 \leq \vartheta \leq \pi/2$.

The simple oscillation formula (2.16) is very useful for drawing concrete conclusions about neutrino oscillations as well as for the analysis of experimental data. Yet, before exploiting Eqn. (2.13) in this sense, some more general implications lend itself to examination.

2.3 Analysis of the oscillation formula

First of all, it is instructive to analyze the dependencies of the flavor oscillation probability (2.13) with respect to “constants of nature”, i.e. quantities which do not depend on the production/detection process, for these quantities require experimental determination.

In detail, Eqn. (2.13) depends on the squared-mass differences of the massive neutrino states, not their absolute value. Hence, measurements of neutrino oscillations do not allow to determine the absolute value of neutrino masses, but their squared-mass differences; the absolute mass scale needs to be set by other experiments. Neutrino oscillation measurements only show that neutrinos are not massless and can provide a lower bound for the neutrino masses.

Besides, Eqn. (2.13) depends on the quartic products of the elements of the lepton mixing matrix U[4],

$$U^*_{\alpha j} U_{\beta j} U_{\alpha k} U^*_{\beta k}, \tag{2.17}$$

which imply that the oscillation probability does not depend on the specific parameterization of the mixing matrix respectively the choice of the phases (see Section 4.2). Admittedly, to allow for a theoretical model involving parameters subject to experimental determination respectively matching, the parameterization needs to be specified.

For a more lucid discussion it is convenient to split the oscillation probability (2.13) into the real and imaginary parts of the mixing matrix components as shown in Ap-

[4] Relating to the statement above about constants of nature, this obviously does not mean that the mixing angles are left untouched by matter effects (cf. Section 4.1), but that they are fixed for a given matter potential, e.g. for vacuum.

pendix A.2. In fact, writing Eqn. (2.13) as

$$P_{\nu_\alpha\to\nu_\beta}(L,E) = \delta_{\alpha\beta} - 4\sum_{j>k} \mathfrak{Re}\left[U^*_{\alpha j}U_{\beta j}U_{\alpha k}U^*_{\beta k}\right]\sin^2\left(\frac{\Delta m^2_{jk}L}{4E}\right) + 2\sum_{j>k} \mathfrak{Im}\left[U^*_{\alpha j}U_{\beta j}U_{\alpha k}U^*_{\beta k}\right]\sin\left(\frac{\Delta m^2_{jk}L}{2E}\right) \tag{2.18}$$

allows for straightforward conclusions in the following considerations.

Except for the transition probability $P_{\nu_\alpha\to\nu_\beta}$, one can also consider the survival probability $P_{\nu_\alpha\to\nu_\alpha}$, linked to the transition probability via (α fixed)

$$P_{\nu_\alpha\to\nu_\alpha} = 1 - \sum_{\beta\neq\alpha} P_{\nu_\alpha\to\nu_\beta}, \tag{2.19}$$

and obtained explicitly by setting $\alpha=\beta$ in Eqn. (2.18),

$$P_{\nu_\alpha\to\nu_\alpha}(L,E) = 1 - 4\sum_{j>k} \mathfrak{Re}|U_{\alpha j}|^2|U_{\alpha k}|^2\sin^2\left(\frac{\Delta m^2_{jk}L}{4E}\right). \tag{2.20}$$

Which of these two probabilities is most relevant for the analysis of experimental data depends on the specific experiment, i.e. whether the experiment is an appearance or disappearance experiment, in the sense as defined in Section 5.1.

The validity of Eqn. (2.19) hinges on the conservation of probability

$$\sum_\beta P_{\nu_\alpha\to\nu_\beta} = 1, \quad \text{respectively} \quad \sum_\alpha P_{\nu_\alpha\to\nu_\beta} = 1, \tag{2.21}$$

for fixed α respectively β. At this point, the conservation of probability (2.21) follows directly from the unitarity of the mixing matrix (2.4) which can be seen by summing first over β (respectively α) and making use of the Kronecker delta. In general, as discussed in Section 3, production and detection processes have to be taken into account to obtain the oscillation probability. If the the production/detection amplitudes now exhibit distinct dependencies on the massive neutrino momenta (e.g. due to a particular energy threshold for the detection process) or the like, the conservation of probability is no longer ensured by the unitarity of the mixing matrix. In this case, in the quantum mechanical approach the constraint (2.21) is usually imposed by hand [5].

Furthermore, neutrino oscillations may exhibit a CP asymmetry. A CP transformation denotes combined charge (C) and parity (P) transformation by which neutrinos and antineutrinos are related via

$$\nu_\alpha \xleftrightarrow{\text{CP}} \bar\nu_\alpha, \quad \text{and consequently} \quad \nu_\alpha\to\nu_\beta \quad \xleftrightarrow{\text{CP}} \quad \bar\nu_\alpha\to\bar\nu_\beta. \tag{2.22}$$

A CP transformation interchanges neutrinos with antineutrinos and reverses their helicity, thus being consistent with the case of Majorana neutrinos whose type does not change by charge conjugation, but is determined by the helicity[5]. Likewise, time re-

[5] To be precise, in the Majorana case neutrino and antineutrino are linked by the Majorana condition $\nu_j = \nu_j^C$. Yet, since neutrinos seem to interact weakly only with definite helicity (cf. Sec. 1), the type of a neutrino is rather given by its helicity, so the states with negative helicity are conventionally called *neutrinos* whereas *antineutrinos* when having positive helicity [2]. However, definite helicity exists only for massless neutrinos.

versal (T) interchanges initial and final states.

In order to quantify CP violation, denoting an asymmetry of the oscillation probabilities with respect to CP transformations, one defines

$$A_{\alpha\beta}^{\mathrm{CP}} = P_{\nu_\alpha \to \nu_\beta} - P_{\bar{\nu}_\alpha \to \bar{\nu}_\beta}, \tag{2.23}$$

yielding by the help of Eqn. (2.18)

$$A_{\alpha\beta}^{\mathrm{CP}} = 4 \sum_{j>k} \mathfrak{Im} \left[U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^* \right] \sin \left(\frac{\Delta m_{jk}^2 L}{2E} \right). \tag{2.24}$$

In the case of two-neutrino mixing, the asymmetry term (2.24) is identical to zero since the mixing matrix is real. Though, this must not hold in general, as can be seen for the realistic case of three neutrino mixing (cf. Section 4.2).

Since CPT transformations (combined charge, parity and time inversions) are a symmetry of any local quantum field theory [2, p.257], we have $P_{\nu_\alpha \to \nu_\beta} = P_{\bar{\nu}_\beta \to \bar{\nu}_\alpha}$, showing explicitly that only appearance experiments, respectively the measurement of flavor transitions can yield information about CP violation in neutrino oscillations.

A so far disregarded case is the possibility of incoherent production or detection (also discussed in more detail in Section 3) which does not allow for the interference in Eqn. (2.13). In fact, the incoherent oscillation probability is given by the constant term

$$P_{\nu_\alpha \to \nu_\beta}^{\text{incoherent}} = \sum_j |\langle \nu_\beta | \nu_j \rangle e^{-iE_j t} \langle \nu_j | \nu_\alpha \rangle|^2 = \sum_j |U_{\alpha j}|^2 |U_{\beta j}|^2, \tag{2.25}$$

coinciding with the the incoherent average $\langle P_{\nu_\alpha \to \nu_\beta} \rangle$ over detector uncertainties (e.g. energy or distance) due to decoherence effects.

Let us now define the characteristic length

$$L_{jk}^{\text{osc}} = \frac{4\pi E}{\Delta m_{jk}^2}, \tag{2.26}$$

which denotes the distance at which the phase generated by Δm_{jk}^2 becomes equal to 2π. With the help of this definition, the sensitivity of neutrino oscillation experiments to Δm_{jk}^2, given a fixed ratio L/E, can be discussed qualitatively. This discussion is most stringent for two neutrino mixing, which is why we adopt to this case.

It can be seen from Eqn. (2.16) that no oscillation can be observed for

$$L \ll \frac{L^{\text{osc}}}{2\pi}, \tag{2.27}$$

due to the negligible variation in phase of the different massive neutrino states. Nor can any oscillation be measured if

$$L \gg \frac{L^{\text{osc}}}{2\pi}, \tag{2.28}$$

owing to the fact that as a result of detector uncertainties, only the averaged probability $\langle P_{\nu_\alpha \to \nu_\beta} \rangle$ is measured, yielding information only about the mixing angle

$$\langle P_{\nu_\alpha \to \nu_\beta} \rangle = \frac{1}{2} \sin^2 2\vartheta \quad (\alpha \neq \beta), \tag{2.29}$$

in the case of two neutrino mixing. The averaged probability (2.29) results from Eqn. (2.25), specified with the mixing matrix (2.15).

In contrast, if

$$L \sim \frac{L^{\mathrm{osc}}}{2\pi}, \tag{2.30}$$

the experiment is able to measure oscillations generated by the squared-mass difference Δm^2, so by means of (2.30) one can determine the order of sensitivity to the squared-mass difference Δm^2, given a ratio L/E.

As afore mentioned, it is impossible to determine L respectively E without uncertainties in practice, for both source and detector have spatial uncertainties and uncertainties in energy, i.e. the emission or detection process takes finite time. Hence, it is natural for the source to possess an energy spectrum respectively for the detector to possess a finite energy resolution.

In the case of two-neutrino mixing, one can account for this uncertainties by averaging the cosine in the transition probability (2.16) over an appropriate distribution $\phi(L/E)$ of L/E. For simplicity, we assume the distribution to have Gaussian shape,

$$\phi\left(\frac{L}{E}\right) = \frac{1}{\sqrt{2\pi\sigma^2_{L/E}}} \exp\left[-\frac{(L/E - \langle L/E\rangle)^2}{2\sigma^2_{L/E}}\right], \tag{2.31}$$

with $\sigma_{L/E}$ being the standard deviation and $\langle L/E\rangle$ the mean. Calculating the average

$$\left\langle \cos\left(\frac{\Delta m^2 L}{2E}\right)\right\rangle = \int \cos\left(\frac{\Delta m^2 L}{2E}\right) \phi\left(\frac{L}{E}\right) \mathrm{d}\frac{L}{E} \tag{2.32}$$

as shown in Appendix A.3, one obtains as result (A.8),

$$\left\langle \cos\left(\frac{\Delta m^2 L}{2E}\right)\right\rangle = \cos\left(\frac{\Delta m^2}{2}\left\langle\frac{L}{E}\right\rangle\right) \exp\left[-\frac{1}{2}\left(\frac{\Delta m^2}{2}\sigma_{L/E}\right)^2\right], \tag{2.33}$$

which can be inserted into Eqn. (2.16), yielding the averaged transition probability

$$\langle P_{\nu_\alpha\to\nu_\beta}\rangle = \frac{1}{2}\sin^2 2\vartheta \left[1 - \cos\left(\frac{\Delta m^2}{2}\left\langle\frac{L}{E}\right\rangle\right) \exp\left[-\frac{1}{2}\left(\frac{\Delta m^2}{2}\sigma_{L/E}\right)^2\right]\right] \quad (\alpha \neq \beta). \tag{2.34}$$

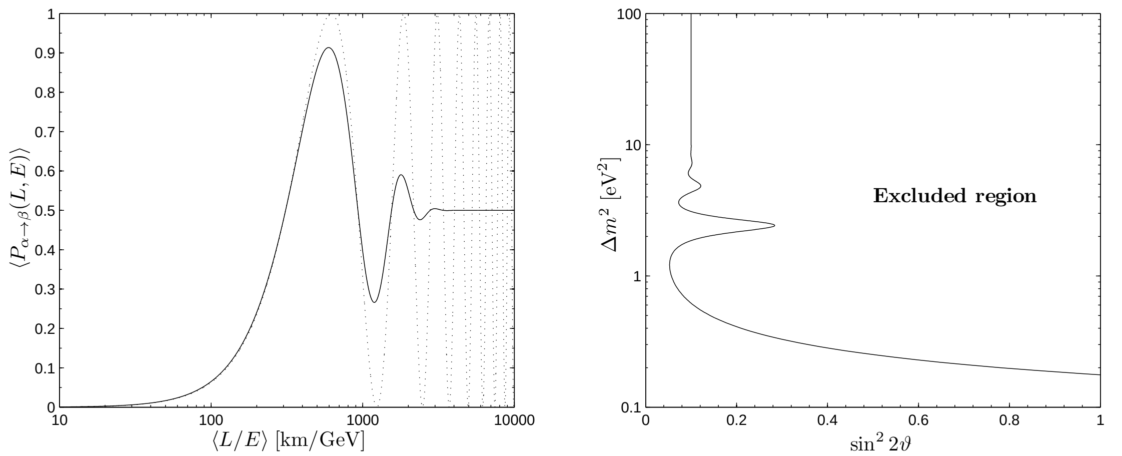
Eqn. (2.34) indicates that the standard deviation $\sigma_{L/E}$ causes an attenuation of the oscillatory behaviour of the transition probability. In fact, due to a stronger averaging out, the larger the uncertainty $\sigma_{L/E}$ becomes, the more the oscillation amplitude is suppressed, leading in the limit of large uncertainties $\sigma_{L/E}$ to (2.29).

Using $\Delta m^2 = 2 \times 10^{-3}$ eV^2, $\sigma_{L/E} = 0.2\,\langle L/E\rangle$ and $\vartheta = \pi/4$, the averaged transition probability (solid line) is plotted over $\langle L/E\rangle$ in logarithmic scale in Figure 2.1a; the dotted line indicates the unaveraged transition probability. One can clearly see that for a large ratio $\langle L/E\rangle$, only the averaged probability (2.29) is observed, originated in the larger uncertainty $\sigma_{L/E}$. This implies that even if the experiment does not observe any neutrino oscillation, the data can be used to set an upper bound on the averaged transition probability,

$$\langle P_{\nu_\alpha\to\nu_\beta}(L,E)\rangle \leq P^{\max}_{\nu_\alpha\to\nu_\beta}, \tag{2.35}$$

where $P^{\max}_{\nu_\alpha\to\nu_\beta}$ denotes the measured transition probability. In case of two-neutrino mixing, this can be rewritten explicitly, yielding an upper limit of $\sin^2 2\vartheta$ as a function of Δm^2 for fixed $\langle L/E\rangle$, $\sigma_{L/E}$:

$$\sin^2 2\vartheta \leq \frac{2P^{\max}_{\nu_\alpha\to\nu_\beta}}{1-\left\langle \cos\left(\frac{\Delta m^2 L}{2E}\right)\right\rangle}. \tag{2.36}$$



(a) Averaged transition probability as a function of $\langle L/E\rangle$.

