Mechanics and kinetics in the Friedmann–Lemaître–Robertson–Walker space-times

S.R. Kelner
Max-Planck-Institut für Kernphysik, Saupfercheckweg 1, D-69117 Heidelberg, Germany *

A.Yu. Prosekin
Max-Planck-Institut für Kernphysik, Saupfercheckweg 1, D-69117 Heidelberg, Germany†

F.A. Aharonian
Dublin Institute for Advanced Studies, 31 Fitzwilliam Place, Dublin 2, Ireland‡

(Dated: May 13, 2011)

Using the standard canonical formalism, the equations of mechanics and kinetics in the Friedmann–Lemaître–Robertson–Walker (FLRW) space-times in Cartesian coordinates have been obtained. The transformation law of the generalized momentum under the shift of the origin of the coordinate system has been found, and the form invariance of the Hamiltonian function relative to the shift transformation has been proved. The general solution of the collisionless Boltzmann equation has been found. In the case of the homogeneous distribution the solutions of the kinetic equation for several simple, but important for applications, cases have been obtained.

PACS numbers: 04.20.-q, 05.20.Dd , 05.60.Cd, 45.20.Jj

I. INTRODUCTION

The covariant general relativistic Boltzmann equation for the one-particle distribution function has been found in Ref. [1]. Eq. (16) from Ref. [1] reads

$$\frac{\partial \chi}{\partial x^i}\, p^i - \frac{\partial \chi}{\partial p^i}\, \Gamma^i_{jk} p^j p^k = 0\,. \qquad (1)$$

However, the physical meaning of the function χ and correct interpretation of the equation had been revealed much later (see Refs. [2, 3] and references cited there, as well as Ref. [4], where a critical review of previous studies conducted for the basic plasma modes in the expanding Universe is given). Throughout the present paper, except for Appendix A, we consider the motion of particles and the kinetics in the FLRW space-times in Cartesian coordinates. The uniqueness of the FLRW metrics [5], its homogeneity and isotropy are more clearly exposed in these coordinates, therefore we do not use generally covariant notations.

Our approach is based on the standard scheme of the classical mechanics: the generalized coordinates and Lagrangian function → the generalized momentum → the Hamiltonian function → the phase space. Thus there is no problem to derive the collisionless Boltzmann equation and to interpret the distribution function. Moreover, the Hamilton-Jacobi equation allows us, as in the case of the conventional space, to find in the explicit form all six integrals of motion and thereby to obtain the general solution of the Boltzmann equation. The solution is particularly simple in the spherically symmetric case.

It is well known that the metrics FLRW is form-invariant relative to the shift of the origin of the coordinate system, and it is known how the Cartesian coordinates are transformed under the shift [6]. The mechanics also appears to be form-invariant. This circumstance is always implicitly assumed, however, to our knowledge, the direct prove of the invariance has not been demonstrated. In this paper we prove the form-invariance of the Hamiltonian function and find the law of the momentum transformation under the shift of the origin of coordinates. If one moves the reference point to the point where the observer is located, we can easily interpret the results of calculations since in the vicinity of the observation point the space can be considered as Euclidean one.

In the curved space, if the distribution of particles is homogenous, it is isotropic as well (see Appendix B). In this case, the derivation of the collisionless Boltzmann equation is trivial, and the collision integrals can be described in the same way as in the flat space. It is impossible to find analytical solutions to the Boltzmann equation with collision integrals, therefore we restricted ourselves by several simple, but important for applications, cases when the solution can be obtained by quadratures.

For convenience, some of the principal calculations are presented in four Appendices. In Appendix A we give a simple derivation of the Boltzmann equation for space with an arbitrary metric written in arbitrary coordinates. The Hamilton function has been obtained, and it has been shown that the equation of motion and the Boltzmann equation can be written in the standard form. Finally, in Appendix D we note the surprising effect that a photon, emitted by the source with superluminal recession velocity in the direction to the observer, during a certain time interval moves away from the observer.

*Electronic address: Stanislav.Kelner@mpi-hd.mpg.de
†Electronic address: Anton.Prosekin@mpi-hd.mpg.de
‡Max-Planck-Institut für Kernphysik, Saupfercheckweg 1, D-69117 Heidelberg, Germany; Electronic address: Felix.Aharonian@mpi-hd.mpg.de

II. MECHANICS OF FREE PARTICLES IN FLRW SPACE-TIMES

In Cartesian coordinates the Friedmann–Lemaître–Robertson–Walker metric can be represented in the form

$$ds^2 = dt^2 - a^2(t)\left(d\boldsymbol{r}^2 + \kappa\frac{(\boldsymbol{r}d\boldsymbol{r})^2}{1-\kappa\boldsymbol{r}^2}\right), \tag{2}$$

where κ is the discrete quantity which describes possible isotropic models and has the following values: $\kappa = 1$ for the closed model (positive curvature), $\kappa = -1$ for the open model (negative curvature), and $\kappa = 0$ for the flat space. The function $a(t)$, which is determined by Friedmann equation, is assumed to be given. We use the three-dimensional vector notations, $\boldsymbol{r} = (x, y, z)$, $(\boldsymbol{r}d\boldsymbol{r}) = x\,dx + y\,dy + z\,dz$, etc., where $\boldsymbol{r}$ is considered to be vector in the sense that under rotations relative to the origin of coordinates the components of $\boldsymbol{r}$ are transformed the same way as the vector components in Euclidean space.

The action functional of a particle in the gravitational field is given by (assuming $c = 1$)

$$S = -m\int ds = \int L\,dt\,, \tag{3}$$

where Lagrangian function is

$$L = -m\sqrt{1 - a^2(t)\left(\boldsymbol{v}^2 + \kappa\frac{(\boldsymbol{r}\boldsymbol{v})^2}{1-\kappa\boldsymbol{r}^2}\right)}\,. \tag{4}$$

Considering $\boldsymbol{r}$ and $\boldsymbol{v} = d\boldsymbol{r}/dt$ as generalized coordinates and velocities, we can make use of the formalism of the classical mechanics (see, e.g., [7], [8]). Then the generalized momentum is expressed as

$$\boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}} = \frac{ma^2\left(\boldsymbol{v} + \kappa\dfrac{\boldsymbol{r}(\boldsymbol{r}\boldsymbol{v})}{1-\kappa\boldsymbol{r}^2}\right)}{\sqrt{1 - a^2(t)\left(\boldsymbol{v}^2 + \kappa\dfrac{(\boldsymbol{r}\boldsymbol{v})^2}{1-\kappa\boldsymbol{r}^2}\right)}}\,, \tag{5}$$

and the energy is

$$E = \boldsymbol{v}\,\frac{\partial L}{\partial \boldsymbol{v}} - L = \frac{m}{\sqrt{1 - a^2(t)\left(\boldsymbol{v}^2 + \kappa\dfrac{(\boldsymbol{r}\boldsymbol{v})^2}{1-\kappa\boldsymbol{r}^2}\right)}}\,. \tag{6}$$

Note that $E \geq m$, as it is in Minkowski space. It is easy to ascertain by direct check that ($a \equiv a(t)$)

$$E^2 - \frac{1}{a^2}\left(\boldsymbol{p}^2 - \kappa(\boldsymbol{p}\boldsymbol{r})^2\right) = m^2\,, \tag{7}$$

therefore the Hamiltonian function has the following form

$$\mathscr{H}(\boldsymbol{p}, \boldsymbol{r}, t) = \frac{1}{a}\sqrt{\boldsymbol{p}^2 - \kappa(\boldsymbol{p}\boldsymbol{r})^2 + m^2a^2}\,. \tag{8}$$

The dependence of generalized coordinates and momenta on time is found from canonical equations of motion

$$\dot{\boldsymbol{r}} = \frac{\partial \mathscr{H}}{\partial \boldsymbol{p}} = \frac{1}{a}\,\frac{\boldsymbol{p} - \kappa\boldsymbol{r}\,(\boldsymbol{p}\boldsymbol{r})}{\sqrt{\boldsymbol{p}^2 - \kappa(\boldsymbol{p}\boldsymbol{r})^2 + m^2a^2}}\,, \tag{9}$$

$$\dot{\boldsymbol{p}} = -\frac{\partial \mathscr{H}}{\partial \boldsymbol{r}} = \frac{1}{a}\,\frac{\kappa\boldsymbol{p}\,(\boldsymbol{p}\boldsymbol{r})}{\sqrt{\boldsymbol{p}^2 - \kappa(\boldsymbol{p}\boldsymbol{r})^2 + m^2a^2}}\,, \tag{10}$$

which admit an exact analytical solution in general case (see below). For massless particles the Hamiltonian function becomes

$$\mathscr{H} = \frac{1}{a}\sqrt{\boldsymbol{p}^2 - \kappa(\boldsymbol{p}\boldsymbol{r})^2}\,. \tag{11}$$

This formula can also be applied to ultrarelativistic particles.

