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The horizon-entropy increase law for causal and quasi-local horizons and conformal field redefinitions

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Abstract

We explicitly prove the horizon-entropy increase law for both causal and quasi-locally defined horizons in scalar–tensor and $f(R)$ gravity theories. Contrary to causal event horizons, future outer trapping horizons are not conformally invariant and we provide a modification of trapping horizons to complete the proof, using the idea of generalized entropy. This modification means that they are no longer foliated by marginally outer trapped surfaces but fixes the location of the horizon under a conformal transformation. We also discuss the behaviour of horizons in 'veiled' general relativity and show, using this new definition, how to locate cosmological horizons in flat Minkowski space with varying units, which is physically identified with a spatially flat FLRW spacetime.

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1. Introduction

The entropy of a black hole is not always given simply by one quarter of its area. In alternative theories of gravity, such as Brans–Dicke or $f(R)$ theories, the horizon-entropy of the black hole is given by a more complicated function of the black hole geometry and possible horizon fields. In such cases, ensuring that the entropy of the black hole is non-decreasing is not equivalent to ensuring that the area is non-decreasing. A number of authors have been able to prove an equivalent of Hawking's area increase theorem for black hole event horizons in several alternative theories [1–3].

Quasi-local horizons also have an area increase law [4]. The thermodynamic properties of apparent horizons and their quasi-local associates, dynamical and trapping horizons, have been investigated in [4, 5] and [6]. In Einstein gravity, the area of a trapping horizon is

guaranteed to be non-decreasing if the null energy condition is satisfied. This result for the area is true even in alternative gravity situations, provided the null convergence condition is substituted for the null energy condition. But this does not guarantee that the horizon-entropy of the trapping horizon is non-decreasing.

In this paper, we examine situations where the horizon-entropy is not one quarter of the area. We examine both causal horizons and quasi-local horizons. Causal horizons are the null causal boundaries of a given spacetime region and include event horizons, which are the past causal boundary of future null infinity. Quasi-local horizons include dynamical and trapping horizons, but we also investigate a new definition, closely related to that of a trapping horizon, that satisfies a horizon-entropy increase law in a range of situations [7]. This new surface has the important property that it reduces to that of a trapping horizon in cases where the horizon-entropy is one quarter of the area. It therefore retains all of the previous results for trapping horizons in the case of Einstein gravity and extends their validity to other theories. We extend the results in [7] to a much wider class of gravity theories, including scalar–tensor theories and $f(R)$ theories and also extend the results to a much wider class of horizons, including ones that are not necessarily spherically symmetric. In addition, we derive a corresponding horizon-entropy equation for causal horizons that unifies many of the previous results that have appeared in the literature.

This new horizon definition has the property that under a conformal transformation of the metric, its location and in particular its relation to the event horizon are unchanged. This is not true of trapping horizons. The use of conformal transformations is fairly common in the study of gravity theories. This is particularly true in string theory where conformal transformations are used to relate the string frame, with a non-minimally coupled dilaton field, to the Einstein frame[3]. It has been argued in the literature that, classically, the two frames are physically equivalent [8–11]. This physical equivalence suggests that the new horizons should be preferred to trapping horizons if these surfaces are to have physical significance, such as a role in black hole thermodynamics and Hawking radiation.

The conformal transformation rescales lengths and areas as measured by the metric. The physical effect of this rescaling is, for example, to change the meaning of mass since the norm of the four-momentum $p^a p_a$ will no longer be constant from point to point or from time to time. The importance of running units in making the correspondence was emphasized in [9, 11]. The example of Einstein gravity in a frame where gravity is not minimally coupled to the matter fields was explicitly examined in [8]. In this case, where there are no 'fundamental' scalar fields, the observational predictions are still exactly the same in two different frames. The authors of [8] use the term 'veiled general relativity' to describe this situation.

The plan of this paper is as follows. Section 2 provides background material on horizon-entropy in modified theories of gravity. Section 3 examines the various proofs for the increase of this horizon-entropy for both causal horizons and quasi-local horizons such as trapping horizons. The proofs are discussed for Einstein gravity, Brans–Dicke gravity and general scalar–tensor and $f(R)$ gravity theories. Here we see that trapping horizons, as commonly defined, can only guarantee increase of horizon-entropy in the case of Einstein gravity. However, for the modification given in equations (34), the horizon-entropy law can be guaranteed in a large class of other theories. This modification makes the location of the geometrically defined horizon invariant under a conformal transformation, as we discuss in section 4. This allows us to locate invariantly defined horizons in conformally equivalent

[3] Several authors have already noted that they should more properly be called 'representations' rather than 'frames' [3, 8].

spacetimes and we demonstrate this for cosmological horizons in section 5. Section 6 contains a discussion and the conclusions.

2. Horizon-entropy for general gravity theories

There are several ways to derive the entropy that should be associated with a black hole horizon. For a static spacetime, one can make use of the Euclideanized action. This was used in [12] to show that static black holes do not obey $S = A/4$ to linear order in a particular model of second-order curvature corrections derived from string theory. This technique was later generalized to all orders for Lagrangians that are an arbitrary function of the Riemann tensor by Visser [13, 14] who derived the formula

$$S = \frac{A_H}{4} + 4\pi \int_H \frac{\partial \mathscr{L}_m}{\partial R_{abcd}} g^{\perp}_{ac} g^{\perp}_{bd} \sqrt{q}\, \mathrm{d}^2 x, \tag{1}$$

where integrations should be taken over closed two-spheres, H, with metric q_{ab}, while $g^{\perp}_{ab}$ is the symmetric metric of the two-dimensional subspace orthogonal to these surfaces, spanned by null vectors l^a and n^a such that $g^{\perp}_{ab} = -l_a n_b - n_a l_b$ with $n^a l_a = -1$. $\mathscr{L}_m$ is the 'matter' Lagrangian density, which can be constructed as the total Lagrangian density minus the Einstein–Hilbert term.

Alternatively, one can require the validity of the first law for Killing horizons of any diffeomorphism-invariant theory. This was done in [15] and gives the result

$$S = -2\pi \int_H \frac{\partial \mathscr{L}}{\partial R_{abcd}} \hat{\varepsilon}_{ab} \hat{\varepsilon}_{cd} \sqrt{q}\, \mathrm{d}^2 x + \text{higher derivative terms}, \tag{2}$$

where $\hat{\varepsilon}_{ab}$ is the antisymmetric binormal form for the surface H, $\hat{\varepsilon}_{ab} = l_a n_b - n_a l_b$ and $\mathscr{L}$ is the full Lagrangian density. The higher derivative terms arise for theories that depend on derivatives of the Riemann tensor and we will ignore them here. The equivalence of this formula with (1) is obtained by the relation $\hat{\varepsilon}_{ab}\hat{\varepsilon}_{cd} = g^{\perp}_{ad} g^{\perp}_{bc} - g^{\perp}_{ac} g^{\perp}_{bd}$.

What is needed for these formulae is a choice of spacelike surface H, knowledge of how the Lagrangian density $\mathscr{L}$ depends on the Riemann tensor and knowledge of the local geometry and fields at the surface H. The horizon-entropy has the form of an integral over the two-dimensional surface of a two-form, $S = \int_H s_{ab}$, with

$$s_{ab} = -2\pi \frac{\partial \mathscr{L}}{\partial R_{cdef}} \hat{\varepsilon}_{cd} \hat{\varepsilon}_{ef} \varepsilon_{ab}, \tag{3}$$

which is just a scalar quantity times the area two-form ε_{ab} of the surface H. In principle, a horizon-entropy two-form can be associated with each point of the horizon, although it depends on which two-surface it is associated with. For the normal Einstein–Hilbert action of Einstein gravity, where the 'matter' Lagrangian is zero and hence the Visser horizon-entropy is trivial, we have

$$\mathscr{L} = \frac{R}{16\pi}, \tag{4}$$

$$\frac{\partial \mathscr{L}}{\partial R_{abcd}} = \frac{1}{16\pi} \frac{1}{2} (g^{ac} g^{bd} - g^{ad} g^{bc}); \tag{5}$$

thus,

$$s_{ab} = -2\pi \left(\frac{1}{16\pi} \right) \hat{\varepsilon}^{cd} \hat{\varepsilon}_{cd} \varepsilon_{ab}, \tag{6}$$

and therefore, since $\hat{\varepsilon}^{cd}\hat{\varepsilon}_{cd} = -2$,

$$S = \frac{A}{4}, \tag{7}$$

with A being the area of H. In the case of scalar–tensor gravity [16], we have

$$\mathscr{L} = F(\phi)\frac{R}{16\pi} + \text{other terms independent of Riemann} \tag{8}$$