(b) Exclusion plot in the $\sin^2 2\vartheta$-Δm^2 plane.

Figure 2.1: Analysis of the two-neutrino flavor transition probability, after Fig.s 7.2, 7.4 in Ref. [2].

Plotting reversely the squared-mass difference Δm^2 as a function of $\sin^2 2\vartheta$ for a fixed value $\langle L/E\rangle$, a so called *exclusion plot* is obtained as shown in Figure 2.1b, where the region to the right of the solid line corresponds to the excluded region of the oscillation parameters. For Figure 2.1b, the values $P^{\max}_{\nu_\alpha\to\nu_\beta} = .05$, $\langle L/E\rangle = 1$ km/GeV and $\sigma_{L/E} = 0.15$ km/GeV were used. Figure 2.1b depicts also the limits on the significance of the exclusion plot with respect to $\sin^2 2\vartheta$ for large and small squared-mass differences: for small Δm^2, the variation of the cosine function in (2.34) is negligible, so no constraint on $\sin^2 2\vartheta$ can be set. Likewise, for large Δm^2 the oscillation length (2.26) is small compared to the uncertainty $\sigma_{L/E}$, giving rise to a vanishing average. The bound on $\sin^2 2\vartheta$ is most confining if the average of the cosine function in Eqn. (2.36) is equal to minus one, which is the case for

$$\frac{\Delta m^2}{2}\left\langle\frac{L}{E}\right\rangle \simeq \pi. \tag{2.37}$$

It is clear that by superposing the exclusion curves of several measurements, the allowed region for $\sin^2 2\vartheta$, Δm^2 is gradually restricted, leading ideally to a clearly localized, small region for the allowed parameters. A positive measurement of flavor transitions allows for an exclusion plot as well, with the only difference that in this case both lower and upper bound can be set on the averaged transition probability, leading to a band of allowed parameters in the $\sin^2 2\vartheta$-Δm^2 plane. In general, however, there is no global L/E distribution, so the exclusion plots have to be generated subsequently, taking the distribution function of each interval L/E into account.

3 A more proper discussion

3.1 Wave packet approach

As has become clear at the end of Section 2.1, the assumptions made in the standard derivation of the neutrino oscillation probability are not at all *a priori* justified, but rather stress the need for an accurate treatment providing some hints why in this special case an improper derivation yields a correct result.

Indeed, the assumption of propagation is inconsistent with the plane wave solution (2.8), using $t \simeq L$. This can be seen by asking the simple question how a quantum mechanical neutrino state with definite momentum would look like in position space. There, its spatial propagation is given by the phase factor e^{ipx} which implies that no localization is possible, in contrast to the assumption of propagation between production and detection.

In addition, the equal momentum (or energy) assumption is not tenable. Approximating the energy analogous to (2.6) and considering a Lorentz frame $\mathcal{O}$ in which the neutrinos have equal momentum, yet different energy due to the mass contribution, one can consider another Lorentz frame $\mathcal{O}'$, having velocity v with respect to $\mathcal{O}$ along the neutrino path (in opposite direction of their motion) [7]. In this frame, the massive neutrino state j has momentum p'_j,

$$p'_j = \gamma p + v\gamma \left(p + \frac{m_j^2}{2p}\right) = \sqrt{\frac{1+v}{1-v}}p + \frac{v}{\sqrt{1-v^2}}\frac{m_j^2}{2p} = p' + \frac{v}{1-v}\frac{m_j^2}{2p'}, \tag{3.1}$$

with $p' = \sqrt{\frac{1+v}{1-v}}\,p$ and γ being the Lorentz factor $\gamma \equiv 1/\sqrt{1-v^2}$. Calculating the difference in momentum in frame $\mathcal{O}'$, one obtains

$$\Delta p'_{jk} = \frac{v}{1-v}\frac{\Delta m_{jk}^2}{2p'}, \tag{3.2}$$

showing that even if neutrinos had equal momentum[6] in one Lorentz frame, this does not hold any more in another frame.

Besides, even the assumption that the massive neutrino states have definite momenta respectively energies is untenable. From energy-momentum conservation follows that the kinematic properties of massive neutrinos are determined by the kinematics of the production process, i.e. the particles involved in the production process need to have definite momenta respectively energies. In this case, since flavor neutrinos do not have definite kinematic properties (consisting of massive neutrinos), energy-momentum conservation must hold for all massive neutrino states simultaneously. This, however, does not comply with the definite kinematic properties of the initial particles, so the whole concept of definite energy/momentum of the massive neutrino states has to be abandoned in a proper treatment. Furthermore, the stipulation of definite kinematic properties of the initial particles would constrain them to be completely delocalized in time and space, thus contradicting the assumption of a localized process. Though, as stressed in Refs. [4, 8], exact energy-momentum conservation *does* hold when applied to

[6] The same argument contradicts analogous the equal energy assumption; one needs just to replace approximation (2.6) by (2.14) and consider ΔE_{jk} [7].

all particles in the system and with respect to each individual momentum component contributing to a wave packet, whereas with respect to a specific process involving wave packets, energy-momentum conservation is fulfilled only up to some quantum mechanical uncertainties.

In the framework of quantum mechanics, a more proper treatment of neutrino oscillations is based on the wave packet approach. In this approach, neutrinos are described by *wave packets*, i.e. (normalized) superpositions of plane waves, allowing for the description of real localized particles. Instead of a unique momentum, a wave packet is described by a peaked momentum distribution, thus having a mean, so to say "effective" momentum and a momentum uncertainty, quantified for Gaussian shape by the standard deviation. The velocity of a neutrino is consequently given by the group velocity of the wave packet which in general differs for distinct massive neutrinos, giving rise to (de)coherence effects discussed below. Moreover, the uncertainties in energy/momentum depend on the production/detection process and need therefore to be taken into account jointly.

To allow for a more accurate discussion, which follows Ref. [2], let us at first consider an asymptotic final state $|f\rangle$ resulting from an interaction with the asymptotic initial state $|i\rangle$, which is given in the framework of quantum field theory by the action of the S-matrix operator $\mathcal{S}$ on the initial state,

$$|f\rangle = \mathcal{S}\,|i\rangle\,. \tag{3.3}$$

Regarding a neutrino produced in the generic decay process

$$P_I \to P_F + \ell_\alpha^+ + \nu_\alpha, \tag{3.4}$$

the production amplitude of state $|\nu_j, \ell_\alpha^+, P_F\rangle$ in the decay above is given by

$$\mathcal{A}_{\alpha j}^P = \langle \nu_j, \ell_\alpha^+, P_F | f\rangle = \langle \nu_j, \ell_\alpha^+, P_F|\,\mathcal{S}\,|P_I\rangle\,. \tag{3.5}$$

Likewise, the detection amplitude $\mathcal{A}_{\alpha j}^D$ indicating the probability with which the charged lepton ℓ_α^- is released through CC interaction of ν_j with D_I is given by

$$\mathcal{A}_{\alpha j}^D = \langle \nu_j, D_I|\,\mathcal{S}^\dagger\,|D_F, \ell_\alpha^-\rangle\,. \tag{3.6}$$

Taking into account leptonic mixing in CC interactions (cf. Eqn. (4.29)), the amplitudes $\mathcal{A}_{\alpha j}^P$, $\mathcal{A}_{\alpha j}^D$ can be written as

$$\mathcal{A}_{\alpha j}^P = U_{\alpha j}^* \mathcal{M}_{\alpha j}^P, \quad \mathcal{A}_{\alpha j}^D = U_{\alpha j}^* \mathcal{M}_{\alpha j}^D, \tag{3.7}$$

with the interaction matrix elements $\mathcal{M}_{\alpha j}^{P,D}$.

The neutrino states relevant for the discussion of neutrino oscillations are described by the wave packet states

$$|\nu_\alpha^{P,D}\rangle = N_\alpha^{P,D} \sum_j \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_j} \sum_h \mathcal{A}_{\alpha j}^{P,D}(\vec{p}, h)\,|\nu_j(\vec{p}, h)\rangle\,, \tag{3.8}$$

where h denotes the helicity of the states and $N_\alpha^{P,D}$ the normalization factors

$$N_\alpha^{P,D} = \left(\sum_j \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_j} \sum_h |\mathcal{A}_{\alpha j}^{P,D}(\vec{p}, h)|^2 \right)^{-1/2} . \tag{3.9}$$

Neglecting the normalization factors, the flavor transition amplitude $A_{\nu_\alpha \to \nu_\beta}$ reads

$$\begin{aligned} A_{\nu_\alpha \to \nu_\beta}(\vec{L}, T) &= \langle \nu_\beta^D | \, e^{-i(E_j T - \vec{p} \cdot \vec{L})} \, | \nu_\alpha^P \rangle \\ &\propto \sum_j \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_j} \sum_h \mathcal{A}_{\beta j}^{D*}(\vec{p}, h) \mathcal{A}_{\alpha j}^{P}(\vec{p}, h) e^{-i(E_j T - \vec{p} \cdot \vec{L})}. \end{aligned} \tag{3.10}$$

It is important to stress at this point the reason for the phase factor $e^{-i(E_j T - \vec{p} \cdot \vec{L})}$ looking slightly different than the one used before in Eqn. (2.8). The difference originates in the fact that as a result of a more accurate treatment, spatial separation of production and detection have been explicitly taken into account in Eqn. (3.10) (i.e. production and detection are separated by a space-time interval $(\vec{L}, T)$), evoking the additional phase, causing Lorentz invariance of the evolution factor[7].

As has been stressed in Ref. [2, p.297], the Lorentz invariance of the evolution phase implies that the assumption of neutrinos being ultrarelativistic is justified because an appropriate Lorentz frame can always be chosen in which this assumption holds, having no influence on the phase of the (Lorentz invariant) oscillation probability.

For simplicity, aiming to derive the oscillation probability, let us approximate the production and detection amplitudes by

$$\mathcal{A}_{\alpha j}^{P}(\vec{p}, h) \mathcal{A}_{\beta j}^{D*}(\vec{p}, h) \propto U_{\alpha j}^* U_{\beta j} \exp \left[-\frac{(\vec{p} - \vec{\bar{p}}_j)^2}{4\sigma_p^2} \right], \tag{3.11}$$

where $\vec{\bar{p}}_j$ denotes the average mean momentum of the massive neutrino j and σ_p denotes the combined momentum uncertainty; again, only neutrinos are being considered.

Though being discussed below, it is important to realize at this point that σ_p depends on both production *and* detection uncertainties. The same applies to $\vec{\bar{p}}_j$: this average momentum is due to the fact that the detection process might single out slightly different momenta of the massive neutrino states than the ones with which they were produced. This follows from energy–momentum conservation and is accounted for by $\vec{\bar{p}}_j$: only flavor neutrinos can be detected which do not have definite kinematic properties, so production and detection momenta do not need to coincide exactly, particularly if the processes have different energy thresholds or the like. Hence, the wave packet treatment discussed in this section is an effective treatment but requires additional input in order to yield meaningful results.

Furthermore, in order to obtain Eqn. (3.11), additional approximations had to be made. In particular, the detection experiment is assumed not to be sensitive to the dependence of the interaction matrix elements (which are assumed to be smooth functions of the massive neutrino momenta) on the neutrino masses and for σ_p is assumed to hold: $\sigma_p \ll \langle p_j \rangle = |\vec{\bar{p}}_j|$. Please note that the validity of the above assumptions is crucial for the correctness of the neutrino states used for the subsequent derivation in the

[7] The Lorentz invariance of the phase is easily seen by writing $-i(E_j T - \vec{p} \cdot \vec{L}) = -ip_\mu x^\mu$ in an obvious notation.

framework of quantum mechanics. Under these assumptions, however, the interaction matrix elements can be factored out of Eqn. (3.10), as shown in Eqn. (3.11).

Expanding $E_j(\vec{p})$ around $\vec{\tilde{p}}$,

$$E_j(\vec{p}) \simeq E_j(\vec{\tilde{p}}) + \left.\frac{\partial E_j(\vec{p})}{\partial \vec{p}}\right|_{\vec{p}=\vec{\tilde{p}}_j} (\vec{p} - \vec{\tilde{p}}_j) = \tilde{E}_j + \vec{v}_j(\vec{p} - \vec{\tilde{p}}_j), \tag{3.12}$$

one can rewrite the amplitude $A_{\nu_\alpha \to \nu_\beta}$ as

$$A_{\nu_\alpha \to \nu_\beta}(\vec{L}, T) \propto \sum_j U^*_{\alpha j} U_{\beta j} e^{-i\tilde{E}_j T + i\vec{\tilde{p}}_j \cdot \vec{L}} \int \mathrm{d}^3 p \; e^{i(\vec{p}-\vec{\tilde{p}}_j)\cdot(\vec{L}-\vec{v}_j T) - \frac{(\vec{p}-\vec{\tilde{p}}_j)^2}{4\sigma_p^2}}. \tag{3.13}$$

In Eqn. (3.12), $\tilde{E}_j = \sqrt{(\vec{\tilde{p}}_j)^2 + m_j^2}$ and $\vec{v}_j = \vec{\tilde{p}}_j / \tilde{E}_j$ is a sort of an "effective" group velocity since it depends on both production and detection process.