Since the Hamiltonian function explicitly depends on time, the energy of particle is not conserved. To determine the time dependence of the energy, let us consider the differential equation which at any κ has the form

$$\frac{dE}{dt} = \frac{\partial \mathscr{H}}{\partial t} = -\frac{\dot{a}}{a}\left(E - \frac{m^2}{E}\right). \tag{12}$$

The solution of this equation gives the relation between values of the energy of a freely moving particle at different moments of time:

$$E(t) = \left[\left(\frac{a(t')}{a(t)}\right)^2\left(E^2(t') - m^2\right) + m^2\right]^{1/2}. \tag{13}$$

For photons (or ultrarelativistic particles) this leads to the well known relation between the energy and the scale factor (redshift):

$$a(t)\,E(t) = \text{const}\,. \tag{14}$$

Often this relation is interpreted as a consequence of the Doppler effect. However this seems to us not correct since Eq. (13) for $m \neq 0$ cannot be obtained using Lorentz transformation. In the nonrelativistic case, denoting $E = m + E_{\text{kin}}$ and assuming that $E_{\text{kin}} \ll m$, we get

$$a^2(t)\,E_{\text{kin}}(t) = \text{const}\,, \tag{15}$$

which means that for nonrelativistic particles the decrease of kinetic energy with time is faster.

III. FORM-INVARIANCE OF MECHANICS

The space with metric given by Eq. (2) is homogeneous and isotropic. The isotropy of the space is obvious since the quantities $d\boldsymbol{r}^2$, $\boldsymbol{r}^2$ and $(\boldsymbol{r}\,d\boldsymbol{r})$ in Eq. (2) do not change under rotation relative to the origin of coordinates. The homogeneity implies that the origin of coordinates can be chosen at any point of the space, and the metric would have the same form of Eq. (2). The proof of homogeneity can be found in Ref. [6]. If the origin of coordinates is shifted from the point $\boldsymbol{r}=0$ to the point $\boldsymbol{r}=\boldsymbol{b}$, then the new coordinates $\boldsymbol{r}'$ are expressed through the old coordinates in the following way [6]:

$$\boldsymbol{r}'=\tilde{\boldsymbol{r}}(\boldsymbol{r},\boldsymbol{b})\equiv\boldsymbol{r}-\boldsymbol{b}\left(\sqrt{1-\kappa r^2}+\frac{\kappa(\boldsymbol{br})}{\sqrt{1-\kappa b^2}+1}\right). \quad (16)$$

If $\boldsymbol{r}=\boldsymbol{b}$, we have $\boldsymbol{r}'=0$. The inverse transformation

$$\boldsymbol{r}=\tilde{\boldsymbol{r}}(\boldsymbol{r}',-\boldsymbol{b}) \quad (17)$$

is obtained from Eq. (16) by replacing $\boldsymbol{b}\to-\boldsymbol{b}$. The "volume" elements in the new and old coordinates are connected by relations

$$d^3r'=\left(\sqrt{1-\kappa b^2}+\frac{\kappa(\boldsymbol{br})}{\sqrt{1-\kappa r^2}}\right)d^3r\,, \quad (18)$$

$$d^3r=\left(\sqrt{1-\kappa b^2}-\frac{\kappa(\boldsymbol{br}')}{\sqrt{1-\kappa r'^2}}\right)d^3r' \quad (19)$$

(note that Eqs. (18) and (19) are equivalent).

Let us find the transformation laws for the velocity and momentum under shifts. The velocity is transformed as contravariant vector. Assuming $\boldsymbol{r}=\boldsymbol{r}(t)$, $\boldsymbol{r}'=\boldsymbol{r}'(t)$ in Eq. (16) and differentiating it with respect to t, we find

$$\boldsymbol{v}'=\tilde{\boldsymbol{v}}(\boldsymbol{v},\boldsymbol{r},\boldsymbol{b})\equiv\boldsymbol{v}+\kappa\boldsymbol{b}\left(\frac{(\boldsymbol{rv})}{\sqrt{1-\kappa r^2}}-\frac{(\boldsymbol{bv})}{\sqrt{1-\kappa b^2}+1}\right). \quad (20)$$

The momentum, as it follows from Eq. (5), is a covariant vector, therefore the transformation law of $\boldsymbol{p}$ is

$$p'_\alpha=(M^{-1})_\alpha{}^\beta\,p_\beta\,, \quad (21)$$

where matrix M^{-1} is inverse to $M_\alpha{}^\beta=\partial v'_\beta/\partial v_\alpha$. Eq. (21) can be written in the explicit form :

$$\boldsymbol{p}'=\tilde{\boldsymbol{p}}(\boldsymbol{p},\boldsymbol{r},\boldsymbol{b})\equiv\boldsymbol{p}-\frac{\kappa(\boldsymbol{bp})}{\sqrt{(1-\kappa b^2)(1-\kappa r^2)}+\kappa(\boldsymbol{br})}\times\left(\boldsymbol{r}-\frac{\boldsymbol{b}\sqrt{1-\kappa r^2}}{\sqrt{1-\kappa b^2}+1}\right). \quad (22)$$

As in the case of the transformation Eq. (16), the inverse transformations of Eqs. (20) and (22) can be obtained by replacement of $\boldsymbol{b}$ with $-\boldsymbol{b}$ and interchange of primed and unprimed quantities:

$$\boldsymbol{v}=\tilde{\boldsymbol{v}}(\boldsymbol{v}',\boldsymbol{r}',-\boldsymbol{b})\,,\quad \boldsymbol{p}=\tilde{\boldsymbol{p}}(\boldsymbol{p}',\boldsymbol{r}',-\boldsymbol{b})\,. \quad (23)$$

From Eqs. (22) and (23) the following relations between the volume elements in the momentum space can be found:

$$d^3p'=\frac{\sqrt{1-\kappa r^2}}{\sqrt{(1-\kappa r^2)(1-\kappa b^2)}+\kappa(\boldsymbol{br})}\,d^3p\,, \quad (24)$$

$$d^3p=\frac{\sqrt{1-\kappa r'^2}}{\sqrt{(1-\kappa r'^2)(1-\kappa b^2)}-\kappa(\boldsymbol{br}')}\,d^3p'\,. \quad (25)$$

The change of variables from $(\boldsymbol{p},\boldsymbol{r})$ to $(\boldsymbol{p}',\boldsymbol{r}')$ is a canonical transformation that can be proved by direct computation of Poisson brackets. However, it can be demonstrated much easier by noting that the transformation can be carried out by the generating function

$$\mathscr{S}(\boldsymbol{p},\boldsymbol{r}')=(\boldsymbol{pr}')+(\boldsymbol{pb})\left(\sqrt{1-\kappa r'^2}-\frac{\kappa(\boldsymbol{br}')}{\sqrt{1-\kappa b^2}+1}\right). \quad (26)$$

The equations

$$\boldsymbol{r}=\frac{\partial\mathscr{S}}{\partial\boldsymbol{p}}\,,\qquad \boldsymbol{p}'=\frac{\partial\mathscr{S}}{\partial\boldsymbol{r}'} \quad (27)$$

following from this are equivalent to Eqs. (16) and (22).

Since $\mathscr{S}$ does not depend on time, the old and new Hamiltonian functions are equal: $\mathscr{H}'=\mathscr{H}$. By the direct check one can ascertain the validity of the equation

$$\boldsymbol{p}'^2-\kappa(\boldsymbol{p}'\boldsymbol{r}')^2=\boldsymbol{p}^2-\kappa(\boldsymbol{pr})^2\,. \quad (28)$$

Therefore, in the new reference system

$$\mathscr{H}'(\boldsymbol{p}',\boldsymbol{r}',t)=\mathscr{H}(\boldsymbol{p}',\boldsymbol{r}',t)=\frac{1}{a}\sqrt{\boldsymbol{p}'^2-\kappa(\boldsymbol{p}'\boldsymbol{r}')^2+m^2a^2}\,, \quad (29)$$

i.e. the Hamiltonian function is form-invariant. Thus we arrive at the natural conclusion that not only the geometry but also the mechanics in FLRW space–times does not change under the shift of the origin of coordinates. It should be kept in mind that in the curved space not only generalized coordinates but also generalized momenta depends on choice of origin of coordinates.

The transformations given by Eqs. (16), (20) and (22) are convenient for analysis of the results obtained in the curved space. Let us assume that we know the solution of a problem in the reference frame with the origin located at the source, and an observer is located at the point $\boldsymbol{r}$. For analysis of the result it is convenient to move to another coordinate system shifting the origin to the location of observer, i.e. to assume $\boldsymbol{b}=\boldsymbol{r}$. The space in the small neighborhood of $\boldsymbol{r}'=0$ of new reference frame can be considered as Euclidean one that appreciably simplifies the analysis.