and thus

$$s_{ab} = \frac{F(\phi)}{4}\varepsilon_{ab}. \tag{9}$$

When $F(\phi)$ is constant over the horizon, for example, for a spherically symmetric surface, we have

$$S = \frac{F(\phi)A}{4} \tag{10}$$

(cf [17]), while in the case of $f(R)$ gravity [18], we have

$$\mathscr{L} = \frac{f(R)}{16\pi} \tag{11}$$

and thus

$$s_{ab} = \frac{f'(R)}{4}\varepsilon_{ab}. \tag{12}$$

Again, in the case where $f'(R)$ is constant over H, this gives

$$S = \frac{f'(R)A}{4}. \tag{13}$$

In all these cases, the horizon-entropy has the form $S = WA$ for some scalar function W. The horizon-entropy in [15] was explicitly derived to apply to Killing horizons in a stationary spacetime. It was suggested in [19] that in certain cases the entropy could also take this form for non-stationary situations. We will henceforth refer to (2) as the horizon-entropy, without prejudice to the question of whether it represents a true entropy or not in dynamical situations. The question then arises as to what kind of surface this horizon-entropy can be applied to. In non-stationary situations, the event horizon does not in general coincide with the trapping horizon even though both satisfy an area increase law in Einstein gravity. In the following section, we consider to what extent the horizon-entropy satisfies an increase law for non-stationary surfaces.

3. The second law of black hole mechanics

Let us consider a three-dimensional surface that can be foliated by closed spacelike two-dimensional surfaces (such an object could be an event horizon, a trapping horizon or even something else). In a four-dimensional Lorentzian signature spacetime, the spacelike two-surfaces have null normals l^a and n^a that are unique up to scalings. l^a and n^a are conventionally referred to as the outgoing future-directed null normal and ingoing future-directed null normal, respectively. The tangent r^a to the surface, which is normal to the spacelike two-surfaces, can be written everywhere as a linear combination of l^a and n^a,

$$r^a = Bl^a + Cn^a\,. \tag{14}$$

For a Killing horizon, or a non-stationary event horizon, or general causal horizon, r^a would be the generators of the horizon and would therefore be null with either $B = 0$ or $C = 0$ and

in the case of a dynamical horizon, we would have $B > 0$ and $C < 0$. A future outer trapping horizon can have any sign for C. The signature of the three-dimensional surface is just given by the norm squared of r^a,

$$r^a r_a = 2BCl^a n_a, \tag{15}$$

where, for future directed null normals, $l^a n_a$ is negative. The discussion here will follow that of [4] where l^a and n^a can be chosen such that B and C above are constant on the two-dimensional surfaces. To fix a direction on the three-dimensional horizon surface, we can choose $B > 0$. The horizon will then be spacelike if $C < 0$, null if $C = 0$ and timelike if $C > 0$. The horizon-entropy will in all these cases be non-decreasing if

$$\int \mathcal{L}_r s_{ab} \geqslant 0, \tag{16}$$

with $\mathcal{L}_r$ the Lie derivative along r^a. Now one can look at how the horizon-entropy two-form s_{ab} varies as one moves along integral curves of r^a from one spacelike two-surface to another:

$$\mathcal{L}_r s_{ab} = B\mathcal{L}_l s_{ab} + C\mathcal{L}_n s_{ab}. \tag{17}$$

Since the entropy two-form can be written as $s_{ab} = W\,\varepsilon_{ab}$, this equation is equivalent to

$$\mathcal{L}_r s_{ab} = [B\,(\mathcal{L}_l W + W\theta_l) + C\,(\mathcal{L}_n W + W\theta_n)]\,\varepsilon_{ab}, \tag{18}$$

where we define the expansion θ_l by $\mathcal{L}_l \varepsilon_{ab} = \theta_l \varepsilon_{ab}$. Determining, or defining, that the signs of the scalar terms in $\mathcal{L}_r s_{ab}$ combine to give an overall non-negative result implies that the entropy two-form is non-decreasing in the direction of r^a everywhere on H and thus the horizon-entropy is non-decreasing along the three-dimensional surface in question. For causal horizons, this just reduces to the requirement that $\varepsilon^{ab}\mathcal{L}_l s_{ab}$ be non-negative, which can be related to the equations of motion and an energy condition. In situations where the horizon is not null, as we will see below, the sign of the $C\varepsilon^{ab}\mathcal{L}_n s_{ab}$ term can also be evaluated in a similar manner.

3.1. Einstein gravity

In the usual case of Einstein gravity, we have $s_{ab} = \varepsilon_{ab}/4$ and $S = A/4$. In this case, the horizon-entropy increase law for event horizons is just the area theorem of Hawking [20, 21]. Since it does not affect the sign of the change in entropy, henceforth we will incorporate the factor of 4 into A for notational convenience.

For the case of quasi-local horizons, foliated by marginally outer trapped surfaces with outgoing null expansion $\theta_l = 0$ and ingoing null expansion negative, $\theta_n < 0$, the variation of the horizon-entropy two-form is simply

$$\begin{aligned} \mathcal{L}_r \varepsilon_{ab} &= B\mathcal{L}_l \varepsilon_{ab} + C\mathcal{L}_n \varepsilon_{ab} \\ &= C\theta_n \varepsilon_{ab}. \end{aligned} \tag{19}$$

If C is assumed to be negative, the area-entropy is non-decreasing without further assumptions. This is the case considered for dynamical horizons in [5] since dynamical horizons are required to be spacelike and by equation (15) this guarantees $C < 0$.

In the more general case of a future outer trapping horizon, which can have any signature, the sign of C can be related to the energy conditions via the condition that θ_l should be zero everywhere on the trapping horizon. The conditions for a future outer trapping horizon are [4]

$$\begin{aligned} &\theta_l = 0, \\ &\theta_n < 0, \\ &\mathcal{L}_n \theta_l < 0. \end{aligned} \tag{20}$$

For a past inner trapping horizon (PITH), one would interchange the ns and the ls and reverse the sign of the inequalities. The third condition distinguishes trapping horizons from dynamical horizons. The constancy of the expansion θ_l on the horizon gives the condition

$$\mathcal{L}_r\theta_l = B\mathcal{L}_l\theta_l + C\mathcal{L}_n\theta_l = 0. \tag{21}$$

The Raychaudhuri equation for null geodesic congruences is

$$\mathcal{L}_l\theta_l = \kappa_l\theta_l - \tfrac{1}{2}\theta_l^2 - \sigma_l^2 + \omega_l^2 - R_{ab}l^a l^b, \tag{22}$$

where κ_l is a measure of the failure of l^a to be affinely parameterized (a 'surface gravity' [22]), σ_l is the shear and ω_l is the vorticity. If the null vectors used to define the horizon are derived from a double-null foliation (this construction is used in [4]), then the vorticity vanishes identically and for $\theta_l = 0$ we have

$$\mathcal{L}_l\theta_l = -\sigma_l^2 - R_{ab}l^a l^b, \tag{23}$$

and we obtain

$$C = \frac{B}{\mathcal{L}_n\theta_l}\left(\sigma_l^2 + R_{ab}l^a l^b\right). \tag{24}$$

For situations satisfying the null curvature condition, $R_{ab}l^a l^b \geqslant 0$, which can be related to the null energy condition, $T_{ab}l^a l^b \geqslant 0$, by the Einstein equations, C is seen to be negative and thus by (19) the area-entropy of the future outer trapping horizon is non-decreasing, in which case it is also spacelike. By equivalent reasoning, an area-entropy law can be derived for PITHs. In the case where a normalization $l^a n_a = -1$ is imposed, the same conclusion about area increase can be reached using a minimum principle [23].

The existence of the $R_{ab}l^a l^b$ term in the area law gives a direct local relation between the curvature at a point and the rate of increase of an area element at that point. However, the shear and vorticity terms, although locally defined, are related not only to the local properties of the geometry, but also to the choice of surface passing through the geometry, i.e. of the choice of null normals l^c and n^c. It is perfectly possible, for example, that a portion of the horizon can be growing locally in Schwarzschild spacetime, because of the non-local influence on the shear and vorticity. This is encapsulated in FOTS property 5 of [23]. In vacuum spacetimes, the shear can only increase the area of the trapping horizon and the only way for the horizon to shrink is to develop non-zero vorticity.