Performing the integral over the momentum space as shown in (A.9), we obtain

$$A_{\nu_\alpha \to \nu_\beta}(\vec{L}, T) \propto \sum_j U^*_{\alpha j} U_{\beta j} \exp\left[-i\tilde{E}_j + i\vec{\tilde{p}}_j \cdot \vec{L} - \frac{(\vec{L} - \vec{v}_j T)^2}{4\sigma_x^2}\right], \tag{3.14}$$

which allows for the calculation of the oscillation probability. Yet, since most experiments do not measure production and detection times, the oscillation probability must be integrated over the unmeasured time T. This integration is not invalidated even in cases when the neutrino time of flight is measured, since the typical time extension σ_t of the wave packets is much smaller than the ordinary time scale of the measurement [4]. Accordingly,

$$P_{\nu_\alpha \to \nu_\beta}(\vec{L}) \propto \int \mathrm{d}T \, |A_{\nu_\alpha \to \nu_\beta}(\vec{L}, T)|^2, \tag{3.15}$$

where the proportionality sign indicates, inter alia, that the proper normalization factor has been omitted. Usually, to avoid the calculation of the proper normalization factor, one imposes the unitarity condition (2.21) by hand, which is an *ad hoc* procedure and should not happen in a consistent framework. However, in the framework of QM this potential inconsistency cannot be avoided easily [5].

The evaluation of Eqn. (3.15) yields after a slightly lengthy but straightforward calculation shown in A.4:

$$P_{\nu_\alpha \to \nu_\beta}(\vec{L}) \propto \sum_{j,k} U^*_{\alpha j} U_{\beta j} U_{\alpha k} U^*_{\beta k} \exp\left\{-i\left[(\tilde{E}_j - \tilde{E}_k)\frac{\vec{v}_j + \vec{v}_k}{v_j^2 + v_k^2} - (\vec{\tilde{p}}_j - \vec{\tilde{p}}_k)\right] \cdot \vec{L}\right\}$$
$$\times \exp\left\{-\frac{L^2}{2\sigma_x^2} + \frac{(\vec{v}_j \cdot \vec{L})^2 + (\vec{v}_k \cdot \vec{L})^2}{2\sigma_x^2(v_j^2 + v_k^2)} - \frac{\left[(\vec{v}_j - \vec{v}_k)\cdot \vec{L}\right]^2}{4\sigma_x^2(v_j^2+v_k^2)} - \frac{(\tilde{E}_j - \tilde{E}_k)^2}{4\sigma_p^2(v_j^2+v_k^2)}\right\}. \tag{3.16}$$

Eqn. (3.16) clearly shows that the neutrino oscillation probability is given by an oscillating phase and an attenuating term, both with various dependencies. To simplify the inspection of the oscillation probability, it is useful to introduce some approximations to allow for clear-cut conclusions.

At first, it is easily seen that the first two terms in the damping term cancel if the velocities $\vec{v}_j, \vec{v}_k$ and $\vec{L}$ are collinear. On the other hand, if the deviation from collinearity increases, the damping increases as well. This fact is brought about by the physical reason that the trajectories of the massive neutrinos don't "hit" the detector even in the present spatial and momentum uncertainties if the deviation of the neutrino momenta from collinearity with $\vec{L}$ is not extremely small, thus preventing unsuppressed interference.

In practice, where for the wave packet length usually $\sigma_x^\nu \ll L$ holds, the effect of deviation from collinearity is negligible, reducing the three dimensional expression (3.16) to a one dimensional one. This is assumed to be the case in the following since a more general treatment does not shed much more light on the conditions for observable neutrino oscillations (for a slightly more detailed discussion on this topic see Ref. [2]).

Hence, taking into account the (first order) deviation of massive neutrino momenta due to the neutrino mass, the average momenta can be approximated by

$$\tilde{p}_j \simeq E - \xi \, \frac{m_j^2}{2E}, \quad \text{with} \quad \xi = -2E \left. \frac{\partial p_j}{\partial m_j^2} \right|_{m_j=0}, \tag{3.17}$$

where $E = p$ denotes momentum respectively energy of a massless neutrino and the value of ξ the "sensitivity" to squared–mass deviation, depending on the production process, particularly on the energy released in the process.

The energy $\tilde{E}_j$ and the velocity v_j can be similarly approximated, yielding

$$\tilde{E}_j \simeq \tilde{p}_j + \frac{m_j^2}{2\tilde{p}_j} \simeq E + (1-\xi)\frac{m_j^2}{2E}, \tag{3.18}$$

$$v_j = \frac{\tilde{p}_j}{\tilde{E}_j} \simeq \frac{E - \xi\frac{m_j^2}{2E}}{E + (1-\xi)\frac{m_j^2}{2E}} \simeq 1 - \frac{m_j^2}{2E^2}, \tag{3.19}$$

at first order in m_j^2.

Respecting the approximations (3.17), (3.18) and (3.19), the oscillation phase in Eqn. (3.16) reads, at first order in $m_{j,k}^2$,

$$\begin{aligned} \left[(\tilde{E}_j - \tilde{E}_k)\frac{v_j + v_k}{v_j^2 + v_k^2} - (\tilde{p}_j - \tilde{p}_k)\right] L &\simeq \frac{\Delta m_{jk}^2 L}{2E}\left[(1-\xi)\,\frac{v_j + v_k}{v_j^2 + v_k^2} + \xi\right] \\ &\simeq \frac{\Delta m_{jk}^2 L}{2E}\left[(1-\xi)\left(1 + \frac{m_j^2 + m_k^2}{4E^2}\right) + \xi\right] \\ &\simeq \frac{\Delta m_{jk}^2 L}{2E}, \end{aligned} \tag{3.20}$$

exhibiting the well–known dependency on the squared-mass differences. It is indeed very remarkable that the deviations in energy and momentum which depend on the production process cancel in the oscillation phase, since these deviations are by no means negligible. Considering the pion decay $\pi^+ \to \mu^+ + \nu_\mu$, we have for the energy of the massive neutrino ν_j in the rest frame of the pion according to energy-momentum conservation[8]

$$E_j = \frac{m_\pi}{2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right) + \frac{m_j^2}{2m_\pi}. \tag{3.21}$$

Splitting Eqn. (3.21) analogous to Eqn. (3.18), the deviation ξ can be quantified for a

[8] Respecting $E_\pi = E_\mu + E_j$, $\vec{p}_\mu = -\vec{p}_j$ and $E_\pi^2 = m_\pi^2$ in the rest frame of the pion, one obtains Eqn. (3.21) after a simple calculation, starting from $E_\pi^2 = (E_\mu + E_j)^2$.

pion decay at rest,

$$\xi = \frac{1}{2}\left(1 + \frac{m_\mu^2}{m_\pi^2}\right) \simeq 0.8, \tag{3.22}$$

with $m_\pi \simeq 139.57$ MeV, $m_\mu \simeq 105.66$ MeV. Accordingly, though the massive neutrino energies/momenta *do* exhibit a strong dependency on the production process, the oscillation phase does not, being universal in that sense.

In the same way, the damping term can be rewritten, yielding

$$-\frac{(\tilde{E}_j - \tilde{E}_k)^2}{4\sigma_p^2(v_j^2 + v_k^2)} \simeq -\frac{\sigma_x^2}{2}(1-\xi)^2\left(\frac{\Delta m_{jk}^2}{2E}\right)^2 = -2\pi^2(1-\xi)^2\left(\frac{\sigma_x}{L_{jk}^{\text{osc}}}\right)^2, \tag{3.23}$$

with L_{jk}^{osc} as defined in Eqn. (2.26), and

$$-\frac{[(v_j - v_k)L]^2}{4\sigma_x^2(v_j^2 + v_k^2)} \simeq -\frac{L^2}{8\sigma_x^2}\left(\frac{-\Delta m_{jk}^2}{2E^2}\right)^2 = -\left(\frac{L}{L_{jk}^{\text{coh}}}\right)^2, \tag{3.24}$$

using the coherence lengths

$$L_{jk}^{\text{coh}} = \frac{4\sqrt{2}E^2}{|\Delta m_{jk}^2|}\sigma_x. \tag{3.25}$$

Consequently, Eqn. (3.16) reduces to

$$P_{\nu_\alpha \to \nu_\beta}(L) \propto \sum_{j,k} U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^* \exp\left[-i\frac{\Delta m_{jk}^2 L}{2E} - \left(\frac{L}{L_{jk}^{\text{coh}}}\right)^2 - 2\pi^2(1-\xi)^2\left(\frac{\sigma_x}{L_{jk}^{\text{osc}}}\right)^2\right]. \tag{3.26}$$

The last term (3.23) in Eqn. (3.26) is called the *localization* term while (3.24) is called the *coherence* term. In brevity, the localization term accounts for the damping if the oscillation length L_{jk}^{osc} is not small compared to σ_x: If the (spatial) regions of production and detection are not small compared to the oscillation length, the oscillations are averaged out in the detection/production process. In addition, (3.23) guarantees energy conservation within the given energy uncertainty [9]. The coherence term on the other hand accounts for the spatial separation of the different massive neutrinos due to their variation in group velocity as can be seen from Eqn. (3.24), leading to a smaller overlap of the wave packets, thus preventing coherent detection.

Consequently, the localization term allows for decoherence effects as to production/detection whereas the coherence term with respect to the propagation of the neutrino wave packets. A more detailed analysis of the conditions for neutrino oscillations is carried out below, yet it can be seen that if the coherence condition as well as the localization condition are satisfied, the oscillation probability reduces to the Standard expression (2.13).

Before presenting a non-Gaussian wave packet approach, it is worth commenting on an important aspect of the oscillation probability which could not be consistently derived in Section 2.1: Lorentz invariance. In contrast, in the wave packet approach beginning with Eqn. (3.10), Lorentz invariance is obtained consistently. This is of crucial importance since flavor is a Lorentz-invariant quantity which does not depend on the frame of reference. Indeed, considering a similar set-up as described in the beginning of Section 3.1, i.e. a Lorentz frame $\mathcal{O}'$ moving with velocity v with respect to $\mathcal{O}$ along the direction of neutrino propagation, the transformations of L and T are given by[9]

$$L' = \gamma(L - vT), \quad T' = \gamma(T - vL). \tag{3.27}$$

For $L = T$[10] in $\mathcal{O}$ it follows in $\mathcal{O}'$: $L' = T' = \gamma(1-v)L$. Similarly, for E, p:

$$E' = \gamma(E - vp), \quad p' = \gamma(p - vE), \tag{3.28}$$

implying that in the massless limit $E = p$ we have $E' = p' = \gamma(1-v)E$ and hence the same ratio $L'/E' = L/E$, demonstrating the Lorentz invariance of the oscillation probability [2, p.295],[10].

One could ask whether the conclusions drawn from Eqn. (3.16) could be obtained without assuming a Gaussian wave packet. This is in fact the case, as can be seen from the following reasoning which is based on Ref. [4].

Starting with the (production) flavor neutrino state, using customary notation[11],

$$|\nu_\alpha(x,T)\rangle = \sum_j U^*_{\alpha j} \Psi^P_j(x,T)\,|\nu_j\rangle, \tag{3.29}$$

with $\Psi^P_j(x,t)$ describing the wave packet of the massive neutrino j,

$$\Psi^P_j(x,T) = \int \frac{\mathrm{d}\tilde{p}}{\sqrt{2\pi}} f^P_j(\tilde{p} - p_j) e^{i(\tilde{p}x - E_j(p)T)}, \tag{3.30}$$

and $f^P_j(\tilde{p} - p_j)$ denoting the production momentum distribution function, sharply peaked around its mean momentum p_j with width $\sigma^P_p \ll |p_j|$. Expanding E_j around p_j as in Eqn. (3.12) and shifting the integration variable, $p = \tilde{p} - p_j$, Eqn. (3.30) can be rewritten:

$$\Psi^P_j(x,T) \simeq e^{ip_jx - iE_j(p_j)T} \left\{ \int \frac{\mathrm{d}p}{\sqrt{2\pi}} f^P_j(p) e^{ip(x - v_jT)} \right\}. \tag{3.31}$$

For further use, we denote the term in curly brackets as $g^P_j(x - v_jT)$. The detecting flavor neutrino state be described as peaked around the detector coordinate,

$$|\nu_\beta(x - L)\rangle = \sum_j U^*_{\beta j} \Psi^D_j(x - L)\,|\nu_j\rangle, \tag{3.32}$$

having the wave function

$$\Psi^D_j(x - L) = \int \frac{\mathrm{d}\tilde{p}}{\sqrt{2\pi}} f^D_j(\tilde{p} - p'_j) e^{i\tilde{p}(x-L)}, \tag{3.33}$$

with p'_j being the mean momentum of the detection process. By shifting the integration variable, $p = \tilde{p} - p'_j$, we find

$$\Psi^D_j(x - L) = e^{ip'_j(x-L)} \left\{ \int \frac{\mathrm{d}p}{\sqrt{2\pi}} f^D_j(p) e^{ip(x-L)} \right\}, \tag{3.34}$$

[9] L does not obey the ordinary length contraction since in general, it does not correspond to the instantaneous source-detector distance, but is determined by the condition $L = T$.

[10] If another relation holds, the same applies to the relation regarding E, p since the phase is Lorentz invariant (see Footnote 7 on page 13), so the following argument remains valid.