The replacement of $\boldsymbol{b}=\boldsymbol{r}$ in Eqs. (20) and (22) gives the generalized velocity and momentum in the observation point:

$$\boldsymbol{u}=\tilde{\boldsymbol{v}}(\boldsymbol{v},\boldsymbol{r},\boldsymbol{r})=\boldsymbol{v}+\frac{\boldsymbol{r}(\boldsymbol{rv})}{r^2}\left(\frac{1}{\sqrt{1-\kappa r^2}}-1\right), \quad (30)$$

$$\boldsymbol{q}=\tilde{\boldsymbol{p}}(\boldsymbol{p},\boldsymbol{r},\boldsymbol{r})=\boldsymbol{p}+\frac{\boldsymbol{r}(\boldsymbol{rp})}{r^2}\left(\sqrt{1-\kappa r^2}-1\right). \quad (31)$$

The vectors $\boldsymbol{u}$ and $\boldsymbol{q}$ are parallel, while their squares are

$$\boldsymbol{u}^2=\boldsymbol{v}^2+\frac{\kappa(\boldsymbol{vr})^2}{1-\kappa r^2}\,,\quad \boldsymbol{q}^2=\boldsymbol{p}^2-\kappa(\boldsymbol{pr})^2\,. \quad (32)$$

It should be noted that quantities $\boldsymbol{u}^2$ и $\boldsymbol{q}^2$ are invariants relative to shift. Multiplying Eq. (31) by vector $\boldsymbol{r}$, we find

$$(\boldsymbol{qr}) = (\boldsymbol{pr})\sqrt{1-\kappa r^2}\,. \tag{33}$$

Rewriting $(\boldsymbol{pr}) = pr\cos\theta$, $(\boldsymbol{qr}) = qr\cos\theta'$ and taking into account that $q = p\sqrt{1-\kappa r^2\cos^2\theta}$ in such notation, we obtain the following relation between the angles in the new and old coordinate systems:

$$\cos\theta' = \cos\theta\sqrt{\frac{1-\kappa r^2}{1-\kappa r^2\cos^2\theta}}\,, \tag{34}$$

and

$$\cos\theta = \frac{\cos\theta'}{\sqrt{1-\kappa r^2\sin^2\theta'}}\,, \tag{35}$$

$$\sin\theta = \sin\theta'\sqrt{\frac{1-\kappa r^2}{1-\kappa r^2\sin^2\theta'}}\,. \tag{36}$$

Let us denote by $\boldsymbol{V}$ and $\boldsymbol{P}$ the usual (not generalized) velocity and momentum of the particle registered by the observer. Then

$$\boldsymbol{V} = a\,\boldsymbol{u}\,, \quad \boldsymbol{P} = \boldsymbol{q}/a\,. \tag{37}$$

Note that $(\boldsymbol{PV}) = (\boldsymbol{qu}) = (\boldsymbol{pv})$. From Eqs. (30) and (31) one can find the generalized velocity and momentum at the point $\boldsymbol{r}$ expressed in terms of $\boldsymbol{V}$ and $\boldsymbol{P}$:

$$\boldsymbol{v} = \frac{1}{a}\left[\boldsymbol{V} + \frac{\boldsymbol{r}(\boldsymbol{rV})}{r^2}\left(\sqrt{1-\kappa r^2}-1\right)\right], \tag{38}$$

$$\boldsymbol{p} = a\left[\boldsymbol{P} + \frac{\boldsymbol{r}(\boldsymbol{rP})}{r^2}\left(\frac{1}{\sqrt{1-\kappa r^2}}-1\right)\right]. \tag{39}$$

The relations between quantities $\boldsymbol{V}$, $\boldsymbol{P}$ and E are the same as in special relativity:

$$E = \frac{m}{\sqrt{1-V^2}} = \sqrt{P^2+m^2}\,, \quad \boldsymbol{V} = \frac{\boldsymbol{P}}{E}\,. \tag{40}$$

As an example, let us consider the solution given by Eqs. (9), (10) in the case of negative curvature ($\kappa = -1$). The origin of coordinates is taken at the position of the particle at the initial moment of time t_i, i.e. $\boldsymbol{r}(t_i) = 0$. Then the motion is radial and we seek the solution in the following form

$$\boldsymbol{r}(t) = \boldsymbol{n}\rho(t)\,, \qquad \boldsymbol{p}(t) = \boldsymbol{n}\varpi(t) \tag{41}$$

with initial condition $\rho\big|_{t=t_i} = 0$, $\varpi\big|_{t=t_i} = \varpi_0$, where $\boldsymbol{n}$ is an arbitrary unit vector, ϖ_0 is an arbitrary constant. The differential equations for ρ and ϖ are given by

$$\frac{d\rho}{dt} = \frac{1}{a}\,\frac{\varpi\,(1+\rho^2)}{\sqrt{\varpi^2(1+\rho^2)+m^2a^2}}\,, \tag{42}$$

$$\frac{d\varpi}{dt} = -\frac{1}{a}\,\frac{\varpi^2\rho}{\sqrt{\varpi^2(1+\rho^2)+m^2a^2}}\,. \tag{43}$$

Dividing one equation to another, we find

$$\frac{d\varpi}{d\rho} = -\frac{\varpi\rho}{1+\rho^2}\,, \tag{44}$$

from where it follows that $\varpi^2(1+\rho^2) = \varpi_0^2 = \text{const}$.

It is convenient to introduce a new function η defined as

$$\eta(t,t_i) = \int_{t_i}^{t}\frac{dt'}{a(t')\sqrt{1+(ma(t')/\varpi_0)^2}}\,. \tag{45}$$

Then the functions from Eq. (41) can be expressed through η:

$$\rho(t) = \sinh\eta\,, \qquad \varpi(t) = \varpi_0/\cosh\eta\,. \tag{46}$$

The generalized velocity is

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{d\eta}\,\frac{d\eta}{dt} = \frac{\boldsymbol{n}\cosh\eta}{a\sqrt{1+(ma/\varpi_0)^2}}\,. \tag{47}$$

Substituting the solution into Eqs. (30) and (31) and assuming $\kappa = -1$, we find the velocity and momentum at an arbitrary moment of time

$$\boldsymbol{P} = \frac{\boldsymbol{n}\varpi_0}{a}\,, \quad \boldsymbol{V} = \frac{\boldsymbol{n}}{\sqrt{1+(ma/\varpi_0)^2}}\,. \tag{48}$$

If at the initial moment of time the particle is at the point $\boldsymbol{r} = \boldsymbol{b}$, the solutions have the form

$$\boldsymbol{r}(t) = \tilde{\boldsymbol{r}}(\boldsymbol{n}\sinh\eta, -\boldsymbol{b})\,, \tag{49}$$

$$\boldsymbol{p}(t) = \tilde{\boldsymbol{p}}(\boldsymbol{n}\varpi_0/\cosh\eta, \boldsymbol{n}\sinh\eta, -\boldsymbol{b})\,. \tag{50}$$

This expressions are obtained from Eq. (41) by shifting the origin of coordinates to $-\boldsymbol{b}$. They describe the general solution of Hamilton equations which depends on six arbitrary constants: three components of vector $\boldsymbol{b}$, two angles defining the direction of $\boldsymbol{n}$, and ϖ_0.

For the space with positive curvature ($\kappa = +1$) the calculations are similar. Eq. (48) remains correct in this case, but instead of Eq. (46) we have

$$\rho = \sin\eta\,, \qquad \varpi = \varpi_0/\cos\eta\,, \tag{51}$$

where η is defined as before by Eq. (45).

IV. DISTRIBUTION FUNCTION

Let $f(\boldsymbol{p},\boldsymbol{r},t)$ is a distribution function of particles in the phase space. By definition, the quantity

$$dN = f(\boldsymbol{p},\boldsymbol{r},t)\,d^3p\,d^3r \tag{52}$$

implies the number of particles found at the moment t in the volume element $d^3p\,d^3r$ of the phase space. The Jacobian of the canonical transformation equals unity, therefore the phase volume does not change under shift, i.e.

$$d^3p\,d^3r = d^3p'\,d^3r'\,, \tag{53}$$

where the canonical variables $(\boldsymbol{p}', \boldsymbol{r}')$ are connected to $(\boldsymbol{p}, \boldsymbol{r})$ through Eqs. (16) and (22). One can directly verify Eq. (53) by multiplying Eqs. (18) and (24), or (19) and (25). Since dN and $d^3p\,d^3r$ are invariants, the distribution function is also invariant relative to the shift of the origin of coordinates:

$$f(\boldsymbol{p}, \boldsymbol{r}, t) = f'(\boldsymbol{p}', \boldsymbol{r}', t)\,. \tag{54}$$

The Boltzmann equation for f has the following standard form (see Appendix A)

$$\hat{L}f \equiv \left(\frac{\partial}{\partial t} + \frac{\partial \mathscr{H}}{\partial \boldsymbol{p}}\frac{\partial}{\partial \boldsymbol{r}} - \frac{\partial \mathscr{H}}{\partial \boldsymbol{r}}\frac{\partial}{\partial \boldsymbol{p}}\right) f = 0\,. \tag{55}$$

This equation describes the evolution of the distribution function of free-moving particles[1]. To take into account interactions, the collision integral should be added to the right part.