3.2. Brans–Dicke theory

Brans–Dicke theory is the prototype alternative theory of gravity with scalar and tensor modes. The theory was first expressed in a frame in which the particle masses remain constant, the effective gravitational constant varies from point to point and massive test particles follow timelike geodesics (Jordan or string frame). In this frame, the action is given by the Lagrangian density

$$\mathscr{L} = \frac{1}{16\pi}\left(\phi R - \frac{\omega}{\phi}\nabla_a\phi\nabla^a\phi\right) + \mathscr{L}_{\text{matter}}, \tag{25}$$

where ω is the Brans–Dicke parameter, not to be confused with the vorticity ω_l. Such actions include for example the tree-level low-energy effective actions of certain string theories where $\omega = -1$. Variation of this action with respect to the metric gives the gravitational field equations

$$G_{ab}\phi = 8\pi T_{ab} + \frac{\omega}{\phi}\left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla_c\phi\nabla^c\phi\right) + \nabla_a\nabla_b\phi - g_{ab}\nabla_c\nabla^c\phi, \tag{26}$$

where T_{ab} is the energy–momentum tensor of the matter fields. The last two terms arise from an integration by parts in the variation of the ϕR term. Contracting the Einstein tensor with l^a twice for the above yields

$$R_{ab}l^a l^b = \frac{8\pi}{\phi} T_{ab}l^a l^b + \frac{\omega}{\phi^2}(l^a\nabla_a\phi)^2 + \frac{l^a l^b \nabla_a\nabla_b\phi}{\phi}. \tag{27}$$

In Brans–Dicke theory, the horizon-entropy two-form is given by $s_{ab} = \phi\varepsilon_{ab}$. The variation of horizon-entropy in the outgoing null direction is then

$$\mathcal{L}_l s_{ab} = \left(\theta_l + \frac{l^a\nabla_a\phi}{\phi}\right)\phi\varepsilon_{ab}. \tag{28}$$

Since we require $\phi > 0$ for the horizon-entropy to be positive, the term $(\theta_l + l^a\nabla_a\phi/\phi)$ must be positive for the horizon-entropy to be increasing for a causal horizon generated by l^a. The sign of the $l^a\nabla_a\phi$ term though cannot immediately be established for a causal horizon. But, by extending a method used in [3], taking another derivative gives

$$\mathcal{L}_l\left(\theta_l + \frac{l^a\nabla_a\phi}{\phi}\right) = \mathcal{L}_l\theta_l + \frac{l^b\nabla_b\left(l^a\nabla_a\phi\right)}{\phi} - \frac{1}{\phi^2}(l^a\nabla_a\phi)^2. \tag{29}$$

Using the Raychaudhuri equation (22), the equations of motion (26) and

$$\begin{aligned} l^b\nabla_b(l^a\nabla_a\phi) &= (l^b\nabla_b l^a)\nabla_a\phi + l^a l^b\nabla_a\nabla_b\phi \\ &= \kappa_l l^b\nabla_b\phi + l^a l^b\nabla_a\nabla_b\phi, \end{aligned} \tag{30}$$

where κ_l is again a measure of the failure of l^a to be affinely parameterized, we obtain

$$\mathcal{L}_l\left(\theta_l + \frac{l^a\nabla_a\phi}{\phi}\right) = \kappa_l\left(\theta_l + \frac{l^a\nabla_a\phi}{\phi}\right) - \frac{\theta_l^2}{2} - \sigma_l^2 + \omega_l^2 - \frac{(\omega+1)}{\phi^2}(l^a\nabla_a\phi)^2 - \frac{8\pi}{\phi}T_{ab}l^a l^b. \tag{31}$$

For a causal horizon with $\omega_l = 0$, affinely parameterized ($\kappa_l = 0$) and $\omega + 1 \geqslant 0$ Brans–Dicke theory, this quantity will be negative provided the matter T_{ab} satisfies the null energy condition. The condition $\omega_l = 0$ is guaranteed because the horizon generators are hypersurface orthogonal to the null horizon and a normalization of the generators can always be chosen so that $\kappa_l = 0$. If we then assume that the horizon settles down at late times to a Killing horizon, such that $\mathcal{L}_l s_{ab} = 0$ at late times, then the term $\left(\theta_l + \frac{l^a\nabla_a\phi}{\phi}\right)$ cannot ever be negative, because to go from a negative value to zero, its derivative must be positive somewhere in between, which is excluded by equation (31). Thus, the horizon-entropy must be non-decreasing for a causal horizon, provided it settles down at late times to a Killing horizon. Noting that this requires us to assume that the horizon settles down at late times to a Killing horizon, but that this is sufficient, we do not need to assume that the horizon forms the causal past of future null infinity. This assumption is not needed in the case of Einstein gravity and can be replaced by the assumption that the spacetime contains no naked singularities.

In the general case, equation (31) implies that if $\theta_l + \frac{l^a\nabla_a\phi}{\phi}$ were anywhere negative on the horizon, it would reach an infinite value in a finite parameter distance. Thus, either θ_l would become infinite, implying a focal point, or $\frac{l^a\nabla_a\phi}{\phi}$ would become infinite, implying a discontinuity in ϕ. In the former case, a focal point for the null generators of the horizon is forbidden since the generators of the event horizon can have no future end points. We can therefore conclude that if ϕ is continuous, $\theta_l + \frac{l^a\nabla_a\phi}{\phi}$ cannot be negative anywhere on the horizon. If the causal horizon is the past causal boundary of some set other than future null infinity, then its generators can only have future end points on the set itself.

For a causal horizon, the change of the horizon-entropy cannot be taken arbitrarily close to zero in the past if the area remains non-zero. If a null surface is initially a Killing horizon

with zero horizon-entropy change, it cannot return to a Killing horizon after a perturbation. The equivalent statement in the Einstein case is that the expansion of the horizon is always decreasing even though it is always positive, so its initial value must be larger than any subsequent value. The moment at which the logarithm of the horizon-entropy of a causal horizon is changing, the most lies in the infinite past even though the moment at which the horizon-entropy itself is changing, the most is not necessarily in the infinite past.

For a trapping horizon, we can again use equation (24) but now, instead of (19), we have

$$\mathcal{L}_r s_{ab} = [Bl^c\nabla_c\phi + C(n^c\nabla_c\phi + \phi\theta_n)]\varepsilon_{ab}. \tag{32}$$

The signs of the terms $l^a\nabla_a\phi$ and $n^a\nabla_a\phi$ cannot be guaranteed from the equations of motion. Ultimately, this is related to the value of $r^a\nabla_a\phi$ on the horizon. Because of this, the horizon-entropy can decrease for a trapping horizon [3], even in situations where the matter fields obey the null energy condition such as considered in [24].

Because the expansion θ_p of a null congruence with tangent p^a is related to the variation of the cross-sectional area two-form, $\mathcal{L}_p\varepsilon_{ab} = \theta_p\varepsilon_{ab}$, the conditions for a future outer trapping horizon (20) can be re-written as

$$\begin{aligned} &\varepsilon^{ab}\mathcal{L}_l\varepsilon_{ab} = 0, \\ &\varepsilon^{ab}\mathcal{L}_n\varepsilon_{ab} < 0, \\ &\mathcal{L}_n(\varepsilon^{ab}\mathcal{L}_l\varepsilon_{ab}) < 0. \end{aligned} \tag{33}$$

Consider now, instead of future outer trapping horizons, the following conditions:

$$\begin{aligned} &\varepsilon^{ab}\mathcal{L}_l s_{ab} = 0, \\ &\varepsilon^{ab}\mathcal{L}_n s_{ab} < 0, \\ &\mathcal{L}_n(\varepsilon^{ab}\mathcal{L}_l s_{ab}) < 0. \end{aligned} \tag{34}$$

In ordinary Einstein gravity, this would reduce to the requirements on the null expansions for a trapping horizon given in (33) since, in this case, $s_{ab} = \varepsilon_{ab}$ up to a constant factor. But in cases where the horizon-entropy is not simply the area, these conditions will in general be satisfied at different locations of the spacetime. In Brans–Dicke theory, the horizon-entropy two-form is just $s_{ab} = \phi\varepsilon_{ab}$ in which case the first condition is satisfied where $\phi\theta_l + l^a\nabla_a\phi = 0$ and the second condition when $\phi\theta_n + n^a\nabla_a\phi < 0$.

The variation of the horizon-entropy two-form is now

$$\begin{aligned} \mathcal{L}_r s_{ab} &= B\mathcal{L}_l s_{ab} + C\mathcal{L}_n s_{ab} \\ &= C\mathcal{L}_n s_{ab} \\ &= C(\phi\theta_n + n^c\nabla_c\phi)\varepsilon_{ab}. \end{aligned} \tag{35}$$

The first term on the right-hand side of the first line is now zero by assumption. The term $(\phi\theta_n + n^c\nabla_c\phi)$ is negative by assumption and so the sign of the change in horizon-entropy along the horizon is given by the sign of C again. If C is negative, the horizon is spacelike and the horizon-entropy increases.

