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When do measures on the space of connections support the triad operators of loop quantum gravity?

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In this work we investigate the question under what conditions Hilbert spaces that are induced by measures on the space of generalized connections carry a representation of certain non-Abelian analogues of the electric flux. We give the problem a precise mathematical formulation and start its investigation. For the technically simple case of U(1) as gauge group, we establish a number of "no-go theorems" asserting that for certain classes of measures, the flux operators can not be represented on the corresponding Hilbert spaces. The flux-observables we consider, play an important role in loop quantum gravity since they can be defined without recurse to a background geometry and they might also be of interest in the general context of quantization of non-Abelian gauge theories. © *2011 American Institute of Physics.*
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I. INTRODUCTION

Loop quantumgravity (LQG) is a promising approach to the problem of finding a quantum theory of gravity and has led to many interesting insights (see Ref. 1 for an extensive and Ref. 2 for a shorter nontechnical review). It is based on the formulation of gravity as a constrained canonical system in terms of the Ashtekar variables,[3] a canonical pair of an SU(2)-connection A (in its real formulation), and a triad field E with a nontrivial density weight. Both of these take values on a spacial slice Σ of the spacetime. A decisive advantage of these new variables is that both connection and triad allow for a metric-independent way of integrating them to form more regular functionals on the classical phase space and hence make a quantization feasible.

Being a one-form, A can be integrated naturally (that is, without recurse to background structure) along differentiable curves e in Σ to form *holonomies,*

$$h_e[A] = \mathcal{P} \exp\left[i \int_e A\right] \qquad \in SU(2). \tag{1}$$

The density weight of E on the other hand is such that using an additional real covector field f^i it can be naturally integrated over oriented surfaces S to form a quantity,

$$E_{S,f}[E] = \int_S f^i (*E)_i, \tag{2}$$

analogous to the electric flux through S. Since the variables $h_e[A]$ and $E_{S,f}[E]$ do not rely on any background geometry for their definition, they are very natural in the context of diffeomorphism invariant theories, and many important results of LQG such as the quantization of area and volume are related to the choice of these variables as basic observables. With this choice, however, LQG is in sharp contrast to the usual formulation of (quantum) gauge theories in which it is assumed that only when integrated over three or even four dimensional regions in the spacetime, the quantum fields make sense as operators on some Hilbert space.

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All of this makes it worthwhile, to study the representation theory of the observables (1) and (2) in somewhat general terms. Indeed, the representations of the algebra of holonomies (1) is well studied and powerful mathematical tools have been developed.[4–7] It turns out that the cyclic representations are in one-to-one correspondence with measures on the space $\overline{\mathcal{A}}$ of (generalized) connections. We will briefly review some results of these works in Sec. II. About the representations of the flux variables on the other hand, not so much is known. Therefore, in Ref. 8 we considered the representation theory of the holonomies (or more precisely, a straightforward generalization, the cylindrical functions) *together* with the momentum variables $E_{S,f}$ in rather general terms. In the present paper, we continue this work by focusing on a specific aspect: given a cyclic representation of the cylindrical functions, is it possible to also represent the momentum variables on the same Hilbert space? It is well known that this is possible for a specific measure on the space of generalized SU(2) connections, the Ashtekar–Lewandowski measure, μ_{AL}. It is distinguished by its simple and elegant definition and by its invariance under diffeomorphisms of the spacial slice Σ. The representation it induces is therefore considered as the fundamental representation for LQG.

We will see below that at least in the somewhat simpler case when the gauge group is U(1) instead of SU(2), the Ashtekar–Lewandowski representation is not only distinguished, it is unique: μ_{AL} induces the only diffeomorphism invariant cyclic representation of the algebra of cylindrical functions which also carries a representation of the flux observables $E_{S,f}$. A more general result of this kind was recently established in Ref. 9.