For a single particle moving according to Eq. (41), the distribution function is

$$f(\boldsymbol{p}, \boldsymbol{r}, t) = \delta(\boldsymbol{p} - \boldsymbol{n}\varpi_0/\cosh\eta)\,\delta(\boldsymbol{r} - \boldsymbol{n}\sinh\eta)\,. \tag{56}$$

If at the initial moment of time a particle is found at the point $\boldsymbol{r} = \boldsymbol{b}$ and has momentum $\boldsymbol{P}_0 = \boldsymbol{n}\varpi_0/a(t_i)$, then the distribution function is

$$f(\boldsymbol{p}, \boldsymbol{r}, t) = \delta\big(\tilde{\boldsymbol{p}}(\boldsymbol{p}, \boldsymbol{r}, \boldsymbol{b}) - \boldsymbol{n}\varpi_0/\cosh\eta\big) \times \delta\big(\tilde{\boldsymbol{r}}(\boldsymbol{r}, \boldsymbol{b}) - \boldsymbol{n}\sinh\eta\big)\,, \tag{57}$$

where $\tilde{\boldsymbol{r}}$ and $\tilde{\boldsymbol{p}}$ is defined in Eqs. (16) and (22). Here the invariance of f relative to the shift has been used. Eqs. (17) and (23), as well as the fact that Jacobian is unity, allow us to write f in the form:

$$f(\boldsymbol{p}, \boldsymbol{r}, t) = \delta\big(\boldsymbol{p} - \tilde{\boldsymbol{p}}(\boldsymbol{n}\varpi_0/\cosh\eta, \boldsymbol{n}\sinh\eta, -\boldsymbol{b})\big) \times \delta\big(\boldsymbol{r} - \tilde{\boldsymbol{r}}(\boldsymbol{n}\sinh\eta, -\boldsymbol{b})\big)\,. \tag{58}$$

The general solution of the Eq. (55) can be found in the case of free-moving particles. To find the characteristics of the equation, let us return to the problem considered in the previous section and solve it by the use of Hamilton-Jacobi equation, which for the Hamiltonian function of Eq. (8) has the form (for sake of definiteness we restrict our considerations to the case of $\kappa = -1$)

$$\frac{\partial S}{\partial t} + \left[\frac{1}{a^2(t)}\left((\nabla S)^2 + (\boldsymbol{r}\nabla S)^2\right) + m^2\right]^{1/2} = 0. \tag{59}$$

[1] In the absence of collisions this equation is equivalent to the Vlasov equation or Liouville equation for one particle.

As usual, for integrable systems the complete integral of the equation can be found by separation of variables [7]. Solving Eq. (59), we find

$$S(\boldsymbol{s}, \boldsymbol{r}, t) = s\,\mathrm{arsinh}(\boldsymbol{\nu}\boldsymbol{r}) - \int^t \sqrt{s^2/a^2(t') + m^2}\,dt'\,. \tag{60}$$

Here s is an arbitrary constant, $\boldsymbol{\nu}$ is an arbitrary unit vector. Without loss of generality it can be assumed that $s \geq 0$. It is convenient to consider the expression in the right part of Eq. (60) as a function of t and the vectors $\boldsymbol{s} = s\boldsymbol{\nu}$ and $\boldsymbol{r}$. The lower limit of integration over dt' is taken for convenience. It is easy to show directly that Eq. (60) satisfies to Eq. (59).

The momentum is

$$\boldsymbol{p} = \frac{\partial S}{\partial \boldsymbol{r}} = \frac{\boldsymbol{s}}{\sqrt{1 + (\boldsymbol{\nu}\boldsymbol{r})^2}}\,. \tag{61}$$

From this it follows that $\boldsymbol{\nu} = \boldsymbol{s}/s = \boldsymbol{p}/p$, thus

$$\boldsymbol{s} = \boldsymbol{p}\sqrt{1 + (\boldsymbol{\nu}\boldsymbol{r})^2}\,. \tag{62}$$

To determine the particle motion, let us differentiate S with respect to arbitrary constants and equate the result to another constants:

$$\frac{\partial S}{\partial \boldsymbol{s}} = \boldsymbol{\nu}\,\mathrm{arsinh}(\boldsymbol{\nu}\boldsymbol{r}) + \frac{\boldsymbol{r} - \boldsymbol{\nu}(\boldsymbol{\nu}\boldsymbol{r})}{\sqrt{1 + (\boldsymbol{\nu}\boldsymbol{r})^2}} - \eta\boldsymbol{\nu} = \boldsymbol{\xi}\,, \tag{63}$$

where

$$\eta = \int^t \frac{dt'}{a(t')\sqrt{1 + (ma(t')/s)^2}}\,. \tag{64}$$

The six quantities $\boldsymbol{s}$ and $\boldsymbol{\xi}$ are integrals of motion of the problem and at the same time they are characteristics of Eq. (55). Therefore the general solution of the Boltzmann equation can be presented in the form

$$f(\boldsymbol{p}, \boldsymbol{r}, t) = \Phi(\boldsymbol{s}, \boldsymbol{\xi})\,, \tag{65}$$

where Φ is an arbitrary function[2]. Here it is implied that $\boldsymbol{s}$ and $\boldsymbol{\xi}$ are expressed via generalized coordinates and momenta via Eqs. (62) and (63).

Assuming $m = 0$, the solution given by Eq. (65) can be applied to study the evolution of distribution function of photons and neutrinos in FLRW space-time in the case of inhomogeneous and anisotropic distribution.

For the collision integrals it is convenient to use, instead of generalized momentum, the energy E and unit vector $\boldsymbol{n} = \boldsymbol{P}/P = \boldsymbol{q}/q$ in the direction of momentum.

[2] The general solution based on Killing vector constants of the motion is obtained as well in Ref. [9]. However, the solution is expressed as a function of arguments which are not canonically conjugated quantities. Therefore one needs an additional analysis to find the relationship between the solution and the distribution function.

Let us introduce the distribution function F according to the relation

$$dN = F(E, \boldsymbol{n}, \boldsymbol{r}, t)\, dE\, \frac{d\Omega}{4\pi}\, dV\,. \tag{66}$$

dN is the number of particle at the moment t located in the volume dV with the energy restricted in the interval dE and the direction of the momentum $\boldsymbol{P}$ enclosed in the solid angle $d\Omega$. It is follows from the metric of Eq. (2) that the volume element is

$$dV = \frac{a^3(t)\, d^3r}{\sqrt{1-\kappa r^2}}\,. \tag{67}$$

Let us denote $dV_a = dV/a^3$ which is dimentionless volume element (in the units of a^3). It is easy to find from Eq. (31) that

$$d^3q = \sqrt{1-\kappa r^2}\, d^3p\,, \tag{68}$$

therefore the volume element of the phase space is

$$d^3p\, d^3r = d^3q\, dV_a = d^3P\, dV\,. \tag{69}$$

Using this equation and definitions given by Eqs. (52) and (66), we find the relation between functions f and F:

$$f(\boldsymbol{p}, \boldsymbol{r}, t) = F(E, \boldsymbol{n}, \boldsymbol{r}, t)/(4\pi PE)\,. \tag{70}$$

Since E and $|\boldsymbol{P}|$ do not change under shifts, F is an invariant as is f

$$F(E, \boldsymbol{n}, \boldsymbol{r}, t) = F'(E, \boldsymbol{n}', \boldsymbol{r}', t)\,, \tag{71}$$

where $\boldsymbol{n}'$ is unity vector in the direction of $\boldsymbol{q}' = \tilde{\boldsymbol{p}}(\boldsymbol{p}', \boldsymbol{r}', \boldsymbol{r}')$. Further in this section we use both distribution functions.

To derive the collision integral in the FLRW space-time it is convenient to shift the origin of the reference frame to the collision point. Then the collision integral can be written as in the flat space. After that one should move to the initial reference frame.

A. Homogeneous distribution

Let us consider the case of the homogeneous space distribution of particles. In this case the distribution function should be form-invariant relative to shift, i.e. it has an identical form in different reference frames. Moreover, along with Eq. (54) the stronger condition should be fulfilled:

$$f(\boldsymbol{p}, \boldsymbol{r}, t) = f(\boldsymbol{p}', \boldsymbol{r}', t)\,. \tag{72}$$

If $\boldsymbol{p}'$ and $\boldsymbol{r}'$ are expressed in accordance with Eqs. (16) and (22) in terms of $\boldsymbol{p}$, $\boldsymbol{r}$ and $\boldsymbol{b}$, then the right part should not depend on $\boldsymbol{b}$. Therefore the condition given by Eq. (72) imposes severe restrictions on the form of function f. Assuming $\boldsymbol{b} = \boldsymbol{r}$, we find as the necessary condition

$$f(\boldsymbol{p}, \boldsymbol{r}, t) = f(\boldsymbol{q}, 0, t)\,, \tag{73}$$

where $\boldsymbol{q}$ is defined in Eq. (31). In the flat space $\boldsymbol{q}$ does not change under shift, therefore any function of the form of Eq. (73) describes a homogeneous distribution. This is not the case for $\kappa = \pm 1$. In the curved space the homogeneous distribution is also isotropic, i.e. f depends only on $|\boldsymbol{q}|$ (see Appendix B). Then in the case of homogeneous distribution one can write

$$f(\boldsymbol{p}, \boldsymbol{r}, t) = g(q, t)\,. \tag{74}$$

We assume further that at $\kappa = 0$ the distribution function is also isotropic.

The substitution of Eq. (74) into Eq. (55) cancels the last two terms that results in the simple equation

$$\left(\frac{\partial g}{\partial t}\right)_q = 0\,. \tag{75}$$

Here for clarity we introduce the notation used in thermodynamics to stress that the time derivative is taken at constant q. Further we use this notation as well. The result means that for noninteracting particles, in the case of homogeneous and isotropic distribution, the distribution function *in phase space* does not depend on time. It should be noted that in the case under consideration Eq. (75) retains its form also in the presence of arbitrary magnetic field if one abstract from energy losses due to synchrotron radiation.

