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Alina Wilhelm

geboren in Omsk, Russland

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Die Diplomarbeit wurde von Alina Wilhelm

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Herrn Prof. Manfred Lindner

Department of Physics and Astronomy

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Neutrino oscillations in non-uniform matter in quantum field theory framework

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Neutrino-Oszillationen in inhomogener Materie im Rahmen der Quantenfeldtheorie:

Neutrino-Oszillationen sind ein Phänomen jenseits des Standard Modells, welches experimentell sehr gut untersucht ist. Sie wurden beobachtet in Experimenten zu atmosphärischen und solaren Neutrinos, sowie in Reaktor- und Beschleunigerexperimenten. Neutrino-Oszillationen sind eine wichtige Beobachtung für die moderne Physik, die zeigt, dass Neutrinos Masse besitzen. In der vorliegenden Arbeit beschreiben wir Neutrino-Oszillationen in inhomogener Materie, ausgehend von einem Ansatz, in dem die Neutrinoerzeugung, Neutrinopropagation und der Neutrinonachweis als einzelne Prozesse betrachtet werden. In diesem Ansatz sind die Neutrinos durch Propagatoren, die in einem allgemeinen Feynman-Diagramm die Erzeugungs- und Nachweisvertices verbinden, beschrieben. Dabei enthält der Propagator die gesamte Information über die Neutrinowechselwirkung mit der Materie durch ein effektives Potenzial. Wir zeigen, wie man mithilfe von Feynmanregeln und experimentellen Beobachtungen eine sinnvolle Oszillationswahrscheinlichkeit definiert. Von dieser Grösse ausgehend leiten wir die Amplitude für den Oszillationsprozess her und bestimmen unter welchen Bedingungen diese mit dem üblichen Resultat, das von der Lösung der Schrödinger-artigen Differentialgleichung kommt, übereinstimmt. Um die Näherungen, die wir für die Rechnungen benutzt haben, zu verdeutlichen, diskutieren wir ein Beispiel, das Oszillationen zwischen zwei Neutrinoarten in der adiabatischen Näherung beschreibt.

Neutrino oscillations in non-uniform matter in quantum field theory framework:

Neutrino oscillations are a phenomenon beyond the Standard Model that is very well established experimentally. They were observed in the atmospheric, solar, accelerator and reactor neutrino experiments. This is an important fact for modern physics, since it demonstrates that neutrinos are massive. In the present work we describe neutrino oscillations in non-uniform matter, using an approach based on Quantum Field Theory, in which neutrino production, propagation and detection are considered as a single process. In this approach neutrinos are described through propagators connecting the production and detection vertices in a general Feynman diagram. In our treatment the information about neutrino-matter interaction is contained in the neutrino propagator through an effective matter potential. We present a way to define a meaningful oscillation probability using the Feynman rules and experimental considerations. From this quantity we derive the amplitude for the oscillation process and determine under which conditions it coincides with the result predicted by the standard approach, where the amplitude is found from a Schrödinger-like evolution equation. To illustrate the approximations used in the calculations we present an example of two-flavour oscillations in the adiabatic limit.

Contents

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1 Introduction

1.1 Motivation

Neutrino oscillations is a phenomenon beyond the Standard Model (SM) that is firmly established experimentally nowadays. Neutrino oscillations mean a change of the probability to find a neutrino in a certain flavour state. This phenomenon was observed in atmospheric and solar neutrino experiments (such as SuperKamiokande, Davis experiment, Gallex, SAGE, SNO and Borexino) as well as in experiments of nuclear reactors and particle accelerators (e.g. KamLAND, K2K, Minos and T2K) [1]. The first one who put forward the idea of neutrino oscillations was Bruno Pontecorvo in 1957 [2]. Inspired by kaon oscillations $K^0 \leftrightarrow \bar{K}^0$ he suggested in his work the possibility of the oscillations of a neutrino ν into its antineutrino $\bar{\nu}$ and discussed the case of flavour oscillations $\nu_\mu \leftrightarrow \nu_e$ ten years later [3]. The same idea was independently put forward by Maki, Nakagawa and Sakata [4]. Lincoln Wolfenstein was the first to extend this framework to describe neutrino oscillations in the presence of matter [5]. To derive the oscillation probability he used a Schrödinger-like equation, which is known today as the Wolfenstein evolution equation. A crucial physical insight into the theory of neutrino oscillations was provided by Stanislav Mikheev and Alexei Smirnov in 1985 [6], who recognized that the oscillation probability can be resonantly enhanced in the presence of matter. This was an excellent explanation for the solar neutrino deficit problem: the deficit of the observed solar neutrino flux compared to the flux predicted by the standard solar model and the SM of particle physics. Today this problem is properly solved and we know that the missing neutrinos are merely converted into another flavor due to the flavour transitions in the solar matter. The effect of modification of neutrino oscillations by matter can play a crucial role in oscillations of solar neutrinos, atmospheric and accelerator neutrinos inside the Earth and neutrinos from supernova and early universe. Today neutrino oscillations in matter are known as the Mikheev-Smirnov-Wolfenstein (MSW) effect. They are the subject of the present work.

We will consider neutrino oscillations in vacuum before we switch to the description of the MSW effect. A necessary condition for neutrino oscillations is that neutrinos have mass, which is absent in the Standard Model. Neutrinos can be easily made massive by introducing right-handed partners, which are gauge-singlets and therefore hard to detect. There is no unique way to introduce a mass for the neutrinos, and concrete models can vary in their content and implications. Many models, such as those based on the seesaw mechanism, have been brought forward to explain the smallness of the neutrino masses compared to all other particle masses.

The neutrino flavour eigenstates and the neutrino mass eigenstates are related by the

Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix, also called the leptonic mixing matrix, which is analogous to the Cabbibo-Kobayashi-Maskawa (CKM) matrix in the quark sector. The PMNS matrix is a unitary 3×3 matrix, that can be parameterized as

$$\begin{aligned} U &= \begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix} \\ &= \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{bmatrix} \begin{bmatrix} e^{i\alpha_1/2} & 0 & 0 \\ 0 & e^{i\alpha_2/2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \end{aligned} \tag{1.1}$$

where $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$. Two of the three mixing angles θ_{ij} have been rather precisely measured experimentally (see Table (1.1)), while a clear determination of θ_{13} has yet to succeed. However, recently there have been indications of non-zero value of θ_{13} coming from the T2K, Minos and Double Chooz experiments [7]. Only for nonzero θ_{13} is the CP-violating Dirac phase δ measurable. The phases α_1 and α_2 can be introduced only if neutrinos are Majorana particles, i.e. are their own antiparticles. Note that the nature of neutrinos, whether they are Dirac or Majorana particles, does not influence neutrino oscillations.
Throughout this work we denote the state vectors of the neutrino mass eigenstates as kets with Latin letter indices $|\nu_j\rangle$, and those of flavour eigenstates as kets with Greek letter indices $|\nu_\alpha\rangle$. They are related through the PMNS matrix U:

$$|\nu_\alpha\rangle = \sum_j U^*_{\alpha j} |\nu_j\rangle \,. \tag{1.2}$$

The main idea behind neutrino oscillations from the Quantum Mechanics point of view is the following: Consider a system, whose Hamiltonian possesses eigenstates $|\Psi_i\rangle$. When this system resides in the state $|\Psi_i\rangle$ at some time $t = 0$, its time evolution is described through $|\Psi(t)\rangle = e^{-iE_i t}|\Psi_i\rangle$, where E_i is the corresponding energy eigenvalue. Note that we use the natural units $c = \hbar = 1$ throughout this thesis. Imagine one succeeds to prepare states which are not eigenvectors of the Hamiltonian (in our case they are flavour eigenstates). The probability to detect one of them will then oscillate with time.

1.2 Neutrino oscillations in vacuum

We will now present a simple way to calculate the neutrino oscillation probability in vacuum which can be found in almost every textbook on neutrinos (see, for instance, Ref. [8]). Imagine that at some time $t = 0$ a certain flavour state $|\nu_\alpha\rangle$ is prepared, which can be expressed as a superposition of mass eigenstates $|\nu(t=0)\rangle = |\nu_\alpha\rangle = \sum_j U^*_{\alpha j}|\nu_j\rangle$. After some time interval t the system evolves into the state $|\nu(t)\rangle = \sum_j U^*_{\alpha j} e^{-iE_j t}|\nu_j\rangle$. To obtain the probability amplitude to find the neutrino in a certain flavour state $|\nu_\beta\rangle$ after time t, we should project the evolved state on this flavour state:

parameter	**best fit $\pm 1\sigma$**	**$\pm 2\sigma$**	**$\pm 3\sigma$**
Δm^2_{21} $[10^{-5}\mathrm{eV}^2]$	$7.59^{+0.20}_{-0.18}$	7.24–7.99	7.09–8.19
Δm^2_{31} $[10^{-3}\mathrm{eV}^2]$	$2.50^{+0.09}_{-0.16}$ $-(2.40^{+0.08}_{-0.09})$	$2.25 - 2.68$ $-(2.23 - 2.58)$	$2.14 - 2.76$ $-(2.13 - 2.67)$
$\sin^2\theta_{12}$	$0.312^{+0.017}_{-0.015}$	0.28–0.35	0.27–0.36
$\sin^2\theta_{23}$	$0.52^{+0.06}_{-0.07}$ 0.52 ± 0.06	0.41–0.61 0.42–0.61	0.39–0.64
$\sin^2\theta_{13}$	$0.013^{+0.007}_{-0.005}$ $0.016^{+0.008}_{-0.006}$	0.004–0.028 0.005–0.031	0.001–0.035 0.001–0.039
δ	$\left(-0.61^{+0.75}_{-0.65}\right)\pi$ $\left(-0.41^{+0.65}_{-0.70}\right)\pi$	$0 - 2\pi$	$0 - 2\pi$

Table 1.1: Neutrino oscillation parameters, taken from Ref. [7]. For Δm^2_{31}, $\sin^2\theta_{23}$, $\sin^2\theta_{13}$, and δ the upper (lower) row corresponds to normal (inverted) neutrino mass hierarchy. Normal mass hierarchy means: $\Delta m^2_{31} > 0$ and inverted mass hierarchy: $\Delta m^2_{31} < 0$. The definitions of these parameters are given in the main text.

$$\mathcal{A}(\nu_\alpha \to \nu_\beta, t) = \langle \nu_\beta | \nu(t) \rangle = \sum_j U^*_{\alpha j} e^{-iE_j t} \langle \nu_\beta | \nu_j \rangle \;\; = \sum_{j,k} U^*_{\alpha j} e^{-iE_j t} U_{\beta k} \langle \nu_k | \nu_j \rangle$$
$$= \sum_j U^*_{\alpha j} e^{-iE_j t} U_{\beta j} \,. \tag{1.3}$$

The probability for the process of oscillation of the neutrino ν_α into ν_β after time t is then:

$$P(\nu_\alpha \to \nu_\beta, t) = |\mathcal{A}(\nu_\alpha \to \nu_\beta, t)|^2 = |\sum_j U^*_{\alpha j} e^{-iE_j t} U_{\beta j}|^2 \,. \tag{1.4}$$

Since we consider relativistic particles we can write the energy of such a particle as

$$E_i = \sqrt{p^2 + m_i^2} \approx p + \frac{m_i^2}{2p} \,, \tag{1.5}$$

where m_i is the mass and p the momentum of the corresponding mass eigenstate. It is important to note that the particles of different mass are assumed to have the same momentum here. The next assumption we use is that for relativistic pointlike particles the distance L that they propagate and the time t they need for this are nearly the same:

$$L \approx t \,. \tag{1.6}$$

With these two approximations the oscillation probability becomes

$$P(\nu_\alpha \to \nu_\beta, L) = |\sum_j U^*_{\alpha j} e^{-i\frac{\Delta m^2_{jk}}{2p}L} U_{\beta j}|^2 \,, \tag{1.7}$$

where $\Delta m^2_{jk} := m^2_j - m^2_k$ and k is an arbitrary index corresponding to a mass eigenstate. This is a very simple and transparent derivation but it has a problem, namely the assumption that different mass eigenstates have the same momenta, which can not be justified. Surprisingly the result (1.7) is nevertheless correct. The explanation for this contradiction can be found in Ref. [9].
Let us consider the simple example of two-flavour mixing, which describes oscillations between electron neutrino and muon neutrino. The PMNS matrix takes then the simple form

$$U = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \,, \tag{1.8}$$

where θ is the vacuum mixing angle. The flavour eigenstates and mass eigenstates are related through

$$\begin{aligned} |\nu_e\rangle &= \cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle \,, \\ |\nu_\mu\rangle &= -\sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle \,. \end{aligned} \tag{1.9}$$

Using eq. (1.7) we obtain the oscillation probability

$$P(\nu_e \to \nu_\mu, L) = \sin^2 2\theta \sin^2\left(\pi \frac{L}{l_{osc}}\right) \,, \tag{1.10}$$

where l_{osc} is the oscillation length defined as

$$l_{osc} = \frac{4\pi p}{\Delta m^2} \,. \tag{1.11}$$

From this formula it is clear that in vacuum the probability for converting an electron neutrino into a muon neutrino oscillates with propagation distance L with the constant amplitude $\sin^2 2\theta$.

1.3 Effective matter potential

In this section we want to consider neutrino interactions with matter before we turn to the description of the neutrino oscillations in it. In the coming chapters we will simply speak about some general effective matter potential $V(\boldsymbol{x})$ without specifying its exact form.
How do the neutrinos interact with other particles that are present in matter? There are neutral current interactions of all flavoured neutrinos (electron neutrino ν_e, muon

neutrino ν_μ and tau neutrino ν_τ) with neutrons, protons and electrons of the medium by means of exchange of the Z^0 bosons. In addition, there are charged current interactions via the $W^\pm$ bosons exchanges between electrons and the electron neutrinos ν_e (see Figure (1.1)). At low energies the charged current interaction can be described by the effective Hamiltonian

$$H_{CC} = \frac{G_F}{\sqrt{2}}[\bar{e}\gamma_\mu(1-\gamma_5)\nu_e][\bar{\nu}_e\gamma^\mu(1-\gamma_5)e]\,, \tag{1.12}$$

where G_F is the Fermi constant and γ^μ are the gamma matrices. Using the Fierz transformation we can bring the Hamiltonian to the form

$$H_{CC} = \frac{G_F}{\sqrt{2}}[\bar{e}\gamma_\mu(1-\gamma_5)e][\bar{\nu}_e\gamma^\mu(1-\gamma_5)\nu_e] = \frac{G_F}{\sqrt{2}}\nu_e^\dagger\gamma_0\gamma_\mu(1-\gamma_5)[\bar{e}\gamma^\mu(1-\gamma_5)e]\nu_e\,. \tag{1.13}$$