[11] I.e. the source is localized at $x = 0$ and the neutrino is produced at $t = 0$ with an appropriate time uncertainty. Only one dimensional description is used, justified by the reason mentioned above.

with the term in the curly brackets, $g_j^D(x-L)$, similarly denoting the shape factor of the detecting wave packet. The oscillation amplitude is given by the projection of the detection neutrino state on the production neutrino state,

$$A_{\nu_\alpha\to\nu_\beta}(L,T) = \int \mathrm{d}x \, \langle \nu_\beta(x-L)|\nu_\alpha(x,T)\rangle, \tag{3.35}$$

yielding by substitution of (3.29) and (3.32)

$$A_{\nu_\alpha\to\nu_\beta}(L,T) = \sum_j U_{\alpha j}^* U_{\beta j} \left\{ \int \mathrm{d}x \; g_j^P(x - v_j T) g_j^{D*}(x-L) e^{i(p_j - p_j')(x-L)} \right\} e^{ip_j L - iE_j(p_j)T}, \tag{3.36}$$

where the curly brackets this time embrace the effective shape factor $G_j(L-v_jT)$. This shape factor can be rewritten as shown in Appendix A.5, leading to

$$G_j(L-v_jT) = \int \mathrm{d}p \; f_j^P(p) f_j^{D*}(p+\delta_j) e^{ip(L-v_jT)}. \tag{3.37}$$

The oscillation probability $P_{\nu_\alpha\to\nu_\beta}(L)$ is given as in Eqn. (3.15) by

$$P_{\nu_\alpha\to\nu_\beta}(L) \propto \int \mathrm{d}T \, |A_{\nu_\alpha\to\nu_\beta}(L,T)|^2 = \sum_{j,k} U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^* I_{jk}(L), \tag{3.38}$$

with the integral

$$I_{jk}(L) = \int \mathrm{d}T \; G_j(L-v_jT) G_k^*(L-v_kT) e^{-i[(E_j(p_j)-E_k(p_k))T - (p_j-p_k)L]}. \tag{3.39}$$

For a more straightforward calculation, it is useful to expand the differences in momentum around the average momentum $p = (p_j + p_k)/2$, assuming relativistic or quasi-degenerate neutrinos, i.e. $|\Delta E| \ll E$, according to

$$\Delta p_{jk} \simeq \frac{\partial p}{\partial E}\Delta E_{jk} + \frac{\partial p}{\partial m^2}\Delta m_{jk}^2 = \frac{\Delta E_{jk}}{v_g} - \frac{\Delta m_{jk}^2}{2p}, \tag{3.40}$$

with $v_g \equiv (v_j + v_k)/2$ being the "effective" average group velocity. By inserting approximation (3.40) into the phase in Eqn. (3.39), one obtains for the phase

$$-i\left[\Delta E_{jk}T - \Delta p_{jk}L\right] \simeq -i\left[\frac{\Delta m_{jk}^2}{2p}L - \frac{\Delta E_{jk}}{v_g}(L - v_gT)\right]. \tag{3.41}$$

Considering the above simplifications, one finally obtains for the integral $I_{jk}(L)$ the expression calculated in (A.16),

$$\begin{aligned} I_{jk}(L) = e^{-i\frac{\Delta m^2}{2p}L} \, \frac{2\pi}{v_k} \int \mathrm{d}p \; & f_j^P(p) f_j^{D*}(p+\delta_j) \\ & \times f_k^{P*}(rp + \Delta E/v_k) f_k^D(rp + \Delta E/v_k + \delta_k) e^{ip(1-r)L}, \end{aligned} \tag{3.42}$$

with the notation introduced in Appendix A.5. Having come so far, it is now possible to compare the conditions for unsuppressed oscillations in Eqns. (3.26) and (3.42), which have been derived using slightly different approaches.

At first, it is clear from Eqn. (3.42) that if δ_j, δ_k exceed significantly the combined momentum uncertainty of production/detection (corresponding to the widths of the momentum distribution functions), the oscillation amplitude is strongly suppressed since otherwise energy-momentum conservation would be violated in the detection process.

Furthermore, the integrand averages out due to rapid oscillations of the phase if $|p(1-r)L|$ is not small in the effective momentum uncertainty σ_p, quantifying the width of the overlap integral, i.e. if not $|1-r|L\sigma_p \ll 1$. This condition can be rewritten in terms of the average group velocity v_g as

$$L \ll \sigma_x \frac{v_g}{\Delta v_g}, \tag{3.43}$$

with $\Delta v_g = |v_j - v_k|$ and $\sigma_x = 1/2\sigma_p$ being the effective spatial uncertainty. From Eqn. (3.42) can be seen that for uniform effective group velocities no damping occurs due to the phase factor. This implies that condition (3.43) is equivalent to the one embodied in the *coherence* term (3.24) which accounts for the spread of the different massive neutrino wave packets. As a matter of fact, at first order in Δm^2_{jk}, condition (3.43) can be rewritten by means of (3.19) as

$$\frac{v_g}{\Delta v_g}\sigma_x \simeq \left[\frac{2E^2}{|\Delta m^2_{jk}|} - \frac{(m_j^2+m_k^2)}{2|\Delta m^2_{jk}|}\right]\sigma_x \simeq \frac{2E^2}{|\Delta m^2_{jk}|}\sigma_x, \tag{3.44}$$

being obviously in agreement with L^{coh}_{jk} in (3.25). The additional term in Eqn. (3.44) proportional to $(m_j^2+m_k^2)/\Delta m^2_{jk}$ has been neglected, owing to the fact that $E^2 \gg m^2$ for ultrarelativistic or quasi-degenerate neutrinos. In conclusion, the meaning of the coherence condition can be summarized as follows: For unsuppressed oscillations to exist, the separation of the distinct wave packets due to their variation in group velocity needs to be small compared to the spatial uncertainty of production/detection.

In addition, the integral in Eqn. (3.42) vanishes if the split of the arguments of the momentum distribution functions $f^{P,D}_{j,k}$ exceeds the width of the combined momentum uncertainty σ_p, i.e. if

$$\frac{|\Delta E_{jk}|}{v_g}\sigma_x \ll 1, \tag{3.45}$$

where the uncertainty relation as well as a more intuitive parametrization via v_g has been used[12]. Again, this expression can be rewritten using Eqns. (3.18) and (3.19),

$$\frac{|\Delta E_{jk}|}{v_g}\sigma_x \simeq \frac{(1-\xi)\frac{\Delta m^2_{jk}}{2E}}{1-(m_j^2+m_k^2)/4E^2}\sigma_x \simeq (1-\xi)\frac{\Delta m^2_{jk}}{2E}\sigma_x, \tag{3.46}$$

at first order in Δm^2_{jk}, being consistent with the expression derived in Eqn. (3.23).

Consequently, the observability of unsuppressed neutrino oscillations depends mainly on the *localization* and *coherence* condition being satisfied. Since these conditions contain the production/detection uncertainties, it seems worthwhile to take a closer look at the relevant quantum mechanical uncertainties which have only been briefly mentioned up to now.

[12] The parametrization via v_g instead of v_k is possible since v_k can be written as $v_k = v_g + \Delta v_{kj}$, so the deviation is given by $\Delta v_{kj}/v_g$ which is of $\mathcal{O}(\Delta m^2)$ and thus negligible for Eqns. (3.43), (3.45) where the prefactor is already of $\mathcal{O}(\Delta m^2)$.

3.2 QM uncertainties and neutrino oscillations

Neutrino oscillations as introduced above in the framework of quantum mechanics depend vitally on the existence of QM uncertainties since these uncertainties allow for a *coherent superposition* of massive neutrino states in the first place, thus being of crucial importance for the quantum mechanical interference manifesting itself in neutrino oscillations. In fact, only the inevitable ignorance of the specific neutrino mass state renders possible a coherent superposition of neutrino mass states, being the very reason for neutrino oscillations as the distinct mass states evolve differently in space-time, causing oscillations of the probability of flavor neutrino measurements due to the variation in relative phase.

From this point of view, following Refs. [2, 4], it is clear that in order to observe neutrino oscillations, the sensitivity of the experiment to squared-mass must be smaller than the actual squared-mass differences, $\sigma_{m^2} \gtrsim \Delta m^2$ (the indices jk have been omitted since they do not bring further insight to the discussion). Assuming energy and momentum to be independent, the relativistic dispersion relation yields

$$\sigma_{m^2} = \sqrt{(2E\sigma_E)^2 + (2p\sigma_p)^2} \simeq 2\sqrt{2}E\sigma_p, \tag{3.47}$$

in the ultrarelativistic limit. With the help of Eqn. (3.47) and the uncertainty relation, the localization condition contained in Eqn. (3.23) can be rewritten in terms of σ_{m^2},

$$\frac{\sigma_x}{L^{\text{osc}}} = \frac{\Delta m^2}{8\pi E\sigma_p} \simeq \frac{1}{2\sqrt{2}\pi}\frac{\Delta m^2}{\sigma_{m^2}}, \tag{3.48}$$

showing that the coherence condition $\sigma_{m^2} \gtrsim \Delta m^2$ stipulated above is embodied in the localization condition. Conversely, if $\sigma_{m^2} \ll \Delta m^2$, the coherence between the different neutrino mass states is lost, no interference between the mass states can occur, which leads to the incoherent oscillation probability. Though, even then neutrino oscillations can be inferred since for the absence of neutrino oscillations, the survival probability should be equal to one, while the value for the incoherent oscillation probability still depends on the mixing matrix (cf. e.g. Eqn. (2.25)).

However, the question how uncertainties emerge from the production and detection processes has not been addressed yet. To begin with, it is helpful to realize that the energy and momentum uncertainties of both production and detection (to be determined in the laboratory frame for obvious reasons) depend on the time scale and the spatial localization of the corresponding process.

In particular, when considering the production process, the spatial uncertainty is governed by the smallest of the spatial uncertainties of the interacting particles. That is, coherent emission of massive neutrino wave packets is only possible until the first of the particles involved in production interacts with the environment. Hence, the dominant contribution to the spatial uncertainty comes from the smallest one, accounted for by summing the spatial uncertainties of the particles involved in the interaction reciprocal, implying by means of the uncertainty relation the corresponding summation for the momentum uncertainty,

$$\left(\frac{1}{\sigma_x^I}\right)^2 \sim \sum_i \left(\frac{1}{\sigma_x^i}\right)^2, \quad (\sigma_p^I)^2 \sim \sum_i (\sigma_p^i)^2, \tag{3.49}$$

with I standing for the overall interaction process value whereas the i's denote the individual interacting particles.

In principle, as discussed below, σ_p^I and thus the length of the produced neutrino wave packet can be estimated by Eqn. (3.49). It turns out, however, that the energy uncertainty σ_E, depending on the coherence time σ_t^I of the interaction process, is more decisive pertaining to the wave packet size, stressing the necessity to consider the coherence time σ_t^I.

In the case of collision broadening, the coherence time can be estimated by summing up the inverse individual coherence times. As for example in the case when coherence is lost due to the collision of a particle involved in the interaction with surrounding particles as discussed above, one has

$$\left(\frac{1}{\sigma_t^I}\right)^2 \sim \sum_i \left(\frac{v_i}{\sigma_x^i}\right)^2. \tag{3.50}$$

Respecting $v_i < 1$, it can be seen from Eqn. (3.50) that in general $\sigma_t^I > \sigma_p^I$ and thus $\sigma_E^I \simeq 1/\sigma_t^I < \sigma_p^I$.

For the case that no collision broadening occurs, the coherence time is given by the lifetime τ of the decaying particle, implying that the neutrino wave packet size in the rest frame of the decaying particle is determined by its distance travelled: $\sigma_x^\nu \simeq v_g \tau$ with v_g being the average group velocity of the massive neutrino states. In another frame of reference, the distance travelled by the decaying parent particle while decaying has to be taken into account. By doing so, one obtains the physically expected result that the size of the neutrino wave packet is contracted when being emitted in the direction of motion of the parent particle while dilated when being emitted in the opposite direction [4].

Until now, only a single interaction process has been discussed, but, as already stressed above, *effective* uncertainties have been used in the above derivations. Considering the overlap integral in Eqn. (3.37), it becomes clear indeed that the momentum uncertainty of the combined production/detection process is dominated by the *smallest* momentum uncertainty of the two processes. As a result, the effective spatial uncertainty is governed by the largest uncertainty of production/detection and can – for Gaussian wave packets – be estimated by quadratic summation of the production and detection uncertainties.

As mentioned above, the neutrino wave packet size is rather given by

$$\sigma_x^\nu \sim \frac{v_g}{\sigma_E} \tag{3.51}$$

than by the spatial uncertainties of production/detection, since in general $\sigma_E < \sigma_p$ (illustrated in the paragraph about collision broadening and concluded in Ref. [4]), and the two quantities are connected via

$$E\sigma_E = p\sigma_p, \tag{3.52}$$

originated in the relativistic dispersion relation $E^2 = p^2 + m^2$ with m^2 being a definite mass. Eqn. (3.52) is obviously valid only for particles being on-shell, i.e. obeying the above dispersion relation, being derived with its help. Accordingly, massive neutrinos being produced off-shell with uncertainties not obeying (3.52), go on-shell while propagating from their production point, so the larger uncertainty shrinks towards the smaller one, leading to (3.52) being satisfied [4]. This provides an explanation for the remedy of the potential initial mismatch between the wave packet sizes implied by momentum uncertainty and coherence time. Though, it must be stressed that for a comprehensive analysis respecting anisotropy or the like, further analysis is required since the one dimensional approach developed above provides merely basic considerations.

Under these circumstances, one could ask if neutrino oscillations are banned in principle by an extremely small energy uncertainty (i.e. an energy uncertainty smaller than the variation in energy caused by mass differences) in the production process. The answer is no, yet needing some further explanation: If the momentum uncertainty of the detection process is large enough, the coherence of the massive neutrino states, lost soon after production due to the small energy and consequently momentum uncertainty, is restored in the detection process, rendered possible by the increased uncertainty in squared-mass detection according to Eqn. (3.47).

For that reason, production/detection processes need to be taken into account jointly when trying to infer the occurrence of observable neutrino oscillations as done in Section 3.1. Hence, the above procedure for determining the effective uncertainties seems to be slightly *ad hoc* and mark the feet of clay of the QM approach, depending on being given appropriate production/detection states since the underlying processes cannot be described in the framework of quantum mechanics. It is indeed a deficiency of the wave packet approach that these uncertainties cannot be derived in the underlying framework of QM, but it is clear that by using the more proper description of neutrino states by wave packets, some crucial aspects of neutrino oscillations could have been derived and a manageable treatment of neutrino oscillations is made possible. In addition, an instructive analysis of (de)coherence effects is rendered possible by the wave packet approach, thus being expedient for the understanding of neutrino oscillations.