It is possible to determine the sign of C by a similar argument used for trapping horizons. Since we require the tangent r^a to generate evolution along a horizon on which $\mathcal{L}_l s_{ab} = 0$, we have

$$C = -\frac{B\mathcal{L}_l(\varepsilon^{ab}\mathcal{L}_l s_{ab})}{\mathcal{L}_n(\varepsilon^{cd}\mathcal{L}_l s_{cd})}. \tag{36}$$

With the sign of B assumed positive, setting the orientation of r^a, and $\mathcal{L}_n(\varepsilon^{cd}\mathcal{L}_l s_{cd})$ negative by assumption on the horizon, whether the horizon-entropy is increasing or not is just determined by the sign of the term $\mathcal{L}_l(\varepsilon^{ab}\mathcal{L}_l s_{ab})$:

$$\text{sign}(\varepsilon^{ab}\mathcal{L}_r s_{ab}) = -\text{sign}(\mathcal{L}_l(\varepsilon^{ab}\mathcal{L}_l s_{ab})). \tag{37}$$

Using equations (22) and (27), we obtain

$$\mathcal{L}_l(\varepsilon^{ab}\mathcal{L}_l s_{ab}) = 2\phi\left(-\frac{1}{2}\theta_l^2 - \sigma_l^2 - \frac{\omega+1}{\phi^2}(l^a\nabla_a\phi)^2 - \frac{8\pi}{\phi}T_{ab}l^a l^b\right). \tag{38}$$

For $\omega + 1 \geqslant 0$ and matter obeying the null energy condition $T_{ab}l^a l^b \geqslant 0$, the horizon-entropy is guaranteed to be non-decreasing along surfaces satisfying (34).[4] The condition in (38) is very similar to that of (31) for a causal horizon, except now that the term involving κ_l in (31) vanishes on a horizon satisfying (34) anyway, and the horizon-entropy is guaranteed to increase without an assumption that it settles down to a future Killing horizon. We remind the reader that the location of surfaces for which these conditions hold will in general be different from causal horizons. Surfaces satisfying (34) will be spacelike for positive energy.

The similarity of (38) to (31) is not surprising since we have

$$\mathcal{L}_l(\varepsilon^{ab}\mathcal{L}_l s_{ab}) = 2l^a\nabla_a\phi\left(\theta_l + \frac{l^b\nabla_b\phi}{\phi}\right) + 2\phi\mathcal{L}_l\left(\theta_l + \frac{l^c\nabla_c\phi}{\phi}\right). \tag{39}$$

The first term is zero by the assumption $\varepsilon^{ab}\mathcal{L}_l s_{ab} = 0$ and the second term is just equation (31). The right-hand side of (38) is used in the first variation of the horizon-entropy for quasi-local horizons, through equations (35) and (36), but in the second variation of the horizon-entropy for causal horizons, through equations (28) and (31). If the right-hand side of (38) ever becomes positive, then the horizon-entropy of the quasi-local horizons will immediately start to decrease, but the change of horizon-entropy of a causal horizon may still increase because in this case it only influences the second variation of the horizon-entropy.

In the case where we impose a cross-normalization $l^a n_a = -1$ as is done in [23], we do not have complete freedom to rescale l^a and n^a so that B and C in (14) are constant. In this case, the r^a variation, δ_r, as defined in [23], is not equivalent to the Lie derivative with respect to r^a for terms such as θ_l that depend not just on the spacetime point but also on the choice of two-surface for which they are defined. The variation of $\theta_l + \mathcal{L}_l\phi/\phi$ however splits into a variation of θ_l and a part that is equivalent to the Lie derivative because ϕ is a globally defined scalar field. In this case, a maximum principle can still be invoked as in [23] since the variation becomes

$$\delta_r\left(\theta_l + \frac{\mathcal{L}_l\phi}{\phi}\right) = \kappa_r\theta_l + \mathrm{d}^2C - 2\tilde{\omega}^a\,\mathrm{d}_a C + B\mathcal{L}_l\left(\theta_l + \frac{\mathcal{L}_l\phi}{\phi}\right) + C\mathcal{L}_n\left(\theta_l + \frac{\mathcal{L}_l\phi}{\phi}\right), \tag{40}$$

with notation adapted from [23]. The term involving B is once again (38). In the case where this term is negative and l^a and n^a are both derived from a double null foliation so that κ_r vanishes, a maximum principle can be applied (see [23] for further details) to conclude that C is either constant or everywhere negative and the horizon-entropy is non-decreasing.

3.3. Scalar–tensor and $f(R)$ gravity

The scalar–tensor generalizations of Brans–Dicke theory, described by the action

$$S_{\mathrm{ST}} = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{F(\phi)R}{16\pi} - \frac{\omega(\phi)}{\phi}\nabla^a\phi\nabla_a\phi - V(\phi)\right] + S_{\mathrm{matter}}\,, \tag{41}$$

where the Brans–Dicke coupling ω becomes a function of ϕ and a scalar field potential $V(\phi)$ is introduced, can be discussed in the same way as in Brans–Dicke theory. One can consider

[4] In fact, because conditions (34) are satisfied in Brans–Dicke theory by surfaces satisfying $\theta_l = -l^a\nabla_a\phi/\phi$, the θ_l term can be eliminated in equation (38) and the condition on the Brans–Dicke parameter ω becomes $\omega > -3/2$. Thus, this is satisfied by the tree-level effective string actions where $\omega = -1$ and coincides with the lower bound on the Brans–Dicke parameter in weak gravitational field expansion on a negative tension Randall–Sundrum brane $\omega > -3/2$.

a new Brans–Dicke field $\psi \equiv F(\phi)$ provided that the function $F(\phi)$ admits a regular inverse F^{-1} (this is not always the case in the literature, in which $F(\phi)$ is sometimes found in the form of a series of even powers of ϕ [25, 26], but specific choices in the literature are motivated by mathematical, not physical considerations, i.e. by the fact that they allow certain calculations to be performed). Then action (41) can be recast in the form

$$S_{\rm ST} = \int {\rm d}^4x\sqrt{-g}\left[\frac{\psi R}{16\pi} - \frac{\bar{\omega}(\psi)}{\psi}\,\nabla^a\psi\nabla_a\psi - U(\psi)\right] + S_{\rm matter}\,; \tag{42}$$

therefore, we limit ourselves to consider action (41) with $F(\phi) = \phi$, which yields the field equations

$$G_{ab} = \frac{8\pi}{\phi}\,T_{ab} + \frac{\omega(\phi)}{\phi^2}\left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}\,g_{ab}\,\nabla^c\phi\nabla_c\phi\right) + \frac{1}{\phi}\left(\nabla_a\nabla_b\phi - g_{ab}\,\Box\,\phi\right) - \frac{V(\phi)}{2\phi}\,g_{ab}\,, \tag{43}$$

$$\nabla^a\nabla_a\phi = \frac{1}{2\omega+3}\left(8\pi T - \frac{{\rm d}\omega}{{\rm d}\phi}\,\nabla^c\phi\nabla_c\phi + \phi\,\frac{{\rm d}V}{{\rm d}\phi} - 2V\right). \tag{44}$$

The discussion of horizons in scalar–tensor gravity remains the same as in Brans–Dicke theory because, by contracting equation (43) twice with the null vector l^a, one obtains again equation (27) (now with ω dependent on ϕ). Since the horizon-entropy is again $S = \phi A$, one finds again equations (31) and (38).

Metric modified (or $f(R)$) gravity, described by the action

$$S_{\rm MG} = \frac{1}{16\pi}\int {\rm d}^4x\sqrt{-g}\,f(R) + S_{\rm matter}, \tag{45}$$

is equivalent to a Brans–Dicke theory with $\omega = 0$ and a potential [18]. In fact, setting

$$\phi = f'(R), \qquad V(\phi) = \phi R(\phi) - f(R(\phi)) \tag{46}$$

leads to the equivalent action [18]

$$S'_{\rm MG} = \frac{1}{16\pi}\int {\rm d}^4x\sqrt{-g}[\phi R - V(\phi)] + S_{\rm matter} \tag{47}$$

(similarly, Palatini $f(R)$ gravity can be recast as an $\omega = -3/2$ Brans–Dicke theory with a potential, but we will not consider it here because of its well-known problems [18]). Since the potential $V(\phi)$ does not give contributions upon double contraction of equation (43) with the null vector l^a, the considerations on horizons made for Brans–Dicke theory can be immediately extended to $f(R)$ gravity.