Recently it became evident that to investigate the semiclassical regime of LQG it may be useful to also study representations that are not diffeomorphism invariant but encode information about a given classical background geometry. Interesting representations of this type for the algebra of cylindrical functions were discovered in Ref. 10 for the case of U(1) as gauge group and a suitable generalization for the SU(2) case was proposed in Ref. 11. However, these representations do not extend to representations of the cylindrical functions *and* the flux observables $E_{S,f}$. The original motivation of the present work was to remedy this and construct representations of both, cylindrical functions and fluxes which are different from the AL-representation. This turned out to be very difficult, however. Quite contrary to our original goal, the results of the present work show that the constraints put on by requiring the observables $E_{S,f}$ to be represented are quite tight and that consequently it is hard to come up with such a representation that is different from the Ashtekar–Lewandowski representation.

Since the purpose of the present work is to explore the territory, our results are mostly concerned with the case of U(1) as gauge group. This case is technically much less involved than that of a general compact gauge group because the representation theory of U(1) is so simple. We expect, however, that the generalizations of the results to other compact gauge groups are possible.

Our main results for the U(1) case are the following:

- There is no diffeomorphism invariant measure allowing for a representation of the flux observables other then μ_{AL}.
- The r-Fock measures, as well as any other measure obtained by "importing" a regular Borel measure on the space of Schwartz distributions to $\overline{\mathcal{A}}_{U(1)}$ with Varadarajan's method do not support a representation of the flux observables.
- Any measure which is "factorizing" in a certain technical sense will, if it supports a representation of the flux observables, be very close to μ_{AL}. Moreover, the only such measure which additionally is Euclidean invariant, is μ_{AL}.

After work on this paper was finished, new results were obtained that generalize the first point in our list considerably, using ideas presented here and in Ref. 8. The interested reader may want to take a look at Refs. 12 and 13.

Let us finish the introduction with a description of the rest of the present work.

In Sec. II, we prepare the ground by briefly reviewing the projective techniques that are used to define measures on $\overline{\mathcal{A}}$. Also, we give a description of these measures which will be used in establishing our results.

In Sec. III we state and explain a necessary and sufficient condition for a measure μ on $\overline{\mathcal{A}}$ to allow for a representation of the flux observables.

Section IV serves to investigate the condition found in Sec. III in detail in the case the gauge group is U(1). First we introduce the notation necessary for this special case and also specialize the condition. We then proceed to establishing our main results.

With Sec. V we close this work by discussing interpretation and possible consequences of our results and point out problems left open.

II. MEASURES ON THE SPACE OF GENERALIZED CONNECTIONS

We will start by briefly reviewing the projective techniques[5,6] which can be used to construct measures on the space of connections. Using these methods we then introduce a rather explicit representation for such measures which we use in the sequel.

Let Σ be a three dimensional, connected, analytic manifold.

Definition 2.1: By an (oriented) edge e in Σ we shall mean an equivalence class of analytic maps $[0, 1] \longrightarrow \Sigma$, where two such maps are considered equivalent if they differ by an orientation preserving analytic reparametrization.

A graph in Σ is defined to be a union of edges such that two distinct ones intersect at most in their endpoints. The endpoints of the edges contained in the graph will be referred to as its vertices and we will denote the set of edges of a graph γ by $E(\gamma)$.

Analyticity of the edges is required to exclude certain pathological intersection structures of the edges with surfaces which would render the Poisson brackets which will be introduced below ill-defined.

The set of graphs can be endowed with a partial order $\geq$ by stating that $\gamma' \geq \gamma$ whenever γ is contained in γ' in the sense that each edge of γ can be obtained as composition of edges of γ' and each vertex of γ is also a vertex of γ'. Clearly, with this partial order the set of graphs becomes a *directed* set.

Also note that if $\gamma' \geq \gamma$, one can obtain γ' from γ by subdividing edges of γ and adding further edges. Let us denote the graph obtained by subdividing an edge e of a graph γ by adding a vertex v^* by $\mathbf{sub}_{e,v^*}\gamma$ (see Fig. 1) and the graph obtained by adding an edge e to γ by $\mathbf{add}_e\gamma$ (see Fig. 2).