To obtain the matter potential $(V_e)_{CC}$ corresponding to neutrino-electron interactions we integrate H_{CC} over the electron variables while keeping the ν_e variables fixed:

$$\bar{\nu}_e(V_e)_{CC}\nu_e := \langle H_{CC}\rangle_{electron}\,. \tag{1.14}$$

Since we have $\langle\bar{e}\gamma_0 e\rangle = \langle e^\dagger e\rangle = N_e$, where N_e is the electron number density, the charged-current electron potential is

$$(V_e)_{CC} = \sqrt{2}G_F N_e\,, \tag{1.15}$$

because for unpolarized matter with vanishing total electron momentum all other terms of integration are zero. The potential generated through neutral currents can be obtained in the same manner. We take into account that for an electrically neutral medium the densities of electrons and protons are equal, so neglecting tiny loop corrections the corresponding contributions to V_{NC} cancel each other. Therefore only the neutron density N_n contributes to V_{NC}. The final result is

$$(V_{e,\mu,\tau})_{NC} = -\frac{G_F N_n}{\sqrt{2}}\,. \tag{1.16}$$

It is the same for all neutrino flavours, because the tree-level neutral current interactions are flavour-blind. The total potential for the electron neutrinos is the sum of the neutral current and charged current parts:

$$V_e = \sqrt{2}G_F(N_e - \frac{N_n}{2})\,. \tag{1.17}$$

For the tau and muon neutrinos the total potential is given by eq. (1.16). Since we consider non-uniform matter, the neutron and electron densities are coordinate dependent. The matter potential V is a matrix in flavour space, which is diagonal in the flavour

basis. For three flavour case it is

$$V = \begin{bmatrix} V_e & 0 & 0 \\ 0 & V_\mu & 0 \\ 0 & 0 & V_\tau \end{bmatrix}, \tag{1.18}$$

with diagonal terms given by equations (1.16) and (1.17). Since the matter potential is the sum of neutral and charged current terms

$$V = \begin{bmatrix} (V_e)_{CC} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} (V_e)_{NC} & 0 & 0 \\ 0 & (V_\mu)_{NC} & 0 \\ 0 & 0 & (V_\tau)_{NC} \end{bmatrix}, \tag{1.19}$$

and neutral current interaction terms are the same for each of three flavours, one realizes that the second matrix in eq. (1.19) is proportional to the unit matrix and thus does not contribute to the oscillations. In three-flavour case we then can ignore the second term of eq. (1.19) and specify the matter potential matrix to be of the form

$$\hat{V} = \begin{bmatrix} V & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.20}$$

where $V = \sqrt{2} G_F N_e$ is the charged current interaction term of the electron neutrino.
If we extend the Standard Model by including the sterile neutrino ν_s, the oscillations between electron neutrinos and sterile neutrinos will be governed by V_e given by eq. (1.17). The reason for this is the fact that the sterile neutrino has no electro-weak interactions and thus $(V_s)_{CC} = (V_s)_{NC} = 0$. Analogously, $\nu_{\mu,\tau} \leftrightarrow \nu_s$ oscillations will depend on $(V_{\mu,\tau})_{NC}$ given in eq. (1.16).
When we consider the oscillations between the three active neutrino flavours the matter potential V depends only on the electron number density N_e and can be evaluated as

$$V = \sqrt{2} G_F N_e \approx 7.54 \cdot 10^{-14} \cdot \rho \left(\frac{g}{cm^3} \right) \cdot Y_e \cdot eV \,, \tag{1.21}$$

where ρ is the matter density in units of $\left(\frac{g}{cm^3}\right)$ and Y_e is the number of electrons per one nucleon in the medium. Considering neutrinos propagating through the Sun or the Earth we know that the matter potential V is of the order of $(10^{-16} - 10^{-11})\, eV$, because the density for the Sun varies between $\rho_{Sun} \approx (0.005 - 150) \frac{g}{cm^3}$ and for the Earth $\rho_{Earth} \approx (3 - 14) \frac{g}{cm^3}$. Since the typical neutrino energy E is $(0.1 - 20) MeV$ for solar and reactor neutrinos and $(0.1 - 1000) GeV$ for atmospheric and particle accelerator neutrinos, the condition

$$\frac{|V|}{|E|} \ll 1 \tag{1.22}$$

is always fulfilled. We will make use of it in deriving the neutrino propagator in matter in the next chapter.

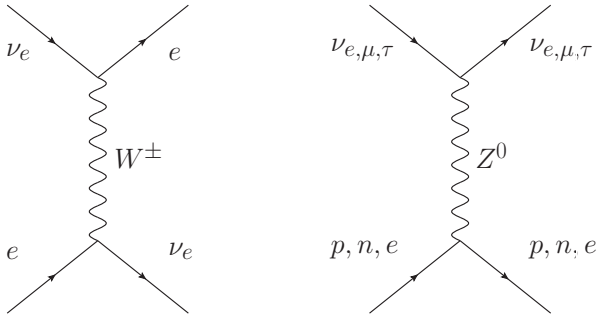

Figure 1.1: Feynman diagrams for neutrino scattering.

1.4 Neutrino oscillations in matter of constant density

Now we turn to neutrino oscillations in matter, usually described by the evolution equation introduced by Wolfenstein [5]. In this approach one defines again the neutrino state $|\nu\rangle$ and applies the Wolfenstein evolution equation to it

$$i\frac{d}{dx}|\nu\rangle = H|\nu\rangle\,, \tag{1.23}$$

where x is the coordinate along the neutrino propagation path. This general equation can be expressed in any basis. When we consider the neutrino system in vacuum in the mass eigenstate basis, the Hamilton operator H is diagonal, $H = H_{mass} = diag(E_1, E_2, E_3)$. Since we are interested in flavour transitions we switch to the flavour basis, in which the Hamilton operator is not diagonal anymore. It takes the form

$$i\frac{d}{dx}|\nu_{fl}\rangle = UH_{mass}U^{\dagger}|\nu_{fl}\rangle = H_{fl}^{vac}|\nu_{fl}\rangle\,, \tag{1.24}$$

where U is the PMNS mixing matrix. In the presence of matter one adds to the Hamilton operator an additional term that describes interactions of neutrinos with matter. The interactions are governed by the electro-weak neutral and charged currents, and therefore are diagonal in the flavour basis. In the case of constant matter density we can find a basis in which the total Hamiltonian is diagonal. For this purpose we introduce the matrix that diagonalizes the total Hamiltonian in matter, which will be called the mixing matrix in matter and denoted $\tilde{U}$. So we have the connection to the flavour basis

$$|\nu_\alpha\rangle = \sum_A \tilde{U}^*_{\alpha A}|\nu_A\rangle\ \ , \tag{1.25}$$

where $|\nu_A\rangle$ are the matter eigenstates, which we will denote with the capital Latin letters to differentiate them from the mass and flavour eigenstates. As we can see the matter eigenstates $|\nu_A\rangle$ and the mixing matrix in matter $\tilde{U}$ play the same role in matter

of constant density as the mass eigenstates and the leptonic mixing matrix in vacuum, respectively. To find the oscillation probability one should use again eq. (1.4) but replace the leptonic mixing matrix by the mixing matrix in matter and free neutrino energies by the eigenvalues of the total Hamiltonian in matter. The formula for the neutrino oscillation probability then takes the form

$$P(\nu_\alpha \to \nu_\beta, L) = |\sum_A \tilde{U}^*_{\alpha A} e^{-iE_A L} \tilde{U}_{\beta A}|^2 \,, \tag{1.26}$$

where we assumed once again $L \approx t$. To illustrate this result let us again consider the two-flavour case discussed above, but with the modification of adding a constant effective matter potential $\hat{V}$. The Wolfenstein evolution equation (1.24) for vacuum in the flavour basis is

$$i\frac{d}{dx}\begin{bmatrix}\nu_e\\ \nu_\mu\end{bmatrix} = \begin{bmatrix} -\frac{\Delta m^2}{4E}\cos 2\theta & \frac{\Delta m^2}{4E}\sin 2\theta \\ \frac{\Delta m^2}{4E}\sin 2\theta & \frac{\Delta m^2}{4E}\cos 2\theta \end{bmatrix} \begin{bmatrix}\nu_e\\ \nu_\mu\end{bmatrix} . \tag{1.27}$$

We derived it from $H_{mass} = diag(E_1, E_2)$ and $E_i \approx p + \frac{m_i^2}{2E}$ for relativistic neutrinos. In the presence of matter we add to the Hamiltonian the matter potential term $\hat{V}$, which is given in eq. (1.20), so that the Wolfenstein evolution equation (1.27) takes the form

$$i\frac{d}{dx}\begin{bmatrix}\nu_e\\ \nu_\mu\end{bmatrix} = \begin{bmatrix} -\frac{\Delta m^2}{4E}\cos 2\theta + V & \frac{\Delta m^2}{4E}\sin 2\theta \\ \frac{\Delta m^2}{4E}\sin 2\theta & \frac{\Delta m^2}{4E}\cos 2\theta \end{bmatrix} \begin{bmatrix}\nu_e\\ \nu_\mu\end{bmatrix} . \tag{1.28}$$

The next step for finding the oscillation probability is to diagonalize the total Hamiltonian in eq. (1.28) and to determine its eigenvalues. We define the mixing matrix in matter $\tilde{U}$ in analogy to the vacuum leptonic mixing matrix:

$$\tilde{U} = \begin{bmatrix} \cos\tilde{\theta} & \sin\tilde{\theta} \\ -\sin\tilde{\theta} & \cos\tilde{\theta} \end{bmatrix} , \tag{1.29}$$

where $\tilde{\theta}$ is the mixing angle in matter. Under the condition that $\tilde{U}$ diagonalizes the total Hamiltonian

$$\tilde{U}^\dagger H_{fl} \tilde{U} = H_{matt} = diag(E_A, E_B) \,, \tag{1.30}$$

we find the equation giving the mixing angle in matter:

$$\tan 2\tilde{\theta} = \frac{\frac{\Delta m^2}{2E}\sin 2\theta}{\frac{\Delta m^2}{2E}\cos 2\theta - V} \,. \tag{1.31}$$

It is obvious that the mixing angle in matter $\tilde{\theta}$ reduces to the vacuum mixing angle θ if the potential V goes to zero.
The difference between the eigenvalues of the effective Hamiltonian in matter given in eq. (1.28) can be easily found:

$$E_A - E_B = \sqrt{\left(\frac{\Delta m^2}{2E}\cos 2\theta - V\right)^2 + \left(\frac{\Delta m^2}{2E}\right)^2 \sin^2 2\theta}\,. \tag{1.32}$$

We find the oscillation probability in constant-density matter from eq. (1.26). The result is

$$P(\nu_e \to \nu_\mu, L) = \sin^2 2\tilde{\theta} \sin^2\left(\pi \frac{L}{l_{matt}}\right)\,, \tag{1.33}$$

where

$$l_{matt} = \frac{2\pi}{E_A - E_B}\,. \tag{1.34}$$

This is of the same form as the oscillation probability in vacuum (see eq. (1.10)), except that the oscillation length l_{osc} is replaced by the oscillation length in matter l_{mat} and the vacuum mixing angle θ by the mixing angle in matter $\tilde{\theta}$. The amplitude of the oscillation probability in eq. (1.33), which is given by

$$\sin^2 2\tilde{\theta} = \frac{\left(\frac{\Delta m^2}{2E}\right)^2 \sin^2 2\theta}{\left(\frac{\Delta m^2}{2E}\cos 2\theta - V\right)^2 + \left(\frac{\Delta m^2}{2E}\right)^2 \sin^2 2\theta}\,, \tag{1.35}$$

depends on the matter potential V. It has a typical resonance form and reaches its maximum, namely $\sin^2 2\theta = 1$, under the condition

$$V_{res} = \frac{\Delta m^2}{2E}\cos 2\theta\,. \tag{1.36}$$

This condition corresponds to the maximal mixing in matter (mixing angle $\tilde{\theta} = 45°$), so the oscillation probability can become large even if the vacuum mixing angle θ is small. This so-called MSW resonance condition is one of the crucial ingredients of the MSW effect. Thus we see that the oscillation probability can be enhanced significantly by matter effects, but it can also be suppressed if the matter potential is much larger than the resonance one given in eq. (1.36), i.e. if $V \gg V_{res}$. In the case $V \ll V_{res}$ the oscillation probability coincides with the oscillation probability in vacuum.

1.5 Neutrino oscillations in matter with varying density

In non-uniform matter the effective matter potential V becomes coordinate dependent and so does the mixing angle in matter $\tilde{\theta}$. Since the matrix $\tilde{U}$ depends on the position in space it is not possible to find a basis in which the total Hamiltonian in eq. (1.28) is diagonal for arbitrary spatial coordinate. Therefore eq. (1.26) cannot be used. One has to start with eq. (1.28), which unfortunately can not be solved analytically in general. An important example, however, which can at least be treated approximately, is the adiabatic regime. In this case one deals with slowly varying matter density. The derivation

of the oscillation probability in this regime proceeds as follows: We define an instantaneous basis, which diagonalize the total Hamiltonian in flavour basis H_{fl} for a definite coordinate x:

$$\tilde{U}^{\dagger}(x)H_{fl}\tilde{U}(x) = \tilde{H}_{matt}(x) = diag(E_A(x), E_B(x))\,. \tag{1.37}$$

Here $E_A(x)$ and $E_B(x)$ are the instantaneous matter eigenvalues. The corresponding instantaneous matter eigenstates ν_{matt} are then given through

$$\nu_{matt}(x) = \tilde{U}^{\dagger}(x)\nu_{fl}\,, \qquad \nu_{matt} = \begin{bmatrix}\nu_A\\ \nu_B\end{bmatrix}\,, \qquad \nu_{fl} = \begin{bmatrix}\nu_e\\ \nu_\mu\end{bmatrix}\,. \tag{1.38}$$

In the instantaneous matter basis the Wolfenstein evolution equation (1.23) takes the form

$$i\frac{d}{dx}\nu_{matt} = [\tilde{H}_{matt}(x) - i\tilde{U}^{\dagger}(x)\frac{d}{dx}\tilde{U}(x)]\nu_{matt}\,, \tag{1.39}$$

which can be written in components as

$$i\frac{d}{dx}\begin{bmatrix}\nu_A\\ \nu_B\end{bmatrix} = \begin{bmatrix}E_A(x) & -i\tilde{\theta}'(x)\\ i\tilde{\theta}'(x) & E_B(x)\end{bmatrix}\begin{bmatrix}\nu_A\\ \nu_B\end{bmatrix}\,, \tag{1.40}$$

where $\tilde{\theta}'(x) := \frac{d}{dx}\tilde{\theta}(x)$ denotes the derivative of the mixing angle in matter.
The adiabatic approximation means that the matter density changes so slowly that the derivative of the mixing angle $\tilde{\theta}'(x)$ is small compared to the difference of the energy eigenvalues. If $\frac{|\tilde{\theta}'(x)|}{|E_A(x)-E_B(x)|} \ll 1$ is fulfilled, the off-diagonal terms in eq. (1.40) can be neglected. The solution can then be easily found:

$$\begin{bmatrix}\nu_A(x)\\ \nu_B(x)\end{bmatrix} = \begin{bmatrix}\exp\left(-i\int\limits_{x_i}^{x} E_A(x')dx'\right)\cdot\nu_A(x_i)\\ \exp\left(-i\int\limits_{x_i}^{x} E_B(x')dx'\right)\cdot\nu_B(x_i)\end{bmatrix}\,, \tag{1.41}$$

where x_i denotes the initial position.
Imagine that at some point with the coordinate $x = x_i$ an electron neutrino was created. It is described through

$$\nu_e = \nu(x_i) = \cos\tilde{\theta}(x_i)\nu_A(x_i) + \sin\tilde{\theta}(x_i)\nu_B(x_i). \tag{1.42}$$

After propagating to a certain point x this state evolves into

$$\nu(x) = \cos\tilde{\theta}(x_i)e^{-i\int\limits_{x_i}^{x} E_A(x')dx'}\nu_A(x_i) + \sin\tilde{\theta}(x_i)e^{-i\int\limits_{x_i}^{x} E_B(x')dx'}\nu_B(x_i). \tag{1.43}$$

To find the transition amplitude for the process in which the electron neutrino changes its flavour and becomes a muon neutrino after propagating through matter with slowly varying density, one has again to project the evolved neutrino state on the muon neutrino one. For the oscillation probability we then obtain

$$P(\nu_e \to \nu_\mu, x) = |\langle \nu_\mu | \nu(x) \rangle|^2 = \frac{1}{2} - \frac{1}{2} \cos 2\tilde{\theta}(x_i) \cos 2\tilde{\theta}(x) - \frac{1}{2} \sin 2\tilde{\theta}(x_i) \sin 2\tilde{\theta}(x) \Phi(x) \,, \tag{1.44}$$

where $\Phi(x) := \exp\left(-i \int_{x_i}^{x} (E_A(x') - E_B(x')) dx' \right)$.