In conclusion, to wrap up the discussion on coherence, two conditions must obviously be satisfied in order to observe neutrino oscillations [8]: the flavor of the neutrino must be known (otherwise the meaning of flavor "oscillation" is lapsed) and both production and detection process must be localized (otherwise, there is no gauge on which oscillations can occur). If, however, these conditions are satisfied, quantum mechanical uncertainties eventually arise (since, what is more, production/detection process exhibit a specific, "microscopic" time scale by nature), requiring a detailed analysis respecting the distinct properties of massive neutrinos in order to conclude the occurrence of observable neutrino oscillations.

4 Generalizing neutrino oscillations

4.1 Neutrino oscillations in matter

Though the knowledge about vacuum neutrino oscillations is of use for the understanding of what happens to the neutrinos while travelling through space (e.g. being emitted by the Sun), it turns out that most of the observed neutrino oscillations (e.g. solar neutrino oscillations) cannot be understood without taking matter effects into account.

In matter, which consists of quarks and electrons, neutrino interaction is dominated by coherent forward elastic scattering[13]. For all flavor neutrinos, but the electron neutrino, the scattering is given solely by the NC interaction (lepton universality), yet for the electron neutrino also the coherent elastic weak CC scattering $\nu_e e^- \to \nu_e e^-$ exists. For that reason, only the CC potential is of importance for the understanding of neutrino oscillations in matter, at least in the two neutrino case if no sterile neutrinos are respected [11, 12], since the NC interaction generates a common phase to all flavors.

Physically, the additional CC potential can be imagined to increase the index of refraction of the wave function of the electron neutrino, leading to an additional phase which requires an adjustment of the flavor Hamiltonian (4.4), leading to a change in relative phase of the neutrino mass states. However, in the following derivation which is based on Refs. [2, 3], the matter potential is taken as a given.

In order to understand the matter effects, let us consider the massive neutrino states $|\nu_j\rangle$ which are eigenstates to the time-independent vacuum Hamiltonian

$$\mathscr{H}_0 |\nu_j\rangle = E_j |\nu_j\rangle \quad \text{with} \quad E_j \simeq p + \frac{m_j^2}{2E}, \tag{4.1}$$

in the ultrarelativistic limit (cf. Eqn. (2.6)). Neglecting the common phase e^{-ipt} in the following, the Schrödinger equation (2.7) can be written in matrix notation $\boldsymbol{\nu}_m = (\nu_1, \dots)^{\mathrm{T}}$ as

$$i\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\nu}_m(t) = \mathcal{H}_m \boldsymbol{\nu}_m(t) \quad \text{with} \quad (\mathcal{H}_m)_{jk} = \frac{m_j^2}{2E}\delta_{jk}, \tag{4.2}$$

which becomes in the flavor basis, $\boldsymbol{\nu}_m = U^\dagger \boldsymbol{\nu}_F$[14],

$$i\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\nu}_F(t) = U\mathcal{H}_m U^\dagger \boldsymbol{\nu}_F(t). \tag{4.3}$$

Taking into account the CC matter potential V_{CC} by means of $A_{CC} \equiv 2EV_{CC}$, one finally obtains

$$\mathcal{H}_F = U\mathcal{H}_m U^\dagger + \frac{A_{CC}}{2E} |\nu_e\rangle \langle\nu_e| \,. \tag{4.4}$$

The matter potential $V_{CC} \equiv \sqrt{2} G_F N_e$ with G_F being the Fermi constant and N_e the electron density of the medium is clearly positive, so we know that also A_{CC} is positive and thus $\mathcal{H}_F$ real. For two neutrino mixing, with U from Eqn. (2.15) and say

[13] The amount of incoherent scattering is usually very small and can thus be neglected [2, p.322].

[14] This relation can be obtained by considering the summation in Eqn. (2.2).

$\boldsymbol{\nu}_F = (\nu_e, \nu_\mu)^{\mathrm{T}}$, the Hamiltonian (4.4) can be explicitly written as

$$\mathcal{H}_F = \frac{1}{2E}\begin{pmatrix} m_1^2\cos^2\vartheta + m_2^2\sin^2\vartheta + A_{CC} & (m_2^2 - m_1^2)\sin\vartheta\cos\vartheta \\ (m_2^2 - m_1^2)\sin\vartheta\cos\vartheta & m_2^2\cos^2\vartheta + m_1^2\sin^2\vartheta \end{pmatrix} \tag{4.5}$$

$$= \frac{1}{4E}\begin{pmatrix} m_1^2\Delta_\vartheta - m_2^2\Delta_\vartheta + 2A_{CC} & \Delta m^2\sin 2\vartheta \\ \Delta m^2\sin 2\vartheta & -m_1^2\Delta_\vartheta + m_2^2\Delta_\vartheta \end{pmatrix} + \frac{1}{4E}(m_1^2 + m_2^2)\mathbb{1} \tag{4.6}$$

$$= \frac{1}{4E}\begin{pmatrix} -\Delta m^2\cos 2\vartheta + A_{CC} & \Delta m^2\sin 2\vartheta \\ \Delta m^2\sin 2\vartheta & \Delta m^2\cos 2\vartheta - A_{CC} \end{pmatrix} + \frac{1}{4E}\left[\Sigma_{m^2} + A_{CC}\right]\mathbb{1}, \tag{4.7}$$

with $\Delta_\vartheta = \cos^2\vartheta - \sin^2\vartheta$, $\Sigma_{m^2} = m_1^2 + m_2^2$ and $\mathbb{1}$ denoting a 2×2 unit matrix. Obviously, $\mathcal{H}_F$ is Hermitian and, even more, real and symmetric which implies that $\mathcal{H}_F$ can be diagonalized by an orthogonal matrix such as

$$U_M = \begin{pmatrix} \cos\vartheta_M & \sin\vartheta_M \\ -\sin\vartheta_M & \cos\vartheta_M \end{pmatrix}, \tag{4.8}$$

where ϑ_M denotes the effective mixing angle in matter. Diagonalizing the non-diagonal left part[15] of $\mathcal{H}_F$ in Eqn. (4.7) with U_M in Eqn. (4.8), one finds for the effective squared-mass difference in matter

$$\Delta m_M^2 = \sqrt{(\Delta m^2\cos 2\vartheta - A_{CC})^2 + (\Delta m^2\sin 2\vartheta)^2}. \tag{4.9}$$

The effective mixing angle ϑ_M is obtained by solving the eigenvalue equations

$$0 = \cos\vartheta_M\left[-\Delta m^2\cos 2\vartheta + A_{CC} - m_{M1}^2\right] - \sin\vartheta_M\sin 2\vartheta\Delta m^2, \tag{4.10}$$

$$0 = -\sin\vartheta_M\left[\Delta m^2\cos 2\vartheta - A_{CC} - m_{M1}^2\right] + \cos\vartheta_M\sin 2\vartheta\Delta m^2, \tag{4.11}$$

with the eigenvector being the first column of U_M in Eqn. (4.8) for $\tan\vartheta_M$, $1/\tan\vartheta_M$ and using the relation

$$\tan 2\vartheta_M = \frac{2}{\dfrac{1}{\tan\vartheta_M} - \tan\vartheta_M} = \frac{\tan 2\vartheta}{1 - \dfrac{A_{CC}}{\Delta m^2\cos 2\vartheta}}. \tag{4.12}$$

Approximating $t \simeq x$ and inserting $\mathcal{H}_F$ from Eqn. (4.4) in the Schrödinger equation (4.3), Eqn. (4.3) can be rewritten in the massive neutrino basis in matter by means of $\boldsymbol{\nu}_F = U_M\boldsymbol{\nu}_m^M$, respecting $U_M^\dagger\mathcal{H}_F U_M = 1/4E\ \mathrm{diag}(-\Delta m_M^2, \Delta m_M^2)$ as well as $U_M^\dagger\frac{\mathrm{d}}{\mathrm{d}x}U_M = \frac{\mathrm{d}\vartheta_M}{\mathrm{d}x}i\sigma_2$[16]:

$$U_M i\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{\nu}_m^M(x) + i\left(\frac{\mathrm{d}}{\mathrm{d}x}U_M\right)\boldsymbol{\nu}_m^M(x) = \mathcal{H}_F U_M\boldsymbol{\nu}_m^M(x) \tag{4.13}$$

$$\Leftrightarrow \quad i\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{\nu}_m^M(x) = \frac{1}{4E}\begin{pmatrix} -\Delta m_M^2 & -4Ei\frac{\mathrm{d}\vartheta_M}{\mathrm{d}x} \\ 4Ei\frac{\mathrm{d}\vartheta_M}{\mathrm{d}x} & \Delta m_M^2 \end{pmatrix}\boldsymbol{\nu}_m^M(x). \tag{4.14}$$

The diagonal, right part of the Hamiltonian in Eqn. (4.7) has been omitted since it

[15] The right part does not need to be taken into account because the diagonalization is performed with an orthogonal matrix and the unit matrix commutates with all matrices.

[16] σ_2 denotes the second Pauli matrix.

can be eliminated by a common phase shift of the flavor neutrino states.

Transforming Eqn. (4.4) back into the basis of the mass eigenstates of the vacuum Hamiltonian,

$$\mathcal{H}'_m = \frac{1}{2E}\begin{pmatrix} m_1^2 + A_{CC}\cos^2\vartheta & A_{CC}\sin\vartheta\cos\vartheta \\ A_{CC}\sin\vartheta\cos\vartheta & m_2^2 + A_{CC}\sin^2\vartheta \end{pmatrix}, \tag{4.15}$$

it is seen that for $A_{CC} \neq 0$ the mass eigenstates of the vacuum Hamiltonian are no longer mass eigenstates of the matter Hamiltonian, i.e. transitions $\nu_1 \leftrightarrow \nu_2$ are possible.

Furthermore, if the matter density is constant, i.e. $\mathrm{d}\vartheta_M/\mathrm{d}x = 0$, the evolution of the "effective massive neutrinos" in Eqn. (4.14) is decoupled, implying

$$\nu_{mj}^M(x) = \exp\left((-1)^{j+1} i \frac{\Delta m_M^2 x}{4E}\right)\nu_{mj}^M(0) \tag{4.16}$$

and allowing for the calculation of the transition probability $P_{\nu_e\to\nu_\mu}$ analogous to the calculation in Appendix A.1, yielding

$$P_{\nu_e\to\nu_\mu} = \sin^2 2\vartheta_M \sin^2\left(\frac{\Delta m_M^2 x}{4E}\right). \tag{4.17}$$

If the matter density is not constant, $\mathrm{d}\vartheta_M/\mathrm{d}x \neq 0$, transitions $\nu_1^M \leftrightarrow \nu_2^M$ can occur. For that reason, the adiabaticity parameter $\gamma = \Delta m_M^2/4E|\mathrm{d}\vartheta_M/\mathrm{d}x|$ is usually introduced, quantifying the amount of these transitions. The evolution of the effective mass states is adiabatic if $\gamma \gg 1$ for the whole neutrino trajectory, meaning that the transitions $\nu_1^M \leftrightarrow \nu_2^M$ are negligible. This is the simplest case, but can be realized for solar neutrinos of a certain energy range. In this case, the mass state evolution in space is given by the phase factor

$$\exp\left((-1)^{j+1} i \int_0^x \frac{\Delta m_M^2(x')}{4E}\mathrm{d}x'\right), \tag{4.18}$$

with j indicating the massive neutrino component as before.

Moreover, for a non-constant matter density the mismatch between the mixing angles at production and detection has to be taken into account. For instance, one can consider an electron neutrino produced in the Sun with the mixing angle ϑ_M^i, evolving adiabatically with the phase factor given in (4.18) to the detection point where the vacuum mixing angle ϑ holds. Hence, the survival amplitude at this point can be written as

$$\psi_{ee}(x) = \sum_j U_{ej} a_j \exp\left((-1)^{j+1} i \int_0^x \frac{\Delta m_M^2(x')}{4E}\mathrm{d}x'\right), \tag{4.19}$$

with a_j denoting the initial mixing angles (i.e. the first column of (4.8)) and U_{ej} the mixing angles at the detection point. Taking the squared modulus of the amplitude and minding the relation (A.2), one obtains the survival probability

$$P_{\nu_e\to\nu_e}^{\text{adiabatic}} = \frac{1}{2} + \frac{1}{2}\cos 2\vartheta\cos 2\vartheta_M^i + \frac{1}{2}\sin 2\vartheta \sin 2\vartheta_M^i \cos\left(\int_0^x \frac{\Delta m_M^2(x')}{2E}\mathrm{d}x'\right). \tag{4.20}$$

In the case $\vartheta_M^i \simeq \pi/2$ the averaged survival probability (where the cosine depending on $\Delta m^2_M(x')/E$ is averaged out) can be approximated by

$$\overline{P}_{\nu_e \to \nu_e} \simeq \frac{1}{2}(1 - \cos 2\vartheta) = \sin^2 \vartheta, \tag{4.21}$$

implying that flavor transitions are favoured by small vacuum mixing angles (provided that the adiabaticity condition holds). However, a global fit to solar neutrino data favours a large vacuum mixing angle (LMA) [2, p.389].

As can be seen from Eqn. (4.12), the mixing angle is largest when the resonance condition $A^R_{CC} = \Delta m^2 \cos^2 2\vartheta$, i.e. $\vartheta^R_M = \pi/4$, holds, allowing for a maximal transition amplitude between the two flavors (cf. Eqn. (4.17)). The resonance, which refers to the resonant behaviour of the oscillation amplitude with respect to $A_{CC}/\Delta m^2_M$, was discovered by Mikheev, Smirnov and Wolfenstein after whom the mechanism is named the *MSW effect*. In order to take effect, the MSW mechanism requires a variable matter density as well as the production of the flavor neutrinos above the resonance region.

Eqn. (4.12) shows further that due to $A_{CC} \to -A_{CC}$ for antineutrinos, flavor transitions of antineutrinos are disfavoured by the MSW effect, resulting in a CP violation with respect to the oscillation probabilities of neutrinos and antineutrinos. In addition, the degeneracy of the transition amplitude regarding $\vartheta \leftrightarrow \pi/2-\vartheta$ is broken since $\cos 2\vartheta$ changes sign at $\vartheta = \pi/4$ while A_{CC} is generally positive, so resonance can only occur for $\vartheta < \pi/4$.