Let us derive the equation for the distribution function F defined in Eq. (66). If distribution is homogeneous, the function does not depend on $\boldsymbol{n}$ and $\boldsymbol{r}$ and one can assume[3]

$$F \equiv F(E, t) = 4\pi PE\, g(q, t)\,. \tag{76}$$

In the new variables (E, t) the derivative $(\partial g/\partial t)_q$ is written in the form

$$\left(\frac{\partial g}{\partial t}\right)_q = \left(\frac{\partial g}{\partial t}\right)_E - \frac{HP^2}{E}\left(\frac{\partial g}{\partial E}\right)_t\,, \tag{77}$$

where $H = \dot{a}/a$ is the Hubble constant. Expressing g through F, we find

$$\left(\frac{\partial g}{\partial t}\right)_q = \frac{1}{4\pi PE}\,\hat{K}F(E, t)\,, \tag{78}$$

where $\hat{K}$ is an operator defined as:

$$\hat{K}F(E, t) \equiv \left(\frac{\partial F}{\partial t}\right)_E - H\left(\frac{\partial}{\partial E}\left(\frac{P^2}{E}\,F\right)_t - 3F\right). \tag{79}$$

[3] It should be noted that in the case of homogeneous distribution the collision integrals are written as in the flat space since $F(E, t) = F'(E, t)$.

The general solution of Eq. (75) is

$$g(q,t) = \Phi(q)\,, \tag{80}$$

where Φ is an arbitrary function of one argument. Then the function F is

$$F(E,t) = 4\pi P E\,\Phi\big(a(t)\,P\big)\,. \tag{81}$$

This expression defines the evolution of the distribution function for noninteracting particles; if the function $F(E,t')$ is known at the moment $t = t'$, Eq. (81) allows us to find F at subsequent (previous) moments of time. One can write $F(E,t')$ in an explicit form:

$$F(E,t) = \frac{a(t')}{a(t)}\frac{E}{\mathscr{E}(E,t,t')}\,F(\mathscr{E}(E,t,t'),t')\,, \tag{82}$$

where

$$\mathscr{E}(E,t,t') = \frac{a(t)}{a(t')}\sqrt{E^2 - m^2\left(1 - \frac{a^2(t')}{a^2(t)}\right)}\,. \tag{83}$$

The quantity $\mathscr{E}(E,t,t')$ has a simple physical meaning; it is the energy which particle should have at the moment t' in order to have the energy E at the moment t. Therefore the following relation takes place

$$\mathscr{E}\big(\mathscr{E}(E,t,t''),t'',t'\big) = \mathscr{E}(E,t,t')\,, \tag{84}$$

For massless particles this equation is simplified:

$$F(E,t) = \left(\frac{a(t')}{a(t)}\right)^2 F(Ea(t)/a(t'),t')\,. \tag{85}$$

The particle number density is

$$N(t) = \int_m^\infty F(E,t)\,dE\,. \tag{86}$$

Writing F in the form of Eq. (82) and introducing new variable of integration $E' = \mathscr{E}(E,t,t')$, we find

$$N(t) = \left(\frac{a(t')}{a(t)}\right)^3 \int_m^\infty F(E',t')\,dE'\,, \tag{87}$$

and thus

$$N(t)\,a^3(t) = N(t')\,a^3(t') = \text{const}\,. \tag{88}$$

In the presence of sources the equation for g can be written in the form

$$\left(\frac{\partial g}{\partial t}\right)_q = s(q,t)\,. \tag{89}$$

For the case under consideration (homogeneous and isotropic space) the source function $s(q,t)$ depends only on q and t as does the distribution function. Assuming that the source is activated at the moment t_i and that $g(q,t_i) = 0$, we obtain

$$g(q,t) = \int_{t_i}^t s(q,t')\,dt'\,, \qquad t \geq t_i\,. \tag{90}$$

The equation for F in the presence of the sources has the form

$$\hat{K}F(E,t) = S(E,t), \quad S(E,t) = 4\pi P E\,s(q,t)\,. \tag{91}$$

The solution of this equation is

$$F(E,t) = E\int_{t_i}^t \frac{a(t')}{a(t)}\,\frac{S\big(\mathscr{E}(E,t,t'),t'\big)}{\mathscr{E}(E,t,t')}\,dt'\,. \tag{92}$$

Eq. (92) can be considered as Eq. (90) written in different notations. Using Eq. (84) it is easy to verify that if the source is active over a finite time $t_i < t < t_f$, then, after the source is switched off ($t > t_f$), Eqs. (92) and (82) are equivalent.

Let us consider the equation

$$\left(\frac{\partial g}{\partial t}\right)_q = -\lambda(q,t)\,g(q,t)\,, \tag{93}$$

which describes absorbtion or decay of particles. Its solution is

$$g(q,t) = g(q,t_i)\,e^{-\tau}\,, \tag{94}$$

where the optical depth is

$$\tau = \int_{t_i}^t \lambda(q,t')\,dt'\,. \tag{95}$$

Denoting

$$\lambda(q,t') = \mu(E',t') \equiv \mu\big(\sqrt{q^2/a^2(t') + m^2},t'\big)\,, \tag{96}$$

and switching from variables (q,t) to (E,t), we obtain

$$\tau = \int_{t_i}^t \mu(\mathscr{E}(E,t,t'),t')\,dt'\,. \tag{97}$$

In the presence of absorbtion, the function F satisfies the equation

$$\hat{K}F(E,t) = -\mu(E,t)\,F(E,t)\,, \tag{98}$$

which has the solution

$$F(E,t) = \frac{a(t_i)}{a(t)}\frac{E}{\mathscr{E}(E,t,t_i)}\,F(\mathscr{E}(E,t,t_i),t_i)\,e^{-\tau}\,. \tag{99}$$

In a more general case, g obey the equation

$$\left(\frac{\partial g}{\partial t}\right)_q = s(q,t) - \lambda(q,t)\,g(q,t)\,, \tag{100}$$

and at $t < t_i$ the functions g and s equal to zero. Then we have

$$g(q,t) = \int_{t_i}^{t} dt'\, s(q,t') \exp\Big(-\int_{t'}^{t} \lambda(q,t'')\, dt''\Big)\,. \qquad (101)$$

The corresponding equation for the function F

$$\hat{K}F(E,t) = S(E,t) - \mu(E,t)\, F(E,t)\,, \qquad (102)$$

has the following solution

$$F(E,t) = E\int_{t_i}^{t} dt'\, \frac{a(t')}{a(t)}\, \frac{S\big(\mathscr{E}(E,t,t'),t'\big)}{\mathscr{E}(E,t,t')} \times \exp\Big(-\int_{t'}^{t} \mu(\mathscr{E}(E,t,t''),t'')\, dt''\Big)\,. \qquad (103)$$

This expression can be derived from Eq. (101) using definitions in Eqs. (70) and (91), and switching from variables (q,t) to (E,t). One can move from the integration over the time dt in Eq. (103) to the integration over the redshift dz (Appendix C).

It should be noted that in the case of homogeneous distribution the kinetic equations for the metrics with $\kappa = 0$, $+1$ and -1 are written identically. However, it does not mean that the curvature does not effect on kinetics. In fact it does, because the behaviour of $a(t)$, which enters into the equations, depends on κ.

B. Energy losses

In the case of presence of a source of particles and continuous energy losses the equation for distribution function has the following form

$$\hat{K}F(E,t) - \frac{\partial}{\partial E}\,(b(E,t)\, F) = S(E,t)\,. \qquad (104)$$

To solve Eq. (104) let us first find the Green function $G(E,E_0,t,t_0)$ which satisfies (by definition) the equation

$$\hat{K}G - \frac{\partial}{\partial E}\,(b(E,t)\, G) = \delta(E-E_0)\,\delta(t-t_0) \qquad (105)$$

and the condition: $G\big|_{t<t_0} = 0$. From Eq. (105) it follows that

$$G(E,E_0,t_0+0,t_0) = \delta(E-E_0)\,. \qquad (106)$$

The Green function is the distribution function for the case when at the moment t_0 one particle with energy E_0 is injected.

The solution is sought in the form

$$G(E,E_0,t,t_0) = u(t,t_0)\,\delta(E - \mathscr{E}(E_0,t,t_0))\,\theta(t-t_0)\,, \qquad (107)$$

where $\mathscr{E}$ is defined in Eq. (83). From Eq. (106) it follows that the functions u and $\mathscr{E}$ satisfy the initial conditions:

$$u(t_0,t_0) = 1\,, \qquad \mathscr{E}(E_0,t_0,t_0) = E_0\,. \qquad (108)$$

After substitution of Eq. (107) into Eq. (105), it is helpful to represent the product $b(E,t)\,G$ as $b(\mathscr{E}(E_0,t,t_0),t)\,G$. Consequently we find the following system of ordinary differential equations

$$\dot{u} + 3Hu = 0\,, \qquad (109)$$

$$\dot{\mathscr{E}} + \Psi(\mathscr{E},t) = 0\,, \qquad (110)$$

where

$$\Psi(\mathscr{E},t) = H(\mathscr{E} - m^2/\mathscr{E}) + b(\mathscr{E},t)\,. \qquad (111)$$

Eq. (109) gives:

$$u(t,t_0) = \left(\frac{a(t_0)}{a(t)}\right)^3\,. \qquad (112)$$

In the general case, Eq. (110) can be solved numerically.