Let us analyze this result now. Imagine the electron neutrino is created at a position $x = x_i$ in a matter whose density is so high that the effective matter potential is far above the one given by the MSW resonance condition ($V \gg V_{res}$) and propagate to very low density regions ($V \ll V_{res}$). Then the mixing angle in matter $\tilde{\theta}$ at the neutrino creation point is approximately $\tilde{\theta}(x_i) \approx \frac{\pi}{2}$, as we can see from eq. (1.35). This means that the third term in eq. (1.44) is strongly suppressed and the flavour transition probability becomes

$$P(\nu_e \to \nu_\mu, x) \approx \frac{1}{2} - \frac{1}{2} \cos 2\tilde{\theta}(x) = \cos^2 \tilde{\theta}(x) \,. \tag{1.45}$$

We realize that this expression exhibits no oscillatory behavior because the mixing angle in matter $\tilde{\theta}(x)$ changes its value smoothly from nearly $\frac{\pi}{2}$ to some value which is smaller than $\frac{\pi}{2}$. If the matter potential increases slowly until it becomes negligible at some final position x, $\tilde{\theta}(x)$ at the final point of neutrino evolution can be replaced by the vacuum mixing angle θ. The flavour transition probability in this case is approximately

$$P(\nu_e \to \nu_\mu) \approx \cos^2 \theta \,. \tag{1.46}$$

This means that in the present case we have no oscillations in the true sense of the word but rather a non-oscillatory flavour transition. The reason for this is that the system remains all the time approximately in the same matter eigenstate during its evolution, but the flavour composition of this matter eigenstate changes. Indeed, consider the flavour eigenstates as superpositions of the matter eigenstates:

$$\begin{aligned} |\nu_e\rangle &= \cos\tilde{\theta} |\nu_A\rangle + \sin\tilde{\theta} |\nu_B\rangle \,, \\ |\nu_\mu\rangle &= -\sin\tilde{\theta} |\nu_A\rangle + \cos\tilde{\theta} |\nu_B\rangle \,. \end{aligned} \tag{1.47}$$

Due to the initial condition $\tilde{\theta}(x_i) \approx \frac{\pi}{2}$ the electron flavour state nearly coincides with the matter eigenstate ν_B while the fraction of the matter eigenstate ν_A in it is suppressed. The adiabatic approximation means that the state ν_B remains itself in the course of the propagation and does not go to ν_A. Propagating through matter with slowly varying density, the neutrino reaches at some point the density which satisfies the MSW resonance condition, where the flavour transition probability becomes maximal. At the final point the matter density is much smaller than the one required by the MSW resonance

condition and the mixing angle nearly coincides with the vacuum one. At this point the matter eigenstate ν_B has the component of ν_μ with the weight $\cos^2\theta$ (see eq. (1.47)). Therefore for small enough vacuum mixing angle ($\cos^2\theta \approx 1$) one would have an almost complete $\nu_e \leftrightarrow \nu_\mu$ transition. It is quite important to notice that the adiabatic case is not the only example where the flavour transition proceeds in a non-oscillatory way. In vacuum one can have a non-oscillatory behavior of the flavour transition probability if the coherent production or detection conditions are violated. Further on we will discuss in more detail what this actually means.

The usual approach, which we presented in this chapter, describes the phenomenon of neutrino oscillations in a simple way. However, this simplified heuristic consideration is lacking a proper justification, which can be obtained in the framework of Quantum Field Theory, and also has a number of shortcomings. First of all, the standard approach does not involve the neutrino production and detection processes. Secondly, it ignores possible decoherence effects. These effects can be properly taken into account in a Quantum Mechanical wave packet approach. However, in that framework the oscillation probabilities are not automatically properly normalized and the correct normalization must be introduced "by hand". Both these problems can be resolved in a Quantum Field Theoretic approach of neutrino oscillations that we develop in the present work. There have been other works on this topic based on Quantum Field Theory. An example is Ref. [10], where the neutrino oscillations are described through the Dirac equation and the solutions are assumed to be plane waves. The production and detection processes are, however, still omitted in that work and the coherence conditions are not discussed. A very detailed and thorough treatment of neutrino oscillations based on a QFT approach can be found in Ref. [11], which includes the proper normalization and to some degree the effects of production and detection processes. Regrettably, the derivation proceeds in a very complicated way and does not take into account possible decoherence effects at neutrino production and detection.

In this work we present a simple description of neutrino oscillations in non-uniform matter, which deals with the aforementioned problems. Ref. [12] already presented the case of neutrino oscillations in vacuum with a proper Quantum Field Theoretic treatment. In this work we will extend this approach to the case of neutrinos propagating in non-uniform matter by introducing an effective coordinate-dependent matter potential $V(\boldsymbol{x})$. To derive the probability of neutrino oscillations in non-uniform matter within a Quantum Field Theoretic treatment we first calculate the amplitude for the overall process of neutrino production, propagation and detection. In the following chapter we extract the probability of neutrino oscillations from the total probability of the overall process. After that we compare the obtained result with the standard one based on the Wolfenstein evolution equation (1.23). And lastly we consider the two-flavour case in the adiabatic regime in order to demonstrate how our assumptions of the previous chapters work.

2 Neutrino propagator in non-uniform matter

Neutrino field in matter satisfies the Dirac equation

$$[\gamma^\mu(i\partial_\mu - V_\mu P_L) - M]\psi = 0\,, \tag{2.1}$$

where ψ is the four-component neutrino spinor field, the mass term M is Lorentz scalar but a matrix in the flavour space, γ^μ are gamma matrices and V_μ is the four-vector which represents the matter potential, which is diagonal in the flavour space. The zero component of V_μ is the matrix discussed in the introduction (see eq. (1.20)). Since only left-handed neutrinos interact with matter we need to multiply V_μ by a projection operator P_L. In the following consideration we will not work with the neutrino field but rather with the neutrino propagator $S(x_1, x_2)$, which satisfies the equation

$$(i\gamma^\mu\partial_\mu - \gamma^\mu V_\mu(\boldsymbol{x}_1)P_L - M)S(x_1, x_2) = \delta^4(x_1 - x_2)\,. \tag{2.2}$$

In this work we will use the chiral (also called Weyl) representation of γ-matrices and spinors. In this basis the γ-matrices take the form

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \tag{2.3}$$

where $\mathbb{1}$ is the 2×2 identity matrix and σ^i $(i = 1, 2, 3)$ are the Pauli matrices.
The field ψ can be decomposed as

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \tag{2.4}$$

where ψ_L and ψ_R are two-component objects called left-handed and right-handed Weyl spinors.
The projection operators $P_{L,R} = \frac{1\mp\gamma_5}{2}$ that in chiral representation are

$$P_L = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}, \quad P_R = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \tag{2.5}$$

when applied to the field ψ give its left- or right-handed component:

$$P_L\,\psi = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}, \quad P_R\,\psi = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}. \tag{2.6}$$

In analogy to the field ψ we decompose the neutrino propagator as follows:

$$S = \begin{pmatrix} S_{LL} & S_{LR} \\ S_{RL} & S_{RR} \end{pmatrix} . \tag{2.7}$$

Here S_{LL}, S_{LR}, S_{RL}, S_{LL} are 2×2 block matrices in the spinor space.
Using the above definition (and $i\gamma^\mu\partial_\mu = i\partial_t + i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}$) we can write the Dirac equation (2.2) for the propagator as

$$\begin{pmatrix} -M & i(\partial_t + \boldsymbol{\sigma}\cdot\boldsymbol{\nabla}) \\ i(\partial_t - \boldsymbol{\sigma}\cdot\boldsymbol{\nabla}) - V^0 - \boldsymbol{V}\cdot\boldsymbol{\sigma} & -M \end{pmatrix} \begin{pmatrix} S_{LL} & S_{LR} \\ S_{RL} & S_{RR} \end{pmatrix} = \delta^4(x_1 - x_2) \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} . \tag{2.8}$$

The chiral structure simplifies our later calculation because we will need only the S_{LR} block matrix of the neutrino propagator. From the definition of the propagator

$$S(x_1, x_2) = -i\langle 0|T\psi(x_1)\bar{\psi}(x_2)|0\rangle , \tag{2.9}$$

one can see that for electro-weak interactions, which couple only to the left-handed components of neutrinos, the propagator takes the form

$$S(x_1, x_2) = -i\langle 0|T(P_L\psi(x_1)\bar{\psi}(x_2)P_R)|0\rangle = S_{LR} . \tag{2.10}$$

That is why our goal is to derive an equation which determines the left-right-handed part S_{LR} of the neutrino propagator. All information about the neutrino interactions with matter are included in this propagator S_{LR}, because it implicitly depends on the effective matter potential $V(\boldsymbol{x}_1)$. We will be assuming throughout this thesis that the matter potential $V(\boldsymbol{x}_1)$ depends on the coordinate along the neutrino trajectory but is time-independent. This means that the propagator is invariant with respect to time translations and can be written as

$$S_{LR}(x_1, x_2) = S_{LR}(t_1, \boldsymbol{x}_1, t_2, \boldsymbol{x}_2) = S_{LR}(t_1 - t_2, \boldsymbol{x}_1, \boldsymbol{x}_2) . \tag{2.11}$$

That is, S_{LR} depends on $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ and the difference between t_1 and t_2. It is convenient to carry out the Fourier transformation

$$S_{LR}(x_1, x_2) = S_{LR}(t_1 - t_2, \boldsymbol{x}_1, \boldsymbol{x}_2) = \int \frac{dE}{2\pi} S_{LR}(E, \boldsymbol{x}_1, \boldsymbol{x}_2) e^{-iE(t_1 - t_2)} , \tag{2.12}$$

where $S_{LR}(E, \boldsymbol{x}_1, \boldsymbol{x}_2)$ is the propagator in the mixed energy-coordinate representation. It is related to the propagator in the energy-momentum space $\tilde{S}_{LR}(E, \boldsymbol{p}_1, \boldsymbol{p}_2)$ (denoted by tilde) by the Fourier transformation with respect to two momenta:

$$S_{LR}(E, \boldsymbol{x}_1, \boldsymbol{x}_2) = \int \frac{d^3p_1 d^3p_2}{(2\pi)^6} \tilde{S}_{LR}(E, \boldsymbol{p}_1, \boldsymbol{p}_2) e^{i\boldsymbol{p}_1\boldsymbol{x}_1} e^{-i\boldsymbol{p}_2\boldsymbol{x}_2} . \tag{2.13}$$

In the rest of this chapter we will briefly review the calculation of the neutrino propagator in matter given in Ref. [11], which we use in this work. More details can be found in Ref. [11]. From the Dirac equation in the chiral form (2.8) we obtain a system of two coupled equations for the S_{LR} and S_{RR} block matrices:

$$-MS_{LR}(x_1,x_2)+i(\partial_t+\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})S_{RR}(x_1,x_2)=0\,, \tag{2.14}$$

$$[i(\partial_t-\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})-V^0-\boldsymbol{V}\cdot\boldsymbol{\sigma}]S_{LR}(x_1,x_2)-MS_{RR}(x_1,x_2)=\delta^4(x_1-x_2)\,. \tag{2.15}$$

We follow the convention of Ref. [11] and define $J(x_1,x_2):=\frac{1}{M}S_{RR}(x_1,x_2)$. Performing the Fourier transformation with respect to time difference according to eq. (2.12) and using the above convention, eq. (2.14) becomes in the mixed energy-coordinate representation

$$S_{LR}(E,\boldsymbol{x}_1,\boldsymbol{x}_2)=(E+i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})J(E,\boldsymbol{x}_1,\boldsymbol{x}_2)\,. \tag{2.16}$$

Plugging this expression into eq. (2.15) in the mixed energy-coordinate representation we obtain an equation for $J(E,\boldsymbol{x}_1,\boldsymbol{x}_2)$

$$\begin{aligned}[E^2+\boldsymbol{\nabla}^2-M^2-EV^0(\boldsymbol{x}_1)-i\boldsymbol{V}(\boldsymbol{x}_1)\cdot\boldsymbol{\nabla}-i\boldsymbol{\sigma}\cdot(V^0(\boldsymbol{x}_1)\boldsymbol{\nabla}-iE\boldsymbol{V}(\boldsymbol{x}_1)+i\boldsymbol{V}(\boldsymbol{x}_1)\times\boldsymbol{\nabla})]\\ \times J(E,\boldsymbol{x}_1,\boldsymbol{x}_2)=\delta^3(\boldsymbol{x}_1-\boldsymbol{x}_2)\,.\end{aligned} \tag{2.17}$$

In the energy-momentum space this equation becomes

$$[E^2+|\boldsymbol{p}_1|^2-M^2-EV^0+\boldsymbol{p}_1\cdot\boldsymbol{V}+\boldsymbol{\sigma}(V^0\boldsymbol{p}_1-E\boldsymbol{V}+i\boldsymbol{V}\times\boldsymbol{p}_1)]\tilde{J}(E,\boldsymbol{p}_1,\boldsymbol{p}_2)=\mathbb{1}\,, \tag{2.18}$$

or

$$D(E,|\boldsymbol{p}_1|,\hat{\boldsymbol{p}}_1=\hat{\boldsymbol{r}})\tilde{J}(E,\boldsymbol{p}_1,\boldsymbol{p}_2)=\mathbb{1}\,, \tag{2.19}$$

where $\hat{\boldsymbol{p}}_1:=\frac{\boldsymbol{p}_1}{|\boldsymbol{p}_1|}$ is the unit vector of $\boldsymbol{p}_1$ and $\hat{\boldsymbol{r}}:=(\boldsymbol{x}_1-\boldsymbol{x}_2)/|\boldsymbol{x}_1-\boldsymbol{x}_2|$. Without loss of generality we choose $\hat{\boldsymbol{r}}=r\cdot\boldsymbol{e}_3$. The elements of J can be expressed explicitly through the components of D by solving eq. (2.19):

$$\tilde{J}_{11}=[D_{11}-D_{12}(D_{22})^{-1}D_{21}]^{-1}\,, \tag{2.20}$$

$$\tilde{J}_{22}=[D_{22}-D_{21}(D_{11})^{-1}D_{12}]^{-1}\,, \tag{2.21}$$

$$\tilde{J}_{12}=-(D_{11})^{-1}(D_{12})J_{22}\,, \tag{2.22}$$

$$\tilde{J}_{21}=-(D_{22})^{-1}(D_{21})J_{11}\,. \tag{2.23}$$

Let us now turn to describing a neutrino experiment, where neutrinos are produced in some region with the center at $\boldsymbol{x}_P$. They are detected in some region with the center at

$\boldsymbol{x}_D$. Let us denote the distance between this two points $L = |\boldsymbol{x}_D - \boldsymbol{x}_P|$; $\hat{\boldsymbol{L}} := \frac{\boldsymbol{x}_D - \boldsymbol{x}_P}{|\boldsymbol{x}_D - \boldsymbol{x}_P|}$ is the unit vector which points from the source to the detector. When we choose $\hat{\boldsymbol{r}}$ to lie along $\pm\hat{\boldsymbol{L}}$ the matrix D takes the explicit form

$$D\left(E, |\boldsymbol{p}_1|, \hat{\boldsymbol{p}}_1 = \frac{E}{|E|}\hat{\boldsymbol{L}}\right) = \begin{bmatrix} d - E\left(1 - \frac{|\boldsymbol{p}_1|}{|E|}\right)(V^0 + V^3) & -E\left(1 + \frac{|\boldsymbol{p}_1|}{|E|}\right)(V^1 - iV^2) \\ -E\left(1 - \frac{|\boldsymbol{p}_1|}{|E|}\right)(V^1 + iV^2) & d - E\left(1 + \frac{|\boldsymbol{p}_1|}{|E|}\right)(V^0 - V^3) \end{bmatrix}, \tag{2.24}$$

where $d := E^2 - |\boldsymbol{p}_1|^2 - M^2$. It is obvious that in the relativistic limit, when $|\boldsymbol{p}_1| \to |E|$, the matrix D reduces to

$$D\left(E, \hat{\boldsymbol{p}}_1 = \frac{E}{|E|}\hat{\boldsymbol{L}}\right) \approx \begin{bmatrix} -M^2 & -2E(V^1 - iV^2) \\ 0 & -M^2 - 2E(V^0 - V^3) \end{bmatrix}. \tag{2.25}$$