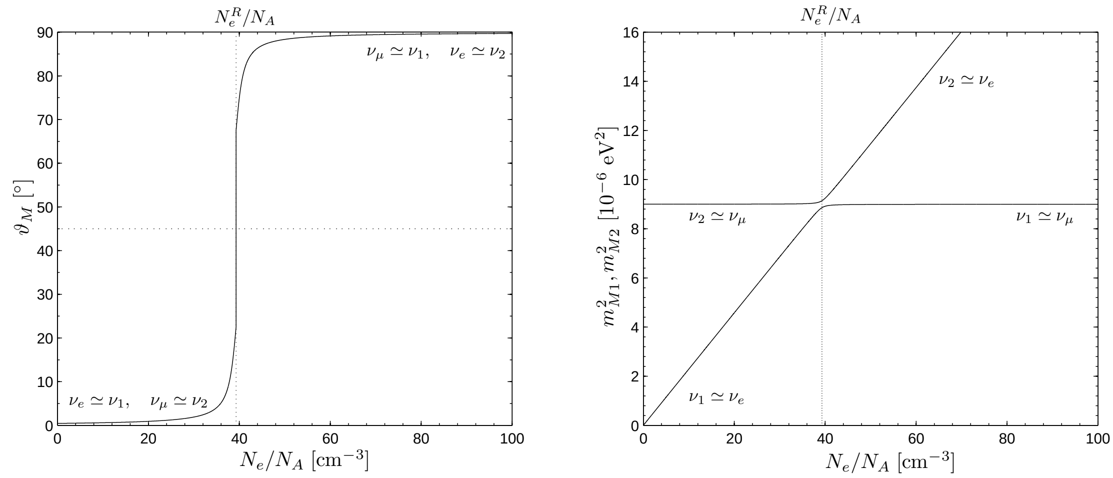
In conclusion, it suggests itself to discuss two-neutrino oscillations in the Sun qualitatively with the help of the MSW effect as introduced above. Writing down the full expression for the effective mass in matter[17],

$$m^2_{M1,2} = \frac{1}{2}[m^2_1 + m^2_2 + A_{CC} \mp \Delta m^2_M], \tag{4.22}$$

and recalling the definition of the electron density, $N_e = A_{CC}/2\sqrt{2}EG_F$, one obtains from Eqn. (4.12) Figure 4.1a while from Eqn. (4.22) Figure 4.1b. For these figures, the parameters $m_1 = 0$, $\Delta m^2 = 9 \times 10^{-6}$ eV^2, $\sin^2 2\vartheta = 10^{-3}$ and $E = 1.5$ MeV were used.

Figure 4.1a shows the variation of the effective mixing angle in matter as a function of the electron density N_e divided by Avogadro's number N_A. Considering an electron neutrino produced in the core of the Sun where a mixing angle of $\vartheta_M \simeq \pi/2$ applies (corresponding to the top right corner of the diagram), leaving the Sun after an adiabatic journey (corresponding to the bottom left corner of the diagram), we see that a conversion of the electron neutrino has taken place due to the change of the mixing angle. Figure 4.1b illustrates the same: When an electron neutrino produced as almost pure ν_2 crosses the resonance, a flavor flip occurs due to the large variation of the mixing angle, for ν_2 remains ν_2 if the crossing is adiabatically. Finally, it can be seen that the squared-mass difference is minimal at resonance.

[17] The additional terms in the effective mass term come from the right hand side of Eqn. (4.7), for the factor 1/2 see right hand side of Eqn. (4.2).



(a) Effective mixing angle ϑ_M as a function of N_e/N_A.

(b) Effective squared-masses m^2_{M1}, m^2_{M2} as a function of N_e/N_A.

Figure 4.1: Matter effects in two neutrino mixing, after Fig. 9.2 in Ref. [2].

4.2 Three neutrino mixing

Experimental findings have shown that the squared-mass difference for solar and atmospheric neutrino oscillation measurements are quite different (cf. Secs. 5.2, 5.3). Thus, the realistic case of neutrino mixing is three neutrino mixing which allows for two independent squared-mass differences.

At this point, it is necessary to stress once again that the knowledge about squared-mass differences does contain only very limited information about the actual neutrino masses, next to the fact that the squared-masses of at least two of them must be at least of the order of the observed squared-mass differences. In particular, the hierarchy of squared-mass differences implied by $\Delta m^2_{SOL} \ll \Delta m^2_{ATM}$ cannot be determined by neutrino oscillation measurements.

In order to understand the consequences of three neutrino mixing, it is essential to study the leptonic mixing matrix in more detail. Yet, before doing so, the origin of the leptonic mixing matrix needs to be discussed briefly, following Ref. [2].

Knowing that in the framework of the SM neutrinos are massless because there are no right-handed (chiral) neutrino fields, it is necessary to introduce right-handed neutrino fields (in the Dirac case) to allow for neutrino masses. These right-handed fields are called *sterile* because they do not participate in weak interaction due to being singlets of $SU(3) \times SU(2)$ and having hypercharge zero[18]. In the following, only Dirac mass is considered.

Introducing three right-handed neutrino fields according to the minimally extended SM, the modified SM Higgs-lepton Yukawa Lagrangian can be written as

$$\mathscr{L}_{H,L} = -\sum_{\alpha,\beta} Y'^{\ell}_{\alpha\beta} \overline{L_{\alpha L}} \Phi \ell'_{\beta R} - \sum_{\alpha,\beta} Y'^{\nu}_{\alpha\beta} \overline{L_{\alpha L}} \tilde{\Phi} \nu'_{\beta R} + \text{ H.c.}, \tag{4.23}$$

where Y' denotes the non-diagonal Yukawa coupling matrix, Φ the Higgs doublet, $L_{\alpha L}$ the weak isospin doublets, $\ell'_{\beta R}$, $\nu'_{\beta R}$ the weak isospin singlets and H.c. the Hermitian

[18] In the SM, weak interaction is described by the symmetry group $SU(2) \times U(1)$ in the framework of electroweak interaction. The generator of the symmetry group $U(1)_Y$ is the hypercharge operator.

conjugate of the whole expression. In unitary gauge, the Lagrangian becomes

$$\mathscr{L}_{H,L} = -\left(\frac{v+H}{\sqrt{2}}\right)\left[\overline{\boldsymbol{\ell}'_L} Y'^{\ell} \boldsymbol{\ell}'_R + \overline{\boldsymbol{\nu}'_L} Y'^{\nu} \boldsymbol{\nu}'_R\right] + \text{ H.c.} \tag{4.24}$$

In this expression, v denotes the vacuum expectation value (VEV) for the Higgs doublet and H the Higgs boson field. The Yukawa coupling matrix Y can be diagonalized through a biunitary transformation,

$$V_L^{\nu\dagger} Y'^{\nu} V_R^{\nu} = Y^{\nu}, \tag{4.25}$$

containing the coefficients determining the neutrino masses on the main diagonal. Eqn. (4.25) can be further used to define the chiral massive neutrino arrays

$$\boldsymbol{n}_L = V_L^{\nu\dagger} \boldsymbol{\nu}'_L, \quad \boldsymbol{n}_R = V_R^{\nu} \boldsymbol{\nu}'_R. \tag{4.26}$$

With this in mind, we can proceed with considering the CC interaction Lagrangian

$$\mathscr{L}^{CC} = -\frac{g}{2\sqrt{2}}\left(j_W^{\eta} W_{\eta} + j_W^{\eta *} W_{\eta}^{\dagger}\right), \tag{4.27}$$

where g denotes the coupling constant, W_η the field creating W^- and annihilating W^+ bosons and j_W^η the leptonic and quark charged-currents $j_W^\eta = j_{W,L}^\eta + j_{W,Q}^\eta$. In particular, the leptonic charged-current is given by

$$j_{W,L}^{\eta} = 2 \sum_{\alpha=e,\mu,\tau} \overline{\nu'_{\alpha L}} \gamma^{\eta} \ell'_{\alpha L}. \tag{4.28}$$

Taking into account the mismatch between the unprimed fields with definite mass and the primed fields participating in weak interaction for both leptons and neutrinos, one finds

$$j_{W,L}^{\eta} = 2\, \overline{\boldsymbol{n}_L} V_L^{\nu\dagger} \gamma^{\eta} V_L^{\ell} \boldsymbol{\ell}_L = 2 \sum_{\alpha,j} \overline{\nu_{jL}} U_{\alpha j}^{*} \gamma^{\eta} \ell_{\alpha L}, \tag{4.29}$$

with $U = V_L^{\ell\dagger} V_L^{\nu}$ being the lepton mixing matrix.

The reason why only the CC and not the NC Lagrangian is considered stems from the fact that due to the unitarity of the matrices $V_{L,R}$, the NC Lagrangian is left untouched by the mixing, i.e. the Lagrangian is the same for primed and unprimed fields. This matter of fact is also called the *GIM mechanism* [2, p.96].

Aiming to study the mixing matrix in more detail one could ask on how many (and what kind of) parameters the matrix U depends. In general, a complex $N \times N$ matrix has $2N^2$ independent real parameters. By imposing the unitarity condition (2.4) only N^2 parameters remain. These parameters can be divided into

$$\frac{N(N-1)}{2} \text{ mixing angles,} \quad \frac{N(N+1)}{2} \text{ phases.} \tag{4.30}$$

However, the electroweak Lagrangian is invariant under global phase transformations of the lepton fields, i.e.

$$\nu_{jL} \to e^{i\psi_j^{\nu}} \nu_{jL}, \quad \ell_{\alpha L} \to e^{i\psi_{\alpha}^{\ell}} \ell_{\alpha L}. \tag{4.31}$$

Hence, the leptonic charged-current (4.29) for which $N = 3$ can be written as

$$j^{\eta}_{W,L} = 2 \sum_{\alpha=e,\mu,\tau} \sum_{j=1,2,3} \overline{\nu_{jL}} \gamma^{\eta} e^{-i\psi^{\nu}_{j}} U^{*}_{\alpha j} e^{i\psi^{\ell}_{\alpha}} \ell_{\alpha L} \tag{4.32}$$

$$= 2 \underbrace{e^{-i(\psi^{\nu}_{2}-\psi^{\ell}_{\mu})}}_{1} \sum_{\alpha=e,\mu,\tau} \sum_{j=1,2,3} \overline{\nu_{jL}} \gamma^{\eta} \underbrace{e^{-i(\psi^{\nu}_{j}-\psi^{\nu}_{2})}}_{N-1} U^{*}_{\alpha j} \underbrace{e^{i(\psi^{\ell}_{\alpha}-\psi^{\nu}_{\mu})}}_{N-1} \ell_{\alpha L}. \tag{4.33}$$

Eqn. (4.33) makes plausible that the physical relevant phases can be reduced by $2N-1$ phases, correcting the amount of relevant (relative) phases in Eqn. (4.30) to

$$\frac{(N-1)(N-2)}{2} \text{ physical phases.} \tag{4.34}$$

This immediately explains the fact why we have considered a *real* 2×2 matrix with one mixing angle for two neutrino mixing. Consequently, we have 3 mixing angles and one physical phase, also called CP-violating phase, for three neutrino mixing.

Regarding Majorana neutrinos, the mixing matrix depends on three physical phases in the case of three neutrino mixing since the Majorana mass term is not invariant under the global $U(1)$ gauge transformation in Eqn. (4.31), so only the lepton fields can be transformed. However, the two additional phases are irrelevant for the oscillation probability because they cancel in the quartic product (2.17)[19]. Therefore, Dirac and Majorana neutrinos cannot be distinguished by neutrino oscillation measurements which implies that the question whether neutrinos are Dirac or Majorana particles cannot be answered by neutrino oscillation measurements. Indeed, the distinct type of Dirac and Majorana neutrinos respectively the likeliness of neutrinos being of Dirac/Majorana type is far beyond the actual topic of this thesis and for that reason not studied any further.

With one physical phase in the mixing matrix U, the CP violation quantifying term in Eqn. (2.24) does not necessarily vanish any more, but can exhibit a deviation dependent on the mixing matrix as well as the ratio $\Delta m^2_{jk} L/E$. This is similar to the quark case, where CP violation is believed to be generated by the physical phase in the quark mixing matrix. The possibility of CP violation in neutrino oscillation is of interest e.g. on account of the apparent particle-antiparticle asymmetry in the Universe and is at present much searched for.

Finally, apart from three neutrino mixing, the possibility of active and sterile neutrino mixing has to be considered in the general analysis of neutrino oscillation measurement data[20]. In any case, the discussion of the nature of sterile neutrinos and their potential impact on neutrino oscillations is out of the scope of this thesis.

[19] As can be seen from Eqn. (4.32), the mixing matrix in the Majorana case can be written as $U^{M}_{\alpha j} = U^{D}_{\alpha j} e^{i\varphi_j}$. These additional phases cancel in (2.17) [2, p.250].

[20] At present, the existence of two additional sterile neutrinos with masses at the eV scale is assumed in the analysis of neutrino oscillation measurement data, based on the measurement results of the LSND and MiniBooNE experiment [13].

5 Neutrino oscillation experiments

5.1 Nomenclature

In general, considering neutrino oscillation experiments, a division into two types is possible, namely *appearance* experiments in which the transition probability is measured, and *disappearance* experiments in which the survival probability is measured. Provided that the measured neutrino flavor is absent in the initial beam, sensitivity to even small mixing angles can be achieved in appearance experiments, while disappearance experiments, not being suited to measure small mixing angles due to statistic fluctuations, allow for the measurement of large mixing angles.

Moreover, as discussed in Section 2.3, the sensitivity to Δm^2 is determined by the ratio L/E. For practical reasons, the various kinds of oscillation experiments can be divided according to their distance L from the neutrino source as shown in Table 5.1, adopted from Table 7.1 in Ref. [2].

With respect to the production process, another distinction can be made. As for "man-made" neutrino sources, one can distinguish *reactor* experiments, which use the flux of antineutrinos produced by β^- decay of heavy fission products in nuclear reactions, and *accelerator* experiments, which use the neutrino flux produced by pion, muon and kaon decay created by fast protons hitting a target.