4. Horizons under conformal transformations

A conformal transformation of the metric will, in general, change the areas of spacelike two-surfaces. This in turn will change the location of the trapping horizons given by the above conditions (33). The conformal transformation relates two different conformal frames if the metric is scaled by a conformal factor that can vary with spacetime point

$$g_{ab} \to \tilde{g}_{ab} = W(x)g_{ab}. \tag{48}$$

The geometric expansion of a null vector l^a in any frame is given by

$$\theta_l = q^{ab}\nabla_a l_b = \left(g^{ab} + \frac{l^a n^b}{(-n^c l^d g_{cd})} + \frac{n^a l^b}{(-n^c l^d g_{cd})}\right)\nabla_a l_b, \tag{49}$$

where $q_a{}^b$ is a projection tensor onto the two-dimensional spacelike surface to which l^a and n^a are normal. (If l^a is defined as globally null, then the third term on the right-hand side vanishes

identically.) This result holds for a Lorentzian signature manifold independently of whether the Einstein equations hold. In general, there is freedom to rescale null vectors even without rescaling the metric. The vanishing of the expansion does not depend on a pure rescaling of the null vector $l^a \to W l^a$, although its value does since under this rescaling, we have

$$\theta_l \to W\theta_l. \tag{50}$$

Under a conformal transformation of the form (48), we have $\tilde{g}^{ab} = W^{-1} g^{ab}$ and $q^{ab} \to \tilde{q}^{ab} = W^{-1} q^{ab}$. We can fix the normalization of l^a by requiring $\tilde{l}^a = l^a$ with $\tilde{l}_a = W l_a$ and thus

$$\tilde{\nabla}_a \tilde{l}_b = W\nabla_a l_b + l_b \nabla_a W - \tfrac{1}{2}(l_a \nabla_b W + l_b \nabla_a W - g_{ab} l^c \nabla_c W); \tag{51}$$

therefore,

$$\tilde{\theta}_l = \theta_l + \frac{l^a \nabla_a W}{W}. \tag{52}$$

The vanishing of θ_l for a given surface is therefore not necessarily invariant under a conformal transformation. Thus, the location of a marginally outer trapped surface satisfying $\theta_l = 0$ is not necessarily invariant. This is despite the fact that the conformal transformation does not change the coordinates of a given spacetime event nor the path of null rays. The location of the event horizon, for example, is unchanged. In one frame, the solution of $\theta_l = 0$ may lie inside the event horizon and in another frame outside, as discussed in [24].

The vanishing of the expansion is equivalent to the statement that the area is unchanged under infinitesimal translations along l^a via the relation $\mathcal{L}_l \, \varepsilon_{ab} = \theta_l \, \varepsilon_{ab}$. Since the conformal factor changes how areas are measured, this criterion no longer selects the same horizon in the two frames. The area two-form changes as $\varepsilon_{ab} \to \tilde{\varepsilon}_{ab} = W\varepsilon_{ab}$. The condition that the Lie derivative of this 'conformally transformed area' be zero is

$$\mathcal{L}_l \, (W\varepsilon_{ab}) = \left(\theta_l + \frac{\mathcal{L}_l W}{W} \right) W\varepsilon_{ab} = 0. \tag{53}$$

This rule is the same as the transformation in (52). This also makes clear what lies behind relations (34). In the Einstein frame, the horizon-entropy is just one quarter of the area but the area is modified in other conformal frames, leading to a modification of the relationship between horizon-entropy and area. Formula (2) gives an explicit way of calculating this new horizon-entropy in the new frame and for the class of theories we have investigated that the entropy is invariant. For example, the gravitational sector in the Einstein frame has the familiar Einstein–Hilbert form of the action

$$S_{\rm action} = \int \mathrm{d}^4x \sqrt{-g} R. \tag{54}$$

By (2), the horizon-entropy is $A/4$ in the Einstein frame. In another frame obtained by $g_{ab} \to \tilde{g}_{ab} = W g_{ab}$, the same action will take the form

$$S_{\rm action} = \int \mathrm{d}^4x \sqrt{-\tilde{g}} \left(\frac{\tilde{R}}{W} - \frac{9}{2} \frac{\tilde{g}^{ab}}{W^3} \tilde{\nabla}_a W \tilde{\nabla}_b W + 3 \frac{\tilde{g}^{ab}}{W^2} \tilde{\nabla}_a \tilde{\nabla}_b W \right), \tag{55}$$

and the horizon-entropy, with W constant on the horizon, will be $\tilde{A}/4W$. But, since the areas are related by $\tilde{A} = AW$, the horizon-entropy will still take the same numerical value in the new frame (the equality between entropies in the Einstein and the Jordan frames extends to all theories with action $\int \mathrm{d}^4x \, \sqrt{-g} \, f \, (g_{ab}, R_{ab}, \phi, \nabla_c \phi)$ [27]). As long as the horizon-entropy transforms in the same way as the metric under a conformal transformation, conditions (34) will give rise to a surface whose location is invariant and for which one can derive a horizon-entropy increase law, exactly as one can derive an area increase law for trapping horizons in the Einstein frame.

The issues that occur can be illustrated with a few examples. One of the cases considered in [8] is the Schwarzschild spacetime under a conformal transformation with conformal factor $W = \Delta^{-1}$. Then the 'veiled general relativity' spacetime becomes

$$d\tilde{s}^2 = -dt^2 + \frac{dr^2}{\Delta^2} + \frac{r^2}{\Delta}\, d\Omega_2^2, \tag{56}$$

where $\Delta = 1 - 2M/r$ and $d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$ is the line element on the unit two-sphere. Like the usual form of the Schwarzschild metric in Schwarzschild coordinates, this metric is valid everywhere in the region $r > 2M$. The radial null vectors in this transformed metric have components

$$l^\mu = (1, \Delta, 0, 0), \qquad n^\mu = (1, -\Delta, 0, 0), \tag{57}$$

and their expansions are, using equation (49),

$$\tilde{\theta}_l = \frac{2}{r^2}(r - 3M), \qquad \tilde{\theta}_n = -\frac{2}{r^2}(r - 3M). \tag{58}$$

The marginally outer trapped surfaces, where θ_l vanishes, are now found at $r = 3M$ instead of $r = 2M$. But these surfaces do not form a trapping horizon because $\theta_n = 0$ here too. The constant r tube at $r = 3M$ is timelike, but its area is not decreasing because $\theta_n = \theta_l = 0$. There are no true spherically symmetric trapping horizons in this metric. In fact, in this metric there are not even any spherically symmetric trapped surfaces, because nowhere do we have $\tilde{\theta}_l\tilde{\theta}_n > 0$.

This result can be generalized by considering conformal factors of the form $W(x) = \Delta^n$, in which case the null vectors are given again by equation (57) and their expansions become

$$\tilde{\theta}_l = \frac{2}{r^2}[r - M(2 - n)], \qquad \tilde{\theta}_n = -\frac{2}{r^2}[r - M(2 - n)]. \tag{59}$$

The marginally outer trapped surfaces can be conformally transformed to any r by a suitable choice of n. But since the area of the spherically symmetric two-spheres is now $A = 4\pi\Delta^n r^2$, the horizon-entropy is $S = A/W = \Delta^{-n}A = 4\pi r^2$, and the conditions (34) just give back trivially the location of the usual horizon, $r = 2M$.

Similar things can occur with coordinates and conformal factors that are perfectly regular on the horizon. For example, in Kerr–Schild coordinates, the Schwarzschild metric takes the form

$$ds^2 = -\Delta\, dt^2 + 2(1 - \Delta)\, dt\, dr + (2 - \Delta)\, dr^2 + r^2\, d\Omega_2^2. \tag{60}$$

In these coordinates, the radial null vectors have components

$$l^\mu = \left(1 - \frac{\Delta}{2}, \frac{\Delta}{2}, 0, 0\right), \qquad n^\mu = (1, -1, 0, 0), \tag{61}$$

where n^μ is affinely parameterized, i.e. $n^a\nabla_a n^b = 0$. In this frame, the expansions are, as expected,

$$\theta_l = \frac{\Delta}{r}, \qquad \theta_n = -\frac{2}{r}. \tag{62}$$

If we choose a conformal factor of the form $W(x) = e^{-\lambda t^2}$, we find

$$\begin{aligned} \tilde{\theta}_l &= \frac{\lambda r t\Delta + \Delta - 2\lambda r t}{r}, \\ \tilde{\theta}_n &= -\frac{2(\lambda r t + 1)\, e^{\lambda t^2}}{r}. \end{aligned} \tag{63}$$

Setting $\tilde{\theta}_l$ equal to zero and expanding for $\lambda t \ll 1/M$ give

$$r = 2M + 8\lambda t M^2 + \mathcal{O}(\lambda^2). \tag{64}$$

In this limit, the trapping horizon is close, but not equal to the $r = 2M$ surface, but it is now also spacelike. The surface $r = 2M$ is still null and is still the location of the event horizon and again is given simply by conditions (34). The physical horizon is located by (34), not by (33).