At this point is important to note that for relativistic neutrinos the component D_{21} goes to zero. From this fact and eq. (2.23) it follows that $\tilde{J}_{21} \approx 0$ and therefore $J_{21} \approx 0$ in the relativistic approximation. Note that tilde denotes the propagator in the energy-momentum space.
Now let us return to eq. (2.17). One can seek its solution in the form

$$J(E, \boldsymbol{x}_1, \boldsymbol{x}_2) = -\frac{e^{i|E||\boldsymbol{x}_1 - \boldsymbol{x}_2|}}{4\pi|\boldsymbol{x}_1 - \boldsymbol{x}_2|} F(E, \boldsymbol{x}_1, \boldsymbol{x}_2)\,. \tag{2.26}$$

With this ansatz we find

$$\begin{aligned}(\boldsymbol{\nabla}^2 + E^2)J =& \delta^3(\boldsymbol{x}_1 - \boldsymbol{x}_2)Fe^{i|E||\boldsymbol{x}_1 - \boldsymbol{x}_2|} - \frac{2|E|e^{i|E||\boldsymbol{x}_1 - \boldsymbol{x}_2|}}{4\pi|\boldsymbol{x}_1 - \boldsymbol{x}_2|}\left[\frac{1}{2|E|}\boldsymbol{\nabla}^2 F\right. \\ &\left. + i(\hat{\boldsymbol{r}}\cdot\boldsymbol{\nabla}F) - \frac{1}{|E||\boldsymbol{x}_1 - \boldsymbol{x}_2|}(\hat{\boldsymbol{r}}\cdot\boldsymbol{\nabla}F)\right],\end{aligned} \tag{2.27}$$

$$\boldsymbol{\nabla}J = -\frac{2|E|e^{i|E||\boldsymbol{x}_1 - \boldsymbol{x}_2|}}{4\pi|\boldsymbol{x}_1 - \boldsymbol{x}_2|}\left[\frac{i\hat{\boldsymbol{r}}}{2}F + \frac{1}{2|E|}\boldsymbol{\nabla}F - \frac{\hat{\boldsymbol{r}}}{2|E||\boldsymbol{x}_1 - \boldsymbol{x}_2|}F\right]. \tag{2.28}$$

Plugging eqs. (2.27) and (2.28) into eq. (2.17) we obtain an equation which contains delta functions on the right-hand and left-hand sides. They cancel each other when we set the boundary condition:

$$F^{\alpha\beta}(E, \boldsymbol{x}_1, \boldsymbol{x}_2)\Big|_{\boldsymbol{x}_1 \to \boldsymbol{x}_2} = \delta^{\alpha\beta}\,, \tag{2.29}$$

where we have restored the flavour indices α and β.
We are primarily interested in well separated $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, since for neutrino oscillation experiments one deals with macroscopic distances that are much larger than the neutrino De Broglie wavelengths. This means $|E||\boldsymbol{x}_1 - \boldsymbol{x}_2| \gg 1$, and the last terms in eqs. (2.27) and (2.28) can be neglected. Plugging eqs. (2.27) and (2.28) into eq. (2.17) and using the boundary condition (2.29), we find

$$i(\hat{\boldsymbol{r}} \cdot \boldsymbol{\nabla} F) + \frac{1}{2|E|}\boldsymbol{\nabla}^2 F - \frac{1}{2|E|}[M^2 + EV^0 - |E|(\hat{\boldsymbol{r}} \cdot \boldsymbol{V}) \\ - \boldsymbol{\sigma} \cdot (V^0|E|\hat{\boldsymbol{r}} - E\boldsymbol{V} + i|E|\boldsymbol{V} \times \hat{\boldsymbol{r}})]F + \mathcal{O}\left(\frac{V|E|}{|E|^2}|\boldsymbol{\nabla} F|\right) = 0\,, \tag{2.30}$$

where we can ignore the last term because of the condition (1.22). By explicit calculation of the term in the square brackets in eq. (2.30) one realizes that this term is the matrix D in the relativistic limit (see eq. (2.25)) in the mixed energy-coordinate representation. Therefore we can rewrite equation (2.30) as

$$i(\hat{\boldsymbol{r}} \cdot \boldsymbol{\nabla} F) + \frac{1}{2|E|}\boldsymbol{\nabla}^2 F - \frac{1}{2|E|}D(E, \boldsymbol{x}_1)F = 0\,, \tag{2.31}$$

where $\boldsymbol{x}_1$ is chosen in the third spatial direction.
Now let us return to eq. (2.16). In the energy-momentum space this equation is

$$\tilde{S}_{LR}(E, \boldsymbol{p}_1, \boldsymbol{p}_2) = (E - \boldsymbol{\sigma} \cdot \boldsymbol{p}_1)\tilde{J}(E, \boldsymbol{p}_1, \boldsymbol{p}_2)\,. \tag{2.32}$$

Making use of the relativistic condition $|\boldsymbol{p}_1| \approx E$ once again, we can rewrite it as

$$\tilde{S}_{LR}(E, \boldsymbol{p}_1, \boldsymbol{p}_2) \approx E(1 - \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}_1}{|\boldsymbol{p}_1|})\tilde{J}(E, \boldsymbol{p}_1, \boldsymbol{p}_2)\,. \tag{2.33}$$

And since we chose $\hat{\boldsymbol{p}}_1 = \hat{\boldsymbol{r}}$ expression (2.33) takes in the mixed energy-coordinate representation the form

$$S_{LR}(E, \boldsymbol{x}_1, \boldsymbol{x}_2) \approx E(1 - \boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}})J(E, \boldsymbol{x}_1, \boldsymbol{x}_2)\,. \tag{2.34}$$

With $\hat{\boldsymbol{r}}$ along the z-axis, we recognize that in the relativistic limit only one component of S_{LR}, namely $(S_{LR})_{22}$, is nonzero

$$S_{LR} \approx 2E \begin{pmatrix} 0 & 0 \\ 0 & J_{22} \end{pmatrix}\,, \tag{2.35}$$

because $J_{21} = 0$. This result plays a crucial role in our later calculations, since we need to deal only with one component of the neutrino propagator and this simplifies our considerations significantly. We can specify eq. (2.30) for the F_{22} component omitting

the spinor indices for simplicity and taking into account that $|V| \ll |E|$:

$$i(\hat{\boldsymbol{r}} \cdot \boldsymbol{\nabla} F) + \frac{1}{2|E|}\boldsymbol{\nabla}^2 F - \frac{1}{2|E|}[M^2 + 2E(V^0 - V^3)]F = 0\,. \tag{2.36}$$

We can already recognize the term in the square brackets as the effective Hamiltonian of the Wolfenstein evolution equation. To simplify this equation even more, let us distinguish between three cases:

$$\begin{aligned} &(1)\, |\boldsymbol{\nabla} F| \gg \epsilon F \\ &(2)\, |\boldsymbol{\nabla} F| \ll \epsilon F \\ &(3)\, |\boldsymbol{\nabla} F| \sim \epsilon F\,, \end{aligned}$$

where $\epsilon := \frac{1}{2|E|}[M^2 + 2E(V^0 - V^3)]$.
In the first case the gradient of the function F is very large, which means that third term in eq. (2.36) is negligible compared to the first and the second ones. The matter potential and the mass matrix hardly contribute and because of that there are no oscillations at all. Thus we are not interested in this case. In the second case the gradient of F is much smaller then ϵF. This means that the first term and also the second term are very small compared to the third one. Therefore it is impossible for the three terms to cancel each other and there is no way to fulfill this equation. Thus the physically interesting case is the third one, $|\boldsymbol{\nabla} F| \sim \epsilon F$, where $\frac{|\boldsymbol{\nabla}^2 F|}{2|E|}$ can be neglected, compared to $|\boldsymbol{\nabla} F|$, provided that $\frac{|\boldsymbol{\nabla} V^0|}{\epsilon^2} \lesssim 1$. Defining $F' := \frac{d}{dx}F := \hat{\boldsymbol{r}} \cdot \boldsymbol{\nabla} F$ we finally obtain

$$iF' = \left[\frac{M^2}{2|E|} \pm (V^0 - V^3)\right] F\,. \tag{2.37}$$

The plus stands for the case of neutrinos since their energy E is positive and minus appears in the case of antineutrinos because of $E < 0$. This result shows that the propagator function F obeys the Schrödinger-like equation (2.37). In this equation it is differentiated with respect to x, where x is the coordinate along the neutrino propagation path. It is an important result, which we will need in our following calculation. The second equation we will need is

$$S_{LR}(E, \boldsymbol{x}_1, \boldsymbol{x}_2) = -2E\frac{e^{i|E||\boldsymbol{x}_1 - \boldsymbol{x}_2|}}{4\pi|\boldsymbol{x}_1 - \boldsymbol{x}_2|}F(E, \boldsymbol{x}_1, \boldsymbol{x}_2)\,, \tag{2.38}$$

where only $(S_{LR})_{22}$ is nonzero. To get expression (2.38) for S_{LR} compare eqs. (2.35) and (2.26).

3 Neutrino oscillations in matter

Our goal is to derive the probability of neutrino oscillations in non-uniform matter. The first step is to calculate the amplitude for the process that consists of neutrino production, propagation and detection. The next step will be to extract from this quantity the probability of neutrino oscillations. For now consider some thought experiment where neutrinos are produced in the decay of an unspecified particle. Neutrinos are produced in some region of space-time centered around x_P. Analogously, we define x_D as the center of detection region, where neutrinos are detected through some mechanism (e.g. inverse beta decay), that we also do not specify in our general discussion.
In Quantum Theory we describe the state of a particle of type A as the wave packet

$$|A\rangle = \int [dp]\, f_A(\boldsymbol{p}, \boldsymbol{P})\, |A, \boldsymbol{p}\rangle\,, \tag{3.1}$$

where $|A, \boldsymbol{p}\rangle$ is the corresponding momentum eigenstate and $f_A(\boldsymbol{p}, \boldsymbol{P})$ is the momentum distribution function with mean momentum $\boldsymbol{P}$. In this work we use the shorthand notation

$$[dp] := \frac{d^3p}{(2\pi)^3\sqrt{2E_A(\boldsymbol{p})}}\,. \tag{3.2}$$

For a free particle on its mass shell the energy of the particle is $E_A(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2 + m_A^2}$.
Consider now the Feynman diagram of Figure (3.1), which shows neutrino production, propagation and detection. We can define the states of external particles in the form

$$|P_i\rangle = \int [dq]\, f_{Pi}(\boldsymbol{q}, \boldsymbol{Q})\, |P_i, \boldsymbol{q}\rangle\,, \qquad |P_f\rangle = \int [dk]\, f_{Pf}(\boldsymbol{k}, \boldsymbol{K})\, |P_f, \boldsymbol{k}\rangle\,, \tag{3.3}$$

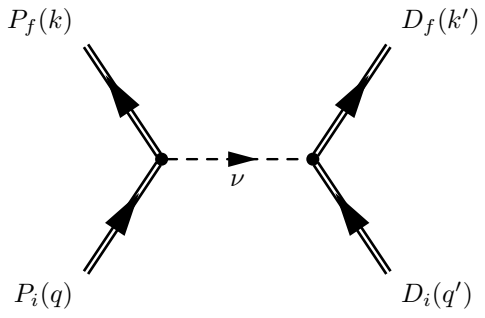

Figure 3.1: Neutrino production, propagation and detection.

where $|P_i\rangle$ describes the particle in the initial state of the production process and $|P_f\rangle$ describes the particle in the final state of the production. The functions f_{Pi} and f_{Pf} denote the corresponding momentum distribution functions. The states of particles involved in the detection process are defined in the same way:

$$|D_i\rangle = \int [dq']\, f_{Di}(\boldsymbol{q'}, \boldsymbol{Q'})\, |D_i, \boldsymbol{q'}\rangle\,, \qquad |D_f\rangle = \int [dk']\, f_{Df}(\boldsymbol{k'}, \boldsymbol{K'})\, |D_f, \boldsymbol{k'}\rangle\,. \tag{3.4}$$

We can find the amplitude $A_{\alpha\beta}$ for the overall neutrino production-propagation-detection process employing the Feynman rules:

$$\begin{aligned} i\mathcal{A}_{\alpha\beta} = &\sum_{j,k} U^*_{\alpha j} U_{\beta k} \int [dq] f_{Pi}(\boldsymbol{q}, \boldsymbol{Q}) \int [dk] f^*_{Pf}(\boldsymbol{k}, \boldsymbol{K}) \int [dq'] f_{Di}(\boldsymbol{q'}, \boldsymbol{Q'}) \\ &\times \int [dk'] f^*_{Df}(\boldsymbol{k'}, \boldsymbol{K'}) \int d^4x_1 \int d^4x_2 \tilde{M}_{kD}(q', k') e^{-i(q'-k')(x_2-x_D)} (S_{LR}(x_2, x_1))_{kj} \\ &\times \tilde{M}_{jP}(q, k) e^{-i(q-k)(x_1-x_P)}\,. \end{aligned} \tag{3.5}$$

Here $U_{\alpha j}$ are the elements of the leptonic mixing matrix, defined in eq. (1.2), $\tilde{M}_{jP}(q,k)$ and $\tilde{M}_{jD}(q',k')$ are the plane-wave amplitudes of the production and detection processes with the neutrino spinors $\bar{u}_j(p)$ and $u_j(p)$ excluded. The full plane-wave amplitudes with the neutrino spinors included will be

$$M_{jP}(q,k) := \frac{\bar{u}_{jL}(p)}{\sqrt{2E_j}} \tilde{M}_{jP}(q,k) \qquad \text{and} \qquad M_{jD}(q',k') := \tilde{M}_{jD}(q',k') \frac{u_{jL}(p)}{\sqrt{2E_j}}. \tag{3.6}$$

The neutrino propagator S_{LR} is in general a matrix in the flavour space and is written in eq. (3.5) in the mass eigenstate basis. In eq. (3.5) we only need the 2×2 matrix S_{LR} due to the chiral nature of weak interactions discussed in Chapter 3.
We perform the Fourier transformation according to eqs. (2.12) and (2.13) and express the propagator in the energy-momentum space. Introducing the shifted coordinates x'_1 and x'_2 according to $x_1 = x_P + x'_1$ and $x_2 = x_D + x'_2$ we obtain for the overall amplitude $\mathcal{A}_{\alpha\beta}$:

$$i\mathcal{A}_{\alpha\beta} = \sum_{j,k} U^*_{\alpha j} U_{\beta k} \int \frac{dE}{2\pi} \int \frac{d^3p d^3p'}{(2\pi)^6} \Phi_{kD}(p') (\tilde{S}_{LR}(E, \boldsymbol{p'}, \boldsymbol{p}))_{kj} \Phi_{jP}(p) e^{-iET + i\boldsymbol{p'x}_D - i\boldsymbol{px}_P}\,. \tag{3.7}$$