Let us consider a smooth principal fiber bundle over Σ with a compact connected structure group G and denote by $\mathcal{A}$ the space of smooth connections on this bundle. It turns out to be convenient to consider a slightly more general class of functions on $\mathcal{A}$ than the holonomies (1).

Definition 2.2: A function c depending on connections A on Σ just in terms of their holonomies along the edges of a graph, i.e.,

$$c[A] \equiv c(h_{e_1}[A], h_{e_2}[A], \ldots, h_{e_n}[A]), \qquad e_1, e_2, \ldots, e_n \quad edges \ of \ some \ \gamma,$$

where $c(g_1, \ldots, e_n)$, viewed as a function on G^n, is continuous, will be called cylindrical.

Now, given a graph γ, any connection of the principal fiber bundle over Σ gives rise to a connection over (the union of) the edges of γ. We denote the space of these connections, modulo gauge transformations that are trivial over the vertices of γ, as $\mathcal{A}_\gamma$. To each of these spaces $\mathcal{A}_\gamma$ there
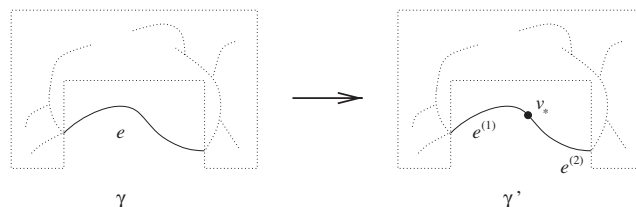

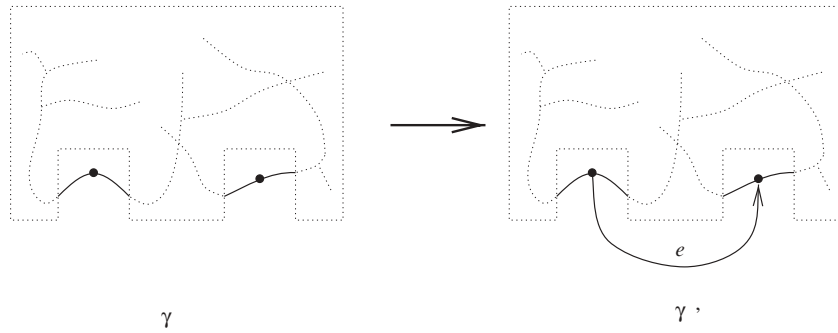FIG. 1. Operation $\mathbf{sub}_{e,v_*}$ subdividing an edge e of a graph.

FIG. 2. Operation **add**$_e$, adding an edge e to a graph (note that e does not necessarily have to begin and end in vertices of γ).

is a surjective map $\pi_\gamma : \mathcal{A} \longrightarrow \mathcal{A}_\gamma$. Moreover, these spaces decompose into the Cartesian product,

$$\mathcal{A}_\gamma = \underset{e \in E(\gamma)}{\times} \mathcal{A}_e.$$

There is a bijection between the elements of $\mathcal{A}_e$ and parallel transport maps between the fibers over the beginning and endpoints of e, and upon fixing an element of those fibers, there are bijections $\Lambda_e : \mathcal{A}_e \longrightarrow G$. Obviously these bijections are not unique. Combining the maps Λ_e of all edges of a graph γ, we also obtain a bijection $\Lambda_\gamma : \mathcal{A}_\gamma \longrightarrow G^{|\gamma|}$.