It has been observed in numerical simulations of black hole collapse in Brans–Dicke theory that the trapping horizon can appear outside the event horizon [24, 28]. This possibility occurs despite the fact that the Jordan frame of Brans–Dicke theory (25) can be related via a conformal transformation to Einstein theory with a scalar field that obeys the null energy condition. In the Einstein frame, the trapping horizon appears exclusively inside the event horizon, in accordance with a theorem of Hawking and Ellis [21].

Two related issues are involved here. First, unlike the event horizon, the location of the trapping horizon changes under a conformal transformation. Second, the null energy condition is not necessarily equivalent to the null curvature condition. (This condition is called the null convergence condition in [21].) The trapping horizon can appear outside the event horizon in the string frame, even if the null energy condition is satisfied, because the Einstein equations do not hold in this frame [24, 28].

The proof that the apparent horizon cannot lie outside the event horizon (the apparent horizon theorem [21]) is purely geometric and relies only on the validity of the null Raychaudhuri equation (22) and the geometrical condition $R_{ab}l^a l^b \geqslant 0$, the null curvature condition. Even if the matter obeys the null energy condition, the sign of the last term in (27) can be negative and, therefore, we may have a violation of the null curvature condition. This is in fact what happens for the surfaces found in [24, 28]. Brans–Dicke theory can be recast in the Einstein frame via the conformal transformation

$$g_{ab} \to \tilde{g}_{ab} = \phi\, g_{ab}, \qquad \phi \to \tilde{\phi} \qquad \text{with} \quad \mathrm{d}\tilde{\phi} = \sqrt{\frac{2\omega+3}{16\pi}}\,\frac{\mathrm{d}\phi}{\phi}. \tag{65}$$

Here $\phi > 0$ in order to guarantee that the effective gravitational coupling $G_{\rm eff} \sim \phi^{-1}$ remains positive. In the Einstein frame, the null tangent vectors are unchanged, $\tilde{l}^a = l^a$, and they are null with respect to the 'new' metric $\tilde{g}_{ab}$ as well as the old one g_{ab}. In the Einstein frame, the gravitational field equations are

$$\tilde{G}_{ab} = 8\pi\left(\tilde{T}_{ab} + \tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi} - \tfrac{1}{2}\tilde{g}_{ab}\tilde{g}^{cd}\tilde{\nabla}_c\tilde{\phi}\tilde{\nabla}_d\tilde{\phi}\right), \tag{66}$$

where $\tilde{T}_{ab} \equiv T_{ab}/\phi^2$ and thus

$$\tilde{g}^{ac}\tilde{g}^{bd}\tilde{R}_{ab}l_c l_d = 8\pi\,\tilde{g}^{ac}\tilde{g}^{bd}\tilde{T}_{ab}l_c l_d + (\tilde{g}^{ab}l_a\tilde{\nabla}_b\tilde{\phi})^2. \tag{67}$$

Provided that the matter obeys the null energy condition and $\omega > 0$, the geometry will also obey the null curvature condition in the Einstein frame. As we have seen, quasi-local horizons satisfying (34) cannot appear outside the event horizon in any conformal frame if they are located entirely inside the event horizon in the Einstein frame.

5. Cosmological horizons

The problem of locating the trapping horizons following a conformal transformation appears not only in black hole spacetimes but also in cosmology in alternative gravity (especially scalar–tensor and $f(R)$ theories, which can be formally reduced to Einstein gravity plus non-minimally coupled scalars by a conformal transformation). The line element of a

spatially homogeneous and isotropic Friedmann–Lemaitre–Robertson–Walker (FLRW) metric is commonly written in comoving coordinates as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\theta^2 + r^2 \sin^2\theta\,\mathrm{d}\phi^2 \right). \tag{68}$$

In the spatially flat case $k = 0$, we have radial null vectors with components

$$l^\mu = (1, 1/a(t), 0, 0), \qquad n^\mu = (1, -1/a(t), 0, 0)\,, \tag{69}$$

which have expansions

$$\theta_l = \frac{2(\dot{a}r + 1)}{ar}, \qquad \theta_n = \frac{2(\dot{a}r - 1)}{ar}. \tag{70}$$

We see that $\theta_n = 0$ when the comoving radius is $r = 1/\dot{a}$ and the physical radius is $r_{\rm physical} = ar = H^{-1}$, the usual Hubble radius[5]. The expansion θ_l is everywhere positive for $r > 0$ and $\dot{a} > 0$. We can also calculate the variation of the expansions with respect to the null directions

$$\mathcal{L}_l\theta_n = \frac{2(-\dot{a}^2r^2 + r\dot{a} + ar^2\ddot{a} + 1)}{a^2r^2}, \tag{71}$$

which equals

$$\mathcal{L}_l\theta_n = 2\left(\frac{\dot{a}}{a}\right)^2 \left(1 + \frac{\ddot{a}a}{\dot{a}^2}\right) = 2H^2(1-q) \tag{72}$$

on the horizon $r = 1/\dot{a}$. The expansion is accelerating if $q < 0$. For de Sitter space, $q = -1$. Thus, for de Sitter space we have a PITH by the classification of [4]. This is reasonable because the region around $r = 0$ is not trapped ($\theta_l\theta_n < 0$), so it should be an inner horizon and a past horizon because beyond the horizon everything must move outwards, nothing can fall back. The components of the normal N_a to the surface $r = 1/\dot{a}$ are

$$N_\mu = \left(\frac{\ddot{a}}{\dot{a}^2}, 1, 0, 0\right). \tag{73}$$

The norm squared of this normal is

$$N^aN_a = -\left(\frac{\ddot{a}}{\dot{a}^2}\right)^2 + \frac{1}{a^2} = \frac{1}{a^2}(1-q^2). \tag{74}$$

For de Sitter spacetime, the horizons are null. In the more general $k \neq 0$ case, we have

$$l^\mu = \left(1, \frac{\sqrt{1-kr^2}}{a(t)}, 0, 0\right), \qquad n^\mu = \left(1, -\frac{\sqrt{1-kr^2}}{a(t)}, 0, 0\right), \tag{75}$$

which have expansions

$$\theta_l = \frac{2(\dot{a}r + \sqrt{1-kr^2})}{ar}, \qquad \theta_n = \frac{2(\dot{a}r - \sqrt{1-kr^2})}{ar}. \tag{76}$$

Here we have horizons at

$$r = \frac{1}{\sqrt{\dot{a}^2 + k}}. \tag{77}$$

(Note that in general relativity, due to the Hamiltonian constraint $H^2 = 8\pi G\rho/3 - k/a^2$, this radius is always real if the energy density ρ is positive-definite.)

[5] The physical horizon radius can also be obtained by rewriting the line element in the form $\mathrm{d}s^2 = -(1-H^2R^2)\,\mathrm{d}t^2 + (1-H^2R^2)^{-1}\,\mathrm{d}R^2 + R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)$, where $R = ar$, and finding the root of $g^{11} = 0$.

By transforming to the conformal time η defined by $a(\eta)\,\mathrm{d}\eta = \mathrm{d}t$, metric (68) can be cast in the form

$$\mathrm{d}s^2 = a^2(\eta)\left[-\mathrm{d}\eta^2 + \frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right]. \tag{78}$$

This line element is manifestly conformally flat for the spatially flat case $k = 0$ but, because the Weyl tensor vanishes in all cases, all FLRW metrics are conformally flat, even for $k \neq 0$. We can relate the metrics to the flat Minkowski metric by a conformal transformation

$$g_{\mathrm{FLRW}} = a^2 g_{\mathrm{flat}}. \tag{79}$$

so that $g^{\mathrm{flat}}_{ab} = \tilde{g}_{ab} = W\, g_{ab}$ with $W = 1/a^2(\eta)$.