Here $T = t_D - t_P$ is the propagation time and the Φ's are the so-called overlap functions defined as

$$\Phi_{jP}(p) = \int d^4x_1' e^{ipx_1'} \int [dq] \int [dk]\, f_{Pi}(\boldsymbol{q}, \boldsymbol{Q})\, f^*_{Pf}(\boldsymbol{k}, \boldsymbol{K})\, e^{-i(q-k)x_1'} \tilde{M}_{jP}(q,k)\,, \tag{3.8}$$
$$\Phi_{kD}(p') = \int d^4x_2' e^{-ip'x_2'} \int [dq'] \int [dk']\, f_{Di}(\boldsymbol{q'}, \boldsymbol{Q'})\, f^*_{Df}(\boldsymbol{k'}, \boldsymbol{K'})\, e^{-i(q'-k')x_2'} \tilde{M}_{kD}(q',k')\,.$$

Note that the overlap functions are in fact two-component spinors because so are $\tilde{M}_P$ and $\tilde{M}_P$. From the previous chapter we know that only one spinor component of S_{LR} is non-zero. This means that the term in the integral in eq. (3.7) has the form

$$\begin{aligned}
&\Phi_{kD}(p')(\tilde{S}_{LR}(E, \boldsymbol{p'}, \boldsymbol{p}))_{kj} \Phi_{jP}(p) \\
&\qquad = \begin{pmatrix} \Phi_{kD_1}(p')\,, & \Phi_{kD_2}(p') \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & (\tilde{S}(E, \boldsymbol{p'}, \boldsymbol{p}))_{kj} \end{pmatrix} \begin{pmatrix} \Phi_{jP_1}(p) \\ \Phi_{jP_2}(p) \end{pmatrix} \\
&\qquad = \Phi_{kD_2}(p')(\tilde{S}(E, \boldsymbol{p'}, \boldsymbol{p}))_{kj} \Phi_{jP_2}(p)\,.
\end{aligned} \tag{3.9}$$

We see that only the second components of the overlap functions contribute to the amplitude $\mathcal{A}_{\alpha\beta}$. In fact the second spinor component of the overlap function $\Phi_{D_2}(p')$ is the wave-packet detection amplitude. We can show this as follows. For relativistic particle the left-handed spinor takes the form

$$u_{jL} = \begin{bmatrix} 0 \\ \sqrt{2E_j} \end{bmatrix}\,. \tag{3.10}$$

Consequently, the full plane-wave detection amplitude given in eq. (3.6) becomes

$$M_{jD}(q',k') := \tilde{M}_D(q',k') \frac{u_{jL}(p)}{\sqrt{2E_j}} = \tilde{M}_{jD_2}(q',k')\,. \tag{3.11}$$

An analogous argument applies for the production process. By integrating the full plane-wave amplitude weighted with the corresponding momentum distribution functions with respect to four-coordinates and four-momenta one obtains the wave-packet amplitude for this process (see eq. (3.8)). From this it is clear that $\Phi_{D_2}(p')$ and $\Phi_{P_2}(p)$ are nothing else but the detection and production amplitudes. Hereafter we will therefore change the notation according to $\Phi_{kD_2}(p') \to \Phi_{kD}(p')$ and $\Phi_{jP_2}(p) \to \Phi_{jP}(p)$. We also omit chiral indices of the propagator and set $S := (S_{22})_{LR}$, since we are going to operate only with this component.
Thus we obtain for the amplitude of the overall process

$$i\mathcal{A}_{\alpha\beta} = \sum_{j,k} U^*_{\alpha j} U_{\beta k} \int \frac{dE}{2\pi} \int \frac{d^3p d^3p'}{(2\pi)^6} \Phi_{kD}(p') \tilde{S}_{kj}(E, \boldsymbol{p'}, \boldsymbol{p}) \Phi_{jP}(p) e^{-iET + i\boldsymbol{p'x}_D - i\boldsymbol{px}_P}\,. \tag{3.12}$$

Having realized that the overlap functions are the detection and production amplitudes containing all the information about the production and detection processes, the next step for simplifying the amplitude $\mathcal{A}_{\alpha\beta}$ is to carry out the integrals over the three-momenta in eq. (3.12). Unfortunately it is not possible to perform the integration analytically. But we can apply the following approximation to the expression. The fact that $\Phi_P(E, \boldsymbol{p})$ and $\Phi_D(E, \boldsymbol{p'})$ are production and detection amplitudes implies that both functions are characterized by the respective momentum widths. We denote the momentum widths of the production and detection amplitudes σ_{pP} and σ_{pD} respectively. Outside the intervals of the momentum widths the overlap functions are strongly suppressed, but inside these regions they change hardly significantly. Now let us study the behavior of the propagator $\tilde{S}(E, \boldsymbol{p}', \boldsymbol{p})$, which in our case has just one non-zero spinor component. The propagator in the energy-momentum space can be written as

$$\tilde{S}(E, \boldsymbol{p}', \boldsymbol{p}) = \int d^3x_1 d^3x_2 S(E, \boldsymbol{x}_2, \boldsymbol{x}_1) e^{i\boldsymbol{p}'\boldsymbol{x}_2} e^{-i\boldsymbol{p}\boldsymbol{x}_1} \,, \tag{3.13}$$

where $S(E, \boldsymbol{x}_2, \boldsymbol{x}_1)$ is the propagator in the mixed energy-coordinate representation. The exponent in the exponential function can be rewritten as

$$i(\boldsymbol{p}'\boldsymbol{x}_2 - \boldsymbol{p}\boldsymbol{x}_1) = \frac{i}{2}(\boldsymbol{p}' - \boldsymbol{p})(\boldsymbol{x}_1 + \boldsymbol{x}_2) - \frac{i}{2}(\boldsymbol{p}' + \boldsymbol{p})(\boldsymbol{x}_2 - \boldsymbol{x}_1) \,. \tag{3.14}$$

We see that only the second term is independent of the absolute coordinate and thus is physically meaningful. The nonphysical term can be absorbed together with $S(E, \boldsymbol{x}_2, \boldsymbol{x}_1)$ into some function $G(E, \boldsymbol{x}_2, \boldsymbol{x}_1)$, so that the propagator takes the form

$$\tilde{S}(E, \boldsymbol{p}', \boldsymbol{p}) = \int d^3x_1 d^3x_2 G(E, \boldsymbol{x}_2, \boldsymbol{x}_1) e^{-\frac{i}{2}(\boldsymbol{p}'+\boldsymbol{p})(\boldsymbol{x}_2 - \boldsymbol{x}_1)} \,. \tag{3.15}$$

The main contribution to the integral (3.12) comes from the region $|\boldsymbol{x}_2 - \boldsymbol{x}_1| \approx L$, where L is the distance between the neutrino source and detector. Now if the value of p changes by some quantity Δp, the additional phase $\Delta\phi$ would be $\Delta\phi = \Delta p L$. To change $\tilde{S}(E, \boldsymbol{p}', \boldsymbol{p})$ significantly, the additional phase needs to be of order 1, $\Delta\phi \sim 1$. Consequently the propagator $\tilde{S}(E, \boldsymbol{p}', \boldsymbol{p})$ changes significantly when Δp changes by $\Delta p \sim L^{-1}$. For $\Delta p'$ the same argumentation can be made. The propagation baseline L is a macroscopic quantity, so it is clear that $\frac{1}{L} \ll \sigma_{pP}$ and $\frac{1}{L} \ll \sigma_{pD}$. It follows that the propagator is a fast oscillating function of the momenta compared to the overlap functions $\Phi_D(p')$ and $\Phi_P(p)$. In the intervals of momenta of the order of the momentum widths σ_{pP}, σ_{pD} the overlap functions can be considered to be nearly constant compared with the propagator. This means that the overlap functions can be pulled out of the integral in eq. (3.12) at those values that give the main contribution to it. So the next question is: what are the values of momenta, that give the main contribution to the integral? Since $\Phi_P(E, \boldsymbol{p})$ is the production amplitude, it is peak-valued at the momentum of the produced neutrino in matter $\boldsymbol{p}_*$. Therefore $\Phi_P(p)$ can be pulled out from the integral at this value. Analogously $\Phi_D(E, \boldsymbol{p}')$ has a maximum at the momentum of the detected neutrino state $\boldsymbol{p}'_*$ and can be evaluated at that value and pulled out from the integral.

Thus, the next task is to find both these peak values $\boldsymbol{p}_*$ and $\boldsymbol{p}'_*$. The processes we consider take place in non-uniform matter. In the introduction we already discussed matter eigenstates, which diagonalize the full Hamiltonian in the case of matter with constant density. Although the potential $V(\boldsymbol{x})$ is position dependent, the condition of constant density is approximately fulfilled in the regions of neutrino production and detection, because they are small compared to the distance over which matter density changes significantly. Therefore we can consider the neutrino production and detection processes as occurring in matter of constant density with the densities corresponding of those at the production and detection points respectively. Neutrino energy and momentum at production and detection satisfy the in-matter dispersion relations. Thus we have to switch from the flavour to the matter basis representation. For this purpose we introduce the mixing matrix in matter $\tilde{U}_{\alpha K}$ in such a way that it converts matter eigenstates into flavour eigenstates:

$$|\nu_\alpha\rangle = \sum_j U^*_{\alpha j}|\nu_j\rangle = \sum_K \tilde{U}^*_{\alpha K}|\nu_K\rangle\,, \tag{3.16}$$

where $|\nu_\alpha\rangle$ are flavour eigenstates, $|\nu_j\rangle$ are mass eigenstates and $|\nu_K\rangle$ are matter eigenstates.
Let us stress that $\tilde{U}_{\alpha K}$ is density (and therefore coordinate) dependent. Note once again that we use Greek letters indices $(\alpha, \beta...)$ to label flavour eigenstates, small Latin letters $(i, j, k...)$ for mass eigenstates and capital Latin letters $(A, B, C, K..)$ for matter eigenstates. Because we consider neutrino momentum in matter we replace the leptonic mixing matrix U by the matrix $\tilde{U}$ of neutrino mixing in matter. From now on the neutrino propagator is also given in the matter eigenstate basis. That is why hereafter we write matter eigenstates indices K and K' and not mass eigenstate indices j and k for it. Also the overlap functions are considered in the matter eigenstate basis and so are labeled by the corresponding indices K, K'. The peak value $\boldsymbol{p}_*$ of the corresponding overlap function in the matter eigenstate basis $(\Phi_P(E, \boldsymbol{p}))_K$ will be simply denoted by $\boldsymbol{p}_K$ and the overlap function evaluated at this maximum by $\Phi_P(E, \boldsymbol{p}_K)$. Analogously we denote $\boldsymbol{p}'_*$ by $\boldsymbol{p}'_{K'}$ and the overlap function at this value in the matter eigenstate basis by $\Phi_P(E, \boldsymbol{p'}_{K'})$. We will illustrate this procedure on an example of two-flavour oscillations in chapter 7. With the above transformations, the transition amplitude takes the following approximate form:

$$\begin{aligned} i\mathcal{A}_{\alpha\beta} \approx \sum_{K,K'} \tilde{U}^*_{\alpha K}\tilde{U}_{\beta K'} \int \frac{dE}{2\pi} e^{-iET}\Phi_D(E, \boldsymbol{p}'_{K'})\Phi_P(E, \boldsymbol{p}_K) \\ \times \int \frac{d^3p d^3p'}{(2\pi)^6}\tilde{S}_{KK'}(E, \boldsymbol{p'}, \boldsymbol{p}) e^{i\boldsymbol{p'x}_D - i\boldsymbol{px}_P} \end{aligned} \tag{3.17}$$

Note that the overlap functions $\Phi_P(E, \boldsymbol{p}_K)$ and $\Phi_D(E, \boldsymbol{p}'_{K'})$ depend on the indices K and K' trough the three-momenta $\boldsymbol{p}_K$ and $\boldsymbol{p}'_{K'}$. One recognizes that in eq. (3.17) the remaining part in the integral is nothing else but the neutrino propagator $S(E, \boldsymbol{x}_D, \boldsymbol{x}_P)$

in the mixed energy-coordinate representation. As discussed in the previous chapter, from Ref. [11] we know that this propagator has the form

$$S(E, \boldsymbol{x}_1, \boldsymbol{x}_2) = -2E \frac{e^{iE|\boldsymbol{x}_1 - \boldsymbol{x}_2|}}{4\pi|\boldsymbol{x}_1 - \boldsymbol{x}_2|} F(E, \boldsymbol{x}_1, \boldsymbol{x}_2)\,, \tag{3.18}$$

where the function $F(E, \boldsymbol{x}_1, \boldsymbol{x}_2)$ obeys the Schrödinger-like equation

$$i\frac{d}{dx}F = \left[\frac{M^2}{2|E|} \pm (V^0(x) - V^3(x))\right] F\,. \tag{3.19}$$

Here M^2 is the mass matrix and V^0 and V^3 are components of the effective matter four-potential. The differential operator $\frac{d}{dx}$ is defined as a derivative with respect to the coordinate in the direction in which the neutrino propagates:

$$\frac{d}{dx} := \hat{\boldsymbol{L}} \cdot \frac{d}{d(\boldsymbol{x}_1 - \boldsymbol{x}_2)}\,, \tag{3.20}$$

where $\hat{\boldsymbol{L}}$ is a unit vector in the direction of neutrino propagation:

$$\hat{\boldsymbol{L}} := \frac{\boldsymbol{L}}{L}\,, \tag{3.21}$$

and $\boldsymbol{L}$ is the baseline vector connecting the source and the detector:

$$\boldsymbol{L} = \boldsymbol{x}_D - \boldsymbol{x}_P\,, \quad L = |\boldsymbol{L}|\,. \tag{3.22}$$

We will assume that the neutrino production coordinate $\boldsymbol{x}_P$ is fixed and $\boldsymbol{x}_D$ is varying, so that the distance L is a variable. For a known matter potential, one can find $F(E, L)$ by solving eq. (3.19) with the boundary condition (2.29). After the propagator function has been found, one can insert it into eq. (3.17), so that the amplitude for the overall neutrino production-propagation-detection process is

$$\begin{aligned} i\mathcal{A}_{\alpha\beta} \approx &- \sum_{K,K'} \tilde{U}^*_{\alpha K}(\boldsymbol{x}_P)\tilde{U}_{\beta K'}(\boldsymbol{x}_D) \\ &\times \int \frac{dE}{2\pi} e^{-iET} \Phi_D(E, \boldsymbol{p}'_{K'})\Phi_P(E, \boldsymbol{p}_K) \times 2E \frac{e^{iEL}}{4\pi L} F_{KK'}(E, L)\,. \end{aligned} \tag{3.23}$$

In this equation the neutrino mixing matrices in matter $\tilde{U}_{\alpha K}(\boldsymbol{x})$, $\tilde{U}_{\beta K'}(\boldsymbol{x})$, which are coordinate-dependent, are taken at the production point $\boldsymbol{x}_P$ and the detection point $\boldsymbol{x}_D$ respectively: $\tilde{U}^*_{\alpha K}(\boldsymbol{x}_P)$, $\tilde{U}_{\beta K'}(\boldsymbol{x}_D)$. The reason for this is the fact that the overlap functions are evaluated at the production and accordingly detection points and also the propagator function $F_{KK'}(E, L)$ depends on these coordinates. In the rest of this work we will simply denote $\tilde{U}_{\alpha K}(\boldsymbol{x}_P)$ as $\tilde{U}_{\alpha K}$ and $\tilde{U}_{\beta K'}(\boldsymbol{x}_D)$ as $\tilde{U}_{\beta K'}$ because we postulate that the unprimed indices always correspond to the neutrino production point and the primed indices to the neutrino detection point, respectively.