With regard to "natural" neutrino sources, the most important are *atmospheric* (ATM) and *solar* (SOL) neutrino experiments. Owing to their specific importance to neutrino physics, these experiments are discussed in more detail below, based as well as the above introduction on Ref. [2].

5.2 Solar neutrino measurements

Anachronistically, one could say since the distance Sun-Earth is quite high ($\sim 10^{11}$ m) and the energies of detectable solar neutrinos are in a range of some few MeV, solar neutrino measurements allow for a high sensitivity to squared-mass differences ($\Delta m^2 \sim 10^{-12}$ eV^2) and are thus crucial for neutrino oscillation research. In addition, the flux of electron neutrinos produced in the thermonuclear fusion reactions (*pp* chain and CNO cycle) in the core of the Sun is extremely large and therefore promising for experiments. Solar neutrino measurements, moreover, allow to obtain a "real-time" image of the Sun, especially of the interior of the Sun, being accessible only through neutrino measurements due to the small cross sections of neutrinos for matter interactions.

Table 5.1: Nomenclature of neutrino oscillation experiments

Naming	Distance L	Sensitivity to Δm^2
Short BaseLine experiments (SBL)	$\sim 10 - 10^3$ m	$\sim 0.1 - 10^2$ eV^2
Long BaseLine experiments (LBL)	$\sim 10^3 - 10^6$ m	$\sim 10^{-3}$ eV^2
Very Long-BaseLine experiments (VLB)	$\sim 10^5 - 10^7$ m	$\sim 10^{-4} - 10^{-5}$ eV^2

Historically, the measurement of solar electron neutrinos gave rise to the Solar Neutrino Problem (SNP) and was likely one of the catalysts for neutrino oscillation research. The solar neutrino problem arose when measurements of the solar electron neutrino flux (e.g. by Homestake, Kamiokande, SAGE) revealed a deficit with respect to the prediction of the Standard Solar Model (SSM).

Ray Davis's famous Homestake experiment in 1970 was the first to monitor solar neutrino flux. The radiochemical experiment was based on the Cl-Ar reaction

$$\nu_e + {}^{37}\mathrm{Cl} \to {}^{37}\mathrm{Ar} + e^- \tag{5.1}$$

which has an energy threshold of $E_{\mathrm{th}} = 814$ keV and is thus able to monitor only intermediate and high-energy neutrinos from the *pp* chain. Such experiments, aiming at measuring solar (or atmospheric) neutrino flux, need to be positioned underground in order to be shielded effectively from cosmic radiation background.

Neutrino oscillations were able to give an explanation for the deficit of measured electron neutrinos relying on the correctness of the SSM. In particular, in the experiments of Sudbury Neutrino Observatory (SNO), total neutrino flux and electron neutrino flux could be distinguished, being compatible with both SSM prediction and the (large mixing angle, LMA) neutrino oscillation solution of the SNP [2, p.379-381,389]. The LMA solution states that the observed neutrino oscillations are caused by matter effects in the Sun as discussed in Section 4.1. Vacuum oscillations, in contrast, are supposed not to affect solar electron neutrino oscillations since neither significant seasonal variation in the oscillation probability has been observed[21] nor has the observed oscillation probability exhibited an energy dependency as implied by the vacuum oscillation formula (2.16) [2, p.382].

It is believed that by means of the LMA neutrino oscillation solution, the SNP has been resolved in 2002, based on the data of the SNO experiment. A best-fit of solar neutrino measurements yields $\Delta m^2 \sim 8 \times 10^{-5}$ eV^2, $\tan^2 \vartheta \sim 0.45$ [2, p.389].

5.3 Atmospheric neutrino measurements

Atmospheric neutrinos are created by the interaction of cosmic rays with atomic nuclei in the upper atmosphere. When cosmic rays, consisting mainly of hadrons (protons), hit atomic nuclei (essentially nitrogen and oxygen), pions and – at high energies – even kaons are produced, generating secondary cosmic rays (air showers). These particles, having a large range of energies, decay subsequently into lighter particles, releasing atmospheric neutrinos with an energy of $0.5 - 10^2$ GeV. Having a mean free path of around 10^{12} m in rock [14], atmospheric neutrinos are not expected to interact noticeable during their travel through the Earth, so a detector positioned anywhere in the Earth should not monitor any anomaly.

Measurement data from SuperKamiokande (SK), in particular, has shown a so-called "up-down anomaly" for muon neutrinos, which means that the measured number of muon neutrinos from above with path lengths of some 10 km is larger than the one from below, having path lengths of 1000s of km. Moreover, the measured muon neutrino flux is in agreement with predictions from Monte Carlo simulations for small ratios L/E,

[21] To be precise, no seasonal variation *additional* to the generic variation $\propto 1/L^2$ due to surface correction has been observed.

while showing a deficit for large ratios [2, p.420]. For electron neutrinos, however, no anomaly has been measured.

The measured deficit gives strong indications for flavor oscillations $\nu_\mu \to \nu_\tau$, which are able to provide an explanation for the decrease of muon neutrino flux with path length. For these oscillations, matter effects do not need to be taken into account[22] since the interaction of both muon and tau neutrino with matter is the same; only electron neutrinos sense matter effects due to CC-interactions.

Though the analysis of the data is not trivial, one can determine mixing angle and squared-mass difference via best-fit, yielding a high probability for maximal mixing, i.e. $\sin^2\vartheta \sim 1$, and a squared-mass difference of $\Delta m^2 \sim 2 \times 10^{-3}\ \mathrm{eV}^2$ [2, p.427]. Especially in respect to neutrino oscillation experiments, the matching of measurement data from various detectors using different kinds of detection processes is necessary to allow for definite interpretations, not biased by a specific – maybe not fully understood – measurement technique.

Hence, the experimental finding $\Delta m^2_{SOL} \ll \Delta m^2_{ATM}$ shows the necessity of considering hierarchic three neutrino mixing as discussed in Section 4.2.

[22] This is, however, only valid for two-neutrino mixing [11].

6 Discussion

Regarding neutrino oscillations, it is an impressive fact that they, though occurring on a macroscopic scale, can be analyzed by using the framework of quantum mechanics, even if quantum mechanics usually gives way to classical mechanics in the macroscopic limit. Neutrino oscillations, being a quantum mechanical interference phenomenon, are a remarkable exception to that rule, which is strongly related to the very properties of neutrinos as to their detection: Neutrino oscillations can only be observed if – for a specified production process – certain general conditions for the detection process (as to measurement uncertainties) are satisfied.

In the framework of quantum mechanics, the wave packet approach is the most consistent one, respecting the localization of the particles involved as well as (de)coherence effects. This is not the case for the standard approach, where an oversimplified assumption of definite massive neutrino momenta is outranged by the additional "same momentum" respectively "same energy" assumption, while the whole derivation is intrinsically inconsistent with respect to the concept of propagation of the introduced massive neutrino states. Nevertheless, the standard oscillation formula is useful for discussion since it allows to study some general aspects of neutrino oscillations and emerges in the limit of negligible decoherence effects from the more consistent wave packet approach. At this point it should not be forgotten to mention that in the above discussion wave packet spread has been neglected since for relativistic wave packets such as relevant for neutrinos, the spread is negligible [9].

Admittedly, the reason for an inadequate approach yielding a correct result is not obvious and can only be understood to be a "fortunate case" where the approximations of the proper approach yield the same expression as used as starting point for the standard derivation. In fact, as has been pointed out in Ref. [4], the wave packet approach allows for the expression obtained in the standard derivation under appropriate conditions (i.e. the coherence conditions discussed in Section 3.1). For instance, by first summing over the plane waves with equal momentum/energy but different mass and afterwards integrating over the whole spectrum, the standard formula is obtained. Likewise, it is a fact that plane wave neutrinos with definite energy and specific overall energy spectrum are physically indistinguishable from a wave packet beam having an energy distribution function whose absolute square is equal to the energy spectrum mentioned above [4, 15]. This helps to understand why with respect to stationary sources, i.e. sources which do not exhibit a certain time dependency, the the "equal energy" approach yields the correct result.

Yet, even the more consistent wave packet approach is limited in its applicability: it turns out that some aspects of neutrino oscillations (regarding the production/detection process, consistent normalization) require QFT treatment which is beyond the scope of this thesis. Being given appropriate neutrino states, however, the QM approach allows for manageable calculations, particularly with respect to matter effects involving a non-trivial density profile [4], thus stressing the necessity of studying neutrino oscillations in the framework of QM.

The vivid picture obtained by imagining a wave packet with "effective shape" propagating through space-time has to be taken with caution since as already discussed, effective quantities depend on both production and detection, causing a causal inconsistency of the above picture. Accordingly, the quantum mechanical description of neutrino oscillations is not complete, hinging on the correctness of the provided (flavor) neutrino states which are not accessible to quantum mechanical derivation.

Contrary to the expectation of a fruitful generalization of neutrino oscillations, oscillations of charged leptons are not possible because the flavor eigenstates necessarily coincide with the mass eigenstates: Charged leptons can be distinguished only by means of their mass (lepton universality), therefore the flavor of a charged lepton is defined by its mass [2, p.317]. Furthermore, a coherent superposition of charged leptons immediately loses coherence due to the separation of the wave packets, since, owing to their large mass differences, the group velocities vary considerably. In addition, in order to observe charged lepton oscillations it must be possible to discriminate coherent superpositions of charged lepton states *as well as* superpositions of flavor neutrino states, which cannot be achieved with known (CC) processes [6].

Nonetheless, this does by no means imply that neutrino physics would run out of questions: Until now, it is not clear yet whether neutrinos have to be regarded as Dirac particles (for which particle and antiparticle do not coincide) or Majorana particles (particle and antiparticle are the same). Experiments with neutrinoless double-β-decay ($0\nu\beta\beta$) aiming to find whether neutrinos are Majorana or Dirac leptons are still going on – as well as experiments aiming to determine the (effective) mass of the known flavor neutrinos. Moreover, it is not clear whether there is firm experimental evidence for sterile neutrinos, i.e. neutral, massive fermions which mix with the active neutrinos but do not participate in weak interaction, having been introduced by the see-saw model in order to explain the smallness of neutrino masses. If there is, one faces the question which consequences this implies for astrophysics and cosmology, respectively.

The above questions show that neutrino physics faces vital challenges, having been – despite the fact that studying neutrinos has ever since been a challenge to experimental physics – boosted by the discovery of neutrino oscillations. Neutrino oscillations have given rise to reasonable doubts about the definitiveness of the present Standard Model and might through an accurate analysis of further experimental measurements give some hints on how the former can be appropriately revised.

A Calculations

A.1 Transition probability for two neutrino mixing

Specifying the mixing matrix in Eqn. (2.13) by the real mixing matrix (2.15) and using the convention $\Delta m^2 \equiv m_2^2 - m_1^2$, $M_{\alpha\beta}^{jk} = U_{\alpha j}U_{\beta j}U_{\alpha k}U_{\beta k}$, one finds for $\alpha \neq \beta$:

$$\begin{aligned}
P_{\nu_\alpha\to\nu_\beta}(L,E) &= \sum_{j>k} M_{\alpha\beta}^{jk}\exp\left(-i\frac{\Delta m^2 L}{2E}\right) + \sum_{j<k} M_{\alpha\beta}^{jk}\exp\left(i\frac{\Delta m^2 L}{2E}\right) + \sum_{j=k} M_{\alpha\beta}^{jk} \\
&= -\cos^2\vartheta\sin^2\vartheta\left[\exp\left(-i\frac{\Delta m^2 L}{2E}\right) + \exp\left(i\frac{\Delta m^2 L}{2E}\right)\right] + 2\cos^2\vartheta\sin^2\vartheta \\
&= 2\cos^2\vartheta\sin^2\vartheta\left[1-\cos\left(\frac{\Delta m^2 L}{2E}\right)\right] \\
&= \sin^2 2\vartheta\sin^2\left(\frac{\Delta m^2 L}{4E}\right). \qquad \text{(A.1)}
\end{aligned}$$

For the calculation of the survival probability in the case of two neutrino mixing in matter, the following relation is worth being kept in mind [12]:

$$\begin{aligned}
\cos^2\vartheta\cos^2\vartheta_M^i + \sin^2\vartheta\sin^2\vartheta_M^i &= \frac{1}{2}(1+\cos 2\vartheta)\cos^2\vartheta_M^i + \frac{1}{2}(1-\cos 2\vartheta)\sin^2\vartheta_M^i \\
&= \frac{1}{2}\left[\cos^2\vartheta_M^i + \sin^2\vartheta_M^i + \cos 2\vartheta(\cos^2\vartheta_M^i - \sin^2\vartheta_M^i)\right] \\
&= \frac{1}{2}\left[1+\cos 2\vartheta\cos 2\vartheta_M^i\right]. \qquad \text{(A.2)}
\end{aligned}$$

A.2 A convenient notation of the oscillation probability

As discussed in Section 2.1, it is often convenient so split the oscillation probability (2.13) into the real and imaginary parts of the mixing matrix components. To maintain clarity, we introduce the notation $M_{\alpha\beta}^{jk} = U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^*$, $\Phi_{jk} = \Delta m_{jk}^2 L/2E$. By splitting the summation and keeping in mind $\mathfrak{Re}[a] = (a+a^*)/2$, one finds

$$\begin{aligned}
\sum_{j,k} M_{\alpha\beta}^{jk} e^{-i\Phi_{jk}} &= \sum_{j=k} M_{\alpha\beta}^{jk} + \sum_{j>k} M_{\alpha\beta}^{jk} e^{-i\Phi_{jk}} + \sum_{j<k} M_{\alpha\beta}^{jk} e^{-i\Phi_{jk}} \\
&= \sum_j |U_{\alpha j}|^2 |U_{\beta j}|^2 + \sum_{j>k}\left[M_{\alpha\beta}^{jk} e^{-i\Phi_{jk}} + M_{\alpha\beta}^{jk*} e^{i\Phi_{jk}}\right] \\
&= \sum_j |U_{\alpha j}|^2 |U_{\beta j}|^2 + 2\sum_{j>k}\mathfrak{Re}\left[M_{\alpha\beta}^{jk} e^{-i\Phi_{jk}}\right]. \qquad \text{(A.3)}
\end{aligned}$$