From the point of view of 'veiled' general relativity [8], an expanding FLRW universe is physically equivalent to its flat conformal cousin. This equivalence is apparently surprising and it helps to consider the variation of units of length l_u, time t_u, mass m_u and of the derived units here [9, 11]. In the flat 'veiled frame' spacetime, these units are not fixed but scale as $\tilde{l}_u \sim W^{1/2} l_u = a^{-1} l_u$, $\tilde{t}_u \sim W^{1/2} t_u = a^{-1} t_u$ and $\tilde{m}_u \sim W^{-1/2} m_u = a m_u$ (the scaling of derived units is argued by straightforward dimensional considerations). Despite appearances, gravity is still present in this space and it acts by shrinking the units $\tilde{l}_u$ and $\tilde{t}_u$ instead of making the universe expand as in the original FLRW space. Thus, we do not have a genuine Minkowski space, but one with time-dependent units, a fact that must be kept in mind at all times. Actual measurements are always made with respect to a unit scale. A given time interval, for example, is recorded by dividing it up into blocks of the time unit t_u.[6] There is no physical difference between a static space with all lengths and times shrinking, or an expanding FLRW space with fixed units. For example, in this static space in the frame with varying units, there is cosmological redshift (which is obviously absent in a genuine Minkowski space with fixed units), caused by the fact that the unit of length $\tilde{l}_u$ assumes different values at the different instants of emission and observation of a light signal. It is instructive to derive this redshift in the flat space with line element

$$\mathrm{d}\tilde{s}^2 = -\mathrm{d}\eta^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2). \tag{80}$$

To keep track of the variation of units in the 'veiled frame', divide by the unit of length squared in this frame and use the fact that $\tilde{l}_u = \tilde{t}_u$ (this is merely a choice of units so that everywhere the measured speed of light is 1); then

$$\frac{\mathrm{d}\tilde{s}^2}{\tilde{l}_u^2} = -\frac{\mathrm{d}\eta^2}{\tilde{t}_u^2} + \frac{\mathrm{d}r^2}{\tilde{l}_u^2} + \frac{r^2}{\tilde{l}_u^2}(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2). \tag{81}$$

The time interval between two equal intervals of η, say $\int_{\eta_1}^{\eta_2}\mathrm{d}\eta$ and $\int_{\eta_3}^{\eta_4}\mathrm{d}\eta$ such that $\eta_2 - \eta_1 = \eta_4 - \eta_3$, will not be measured by a comoving observer to be equal amounts of time, not because the η intervals are different, but because the units with which they are compared are changing with time. In the first term on the right-hand side, the square of the ratio $\mathrm{d}\eta/\tilde{t}_u$ appears, but one must compare t-time intervals with the unit $\tilde{t}_u$ and η-time intervals with the unit $\tilde{\eta}_u$; hence, we convert η to t using an ordinary coordinate transformation (not a transformation of units) $\mathrm{d}t = a\,\mathrm{d}\eta$, obtaining

$$\frac{\mathrm{d}\tilde{s}^2}{\tilde{l}_u^2} = -\frac{\mathrm{d}t^2}{a^2\tilde{t}_u^2} + \frac{\mathrm{d}r^2}{\tilde{l}_u^2} + \frac{r^2}{\tilde{l}_u^2}(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2). \tag{82}$$

[6] In FLRW space, the spatial homogeneity and isotropy select a preferred family of observers, the comoving observers who see the cosmic microwave background homogeneous and isotropic around them (apart from small temperature fluctuations $\delta T/T \sim 5\times 10^{-5}$). The comoving time t is the proper time of these observers; hence, it is a geometrically and physically preferred notion of time.

Equal intervals of the t coordinate will be measured as equal time intervals with respect to the fundamental unit scale. Even though a is a function of t, this line element is still manifestly flat. Consider now a light ray emitted at radius r_e at time t_e, which propagates radially and is received by an observer at $r = 0$ at time t_o. Setting $\mathrm{d}\tilde{s}^2 = 0$ and $\mathrm{d}\theta = \mathrm{d}\varphi = 0$ for radial null geodesics[7], one obtains $\frac{\mathrm{d}t}{a(t)} = \pm \mathrm{d}r$, where the negative sign must be chosen for rays propagating from r_e to $r = 0$ along the direction of decreasing r. Integrating between emission and observation yields

$$\int_{t_e}^{t_o} \frac{\mathrm{d}t}{a(t)} = -\int_{r_e}^{0} \mathrm{d}r. \tag{83}$$

Consider now a second pulse emitted at r_e at time $t_e + \delta t_e$ and received at $r = 0$ at $t_o + \delta t_o$. In the same way, one obtains

$$\int_{t_e+\delta t_e}^{t_o+\delta t_o} \frac{\mathrm{d}t}{a(t)} = -\int_{r_e}^{0} \mathrm{d}r. \tag{84}$$

Since the right-hand sides of equations (83) and (84) are equal, so are their left-hand sides,

$$\int_{t_e+\delta t_e}^{t_o+\delta t_o} \frac{\mathrm{d}t}{a(t)} = \int_{t_e}^{t_o} \frac{\mathrm{d}t}{a(t)}\,, \tag{85}$$

and one can then write

$$\left[\int_{t_e+\delta t_e}^{t_o} + \int_{t_o}^{t_o+\delta t_o} - \left(\int_{t_e}^{t_e+\delta t_e} + \int_{t_e+\delta t_e}^{t_o}\right)\right] \frac{\mathrm{d}t}{a(t)} = 0 \tag{86}$$

and

$$\int_{t_o}^{t_o+\delta t_o} \frac{\mathrm{d}t}{a(t)} = \int_{t_e}^{t_e+\delta t_e} \frac{\mathrm{d}t}{a(t)}\,. \tag{87}$$

Assume now that δt_e and δt_o are very small, so that $a(t)$ does not change appreciably from its value $a(t_e)$ (respectively, $a(t_o)$) in the time interval $(t_e, t_e + \delta t_e)$ (respectively, $(t_o, t_o + \delta t_o)$); then

$$\frac{\delta t_o}{a(t_o)} = \frac{\delta t_e}{a(t_e)} \tag{88}$$

and, assuming now $\nu_e = 1/\delta t_e$ to be the frequency of the signal at emission and $\nu_o = 1/\delta t_o$ the received frequency, both measured with respect to the fundamental frequency unit $1/\tilde{t}_u$, it is

$$\frac{1}{a(t_o)\nu_o} = \frac{1}{a(t_e)\nu_e}\,. \tag{89}$$

The redshift z is then given by

$$z + 1 \equiv \frac{\lambda_o}{\lambda_e} = \frac{\nu_e}{\nu_o} = \frac{a(t_o)}{a(t_e)} \equiv \frac{a_o}{a_e} \tag{90}$$

(where λ_e and λ_o are the wavelengths at emission and observation, respectively). Then there is redshift also in flat 'veiled frame' space and its derivation parallels completely the standard derivation of redshift in FLRW space (e.g., [29]). The result agrees with that of [8], in which the cosmological redshift in the veiled frame is derived in a different way by considering a hydrogen atom and taking into account carefully the local change in the electron mass deriving from the non-trivial coupling of matter to the conformal factor W in the veiled frame. This

[7] A null geodesic ($\mathrm{d}s^2 = 0$) in the original frame corresponds to a null geodesic ($\mathrm{d}\tilde{s}^2 = W\,\mathrm{d}s^2 = 0$) in the 'veiled frame' (it is not so for *timelike* geodesics).

coupling can also be interpreted as a variation of units with the spacetime point [9] and it is the source of redshift. The distance-redshift relation in the veiled frame is also derived in [8], and it coincides, of course, with the one derived in FLRW space with constant units.