4 Probability of neutrino oscillations

In the previous chapter we calculated the amplitude for the process of production, propagation and detection of neutrino. The next problem is to extract the oscillation probability from our previous result. The most expedient way is to ask what is the experimental way to define the oscillation probability. Imagine again that in an experiment neutrinos are created, propagate a certain distance in matter and are then detected. As an experimentalist one can measure the rate Γ^{det} of detected neutrinos and their flavours. We are also able to measure the production rate Γ^{prod} of neutrinos and the cross section of the detection process. These are the quantities we should use to come to a meaningful definition of the oscillation probability. The crucial assumption for the following procedure is that the rate of the overall process can be factorized into the production rate, oscillation probability and detection cross section. The following calculation, which is based on Ref. [12], is valid only if this condition is satisfied. We consider the detection process without specifying the exact way the detection occurs. In general, the detection rate for neutrinos of flavour β is given by

$$\Gamma_{\beta}^{\text{det}} = \int dE\, j_{\beta}(E)\sigma_{\beta}(E)\,, \tag{4.1}$$

where $\sigma_{\beta}(E)$ is the detection cross section and $j_{\beta}(E)$ is the energy density (spectrum) of the ν_{β} flux at the detector. Assuming that the source emits neutrinos of flavour α with the energy spectrum $d\Gamma_{\alpha}^{\text{prod}}(E)/dE$, we can say that the energy density of the flux $j_{\beta}(E)$ amounts to

$$j_{\beta}(E) = \frac{1}{4\pi L^2}\frac{d\Gamma_{\alpha}^{\text{prod}}(E)}{dE}P_{\alpha\beta}(E, \boldsymbol{x}_D, \boldsymbol{x}_P)\,, \tag{4.2}$$

where $L = |\boldsymbol{x}_D - \boldsymbol{x}_P|$ is again the distance between the source and the detector, and the neutrino emission is assumed to be spherically symmetric. $P_{\alpha\beta}(E, \boldsymbol{x}_D, \boldsymbol{x}_P)$ is the oscillation probability that we are looking for. With eq. (4.1) the rate of neutrino production-propagation-detection becomes

$$\Gamma_{\alpha\beta}^{\text{tot}} \equiv \int dE\, \frac{d\Gamma_{\alpha\beta}^{\text{tot}}(E)}{dE} = \frac{1}{4\pi L^2}\int dE\, \frac{d\Gamma_{\alpha}^{\text{prod}}(E)}{dE}\, P_{\alpha\beta}(E, \boldsymbol{x}_D, \boldsymbol{x}_P)\, \sigma_{\beta}(E)\,. \tag{4.3}$$

To find the oscillation probability we must solve equation (4.3) for $P_{\alpha\beta}(E, \boldsymbol{x}_D, \boldsymbol{x}_P)$:

$$P_{\alpha\beta}(E, \boldsymbol{x}_D, \boldsymbol{x}_P) = \frac{d\Gamma_{\alpha\beta}^{\text{tot}}(E)/dE}{\frac{1}{4\pi L^2}\left[d\Gamma_{\alpha}^{\text{prod}}(E)/dE\right]\sigma_{\beta}(E)}\,. \tag{4.4}$$

The next thing to do is to compute the production and detection probabilities. We treat neutrinos as plane waves weighted with the factors Φ. We would like to simplify our calculation even more by assuming that the detection and production processes are isotropic, so that we can average the overlap functions over the direction of the incoming particles P_i and D_i. One can therefore define $\Phi_{P,D}(E, p_K) = \int \frac{d\Omega_{\hat{\boldsymbol{L}}}}{4\pi} \Phi_{P,D}(E, \boldsymbol{p}_K) = \int \frac{d\Omega_{\hat{\boldsymbol{L}}}}{4\pi} \Phi_{P,D}(E, p_K \hat{\boldsymbol{L}})$. If we apply the standard QFT rules but take into account that the neutrino production occurs in matter, so that the vacuum mixing matrices U should be replaced by the mixing matrices in matter $\tilde{U}$, we obtain

$$\begin{aligned} P_\alpha^{\text{prod}} &= \sum_K |\tilde{U}_{\alpha K}|^2 \int \frac{d^3 p_K}{(2\pi)^3} \left|\Phi_P(E, p_K)\right|^2 \\ &= \sum_K |\tilde{U}_{\alpha K}|^2 \frac{1}{8\pi^2} \int dE \left|\Phi_P(E, p_K)\right|^2 4 E p_K \,. \end{aligned} \tag{4.5}$$

Here $p_K(E)$ $(K = A, B, C)$ are momenta of matter eigenstates for $V(\boldsymbol{x}) = V(\boldsymbol{x}_P)$. For 2-flavour case in the adiabatic approximation, we give the explicit formulas for p_K below (see eq. (6.10)).
For the detection probability one can calculate

$$P_\beta^{\text{det}}(E) = \sum_{K'} |\tilde{U}_{\beta K'}|^2 |\Phi_D(E, p'_K)|^2 \frac{1}{V_N} \,, \tag{4.6}$$

where V_N is the normalization volume coming from the plane-wave description of the neutrino.

In experiments we do not deal with single particles, but rather with their fluxes, so we should extend our calculation. Let us define a time interval T_0 that is large compared to the time scales of neutrino production and detection. Let N_P be the number of particles P_i entering the production region in the time interval T_0. For steady fluxes the number of particles P_i entering in time interval dt_P is then $dN_{P_i} = N_P(dt_P/T_0)$, so the neutrino emission probability is

$$\mathcal{P}_\alpha^{\text{prod}}(t) = N_P \int_0^t \frac{dt_P}{T_0} P_\alpha^{\text{prod}} = N_P \, P_\alpha^{\text{prod}} \frac{t}{T_0} \,, \tag{4.7}$$

where P_α^{prod} is the time independent production probability for a single process defined in eq. (4.5). Using eq. (4.7) we are able to define the production rate:

$$\Gamma_\alpha^{\text{prod}} = \frac{d\mathcal{P}_\alpha^{\text{prod}}(t)}{dt} = N_P \, \frac{P_\alpha^{\text{prod}}}{T_0} \,. \tag{4.8}$$

We use a similar argument for the detection, assuming that N_D is the number of particles D_i entering the detection region during the interval T_0, so we obtain for the detection probability

$$\mathcal{P}_\beta^{\rm det}(t) = N_D \int_0^t \frac{dt_D}{T_0}\, P_\beta^{\rm det} = N_D\, P_\beta^{\rm det}\, \frac{t}{T_0}\,, \tag{4.9}$$

and for the detection rate

$$\Gamma_\beta^{\rm det} = \frac{d\mathcal{P}_\beta^{\rm det}(t)}{dt} = N_D\, \frac{P_\beta^{\rm det}}{T_0}\,. \tag{4.10}$$

The cross section results from dividing the detection rate by the flux of incoming neutrinos $j_{\nu K'} = n_{\nu K'} v_{\nu K'}$, with the number density $n_{\nu K'}$ of the detected ν'_K and corresponding velocity $v_{\nu K'}$. With normalization of one particle in the normalization volume $n_{\nu K'} = 1/V_N$ and using eq. (4.10) we get

$$\sigma_\beta(E) = \frac{N_D}{T_0} \sum_{K'} |\tilde{U}_{\beta K'}|^2 |\Phi_D(E, p'_{K'})|^2 \frac{E}{p'_{K'}}\,. \tag{4.11}$$

We also need to calculate the rate of the total production-propagation-detection process $\Gamma_\beta^{\rm tot}$. For this purpose we should, analogously to the computation above, find the time-dependent total probability. Since it involves detection as well as production, we integrate the total probability for each individual single process $P_{\alpha\beta}^{\rm tot}$, which is nothing else but the square modulus of the amplitude which we obtained in the previous chapter (see eq. (3.23)), over t_D and t_P. Therefore we obtain

$$\mathcal{P}_{\alpha\beta}^{\rm tot}(t, L) = \frac{N_P N_D}{T_0^2} \int_0^t dt_D \int_0^t dt_P\, P_{\alpha\beta}^{\rm tot}(T, L)\,. \tag{4.12}$$

When we introduce new variables $\tilde{T} := (t_P + t_D)/2$ and $T = t_D - t_P$, we get

$$\begin{aligned}
\mathcal{P}_{\alpha\beta}^{\rm tot}(t, L) &= \frac{N_P N_D}{T_0^2} \Big[\int_0^t dT\, P_{\alpha\beta}^{\rm tot}(T, L)(t - T) + \int_{-t}^0 dT\, P_{\alpha\beta}^{\rm tot}(T, L)(t + T) \Big] \\
&= \frac{N_P N_D}{T_0^2} \Big[t \int_{-t}^t dT\, P_{\alpha\beta}^{\rm tot}(T, L) - \int_0^t dT\, T P_{\alpha\beta}^{\rm tot}(T, L) + \int_{-t}^0 dT\, T P_{\alpha\beta}^{\rm tot}(T, L) \Big] \\
&=: \frac{N_P N_D}{T_0^2} \Big[t I_1(t) - I_2(t) + I_3(t) \Big]\,.
\end{aligned} \tag{4.13}$$

One can show (see Ref. [12]) that for t much larger than the time scales of the neutrino production and detection processes $I_2(t)$ and $I_3(t)$ are negligible, while $I_1(t)$ is equal to $\tilde{P}^{\rm tot}$, which is defined as

$$\tilde{P}^{\rm tot}_{\alpha\beta}(L) = \int dT |\mathcal{A}_{\alpha\beta}(T,L)|^2 = \sum_{K,K',M,M'} \tilde{U}^*_{\alpha K}\tilde{U}_{\beta K'}\tilde{U}_{\alpha M}\tilde{U}^*_{\beta M'}$$
$$\times \int \frac{dE}{2\pi}(2E)^2 \Phi_D(E,p'_{K'})\Phi_P(E,p_K)\,\Phi^*_D(E,p'_{M'})\Phi^*_P(E,p_M)\frac{F_{KK'}(E,L)F^*_{MM'}(E,L)}{(4\pi)^2 L^2}\,. \tag{4.14}$$

Thus eq. (4.13) reduces to

$$\mathcal{P}^{\rm tot}_{\alpha\beta}(t,L) = \frac{N_P N_D}{T_0^2}\, t\, \tilde{P}^{\rm tot}_{\alpha\beta}(L)\,. \tag{4.15}$$

The rate of the overall process becomes

$$\Gamma^{\rm tot}_{\alpha\beta}(L) = \frac{d\mathcal{P}^{\rm tot}_{\alpha\beta}(t,\boldsymbol{x}_D,\boldsymbol{x}_P)}{dt} = N_P N_D\,\frac{\tilde{P}^{\rm tot}_{\alpha\beta}}{T_0^2}\,. \tag{4.16}$$

We can obtain the oscillation probability formula with the assumption that the factorization of the overall rate into the production rate, oscillation probability and detection cross section for fixed neutrino energy is possible. Inserting equations (4.3), (4.8) and (4.11) into eq. (4.4) we obtain

$$P_{\alpha\beta}(E,L) = \frac{1}{\sum_K |\tilde{U}_{\alpha K}|^2 |\Phi_P(E,p_K)|^2 \sum_{K'} |\tilde{U}_{\beta K'}|^2 |\Phi_D(E,p_{K'})|^2 p_K p_{K'}^{-1}}$$
$$\times\Bigg(\sum_{K,K',M,M'} \tilde{U}^*_{\alpha K}\tilde{U}_{\beta K'}\tilde{U}_{\alpha M}\tilde{U}^*_{\beta M'}\Phi_D(E,p'_{K'})\Phi_P(E,p_K)$$
$$\times\Phi^*_P(E,p_M)\Phi^*_D(E,p'_{M'})F_{KK'}(E,L)F^*_{MM'}(E,L)\Bigg)\,. \tag{4.17}$$

If neutrinos are ultra-relativistic or quasi-degenerate in mass and $|V| \ll |E|$ the conditions

$$\begin{aligned} |p_K - p_M| &\ll p_K, p_M \\ |p'_K - p'_M| &\ll p'_K, p'_M \end{aligned} \tag{4.18}$$

are satisfied. From conditions (4.18) it follows that the production probabilities of different matter eigenstates hardly differ and therefore can be evaluated at some averaged momentum value p. The same argumentation can be applied for the detection probabilities of different matter eigenstates. Therefore, also here we evaluate the detection amplitudes at an averaged momentum value p'. Since in this case the production and

detection amplitudes do not depend on the matter-eigenstate indices they can be pulled out of the sums in the denominator of eq. (4.17). Using the unitarity of the mixing matrices in matter we can simplify

$$\begin{aligned}
&\sum_K |\tilde{U}_{\alpha K}|^2 |\Phi_P(E, p_K)|^2 p_K \to |\Phi_P(E,p)|^2\, p \sum_K |\tilde{U}_{\alpha K}|^2 = |\Phi_P(E,p)|^2\, p\,, \\
&\sum_{K'} |\tilde{U}_{\beta K'}|^2 |\Phi_D(E, {p'}_{K'})|^2 p'^{-1}_{K'} \to |\Phi_D(E,p)|^2\, p^{-1} \sum_{K'} |\tilde{U}_{\beta K'}|^2 = |\Phi_D(E,p')|^2\, p'^{-1}\,.
\end{aligned} \tag{4.19}$$

But this assumption means that the spectral density of the production rate and the detection cross section are independent of the elements of the leptonic mixing matrix, which means that the factorization condition for the total probability of the process is satisfied. Therefore in analogy with eq. (4.4), we have

$$\begin{aligned}
P_{\alpha\beta}(E,L) =& \frac{1}{|\Phi_P(E,p)|^2 |\Phi_D(E,p')|^2} \\
&\times \Bigg(\sum_{K,K',M,M'} \tilde{U}^*_{\alpha K} \tilde{U}_{\beta K'} \tilde{U}_{\alpha M} \tilde{U}^*_{\beta M'} \Phi_D(E, p'_{K'}) \Phi_P(E, p_K) \\
&\times \Phi^*_P(E, p_M) \Phi^*_D(E, p'_{M'}) F_{KK'}(E,L) F^*_{MM'}(E,L) \Bigg)\,,
\end{aligned} \tag{4.20}$$

where in the denominator we canceled p and p' since the mean neutrino momenta at the production and detection coincide to a very good accuracy under the conditions $\frac{\Delta m^2}{2E} \ll E$, $\frac{|V|}{|E|} \ll 1$, which are assumed to be satisfied throughout this work. Note that in general one cannot adopt $p' = p$ in the arguments of Φ_P and Φ_D provided that the peak momenta of these functions are separated by more than $\sigma_{pP} + \sigma_{pD}$.
In general we cannot apply the same approximation (4.19) to the numerator of eq. (4.20) because of the interference terms of different matter eigenstates. The interference terms are proportional to the products of production and detection overlap functions, each function taken at different momentum value. The expression can be simplified only if the coherent production and detection conditions are satisfied. This means

$$\begin{aligned}
&|p_K - p_M| \ll \sigma_{pP}\,, \\
&|p_{K'} - p_{M'}| \ll \sigma_{pD}\,.
\end{aligned} \tag{4.21}$$

In this case the momenta of matter eigenstates at production are sufficiently close to each other and the same applies to neutrino detection. Thus all corresponding overlap functions can be taken at the average momenta p and p':

$$\Phi_P(E, p_K) \to \Phi_P(E, p)\,, \quad \Phi_D(E, p'_{K'}) \to \Phi_P(E, p')\,. \tag{4.22}$$

They can be pulled out from the sum in eq. (4.20) and canceled with the denominator of this expression. Thus eq. (4.20) becomes

$$P_{\alpha\beta}(E, L) = \sum_{K,K',M,M'} \tilde{U}^*_{\alpha K} \tilde{U}_{\beta K'} \tilde{U}_{\alpha M} \tilde{U}^*_{\beta M'} F_{KK'}(E, L) F^*_{MM'}(E, L) = |F_{\alpha\beta}(E, L)|^2\,, \tag{4.23}$$

where $F_{\alpha\beta}(E, L)$ is the propagator function in the flavour basis which obeys eq. (3.19). From eq. (4.23) we see that, when the coherence condition for the neutrino production and detection are satisfied, the oscillation probability becomes independent of the production and detection processes.
On the other hand, if one has

$$|p_K - p_M| \gtrsim \sigma_{pP}\,, \tag{4.24}$$

or

$$|p_{K'} - p_{M'}| \gtrsim \sigma_{pD}\,, \tag{4.25}$$

the difference of momentum values exceeds the momentum width of the corresponding overlap functions, the amplitudes cannot overlap and the interference terms are strongly suppressed. In this case one can speak about the lack of coherence at neutrino production or detection. In the vacuum case the conditions (4.24) and (4.25) imply that one cannot observe any neutrino oscillations at all because the flavour transition probability (4.20) takes its averaged value (see Ref. [12]). In non-uniform matter such an implication is not true. When both conditions (4.24) and (4.25) are satisfied it follows that the flavour transition probability is non-oscillating, analogously to the adiabatic flavour transition discussed in the introduction. However, if only one of the decoherence conditions (4.24) or (4.25), is satisfied, the flavour transition probability will still have an oscillatory behavior, unlike in the case of neutrino oscillations in vacuum (see Appendix A).
If the coherent production and/or detection conditions are violated, the neutrino oscillation probability is described by the general formula (4.20), which is our final result.