The distribution function is expressed via Green function in the following way:

$$F(E,t) = \int_0^t dt_0\, u(t,t_0) \times \int_m^\infty dE_0\, \delta(E - \mathscr{E}(E_0,t,t_0))\, S(E_0,t_0)\,. \qquad (113)$$

The integration over dE_0 gives

$$F(E,t) = \int_0^t dt_0\, u(t,t_0)\, S(E_0,t_0)\, (\partial\mathscr{E}/\partial E_0)^{-1}\,, \qquad (114)$$

where the solution of the equation

$$\mathscr{E}(E_0,t,t_0) = E \qquad (115)$$

should be substituted into integrand instead of E_0.

It is convenient to perform calculations in the following way. Let us denote by $U(E,t_0,t)$ the solution of the equation

$$\frac{\partial U}{\partial t_0} + \Psi(U,t_0) = 0\,, \qquad (116)$$

which satisfy

$$U(E,t,t) = E\,. \qquad (117)$$

In this notations we have $\mathscr{E}(E_0,t,t_0) = U(E_0,t,t_0)$ and the solution of Eq. (115) can be written in the form $E_0 = U(E,t_0,t)$. For calculation of the last factor, we note that

$$\mathscr{D}(E,t_0,t) \equiv \frac{\partial E_0}{\partial E} = \frac{\partial}{\partial E}U(E,t_0,t)\,. \qquad (118)$$

This function obey the linear differential equation

$$\frac{\partial}{\partial t_0}\mathscr{D} + \Psi_1\mathscr{D} = 0\,, \qquad (119)$$

where the function

$$\Psi_1 \equiv \Psi_1(U(E,t_0,t),t_0) = \frac{\partial}{\partial E'}\Psi(E',t_0)\bigg|_{E'=U(E,t_0,t)} \tag{120}$$

can be obtained by differentiation of Eq. (116) and after some obvious redesignations. The initial condition has the form $\mathscr{D}(E,t,t)=1$, therefore

$$\mathscr{D}(E,t_0,t) = \exp\left(\int_{t_0}^{t} \Psi_1(U(E,t_0',t),t_0')\,dt_0'\right). \tag{121}$$

If Ψ does not depend explicitly on time, the equation reduces to

$$\mathscr{D}(E,t_0,t) = \frac{\Psi(E)}{\Psi(E_0)}\,, \tag{122}$$

but in the general case it is necessary to use Eq. (121). Thus, in the case of energy losses, the distribution function is

$$F(E,t) = \int_0^t u(t,t_0)\,S(U(E,t_0,t),t_0)\mathscr{D}(E,t_0,t)\,dt_0\ . \tag{123}$$

For derivation of $\mathscr{D}$ it is easier if one uses Eq. (119) with the initial condition $\mathscr{D}(E,t,t)=1$ instead of the integral representation of Eq. (121). Let us define the function

$$F(E,t_0,t) = \int_{t_0}^t u(t,t_0')\,S(U(E,t_0',t),t_0)\mathscr{D}(E,t_0',t)\,dt_0'\ . \tag{124}$$

It is obvious that $F(E,t) = F(E,0,t)$. The function $F(E,t_0,t)$ can be found solving the differential equation

$$\frac{\partial}{\partial t_0}F(E,t_0,t) + u(t,t_0)\,S(U(E,t_0,t),t_0)\mathscr{D}(E,t_0,t) = 0 \tag{125}$$

with the initial condition $F(E,t_0=t,t)=0$. Thus, the derivation of $F(E,t)$ requires a solution of the system of three ordinary differential equation of first order, namely Eqs. (116), (119) and (125), starting from the point $t_0 = t$, where the values of the functions are known, to the point $t_0 = 0$.

C. Spherically symmetric distribution

In the case of spherical symmetry, the distribution function can be considered as a function of the following arguments

$$f = f(q,r,\mu,t)\,, \tag{126}$$

where $\mu = (\boldsymbol{pr})/(|\boldsymbol{p}||\boldsymbol{r}|) = \cos\theta$. For this function Eq. (55) becomes

$$\frac{\partial f}{\partial t} + \frac{1}{a(t)\sqrt{(1+m^2a^2(t)/q^2)(1-kr^2\mu^2)}} \times\left(\mu(1-kr^2)\frac{\partial f}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial f}{\partial \mu}\right) = 0. \tag{127}$$

The equation does not contain the derivative $\partial f/\partial q$, i.e. q enters in this equation as a parameter. Therefore it is helpful to introduce the new variable instead of time

$$\eta = \int_{t_i}^t \frac{dt'}{a(t')\sqrt{1+m^2a^2(t')/q^2}} \tag{128}$$

and consider f as a function of the arguments (q,r,μ,η). Then it brings us to the equation

$$\frac{\partial f}{\partial \eta} + \frac{1}{\sqrt{1-kr^2\mu^2}}\left(\mu(1-kr^2)\frac{\partial f}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial f}{\partial \mu}\right) = 0, \tag{129}$$

which admits a solution in the general case. The general solution of the equation determined by the method of characteristics has the form

$$f(q,r,\mu,\eta) = \Phi(q,X,Y)\,, \tag{130}$$

where

$$X = \frac{r^2(1-\mu^2)}{1-\kappa r^2}\,, \tag{131}$$

$$Y = \frac{1}{\sqrt{-\kappa}}\ln\left(\sqrt{1-\kappa r^2\mu^2} + r\mu\sqrt{-\kappa}\right) - \eta = \frac{1}{\sqrt{-\kappa}}\,\mathrm{arsinh}\left(r\mu\sqrt{-\kappa}\right) - \eta\,, \tag{132}$$

Φ is an arbitrary function of three arguments[4]. In the case of $\kappa = 1$, we have $\sqrt{-\kappa} = i$, and it is convenient to present Eq. (132) in the form

$$Y = \arcsin(r\mu) - \eta\,. \tag{133}$$

For $\kappa = 0$ we have

$$Y = r\mu - \eta\,. \tag{134}$$

The obtained results are quite convenient to use at least in two cases.

1. *The boundary problem*

Let us assume that the distribution function is known at a certain point $r = r_*$ (on the surface of a "star"):

$$f(q,r_*,\mu,\eta) = f_*(q,\mu,\eta)\,. \tag{135}$$

Then Eq. (130) allows us to derive the distribution function in all space. For instance, in the case $\kappa = -1$ we get

$$f(q,r,\mu,\eta) = f_*(q,\tilde{\mu},\tilde{\eta})\,, \tag{136}$$

[4] This solution is surely a special case of Eq. (65), but it is easier to obtain it by solving directly Eq. (129).

where

$$\tilde{\mu} = \frac{r\mu}{r_* \sqrt{1+r^2}} \sqrt{1 + r_*^2 - \frac{r^2 - r_*^2}{r^2\mu^2}}\,, \tag{137}$$

$$\tilde{\eta} = \eta + \ln \frac{\sqrt{1 + r_*^2\tilde{\mu}^2} + r_*\tilde{\mu}}{\sqrt{1 + r^2\mu^2} + r\mu}\,. \tag{138}$$

Eq. (136) defines the distribution function in the reference frame with the origin of coordinates at the center of the “star”. If the observer is located at the distance r, then, as indicated before, it is convenient to move to the reference frame related to the observer. At the location point of observer, the distribution function is

$$f'(q, \boldsymbol{r}' = 0, \cos\theta', \eta) = f(q, r, \mu, \eta)\,, \tag{139}$$

where μ ($\mu = \cos\theta$) should be represented in the form of Eq. (35). Here, we take into account that q and η do not change under shift.

In the case of $\kappa = 1$, the following relations should be used in Eq. (136):

$$\tilde{\mu} = \frac{r\mu}{r_* \sqrt{1-r^2}} \sqrt{1 - r_*^2 - \frac{r^2 - r_*^2}{r^2\mu^2}}\,, \tag{140}$$

$$\tilde{\eta} = \eta + \arcsin(r_*\tilde{\mu}) - \arcsin(r\mu)\,. \tag{141}$$

2. *Initial value problem*

Let us assume that the distribution function is known at some moment of time $t = t_i$ which corresponds to $\eta = 0$:

$$f(q, r, \mu, \eta = 0) = f_0(q, r, \mu)\,. \tag{142}$$

Then at an arbitrary moment of time

$$f(q, r, \mu, \eta) = f_0(q, r_0, \mu_0)\,. \tag{143}$$

Here

$$r_0 = \left[\frac{r^2(1-\mu^2) + (1-\kappa r^2)\,\zeta^2}{1 - \kappa r^2\mu^2}\right]^{1/2}, \quad \mu_0 = \frac{\zeta}{r_0}\,, \tag{144}$$

where

$$\zeta = r\mu\,\cosh(\eta\sqrt{-\kappa}) - \sqrt{\frac{1-\kappa r^2\mu^2}{-\kappa}}\,\sinh(\eta\sqrt{-\kappa})\,. \tag{145}$$

For analysis of the angular and energy distribution it is convenient, as before, to introduce the new reference frame choosing the origin of coordinates at the point where the observer is located.