In the flat space of veiled FLRW cosmology, the radial null vectors have components (we can use the coordinates η and r from (78) if $k = 0$)

$$l^\mu = (1, 1, 0, 0)\,, \qquad n^\mu = (1, -1, 0, 0)\,. \tag{91}$$

The expansions of these two null vectors in the flat space are

$$\theta_l = \frac{2}{r}, \qquad \theta_n = -\frac{2}{r}. \tag{92}$$

We have $\theta_l\theta_n < 0$ for all finite r, so there are no spherical trapped surfaces in flat space (in fact there are no trapped surfaces entirely contained in flat space at all). But if we instead look at the change of the horizon-entropy in the null directions, we find

$$l^a\nabla_a(a^2A) = \partial_\eta(a^2A) + \partial_r(a^2A) = 8\pi ar[(\partial_\eta a)r + a], \tag{93}$$

$$n^a\nabla_a(a^2A) = \partial_\eta(a^2A) - \partial_r(a^2A) = 8\pi ar[(\partial_\eta a)r - a]. \tag{94}$$

We see that we have a conformal horizon at $r = a/(\partial_\eta a)$. We can convert the coordinates from η to t by writing $\partial_\eta a = a\partial_t a$. Thus, there is a horizon at $r = 1/(\partial_t a)$ just as in the spatially flat FLRW case. We can also compute

$$\begin{aligned} l^a\nabla_a[n^b\nabla_b(a^2A)] &= 8\pi ar^2\partial_\eta\partial_\eta a = 8\pi a^3r^2\left(\ddot{a} + \frac{\dot{a}^2}{a}\right) \\ &= 8\pi a^4r^2H^2(1-q). \end{aligned} \tag{95}$$

We obtain exactly the same kind of horizon as above (a PITH in the case of de Sitter). The signature of the horizon is the same (as expected) since the normal is

$$N_\mu = \left(\frac{a\ddot{a}}{\dot{a}^2}, 1, 0, 0\right), \tag{96}$$

whose norm is given by

$$N^aN_a = 1 - \left(\frac{a\ddot{a}}{\dot{a}^2}\right)^2 = 1 - q^2. \tag{97}$$

It may come to a surprise, especially for an astronomer, that the Hubble parameter (which is also three times the expansion of the timelike worldlines of comoving observers and a scalar quantity) is not a good observable when studying cosmological horizons and their location (see, e.g., [30]). However, from the discussion above, it is clear that H is not a good observable when conformal transformations are used in generalized (and even in Einstein) gravity. H is changed by conformal transformations and so is the location of the cosmological horizon, and a more general quantity is needed.

6. Conclusion

If entropy is a useful quantity in time-dependent situations, and possibly also in non-equilibrium thermodynamics, its applicability may extend beyond event horizons of static or stationary black holes. Dynamical situations are the rule rather than the exception and, in certain theories, stationary situations may not even exist. For example, in the class of $f(R)$ theories designed to explain the present acceleration of the universe without resorting to dark energy, Minkowski space is not a global solution and one cannot contemplate asymptotically

flat black holes in these theories. When the relevant field equations are written in a form that mimics the Einstein equations, a cosmological effective fluid composed of geometric terms is present on the right-hand side of these equations and causes the universe to accelerate its expansion, so that the role of Minkowski space as a global solution giving a static background is played instead by the de Sitter or other FLRW solutions. In this case, black holes are embedded in dynamical (cosmological) backgrounds and one does not have the luxury of considering static horizons in a static background. However, the horizon-entropy formula (2) originally developed for perturbations of stationary systems gives rise to rather generic horizon-entropy increase laws for both causal and quasi-local horizons in general dynamical spacetimes.

Here we have seen how a modification of the trapping horizon conditions can give quasi-local horizons for which a horizon-entropy increase law can be proven in models that are related via field redefinitions to Brans–Dicke theory. The location of these surfaces is invariant under a conformal transformation of the metric, which is not true of ordinary trapping horizons. It is likely that these results will hold for all theories that are conformally related to Einstein gravity and for which the horizon-entropy transforms in the same way as the area, provided the energy conditions hold in the Einstein frame. These conditions can be applied to a variety of situations including finding cosmological horizons in 'veiled' Minkowski space with varying units. It is not known how these surfaces behave in theories that are not conformally related to Einstein gravity, nor in theories that contain higher order curvature corrections which cannot be put in a single scalar–tensor form, such as those derived from higher order corrections in string theory. However, in the low-energy limit of string theories, often the dominant effects are due to the dilaton field φ and the action can be approximated by

$$S = \int \mathrm{d}^4x\sqrt{-g}(\mathrm{e}^{-\varphi}R + \nabla_c\varphi\nabla^c\varphi + \cdots), \tag{98}$$

formally an $\omega = -1$ Brans–Dicke theory with some stringy matter, and the considerations of the previous sections on the location of the (black hole or cosmological) apparent horizon apply as well. This is relevant because of the importance of black holes in string theories.

In a given spacetime, there are many surfaces for which one can define an entropy increase law. We have examined here three different cases, null causal horizons which include global event horizons, locally defined geometric horizons including trapping horizons and the new proposal based on gravitational horizon-entropy. We have derived quasi-local conditions on the rate of increase of the horizon-entropy and shown that this is non-negative for both causal horizons and the new quasi-local horizons. Although the governing equation for these two cases is very similar, compare equations (31) and (38), there are some differences. The main difference is that in the case of causal horizons, it governs the behaviour of the second derivative of the horizon-entropy and for the quasi-local horizons, it governs directly the first derivative of the entropy. The horizon-entropy of both types of horizons can shrink if sufficient negative energy is provided. Both types of horizons can settle down to exact Killing horizons, but only the quasi-local horizons can start from exact Killing horizons. While $\omega_l = 0$ follows trivially for causal horizons, for quasi-local horizons it requires the additional assumption that the null normal l^a is derivable from a double-null foliation.

We have proven that horizon-entropy does not decrease by requiring that all the individual terms in (31) and (38) are negative. In particular, this requires the null energy condition $T_{ab}l^al^b \geqslant 0$ be satisfied for matter fields, rather than the null curvature condition $R_{ab}l^al^b \geqslant 0$. In fact, all that is actually required is that the overall sum of the terms in (31) or (38) be negative. It is possible that in certain specific scenarios some of these terms, in particular $-T_{ab}l^al^b$, are positive, but that overall the horizon-entropy still increases.

The quasi-local surfaces used to derive the horizon-entropy increase law are in general not apparent horizons or trapping horizons. Outside of the Einstein frame, they are not foliated by marginally outer trapped surfaces except in the case where they describe Killing or isolated horizons and in general they do not satisfy $\theta_l = 0$. They will though be spacelike surfaces if the horizon-entropy is increasing and null surfaces if it is constant. In a spacetime that satisfies the null energy condition, they will be located behind the event horizon and so will, in cases like the Brans–Dicke collapse considered in [24], lie inside the apparent horizon.

It was mentioned in [3] that apparent horizons will not satisfy a horizon-entropy increase relation and that the acausal behaviour of event horizons is needed to save this law. The surfaces given by (34) are quasi-locally defined and satisfy a local horizon-entropy increase law of the form used in [3]. In [31], the validity of the generalised second law (GSL) was examined for apparent horizons in a string frame two-dimensional model. This work explicitly included the contribution of both the horizon-entropy and the entropy of fields outside the horizon and concluded that for coherent quantum states, the GSL was valid but possibly violated for non-coherent quantum states. It is not known whether the surfaces satisfying (34) will satisfy the GSL. Since they coincide with trapping horizons in the Einstein frame, if it can be shown that the GSL is violated for trapping horizons in the Einstein frame, then the same will be true for these surfaces.

We have used a dynamical definition of entropy as proposed in [15]. Strictly speaking, this definition is derived only for stationary situations and it is known that its application to non-stationary situations contains several ambiguities [32, 19]. These ambiguities are not essential for our derivation, in fact all we require is a definition of horizon-entropy whose value is invariant under a conformal transformation. Even the association of s_{ab} with entropy is not essential. We only require that it transforms invariantly under a conformal transformation. Throughout this work, we have suppressed the factor of 4 in the area-entropy relation, and our results are independent of the precise numerical value of this factor.

Under a conformal transformation of the metric, the location of the surfaces studied here remains the same. This is not true of trapping horizons. That the surfaces are invariant under a conformal transformation is in a certain sense trivial, because the horizon-entropy definition used is always equal to the area in the Einstein frame and so the definitions always pick out the 'Einstein frame trapping horizon'. Put simply, we have

$$\begin{aligned} & g_{ab} \to \tilde{g}_{ab} = W g_{ab}, \\ & A \to \tilde{A} = W A, \\ & S \to \tilde{S} = \frac{\tilde{A}}{4W} = \frac{A}{4} = S. \end{aligned} \tag{99}$$

But conditions (34) do not make explicit reference to the Einstein frame and thus can be applied simply in non-Einstein frames without the need to transform the metric. While a conformal transformation will always put the theories considered here into the Einstein frame form of Einstein gravity plus matter, and one could proceed with traditional trapping horizons, one must accept that in many alternative theories of gravity, this Einstein frame will not be the standard frame with constant units.

We have argued, along with many other authors, that a conformal transformation of the metric should not change the operationally defined physical features of the spacetime, provided that one redefines standards of length, time and mass in a position-dependent way. This is most easily demonstrated in the case of 'veiled general relativity' where metric solutions of ordinary Einstein gravity are subjected to a conformal transformation. In the simple case of the Schwarzschild spacetime, the usual conditions for a trapping horizon do not always pick

out the $r = 2M$ surface. The modified conditions proposed here do. Thus, the surfaces defined here allow a more operationally physical interpretation than trapping horizons.

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