5 The oscillation amplitude and its evolution equation

Eq. (4.20) gives us the flavour transition probability, i.e. the probability of the process in which neutrino produced in the flavour state α becomes after propagation through matter the flavour state β. In this formula the overlap functions contain all information about the production and detection processes. The function $F(E, L)$ describes the interaction of neutrinos with matter during its propagation. Regrettably, in general it is not possible to give a closed-form expression for $F(E, L)$ with an explicit matter potential dependence. But if we know the potential $V(x)$, we can solve the Schrödinger-like equation (3.19) for the propagator function $F(E, L)$ and then insert the solution into eq. (4.20) to get the oscillation probability. Note that when solving the Schrödinger-like equation (3.19), we use the boundary condition (2.29).
As we get an expression for the probability of neutrino oscillations, it is quite natural to define the corresponding neutrino oscillation amplitude. From the final result (4.20) we find it to be of the form

$$\mathcal{A}_{\alpha\beta}(E, L) = \frac{\sum_{K,K'} \tilde{U}^*_{\alpha K} \tilde{U}_{\beta K'} \Phi_D(E, p'_{K'}) \Phi_P(E, p_K) F_{KK'}(E, L)}{|\Phi_P(E, p)||\Phi_D(E, p')|} \,. \tag{5.1}$$

We have seen that we succeeded in finding a sensible amplitude for the oscillation phenomenon, based on experimental considerations and general Feynman diagram approach. We can now ask, to what extent our general result contains the usual one. In the framework of quantum mechanics, as we already explained in the introduction, the Schrödinger-like equation is used. So we should ask, under which condition the amplitude we derived is the solution of the Schrödinger-like equation

$$i\frac{d}{dL}\mathcal{A}(L) = [U\frac{\Delta m^2}{2E}U^\dagger + V(L)]\mathcal{A}(L)\,, \tag{5.2}$$

which, reintroducing all the indices, is

$$i\frac{d}{dL}\mathcal{A}_{\beta\alpha}(L) = \sum_{j,\gamma}[U_{\beta j}\frac{\Delta m^2_{j1}}{2E}U^\dagger_{j\gamma} + V_{\beta\gamma}(L)]\mathcal{A}_{\gamma\alpha}(L)\,. \tag{5.3}$$

Consider first the case when the coherent production and detection conditions (4.21) are satisfied. Then the probability becomes detection and production independent (see

eq. (4.23)) and the amplitude

$$\mathcal{A}_{\alpha\beta}(E,L) = \sum_{K,K'} \tilde{U}^*_{\alpha K} \tilde{U}_{\beta K'} F_{KK'}(E,L) = F_{\alpha\beta}(E,L) \tag{5.4}$$

is simply the propagator function $F_{\alpha\beta}$ in the flavour basis. As we already know from eq. (3.19), $F_{\alpha\beta}$ satisfies the Schrödinger-like equation which coincides with eq. (5.2). In the case when conditions (4.21) are violated one can convince oneself that such an equation for the amplitude $\mathcal{A}_{\alpha\beta}$ can not be derived. The reason for this is that $F_{\alpha\beta}$ depends on x and is not diagonal for all x in any basis. Thus, we have proven that the usual quantum mechanical approach is valid only when the coherent production and detection conditions (4.21) are satisfied. Then the oscillation amplitude coincides with the propagator function $F_{\alpha\beta}$.
All these considerations imply that if in an experiment the production and/or detection processes are not coherent, the probability differs from the one predicted by the standard approach. Therefore our result could in principle be tested experimentally. If a change of probability in this case could be measured, as predicted by the formula (4.20), it would be a strong indication for the shortcoming of the standard treatment of neutrino oscillations in non-uniform matter based on the Schrödinger-like equation (5.2).
In chapter 6 we will calculate the difference between the momenta of two instantaneous matter eigenstates, p_A and p_B, in the two-flavour case and find (see eq. (6.10))

$$|p_A - p_B| = |2\omega| = \sqrt{\Big(\frac{\Delta m^2}{2E}\cos 2\theta - V(x)\Big)^2 + \Big(\frac{\Delta m^2}{2E}\Big)^2 \sin^2 2\theta}\,. \tag{5.5}$$

From this expression and eq. (4.24) we find that the production process will not satisfy the coherence condition when one deals with very high densities at the production point. Such a case can be realized in the early universe or in supernovae. Another possibility is a relatively large mass squared difference, which can take place if there exist sterile neutrinos. But of course one has to compare the quantity (5.5) with the momentum width of the production amplitude σ_{pP} as well.

6 Oscillations in the adiabatic approximation with two flavours

In this chapter we want to present a simple example of two-flavour oscillations to show why it is possible to pull the overlap functions out of momentum-integral in eq. (3.12). We use the adiabatic approximation for simplicity. Thus we assume that the matter density changes sufficiently slowly with the distance and so does the matter potential. Consider equation (3.19). The operator on the left-hand side is nothing but the momentum operator times minus one, $-\hat{p} = i\frac{d}{dx}$, where x is the coordinate along the neutrino propagation path. So the eigenvalues of the matrix on the right-hand side would be the momentum eigenvalues times minus one. If we apply $\hat{p}$ to S, the eigenvalues would approximately be (see eq. (3.18))

$$p_{A,B} \approx E - p^0_{A,B}\,, \tag{6.1}$$

where $p^0_{A,B}$ are the eigenvalues of the matrix on the right-hand side of equation (3.19). Here we used again $|E||\boldsymbol{x}_1 - \boldsymbol{x}_2| \gg 1$ (see chapter 2). We will calculate these momenta eigenvalues below.
In equation (3.19) the matrix $\frac{M^2}{2E}$ is diagonal in the mass eigenstate basis, while $V(x)$ is diagonal in the flavour eigenstate basis. The sum of them is diagonal neither in the first nor in the second basis but in the instantaneous matter eigenstate basis. We consider eq. (3.19) first in flavour eigenstate basis. It takes the form

$$i\frac{d}{dx}F_{fl} = H_{fl}F_{fl}\,, \tag{6.2}$$

or, written explicitly,

$$i\frac{d}{dx}F_{fl} = \left(U \begin{bmatrix} \frac{m_1^2}{2E} & 0 \\ 0 & \frac{m_2^2}{2E} \end{bmatrix} U^\dagger + \begin{bmatrix} V(x) & 0 \\ 0 & 0 \end{bmatrix}\right) F_{fl}\,, \tag{6.3}$$

where U represents the unitary transformation and is given by the matrix

$$U = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \tag{6.4}$$

where θ is the vacuum mixing angle.
Calculating H_{fl} in eq. (6.3) explicitly, we obtain:

$$H_{fl} = \begin{bmatrix} \frac{m_1^2+m_2^2}{4E} - \cos 2\theta \cdot \frac{\Delta m^2}{4E} + V(x) & \sin 2\theta \cdot \frac{\Delta m^2}{4E} \\ \sin 2\theta \cdot \frac{\Delta m^2}{4E} & \frac{m_1^2+m_2^2}{4E} + \cos 2\theta \cdot \frac{\Delta m^2}{4E} \end{bmatrix} . \tag{6.5}$$

To get the effective Hamilton operator in the instantaneous matter eigenstate basis, which we denote as H_{matt}, we need to diagonalize H_{fl}:

$$H_{matt} = \tilde{U}^\dagger(x) H_{fl} \tilde{U}(x) = diag(p_A^0(x), p_B^0(x)) . \tag{6.6}$$

Here $\tilde{U}$ is the mixing matrix in matter, defined analogously to the vacuum case through

$$\tilde{U} = \begin{bmatrix} \cos\tilde{\theta} & \sin\tilde{\theta} \\ -\sin\tilde{\theta} & \cos\tilde{\theta} \end{bmatrix} , \tag{6.7}$$

where $\tilde{\theta}$ is the mixing angle in matter. It is given by

$$\tan 2\tilde{\theta} = \frac{\frac{\Delta m^2}{2E} \sin 2\theta}{\frac{\Delta m^2}{2E} \cos 2\theta - V(x)} , \tag{6.8}$$

where $\Delta m^2 := m_2^2 - m_1^2$ is mass squared difference of the two mass eigenstates.
It is easy now to find the momentum eigenvalues of eq. (6.3). They are

$$\begin{aligned} p_{A,B}^0(x) &= \pm \frac{1}{2} \sqrt{\Big(\frac{\Delta m^2}{2E} \cos 2\theta - V(x)\Big)^2 + \Big(\frac{\Delta m^2}{2E}\Big)^2 \sin^2 2\theta} - \frac{m_1^2 + m_2^2}{4E} - \frac{V(x)}{2} \\ &=: \pm\omega(x) - \frac{m_1^2 + m_2^2}{4E} - \frac{V(x)}{2} . \end{aligned} \tag{6.9}$$

Therefore the momenta of the neutrino matter eigenstates are

$$p_{A,B} \approx E \mp \omega(x) - \frac{m_1^2 + m_2^2}{4E} - \frac{V(x)}{2} . \tag{6.10}$$

Next, we want to find F in the matter eigenstate basis, like it appears in eq. (3.17). We start with equation (6.2) in the flavour basis. The matrices in this equation can be transformed by unitary matrices $\tilde{U}(x)$ and $\tilde{U}^\dagger(x)$. Here the important point is that they depend on the position x because the mixing angle changes with matter density. So the matrices in matter eigenstate basis H_{matt} and F_{matt} are:

$$H_{matt} = \tilde{U}^\dagger(x) H_{fl} \tilde{U}(x) , \tag{6.11}$$

$$F_{matt} = \tilde{U}^\dagger(x) F_{fl} \tilde{U}(x) . \tag{6.12}$$

Transforming eq. (6.2) into the equation in the matter eigenstate basis and using eqs. (6.11) and (6.12), we get:

$$i\frac{d}{dx}F_{matt} = [H_{matt}(x) - i\tilde{U}^{\dagger}(x)\frac{d}{dx}\tilde{U}(x)]F_{matt}\,. \tag{6.13}$$

As we know, H_{matt} is diagonal:

$$H_{matt} = diag(p_A^0(x), p_B^0(x))\,. \tag{6.14}$$

Using eq. (6.7), eq. (6.13) becomes:

$$i\frac{d}{dx}F_{matt} = \begin{bmatrix} p_A^0(x) & -i\tilde{\theta}'(x) \\ i\tilde{\theta}'(x) & p_B^0(x) \end{bmatrix} F_{matt}\,, \tag{6.15}$$

where the prime denotes the derivative with respect to x. Remember that x means the coordinate in the direction of propagation of neutrino, and only for this direction is this Schrödinger-like equation valid. That is why the momenta and the mixing angle in matter depend on this projected coordinate x and not on a three-vector any more. Now we want to calculate the propagator in the adiabatic approximation. Therefore we assume that the matter density changes very slowly with distance and so does the mixing angle in matter. If $\tilde{\theta}'(x)$ is small compared to the difference of momentum eigenvalues $|p_A^0(x) - p_B^0(x)|$, the off-diagonal terms in (6.15) can be neglected and we can find the propagator simply by solving (6.15) and using the initial condition that $F(E, x, x)$ is just the unit matrix. This gives

$$F_{matt}(E, x', x) = \begin{bmatrix} e^{-i\int_x^{x'} p_A^0(x'')dx''} & 0 \\ 0 & e^{-i\int_x^{x'} p_B^0(x'')dx''} \end{bmatrix}, \tag{6.16}$$

with $p_{A,B}^0(x')$ given by eq. (6.9). Let us now return to eq. (3.12) and illustrate the approximations we used. We rewrite eq. (3.12) in the matter eigenstate basis and insert into it our result for F in the adiabatic case. Thus we obtain

$$\begin{aligned} i\mathcal{A}_{\alpha\beta} = -\sum_{K=A,B} \tilde{U}^*_{\alpha K}\tilde{U}_{\beta K} \int \frac{dE}{2\pi} E e^{-iET} \int \frac{dpdp'}{(2\pi)^2} (\Phi_D(E,p'))_K (\Phi_P(E,p))_K e^{ip'x_D - ipx_P} \\ \times \int dxdx'\, e^{-i\int_x^{x'} p_K^0(x'')dx''} e^{ipx - ip'x'} \frac{e^{iE|x-x'|}}{2\pi|x-x'|}\,, \end{aligned} \tag{6.17}$$

where x, x' and p, p' are the projections of $\boldsymbol{x}$, $\boldsymbol{x}'$ and $\boldsymbol{p}$, $\boldsymbol{p}'$ on the neutrino propagation direction. They should not be confused with the four-momenta and four-coordinates! The integration over x and x' comes from the Fourier transformations, because the propagator in eq. (3.12) is a function of momenta. One can convince oneself that since we chose x as the coordinate of the direction of baseline, the three-dimensional integrations

over the coordinates and momenta reduce to the one-dimensional integrations over x, x' and p, p'. The index K corresponds to the matter eigenstate. Note that we sum over K but not K' because the propagator is diagonal in the adiabatic approximation. The reason for this is the fact that in this approximation matter eigenstates evolve independently and do not go into each other, as we already discussed in the introduction.