V. SUMMARY

In this paper the mechanics and kinetics in the FLRW space-times have been studied on the basis of the standard canonical formalism. The Cartesian coordinates $\boldsymbol{r}$ and the corresponding generalized momenta $\boldsymbol{p}$ are considered as the points $(\boldsymbol{p}, \boldsymbol{r})$ in the phase space where the distribution function $f(\boldsymbol{p}, \boldsymbol{r}, t)$ given by Eq. (52) is introduced. The form-invariance of equations of mechanics and kinetics relative to shift of the origin of reference frame as well as the invariance of the distribution function f have been proved. The transformation of the momentum under shift is described by the quite lengthy equation, Eq. (22). But for applications it is sufficient to use Eq. (31) which defines the transformation of momentum under the shift of the origin to the point $\boldsymbol{r}$ where the observer is located. The collisionless Boltzmann equation admits general solutions for the function f (see Eq. (65)).

Along with the distribution function f, the “conventional” distribution function $F(E, \boldsymbol{n}, \boldsymbol{r}, t)$ given by Eq. (66) is introduced in the phase space. This function is more convenient for inclusion of collision integrals, and defines the relationship between the functions f and F. If the collision integral I for the Minkowski space is known, it can be found also in the FLRW space-time using the following procedure. The origin of coordinates should be shifted to the collision point where I can be written as in the flat space-time. After that the collision integral should be transformed to the initial reference frame using the formulas obtained in this paper.

The equations are considerably simplified in the case of homogeneous and isotropic distribution. For this case the analytical solution of the kinetic equation with the source and absorption processes is given in Section IV A. In the case of energy losses the equation can be no longer solved analytically and determination of F comes to solving the system of three ordinary differential equations.

The results of Section IV C can be quite useful for analysis of angular and energy distribution of particles from sources located at cosmological distances. The distribution function represented by Eq. (130) describes the solution in the reference frame related to the source. This function can be easily transformed to the reference frame associated with observer that gives the spectrum and angular distribution in the observation point.

Appendix A: Relativistic Boltzmann equation in the general form

Below a simple derivation of the collisionless Boltzmann equation for the space with an arbitrary metric $g_{\alpha\beta}(x)$ is presented. It is shown that the equation can be written in the form of Eq. (55) with a relevant Hamiltonian function. Let $x^{(s)i}(t)$ ($i=1,2,3$) are the coordinates of the particle with number s at the moment t, and let $p_i^{(s)}(t)$ are the covariant components of its momentum. It is convenient to consider the single-particle distribution function [2, 3] as a function of the following independent variables: covariant momentum $\boldsymbol{p} = (p_1, p_2, p_3)$, coordinates $\boldsymbol{r} = (x^1, x^2, x^3)$ and time. Let us also introduce the zero components

$$x^0 = t\,, \qquad p_0 \equiv p_0(\boldsymbol{p}, \boldsymbol{r}, t) = \frac{1}{g^{00}}\Big(- g^{0i} p_i + \sqrt{(g^{0i}p_i)^2 - g^{00}(g^{ik}p_i p_k - m^2)}\Big). \tag{A1}$$

Here p_0 is the solution of the quadratic equation $g^{\alpha\beta} p_\alpha p_\beta = m^2$, and the solution should be chosen from two possible ones, so that $p^0 = g^{0\alpha}p_\alpha > 0$.

The microscopic single-particle distribution function is defined from the following relations:

$$f(\boldsymbol{p}, \boldsymbol{r}, t) = \sum_s f^{(s)}(\boldsymbol{p}, \boldsymbol{r}, t)\,, \tag{A2}$$

$$f^{(s)}(\boldsymbol{p}, \boldsymbol{r}, t) = \delta(\boldsymbol{r} - \boldsymbol{r}^{(s)}(t))\,\delta(\boldsymbol{p} - \boldsymbol{p}^{(s)}(t))\,, \tag{A3}$$

where three-dimensional δ-functions are equal to product of three one-dimensional ones. For derivation of the equation let us differentiate $f^{(s)}$ with respect to time,

$$\frac{\partial f^{(s)}}{dt} = -\frac{\partial}{\partial x^i}(\dot{x}^{(s)i} f^{(s)}) - \frac{\partial}{\partial p_i}(\dot{p}_i^{(s)} f^{(s)})\,, \tag{A4}$$

where for the convenience of further transformations the quantities $\dot{x}^{(s)i}$ and $\dot{p}_i^{(s)}$ are inserted under the sign of partial derivatives. From the definition of momentum it is follows that $\dot{x}^{(s)i} = p^{(s)i}/p^{(s)0}$, where $p^{(s)\alpha} = g^{\alpha\beta}p_\beta^{(s)}$ are contravariant components of the 4-momentum. Taking into account that δ-functions entering into $f^{(s)}$ allow to replace $\boldsymbol{r}^{(s)}(t)$ and $\boldsymbol{p}^{(s)}(t)$ by $\boldsymbol{r}$ and $\boldsymbol{p}$, in Eq. (A4) p^i/p^0 can be written instead of $\dot{x}^{(s)i}$ (here the following relation is used: $\delta(x - x_0)\, f(x) = \delta(x - x_0)\, f(x_0)$).

The same operation can be done in the last term of Eq. (A4):

$$\dot{p}_i^{(s)} = \frac{1}{p^{(s)0}}\,\Gamma_{\alpha,\beta i} p^{(s)\alpha} p^{(s)\beta} = \frac{1}{2p^{(s)0}}\,\frac{\partial g_{\alpha\beta}}{\partial x^i} p^{(s)\alpha} p^{(s)\beta}, \tag{A5}$$

where $\Gamma_{\ldots}$ are the Christoffel symbols; the values of the functions should be taken at the point $(\boldsymbol{p}^{(s)}(t), \boldsymbol{r}^{(s)}(t))$. The presence of δ-functions allows the following replacement

$$\dot{p}_i^{(s)} \to \frac{1}{2p^0}\,\frac{\partial g_{\alpha\beta}}{\partial x^i}\, p^\alpha p^\beta\,, \tag{A6}$$

where all functions are evaluated at the point $(\boldsymbol{p}, \boldsymbol{r})$. Thus it brings us to the equation

$$\frac{\partial f^{(s)}}{dt} + \frac{\partial}{\partial x^i}\Big(\frac{p^i}{p^0}\, f^{(s)}\Big) + \frac{\partial}{\partial p_i}\Big(\frac{1}{2p^0}\,\frac{\partial g_{\alpha\beta}}{\partial x^i}\, p^\alpha p^\beta f^{(s)}\Big) = 0. \tag{A7}$$

By differentiating the equation $g^{\alpha\beta}p_\alpha p_\beta = m^2$ with respect to p_i, we find

$$g^{\alpha\beta} p_\alpha \frac{\partial p_\beta}{\partial p_i} = p^0\,\frac{\partial p_0}{\partial p_i} + p^i = 0\,. \tag{A8}$$

By differentiating the equation $g_{\alpha\beta}p^\alpha p^\beta = m^2$ with respect to x^i, we get

$$\frac{1}{2}\,\frac{\partial g_{\alpha\beta}}{\partial x^i} p^\alpha p^\beta = -p_\alpha \frac{\partial p^\alpha}{\partial x^i} = p^\alpha \frac{\partial p_\alpha}{\partial x^i} = p^0 \frac{\partial p_0}{\partial x^i}\,, \tag{A9}$$

where it is taken into account that $\partial p_k / \partial x^i = 0$. Using Eq. (A8) and (A9), Eq. (A7) can be written in the form

$$\frac{\partial f^{(s)}}{\partial t} - \frac{\partial}{\partial x^i}\Big(\frac{\partial p_0}{\partial p_i}\, f^{(s)}\Big) + \frac{\partial}{\partial p_i}\Big(\frac{\partial p_0}{\partial x^i} f^{(s)}\Big) = 0. \tag{A10}$$

It can be seen that the terms, which do not contain derivatives of $f^{(s)}$, are canceled.

Summing over s, we obtain the equation for the function given by Eq. (A2):

$$\Big(\frac{\partial}{\partial t} - \frac{\partial p_0}{\partial p_i}\,\frac{\partial}{\partial x^i} + \frac{\partial p_0}{\partial x^i}\,\frac{\partial}{\partial p_i}\Big) f = 0\,. \tag{A11}$$

This equation can be also written in the form

$$\Big(p^\alpha \frac{\partial}{\partial x^\alpha} + \frac{1}{2}\,\frac{\partial g_{\alpha\beta}}{\partial x^i}\, p^\alpha p^\beta \frac{\partial}{\partial p_i}\Big) f = 0\,. \tag{A12}$$

The macroscopic (averaged over the ensemble) distribution function satisfies the same equations.

Note that the second and third terms enter into Eq. (55) and Eq. (A11) with opposite signs. The reason is the following. Throughout the paper we use the generalized momentum $\partial L / \partial v^i$, while p_i in this Appendix is the covariant components of momentum. One can show that for the Lagrangian of the general form

$$L = -m\sqrt{g_{00} + 2g_{0i}v^i + g_{ik}v^i v^k} \tag{A13}$$

the generalized momentum differs from p_i by sign: $\partial L / \partial v^i = -p_i$. Therefore the momenta used in the main part of the paper and in this appendix have opposite signs. The energy $E = v^i \partial L / \partial v^i - L$ coincides with p_0, and consequently, if one replaces $p_i \to -p_i$ in Eq. (A1), we will get the Hamilton function expressed through generalized momenta and coordinates. In this case the Hamilton equations and the Boltzmann equation have standard forms.