Finally, whenever $\gamma' \geq \gamma$ there is a projection map $p_{\gamma\gamma'} : \mathcal{A}_{\gamma'} \longrightarrow \mathcal{A}_\gamma$ such that $\pi_\gamma = p_{\gamma\gamma'} \circ \pi_{\gamma'}$. The spaces $\{\mathcal{A}_\gamma\}$ together with the maps $\{p_{\gamma\gamma'}\}$ form a *projective family*. Consequently, there is a space $\overline{\mathcal{A}}$, the projective limit of the projective family, containing all the $\mathcal{A}_\gamma$ with the appropriate inclusion relations implied by the projections $p_{\gamma\gamma'}$. The cylindrical functions of Definition 2.2 extend to functions on $\overline{\mathcal{A}}$ in a natural way. Their closure with respect to the sup-norm is an Abelian C* algebra which is usually denoted by $\overline{\text{Cyl}}$. Its spectrum can be identified with $\overline{\mathcal{A}}$, thus endowing it with a Hausdorff topology in which it is a compact space. This shows that $\overline{\mathcal{A}}$ is the natural home for the cylindrical functions. Moreover, as a consequence of the Riesz–Markov theorem, cyclic representations of Cyl are in one to one correspondence with positive Baire measures on $\overline{\mathcal{A}}$.

The projective techniques yield an elegant characterization of measures on $\overline{\mathcal{A}}$: on the one hand, a measure μ gives rise to a family of measures $\{\mu_\gamma\}$, where μ_γ is a measure on $\mathcal{A}_\gamma$ by pushing μ forward with the maps π_γ. The measures $\{\mu_\gamma\}$ bear *consistency relations* among each other: whenever $\gamma' \geq \gamma$ one has

$$p_{\gamma\gamma'*}\mu_{\gamma'} = \mu_\gamma.$$

On the other hand, it was shown in Ref. 7 that the converse also holds true: every consistent family $\{\mu_\gamma\}$ of measures on the $\mathcal{A}_\gamma$ gives rise to a measure μ on $\overline{\mathcal{A}}$. Moreover, properties of the measures translate between these two presentations: if the measure μ is normalized, so are the measures $\{\mu_\gamma\}$ and vice versa. If the measure μ is positive, so are the measures $\{\mu_\gamma\}$ and vice versa.

Finally note that via the maps Λ_γ described above, the family $\{\mu_\gamma\}$ can be pushed forward to obtain a family of measures on Cartesian products of G. We will denote these measures by $\{\tilde{\mu}_\gamma\}$. Everything said about the relation between μ and $\{\mu_\gamma\}$ certainly also holds for μ and $\{\tilde{\mu}_\gamma\}$.

In all of the following we will restrict ourselves to a certain subclass of measures on $\overline{\mathcal{A}}$: we just consider measures μ such that

$$d\tilde{\mu}_\gamma(g_1, \ldots, g_{|E(\gamma)|}) = f_\gamma(g_1, \ldots, g_{|E(\gamma)|}) \, d\mu_\text{H}(g_1) \ldots d\mu_\text{H}(g_{|E(\gamma)|}), \tag{3}$$

where μ_H is the Haar measure on G. Thus we exclude measures whose cylindrical projections pushed forward to $G^{|\gamma|}$, $(\Lambda_\gamma)_*\mu_\gamma$, would also contain a pure point part and a part singular with respect to the product of Haar measures on G. So in all of the following we will consider measures

which can be characterized in terms of a family $\{f_\gamma\}$ of functions:

$$f_\gamma : G^{|E(\gamma)|} \longrightarrow \mathbb{C},$$

via (3). Note that the representation in terms of these functions does, in general, depend on the choice of identifications $\{\Lambda_e\}$. The dependence is, however, a rather simple one. To simplify the notation, in the following we will assume a fixed choice of Λ_γ and do not explicitly distinguish anymore between families $\{\mu_\gamma\}$ and $\{\widetilde{\mu}_\gamma\}$. Note also that there is no dependence on the maps Λ_γ in the case that the measure μ is gauge invariant.