6.1 Stationary phase approximation

For the further calculation we need the stationary phase approximation. In this section we are going to explain briefly what it is. One uses the stationary phase approximation to calculate approximately integrals of fast oscillating functions of the form

$$I = \int_{-\infty}^{\infty} g(t) e^{if(t)}\, dt\,. \tag{6.18}$$

If $g(t)$ is a slowly varying function and $f(t)$ is a large function of t near its stationary point t_0, i.e. the point where $f'(t_0) = 0$, the integral is approximately equal to

$$I \approx \sqrt{\frac{2\pi}{|f''(t_0)|}}\, g(t_0)\, e^{if(t_0)} e^{i\frac{\pi}{4} sgn(f''(t_0))}\,. \tag{6.19}$$

Let us now use this approximation to simplify eq. (6.17). But since we have integrals of momentum and x-coordinate (because of the Fourier transformation), we need to use this approximation twice. More precisely: first, consider the propagator in the energy-momentum space, which is nothing else but the second line of eq. (6.17) times minus one. We apply the stationary phase approximation now only to this quantity. The function f of eq. (6.18) (compare (6.17) and (6.18)) is obviously equal to

$$\begin{aligned} f_{A,B}(x', x) &= E(x' - x) - p'x' + px - \int_x^{x'} p^0_{A,B}(x'')dx'' \\ &= E(x' - x) - p'x' + px \mp \int_x^{x'} \omega(x'')\, dx'' + \frac{m_1^2 + m_2^2}{4E}(x' - x) + \int_x^{x'} \frac{V(x'')}{2}\, dx''\,, \end{aligned} \tag{6.20}$$

with the difference that here it is a function of two variables, x and x'. The indices A and B stand for different momentum eigenvalues p^0_A and p^0_B. The next question is: what are the stationary points of the function $f(x', x)$? We find them in the usual way by differentiating the function first with respect to x and analogously to x' and setting the derivatives equal to zero. This gives us two equations that determine the stationary points for x and x':

$$p' = E \mp \omega(x') + \frac{m_1^2 + m_2^2}{4E} + \frac{V(x')}{2} = E \mp p^0_{A,B}(x')\,, \tag{6.21}$$

$$p = E \mp \omega(x) + \frac{m_1^2 + m_2^2}{4E} + \frac{V(x)}{2} = E \mp p^0_{A,B}(x)\,. \tag{6.22}$$

Notice that these equations should be solved with respect to x and x', i.e. they determine the stationary points as functions of p and p', not the other way around. We just write them in this way since in general it is not possible to solve these equations for x and x' explicitly. The values of x and x' that satisfy eqs. (6.21) and (6.22) we call x_0 and x'_0, respectively. Applying now the stationary phase approximation to eq. (6.17) gives

$$\begin{aligned} i\mathcal{A}_{\alpha\beta} \approx -\sum_K \tilde{U}^*_{\alpha K}\tilde{U}_{\beta K} \int \frac{dE}{2\pi} E e^{-iET} \int \frac{dpdp'}{(2\pi)^2} (\Phi_D(E,p'))_K (\Phi_P(E,p))_K \\ \times e^{if_K(x'_0,x_0)+ip'x_D-ipx_P} \sqrt{\frac{2\pi}{|f''_K(x'_0,x_0)|}} \frac{e^{i\frac{\pi}{4}sgn(f''_K(x'_0,x_0))}}{|x_0 - x'_0|}\,. \end{aligned} \tag{6.23}$$

Next, we want to apply the stationary phase approximation to the calculation of the integrals over the momenta p and p' in eq. (6.23). For this we need to find the stationary points of the exponent in the integrand in the momentum variables. We therefore define

$$F_{A,B}(p',p) = f_{A,B}(x'_0,x_0) + p'x_D - px_P\,, \tag{6.24}$$

where x'_0 and x_0 should be considered as functions of the momenta p' and p, respectively. Differentiating eq. (6.24) with respect to p and p' and setting the resulting expressions equal zero, we obtain

$$\left(\frac{dx'_0(p')}{dp'}\right)\left[E \mp \omega(x'_0) + \frac{m_1^2 + m_2^2}{4E} + \frac{V(x'_0)}{2} - p'\right] - x'_0(p') + x_D = 0\,, \tag{6.25}$$

$$\left(\frac{dx_0(p)}{dp}\right)\left[E \mp \omega(x_0) + \frac{m_1^2 + m_2^2}{4E} + \frac{V(x_0)}{2} - p\right] - x_0(p) + x_P = 0\,. \tag{6.26}$$

The stationary points p_0 and p'_0 are now determined as the solutions of eqs. (6.25) and (6.26). From eqs. (6.21) and (6.22) we know that the terms in the square brackets in eqs. (6.25) and (6.26) are zero. Therefore we find that the stationary value for x is the production point x_P and for x' the detection point x_D, respectively. The points of stationary phase p_0 and p'_0 are now found from the same equations (6.21) and (6.22), which should however now be considered as equations with respect to momenta, and the values of the coordinates should be taken to be x_D and x_P in eqs. (6.21) and (6.22)

respectively. It is clear by looking again at eqs. (6.21) and (6.22) that they are nothing else but the momenta of the produced and detected neutrino matter eigenstates (see eq. (6.1)). Thus we showed that the values at which we evaluate the overlap functions are in fact the momenta of the produced and detected neutrino eigenstates in matter. Therefore in the adiabatic regime the overall amplitude of the process is approximately

$$i\mathcal{A}_{\alpha\beta} = -\sum_{K=A,B} \tilde{U}^*_{\alpha K}\tilde{U}_{\beta K} \int \frac{dE}{2\pi} E e^{-iET} \Phi_D(E, p_{K'})\Phi_P(E, p_K)$$
$$\times \int \frac{dpdp'}{(2\pi)^2} \int dxdx'\, e^{-i\int_x^{x'} p^0_K(x'')dx''} e^{ipx - ip'x'} \frac{e^{iE|x-x'|}}{2\pi|x-x'|} e^{ip'x_D - ipx_P}\,. \qquad (6.27)$$

The important point for us is that it is possible to pull the overlap functions out of the integral evaluated at the vales of the production and detection momenta. The rest of the expression in the integral is simply the propagator in the mixed energy-coordinate representation taken at the the production and detection points. This example explicitly demonstrates how one can pull the overlap function out of the momentum integral using the adiabatic case as an example. As we discussed in Chapter 3, such a procedure is actually justified in the general case as well, provided that σ_{pP}, $\sigma_{pD} \gg L^{-1}$.

7 Summary

The task of the present thesis was to derive the probability of neutrino oscillations in non-uniform matter within the framework of Quantum Field Theory and to define a sensible amplitude for this phenomenon. We used a QFT-approach in which the neutrino is described by a propagator in a general Feynman diagram representing the neutrino production, propagation and detection. The neutrino propagator takes into account the interactions of the neutrino with matter and obeys the Schrödinger-like equation. Without specifying the production and the detection processes, and only assuming that they are isotropic, we calculated the amplitude of the overall process and then studied the way of extracting the probability of neutrino oscillations out of this quantity. We drew on experimental considerations to find a meaningful definition for the oscillation probability, so that we could extract it from the previously calculated total probability for the process. In this treatment we used the quantities we can measure: the cross section, the production and the detection rates. We also identified the conditions under which the overall-process rate factorizes into these quantities and the oscillation probability, which we are looking for. We considered the case of coherent neutrino production and detection and found that when the coherence conditions are satisfied the oscillation probability does not depend on the detection and production mechanisms. Once we had defined the oscillation probability, we were able to derive the oscillation amplitude from our final result. By analyzing this formula, we recognized that in the case of coherent production and detection the derived amplitude coincides with the one which is obtained as the solution of the Wolfenstein evolution equation. However, when the coherent conditions are violated, the oscillation probability differs from the standard result, since it is production and detection dependent. In particular, it does not satisfy the Wolfenstein evolution equation. In the last part of this work we studied the example of two-flavour oscillations in the adiabatic approximation. We showed how to find the momenta of neutrino eigenstates in matter and derived the propagator from the Schrödinger-like equation in the matter eigenstate basis. Furthermore, we illustrated the approximations of our previous calculations. We were also able to identify the difference between two momentum eigenvalues in matter, which enter into the conditions for the non-coherent neutrino production and detection. These conditions could play an important role because if they are satisfied there is a possibility to discriminate between the usual result based on the Wolfenstein evolution equation and the result derived in our work.

A Appendix

In this appendix we illustrate the significance of the coherence conditions for the neutrino oscillation probability by its exact calculations for the two-flavour case. In the general case the evolution matrix of two matter eigenstates T can be written as

$$T = \begin{bmatrix} \sqrt{1-p'}e^{i\alpha} & \sqrt{p'}e^{i\beta} \\ -\sqrt{p'}e^{-i\beta} & \sqrt{1-p'}e^{-i\alpha} \end{bmatrix}, \tag{A.1}$$

where p' is the smooth function which denotes the transition probability of the matter eigenstates while α and β are some phases. The evolution matrix T is chosen in the above form so that the square modulus of its a, b components gives

$$|T_{ab}|^2 = \begin{bmatrix} 1-p' & p' \\ p' & 1-p' \end{bmatrix}. \tag{A.2}$$

Using the definition from eq. (1.29) for the mixing matrix in matter $\tilde{U}(x)$ we obtain the oscillation amplitude $\mathcal{A}_{\alpha\beta} = \tilde{U}(x)T\tilde{U}^{\dagger}(x_i)$ in the general case:

$$\begin{aligned} \mathcal{A}_{\alpha\beta} &= \begin{bmatrix} \cos\tilde{\theta}(x) & \sin\tilde{\theta}(x) \\ -\sin\tilde{\theta}(x) & \cos\tilde{\theta}(x) \end{bmatrix} \begin{bmatrix} \sqrt{1-p'}e^{i\alpha} & \sqrt{p'}e^{i\beta} \\ -\sqrt{p'}e^{-i\beta} & \sqrt{1-p'}e^{-i\alpha} \end{bmatrix} \begin{bmatrix} \cos\tilde{\theta}(x_i) & -\sin\tilde{\theta}(x_i) \\ \sin\tilde{\theta}(x_i) & \cos\tilde{\theta}(x_i) \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{A}_{ee} & \mathcal{A}_{e\mu} \\ \mathcal{A}_{\mu e} & \mathcal{A}_{\mu\mu} \end{bmatrix}, \end{aligned} \tag{A.3}$$

with

$$\mathcal{A}_{ee} = \sqrt{1-p'}(cc_0e^{i\alpha} + ss_0e^{-i\alpha}) + \sqrt{p'}(cs_0e^{i\beta} - sc_0e^{-i\beta}), \tag{A.4}$$

$$\mathcal{A}_{e\mu} = \sqrt{1-p'}(-cs_0e^{i\alpha} + sc_0e^{-i\alpha}) + \sqrt{p'}(cc_0e^{i\beta} + ss_0e^{-i\beta}), \tag{A.5}$$

$$\mathcal{A}_{\mu e} = \sqrt{1-p'}(-sc_0e^{i\alpha} + cs_0e^{-i\alpha}) + \sqrt{p'}(-ss_0e^{i\beta} - cc_0e^{-i\beta}), \tag{A.6}$$

$$\mathcal{A}_{\mu\mu} = \sqrt{1-p'}(ss_0e^{i\alpha} + cc_0e^{-i\alpha}) + \sqrt{p'}(-sc_0e^{i\beta} + cs_0e^{-i\beta}), \tag{A.7}$$

where we simplified the notations by

$$\begin{aligned} c &:= \cos\tilde{\theta}(x)\,, \\ s &:= \sin\tilde{\theta}(x)\,, \\ c_0 &:= \cos\tilde{\theta}(x_i)\,, \\ s_0 &:= \sin\tilde{\theta}(x_i)\,. \end{aligned} \tag{A.8}$$

The transition probability for the process in which the neutrino of flavour α becomes a neutrino of flavour β is just the square modulus of the corresponding component of the oscillation amplitude matrix $\mathcal{A}_{\alpha\beta}$. For example, the probability for the electron neutrino to become a muon neutrino is

$$\begin{aligned} P_{e\mu} = |\mathcal{A}_{e\mu}|^2 &= |\sqrt{1-p'}(-cs_0e^{i\alpha} + sc_0e^{-i\alpha}) + \sqrt{p'}(cc_0e^{i\beta} + ss_0e^{-i\beta})|^2 \\ &= \frac{1}{2} - \left(\frac{1}{2} - p'\right)\cos 2\tilde{\theta}(x)\cos 2\tilde{\theta}(x_i) \\ &- \frac{1}{2}\sin 2\tilde{\theta}(x)\sin 2\tilde{\theta}(x_i)[(1-p')\cos 2\alpha - p'\cos 2\beta] \\ &- \sqrt{p'(1-p')}[\cos 2\tilde{\theta}(x)\sin 2\tilde{\theta}(x_i)\cos(\alpha-\beta) \\ &- \sin 2\tilde{\theta}(x)\cos 2\tilde{\theta}(x_i)\cos(\alpha+\beta)]\,. \end{aligned} \tag{A.9}$$

We realize that in the general case, without considering decoherence effects, this expression shows an oscillatory behavior due to the phases α and β. If the coherence conditions for the production and detection (4.21) are satisfied, the evolution matrix of the matter eigenstates T becomes the propagator in the matter eigenstate basis F_{matt} (see eq. (5.4)).
Now we want to consider the decoherence effects coming from conditions (4.24) and (4.25). In general the oscillation probability has the form

$$P_{\alpha\beta} = \sum_{K,K',M,M'} \tilde{U}^*_{\alpha K}\tilde{U}_{\beta K'}\tilde{U}_{\alpha M}\tilde{U}^*_{\beta M'}T_{KK'}T^*_{MM'}\,. \tag{A.10}$$

Note that the elements of the evolution matrix T already contain the overlap functions. To describe the decoherence in the neutrino production process we multiply expression (A.10) by the Kronecker delta δ_{KM}, as the produced state is a certain matter eigenstate and not a coherent superposition of them. Analogously one multiplies expression (A.10) by the Kronecker delta $\delta_{K'M'}$ to include the effects of decoherence of the detection process. For example, if both conditions (4.24) and (4.25) are satisfied, the oscillation probability takes the form:

$$
\begin{aligned}
P_{\alpha\beta} &= \sum_{K,K',M,M'} \tilde{U}^*_{\alpha K}\tilde{U}_{\beta K'}\tilde{U}_{\alpha M}\tilde{U}^*_{\beta M'}T_{KK'}T^*_{MM'}\delta_{KM}\delta_{K'M'} \\
&= \sum_{K,K'} |\tilde{U}_{\alpha K}|^2|\tilde{U}_{\beta K'}|^2|T_{KK'}|^2 \,. \qquad \text{(A.11)}
\end{aligned}
$$

Rewritten explicitly for the two-flavour case, the flavour transition probability $P_{\alpha\beta}$ is simply

$$
\begin{aligned}
P_{\alpha\beta} &= \begin{bmatrix} \cos^2\tilde{\theta}(x) & \sin^2\tilde{\theta}(x) \\ \sin^2\tilde{\theta}(x) & \cos^2\tilde{\theta}(x) \end{bmatrix} \begin{bmatrix} 1-p' & p' \\ p' & 1-p' \end{bmatrix} \begin{bmatrix} \cos^2\tilde{\theta}(x_i) & \sin^2\tilde{\theta}(x_i) \\ \sin^2\tilde{\theta}(x_i) & \cos^2\tilde{\theta}(x_i) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} + \left(\frac{1}{2}-p'\right)\cos\left(2\tilde{\theta}(x)\right)\cos\left(2\tilde{\theta}(x_i)\right) & \frac{1}{2} - \left(\frac{1}{2}-p'\right)\cos\left(2\tilde{\theta}(x)\right)\cos\left(2\tilde{\theta}(x_i)\right) \\ \frac{1}{2} - \left(\frac{1}{2}-p'\right)\cos\left(2\tilde{\theta}(x)\right)\cos\left(2\tilde{\theta}(x_i)\right) & \frac{1}{2} + \left(\frac{1}{2}-p'\right)\cos\left(2\tilde{\theta}(x)\right)\cos\left(2\tilde{\theta}(x_i)\right) \end{bmatrix} . \qquad \text{(A.12)}
\end{aligned}
$$

We see that the flavour transition probability $P_{\alpha\beta}$ does not oscillate but takes an averaged value as already mentioned in chapter 4. The situation will be different if only one coherence condition (either production or detection) is violated. By multiplying eq. (A.10) with only one Kronecker delta δ_{KM} or $\delta_{K'M'}$ one can convince oneself that some oscillating terms are still present in the oscillation probability. The exception is the adiabatic regime, where only one condition (decoherence in the production or the detection processes) is enough to destroy the oscillatory terms in the flavour transition probabilities.

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Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den 14.12.2011 .....

(Alina Wilhelm)