Multiplying the unitarity relation (2.4) with its complex conjugate, we obtain

$$\begin{aligned}
\delta_{\alpha\beta} &= \sum_{j,k} U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^* \\
&= \sum_j |U_{\alpha j}|^2 |U_{\beta j}|^2 + 2\sum_{j>k}\mathfrak{Re}[M_{\alpha\beta}^{jk}], \qquad \text{(A.4)}
\end{aligned}$$

which can be used to rewrite (2.13) with the help of (A.3) and (A.4):

$$\begin{aligned}
P_{\nu_\alpha\to\nu_\beta}(L,E) &= \delta_{\alpha\beta} - \sum_{j>k}\left[M_{\alpha\beta}^{jk} + M_{\alpha\beta}^{jk*}\right] + \sum_{j>k}\left[M_{\alpha\beta}^{jk}e^{-i\Phi_{jk}} + M_{\alpha\beta}^{jk*}e^{i\Phi_{jk}}\right] \\
&= \delta_{\alpha\beta} - \sum_{j>k}\left[M_{\alpha\beta}^{jk} + M_{\alpha\beta}^{jk*}\right] + \frac{1}{2}\sum_{j>k}\left[\left(M_{\alpha\beta}^{jk} + M_{\alpha\beta}^{jk*}\right)\left(e^{-i\Phi_{jk}} + e^{i\Phi_{jk}}\right)\right] \\
&\quad + \frac{1}{2}\sum_{j>k}\left[\left(M_{\alpha\beta}^{jk} - M_{\alpha\beta}^{jk*}\right)\left(e^{-i\Phi_{jk}} - e^{i\Phi_{jk}}\right)\right] \\
&= \delta_{\alpha\beta} - 2\sum_{j>k}\mathfrak{Re}\left[M_{\alpha\beta}^{jk}\right](1-\cos\Phi_{jk}) - \frac{1}{2}\sum_{j>k}\left(M_{\alpha\beta}^{jk} - M_{\alpha\beta}^{jk*}\right)\left(e^{i\Phi_{jk}} - e^{-i\Phi_{jk}}\right) \\
&= \delta_{\alpha\beta} - 2\sum_{j>k}\mathfrak{Re}\left[M_{\alpha\beta}^{jk}\right](1-\cos\Phi_{jk}) + 2\sum_{j>k}\mathfrak{Im}\left[M_{\alpha\beta}^{jk}\right]\sin\Phi_{jk}. \qquad (A.5)
\end{aligned}$$

A.3 Gaussian average of the cosine function

For a Gaussian distribution, the average (2.32) can be calculated analytically. Substituting

$$x = \frac{L}{E}, \quad y_\pm = x - \left(\langle x\rangle \pm i\frac{\Delta m^2}{2}\sigma_{L/E}^2\right), \quad \lambda = \frac{1}{2\sigma_{L/E}^2}, \qquad (A.6)$$

and respecting

$$\int_{-\infty}^{\infty} e^{-\lambda x^2}\mathrm{d}x = \sqrt{\frac{\pi}{\lambda}}, \qquad (A.7)$$

one finds

$$\begin{aligned}
\left\langle \cos\left(\frac{\Delta m^2 L}{2E}\right)\right\rangle &= \int \cos\left(\frac{\Delta m^2 L}{2E}\right)\phi\left(\frac{L}{E}\right)\mathrm{d}\frac{L}{E} \\
&= \frac{1}{2}\sqrt{\frac{\lambda}{\pi}}e^{-\lambda\langle x\rangle^2}\int \mathrm{d}x\left\{\exp\left[-\lambda\left(x^2 - 2x\left(\langle x\rangle + i\frac{\Delta m^2}{2}\sigma_{L/E}^2\right)\right)\right]\right. \\
&\quad \left. + \exp\left[-\lambda\left(x^2 - 2x\left(\langle x\rangle - i\frac{\Delta m^2}{2}\sigma_{L/E}^2\right)\right)\right]\right\} \\
&= \frac{1}{2}\sqrt{\frac{\lambda}{\pi}}\left\{\exp\left[i\frac{\Delta m^2\langle x\rangle}{2}\right]\int \mathrm{d}y_+\exp\left[-\lambda y_+^2 - \frac{(\Delta m^2)^2}{8}\sigma_{L/E}^2\right]\right. \\
&\quad \left. + \exp\left[-i\frac{\Delta m^2\langle x\rangle}{2}\right]\int \mathrm{d}y_-\exp\left[-\lambda y_-^2 - \frac{(\Delta m^2)^2}{8}\sigma_{L/E}^2\right]\right\} \\
&= \cos\left(\frac{\Delta m^2}{2}\left\langle\frac{L}{E}\right\rangle\right)\exp\left[-\frac{1}{2}\left(\frac{\Delta m^2}{2}\sigma_{L/E}\right)^2\right]. \qquad (A.8)
\end{aligned}$$

A.4 Via Gaussian integrals to the oscillation probability

It is one of the main benefits of functions with Gaussian shape that many integrals can be carried out analytically. In particular, for the integral over $\mathrm{d}p^3$ in (3.13) one obtains:

$$\vec{q} = (\vec{p} - \vec{\tilde{p}}_j), \quad \vec{q}' = \vec{q} - 2i\sigma_p^2(\vec{L} - \vec{v}_j T), \quad \lambda = \frac{1}{4\sigma_p^2},$$

$$\begin{aligned}\int \mathrm{d}^3 p\; e^{i(\vec{p}-\vec{\tilde{p}}_j)\cdot(\vec{L}-\vec{v}_j T) - \frac{(\vec{p}-\vec{\tilde{p}}_j)^2}{4\sigma_p^2}} &= \int \mathrm{d}^3 q\; \exp\left[-\frac{\vec{q}^2}{4\sigma_p^2} + i\vec{q}\cdot(\vec{L} - \vec{v}_j T)\right] \\ &= \int \mathrm{d}^3 q'\; \exp\left[-\lambda \vec{q}'^2 - \sigma_p^2 (\vec{L} - \vec{v}_j T)^2\right] \\ &= \left(\frac{\pi}{\lambda}\right)^{3/2} \exp\left[\frac{(\vec{L} - \vec{v}_j T)^2}{4\sigma_x^2}\right]. \end{aligned} \tag{A.9}$$

In Eqn. (A.9), the uncertainty relation $\sigma_x \sigma_p = \frac{1}{2}$ was used.

For the calculation of the integral in Eqn. (3.15) we ignore the irrelevant mixing matrix components and define the auxiliary variables

$$\begin{aligned} &\vec{\Delta}_p = \vec{\tilde{p}}_j - \vec{\tilde{p}}_k, \quad \Delta_E = \tilde{E}_j - \tilde{E}_k, \quad \lambda = \frac{1}{4\sigma_x^2}, \quad v_{jk}^2 = v_j^2 + v_k^2, \\ &M = e^{i\vec{\Delta}_p \cdot \vec{L} - 2\lambda L^2}, \quad y = T - \left(\frac{(\vec{v}_j + \vec{v}_k)\cdot \vec{L}}{v_{jk}^2} - 2\sigma_x^2 i \frac{\Delta_E}{v_{jk}^2}\right), \end{aligned} \tag{A.10}$$

by whose help we obtain

$$\begin{aligned} \int \mathrm{d}T\; |A_{\nu_\alpha \to \nu_\beta}(\vec{L}, T)|^2 &= \int \mathrm{d}T\; e^{-i\Delta_E T} e^{i\vec{\Delta}_p\cdot\vec{L}} e^{-\lambda[(\vec{L}-\vec{v}_j T)^2 + (\vec{L}-\vec{v}_k T)^2]} \\ &= \int \mathrm{d}T\; M\, e^{-i\Delta_E T} e^{-\lambda[(v_j^2+v_k^2)T^2 - 2\vec{L}\cdot(\vec{v}_j+\vec{v}_k)T]} \\ &= \int \mathrm{d}y\; M\, e^{-\lambda v_{jk}^2 y^2} \exp\left\{\frac{[(\vec{v}_j+\vec{v}_k)\cdot\vec{L}]^2}{4\sigma_x^2 v_{jk}^2} - i\frac{\Delta_E(\vec{v}_j+\vec{v}_k)\cdot\vec{L}}{v_{jk}^2} - \frac{\Delta_E^2}{v_{jk}^2}\sigma_x^2\right\} \\ &= \sqrt{\frac{\pi}{\lambda v_{jk}^2}} \exp\left\{-i\left[\Delta_E \frac{\vec{v}_j+\vec{v}_k}{v_{jk}^2} - \vec{\Delta}_p\right]\cdot\vec{L}\right\} \exp\left\{-\frac{L^2}{2\sigma_x^2} - \frac{\Delta_E^2}{4\sigma_p^2 v_{jk}^2}\right\} \\ &\quad \times \exp\left\{2\frac{(\vec{v}_j\cdot\vec{L})^2 + (\vec{v}_k\cdot\vec{L})^2}{4\sigma_x^2 v_{jk}^2} - \frac{\left[(\vec{v}_j - \vec{v}_k)\cdot\vec{L}\right]^2}{4\sigma_x^2 v_{jk}^2}\right\}, \end{aligned} \tag{A.11}$$

where the uncertainty relation as well as

$$\left[(\vec{v}_j + \vec{v}_k)\cdot\vec{L}\right]^2 = 2\left[(\vec{v}_j\cdot\vec{L})^2 + (\vec{v}_k\cdot\vec{L})^2\right] - \left[(\vec{v}_j - \vec{v}_k)\cdot\vec{L}\right]^2 \tag{A.12}$$

was used.

A.5 A more general approach to neutrino oscillations

To simplify further calculations, it is convenient to rewrite the effective shape factor $G_j(L - v_jT)$. This can be done by inserting the shape factors in Eqns. (3.31), (3.34) into the effective one in Eqn. (3.36), using the definition of the Dirac delta function

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}x\ e^{-ix(p-p_0)} = \sqrt{2\pi}\ \delta(p-p_0), \tag{A.13}$$

and substituting $p' = \tilde{p} - p_j + p'_j$:

$$\begin{aligned} G_j(L - v_jT) &= \int \mathrm{d}x\ g_j^P(x - v_jT) g_j^{D*}(x - L) e^{i(p_j - p'_j)(x-L)} \\ &= \frac{1}{2\pi}\int \mathrm{d}x \int \mathrm{d}p \int \mathrm{d}\tilde{p}\ f_j^P(p) f_j^{D*}(\tilde{p}) e^{ix(p-\tilde{p}+p_j-p'_j)} e^{iL(\tilde{p}-p_j+p'_j)} e^{-ipv_jT} \\ &= \frac{1}{2\pi}\int \mathrm{d}x \int \mathrm{d}p \int \mathrm{d}p'\ f_j^P(p) f_j^{D*}(p' + p_j - p'_j) e^{ix(p-p')} e^{iLp'} e^{-ipv_jT} \\ &= \int \mathrm{d}p\ f_j^P(p) f_j^{D*}(p + \delta_j) e^{ip(L - v_jT)}. \end{aligned} \tag{A.14}$$

In Eqn. (A.14), $\delta_j = p_j - p'_j$ denotes the difference of the mean momenta of production and detection and in the last step, the integration variable has been renamed.

The integral $I_{jk}(L)$ in Eqn. (3.39) can be calculated by substituting

$$p_1 = p, \quad p_2 = \frac{v_j}{v_k} p' + \frac{\Delta E_{jk}}{v_k} \equiv rp' + \frac{\Delta E_{jk}}{v_k}, \tag{A.15}$$

and inserting the approximated phase (3.41) into Eqn. (3.39):

$$\begin{aligned} I_{jk}(L) &= \int \mathrm{d}T \int \mathrm{d}p_1 \int \mathrm{d}p_2\ f_j^P(p_1) f_j^{D*}(p_1 + \delta_j) f_k^{P*}(p_2) f_k^D(p_2 + \delta_k) \\ &\quad \times e^{ip_1(L - v_jT) - ip_2(L - v_kT) - i\left[\frac{\Delta m^2}{2p} L - \frac{\Delta E}{v_g}(L - v_gT)\right]} \\ &= \int \mathrm{d}T \int \mathrm{d}p \int \frac{v_j}{v_k} \mathrm{d}p'\ f_j^P(p) f_j^{D*}(p + \delta_j) f_k^{P*}(rp' + \Delta E/v_k) f_k^D(rp' + \Delta E/v_k + \delta_k) \\ &\quad \times e^{ip(L - v_jT) - i(L - v_kT)(rp' + \Delta E/v_k) - i\left[\frac{\Delta m^2}{2p} L - \frac{\Delta E}{v_g}(L - v_gT)\right]} \\ &= \frac{2\pi}{v_k} \int \mathrm{d}p\ M_{jk}^{PD} e^{ip(1-r)L} e^{-i\frac{\Delta m^2}{2p}L} \left\{ e^{-iL\Delta E\left(\frac{v_j - v_k}{2v_gv_k}\right)} \right\}. \end{aligned} \tag{A.16}$$

Here, M_{jk}^{PD} stands for the shape factors $M_{jk}^{PD} = f_j^P(p) f_j^{D*}(p + \delta_j) f_k^{P*}(rp + \Delta E/v_k)$ $\times f_k^D(rp + \Delta E/v_k + \delta_k)$, $r \equiv v_j/v_k$, and the definition of the Dirac delta function (A.13) has been used in the time integration. The indices jk of energy and squared-mass difference have been omitted for clarity. At first order in Δm^2, the term in the curly brackets is equal to one, since both ΔE and $(v_j - v_k)/(2v_gv_k)$ are of $\mathcal{O}(\Delta m^2)$ (see e.g. Eqn. (3.19), Ref. [4]).

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Erklärung

Ich versichere, dass ich diese Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

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