Appendix B: Isotropy of homogeneous distribution

It is easy to show that in the case of $\kappa = \pm 1$ the direction of the vector $\boldsymbol{q}$ changes under shift. It suffices to consider the infinitesimal shift $\delta\boldsymbol{b}$. Let us denote by $\delta\boldsymbol{q}$ the change of $\boldsymbol{q}$ under the shift to $\delta\boldsymbol{b}$. Since $\boldsymbol{q}^2$ is invariant relative to shift, $\delta\boldsymbol{q}^2 = 2(\boldsymbol{q}\,\delta\boldsymbol{q}) = 0$, i.e. the vectors $\boldsymbol{q}$ and $\delta\boldsymbol{q}$ are orthogonal. Therefore $\delta\boldsymbol{q}$ can be represented as

$$\delta\boldsymbol{q} = (\delta\boldsymbol{\phi} \times \boldsymbol{q})\,, \tag{B1}$$

where $\delta\boldsymbol{\phi}$ depends linearly on $\delta\boldsymbol{b}$ and can be considered as an arbitrary vector. The requirement of invariance of f relative to the shift gives

$$\delta f = \delta\boldsymbol{q}\,\frac{\partial f}{\partial \boldsymbol{q}} = (\delta\boldsymbol{\phi}\times\boldsymbol{q})\,\frac{\partial f}{\partial \boldsymbol{q}} = \delta\boldsymbol{\phi}\left(\boldsymbol{q}\times\frac{\partial f}{\partial \boldsymbol{q}}\right) = 0\,, \quad \text{(B2)}$$

that is equivalent, due to the arbitrariness of $\delta\boldsymbol{\phi}$, to the equality

$$\boldsymbol{q}\times\frac{\partial f}{\partial \boldsymbol{q}} = 0\,. \quad \text{(B3)}$$

This implies that the vector $\boldsymbol{q}$ and $\partial f/\partial\boldsymbol{q}$ are parallel, and

$$\frac{\partial f}{\partial \boldsymbol{q}} = \boldsymbol{q}\,A(\boldsymbol{q},t)\,. \quad \text{(B4)}$$

Writing this equality in spherical coordinates, one can ascertain that f does not depend on angular variables, i.e. it is a function of only $|\boldsymbol{q}|$. Thus, we conclude that the isotropy of the distribution follows from its homogeneity.

Appendix C: Connection between time and redshift

For the flat ΛCDM model, the Friedmann equation has the following form (see, for example, Ref. [10])

$$\left(\frac{\dot a}{a}\right)^2 = H_0^2\left[\Omega_m\left(\frac{a(t_0)}{a(t)}\right)^3 + \Omega_\Lambda\right], \quad \text{(C1)}$$

where $a(t_0)$ is the value of the scale factor a at the present epoch. This equation can be solved analytically. Let us introduce the function $\beta(t) = \big(a(t)/a(t_0)\big)^{3/2}$. For β we have the following equation

$$\dot\beta = \frac{3}{2}\,H_0\sqrt{\Omega_\Lambda\beta^2 + \Omega_m}\,, \quad \text{(C2)}$$

from where, taking into account that $\beta(0) = 0$, we find

$$\frac{1}{\sqrt{\Omega_\Lambda}}\ln\left(\frac{\beta\sqrt{\Omega_\Lambda} + \sqrt{\beta^2\Omega_\Lambda + \Omega_m}}{\sqrt{\Omega_m}}\right) = \frac{3}{2}\,H_0 t\,. \quad \text{(C3)}$$

Solving Eq. (C3) for β, we obtain

$$\frac{a(t)}{a(t_0)} = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3}\left[\sinh\left(\frac{3}{2}\sqrt{\Omega_\Lambda}\,H_0 t\right)\right]^{2/3}. \quad \text{(C4)}$$

This expression is also derived in [10] in a different way (see Eq. (29.131)). According to WMAP [11], the parameters in Eq. (C4) have the following values: $H_0 = 71\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$, $\Omega_m = \Omega_b + \Omega_c = 0.27$, $\Omega_\Lambda = 0.73$. In the model of flat Universe the scale factor a is determined with an accuracy of an arbitrary factor, therefore it is convenient to adopt $a(t_0) = 1$, that is equivalent to the redefining of the comoving coordinates $\boldsymbol{r}$.

The redshift z and $a(t)$ are connected as

$$1 + z = a(t_0)/a(t)\,. \quad \text{(C5)}$$

If $a(t)$ is monotonically increasing, the connection between z and t is unique, and one can move from the integration over dt to the integration over dz. In the case of flat Universe we have

$$dt = -\frac{dz}{H_0(1+z)\sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda}}\,. \quad \text{(C6)}$$

Naturally, the results obtained by integrating over time with Eq. (C4) coincide with ones obtained by integrating over dz.

Appendix D: Superluminal recession velocity

In the expanding Universe distant objects have superluminal recession velocities. This issue is elucidated quite comprehensively in [12]. Here we want to emphasize that superluminal recession velocities always occur in the expanding space with $\kappa = 0$ and $\kappa = -1$, and to call attention to some features of photon propagation from sources with superluminal recession velocities. Let us assume that the observer registers a photon at $t = t_o$. Then, at the moment $t < t_o$ the proper distance from the photon to the observer equals to

$$R_p(t) = a(t)\int_t^{t_o}\frac{c\,dt'}{a(t')}\,. \quad \text{(D1)}$$

If a = const, we would have $R_p(t) = c(t_o - t)$, i.e. $R_p(t)$ decreases linearly. If $a(t)$ increases with time, and $a(0) = 0$, then the propagation of the photon is qualitatively different. In this case, as it follows from Eq. (D1), $R_p(0) = 0$, $R_p(t_o) = 0$, and in the range $0 < t < t_o$ the function $R_p(t) > 0$.

Fig. 1 shows the time dependence $R_p(t)$. The calculations are performed for the values of parameters used in Appendix C, however, it is clear that the qualitative behavior of the curve is defined only by the condition $a(0) = 0$.

Let us call by proper velocity, v_p, the time derivative of the proper distance:

$$v_p(t) \equiv dR_p/dt = v_r(t) - c\,, \quad \text{(D2)}$$

where $v_r(t) = (\dot a(t)/a(t))\,R_p(t) = H(t)\,R_p(t)$ is the recession velocity, $H(t)$ is the Hubble constant. At the point of maximum $t = t_*$ one has $v_p(t_*) = 0$, therefore $v_r(t_*) = c$. In the range $0 < t < t_*$ the quantity $v_p > 0$. It means that the recession velocity is greater than c. The moment t_* corresponds to the redshift $z_* = 1.64$, i.e. all sources with $z > 1.64$ move away from us with velocities greater than the speed of light. Photons emitted by this sources in the direction to the observer initially move away, reaching the maximum proper distance $R_{\max}$, and

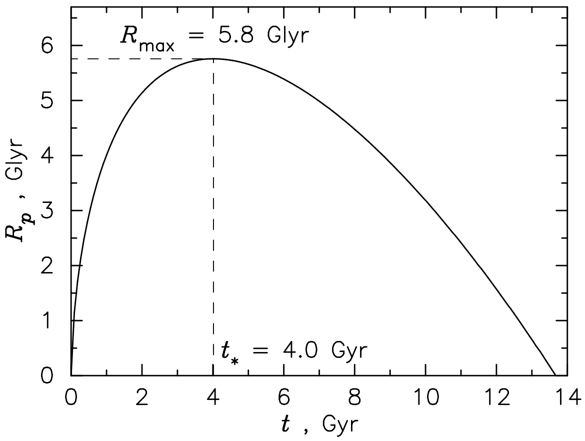

Figure 1: The dependence of proper distance on time

only after that $R_p(t)$ starts to decrease. One can show that the source has a minimal angular size if it is located at the distance $R_{\max} \approx 1.78$ Gpc that corresponds to redshift $z_* = 1.64$.

In the case of the closed space ($\kappa = +1$), sources with superluminal recession velocities are also possible provided that the condition $\dot{a} > c/\pi$ is fulfilled.

[1] A. G. Walker, in *Proceedings of the Edinburgh Mathematical Society (Series 2), 4* (1936), pp. 238–253.
[2] F. Debbasch and W. A. van Leeuwen, Physica A Statistical Mechanics and its Applications **388**, 1079 (2009).
[3] F. Debbasch and W. A. van Leeuwen, Physica A Statistical Mechanics and its Applications **388**, 1818 (2009).
[4] I. Y. Dodin and N. J. Fisch, Phys. Rev. D **82**, 044044 (2010).
[5] C. B. Collins, General Relativity and Gravitation **19**, 493 (1987).
[6] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley and Sons, Inc., 1972).
[7] L. D. Landau and E. M. Lifshitz, *Mechanics* (Butterworth-Heinemann, 1976), 3rd ed.
[8] H. Goldstein, *Classical mechanics* (Addison Wesley Pub. Com., 1972), 2nd ed.
[9] S. D. Maharaj and R. Maartens, General Relativity and Gravitation **19**, 1217 (1987).
[10] B. W. Carroll and D. A. Ostlie, *An Introduction to Modern Astrophysics* (Pearson Addison-Wesley, 2007), 2nd ed.
[11] N. Jarosik, C .L. Bennett, C. L. Dunkley, et al., ApJS **192**, 1 (2011), 1001.4744.
[12] T. M. Davis and C. H. Lineweaver, Publications of the Astronomical Society of Australia **21**, 97 (2004), arXiv:astro-ph/0310808.