Let $\{f_\gamma\}$ be a family of functions defining a positive, normalized measure μ on $\overline{\mathcal{A}}$ by way of (3). Then positivity implies

$$f_\gamma \geq 0, \qquad \text{pointwise on } G^{|E(\gamma)|}, \text{ for all } \gamma. \qquad \text{(pos)}$$

Normalization implies

$$\int_G f_e(g)\, d\mu_\text{H}(g) = 1 \qquad \text{for all edges } e. \qquad \text{(norm)}$$

Consistency implies

$$f_\gamma(g_{e_1}, \ldots, g_{e_n}) = \int f_{\mathbf{add}_e(\gamma)}(g_e, g_{e_1}, \ldots, g_{e_n})\, d\mu_\text{H}(g_e), \qquad \text{(add)}$$

$$f_\gamma(g_{e_1}, \ldots, g_{e_i}, \ldots, g_{e_n}) = \int f_{\mathbf{sub}_{e_i}(\gamma)}(g_{e_1}, \ldots, g, g^{-1}g_{e_i}, \ldots, g_{e_n})\, d\mu_\text{H}(g). \qquad \text{(sub)}$$

It is easy to check, that also the converse holds true.

Proposition 2.3: Let a family $\{f_\gamma\}_\gamma$ of functions f_γ on $G^{|E(\gamma)|}$ be given that fulfills (pos), (norm), (add), and (sub). Then this family defines a positive normalized measure μ on $\overline{\mathcal{A}}$ by virtue of (3).

Let us close this section by pointing out that a wide variety of measures on $\overline{\mathcal{A}}$ has been constructed with the projective techniques reviewed above. Most important is perhaps the Ashtekar–Lewandowski measure μ_AL (Ref. 6) which is obtained by setting all f_γ equal to 1. Other measures are the diffeomorphism invariant Baez measures,[14] the heat kernel measure of Ref. 5, the r-Fock measures constructed from the Gaussian measure of the free electromagnetic field,[10] and the measures obtained with the complexifier method.[15]

III. ADMISSIBILITY

Up to now we have only considered representations of the algebra Cyl of cylindrical functions. Now we will turn to the momentum observables $E_{S,f}$ defined in (2). First, we should note that to avoid pathologies, it is required to put certain restrictions on the surfaces S to be considered. In the following, we will always assume that the surfaces S are analytically embedded in Σ, simply connected and such that $S = S - \partial S$. We caution the reader that we will not always explicitly state this in the following. Also let us restrict the vector fields f used in the definition of the $E_{S,f}$ to be smooth and bounded. Under these assumptions, one can compute the Poisson brackets for the $c \in \text{Cyl}$ with the $E_{S,f}$:[16]

$$\{E_{S,f}, c\} = X_{S,f}[c], \quad \text{where} \quad X_{S,f}[c] = \frac{\kappa}{2} \sum_{v \in S \cap \gamma} \sum_{e \in E(v)} \sigma(v, e) f_i(v) X_e^i[c]. \qquad (4)$$

In this formula we have assumed without loss of generality that all the intersections of γ and S are the vertices of γ. Moreover, X_e denotes the right or left invariant vector-field on SU(2) depending on whether e is ingoing or outgoing, respectively, acting on the entry corresponding to e of c written as a function on SU(2)$^{|E(\gamma)|}$. Finally, $\sigma(v, e)$ is the sign of the natural pairing between orientation two-form on S and tangent of e in v (and 0 if e is tangential). κ is the coupling constant of gravity.

In the present section we are going to consider the following problem: given a measure μ on $\overline{\mathcal{A}}$, what are the conditions μ has to satisfy in order to allow for a representation of the $E_{S,f}$ on the Hilbert space $\mathcal{H} = L^2(\overline{\mathcal{A}}, d\mu)$ by self-adjoint operators?

Since the operators representing the $E_{S,f}$ will in general be unbounded, it is necessary to make the notion "representation" in the question formulated above a bit more precise by putting some requirement on the domains of the operators representing the $E_{S,f}$. In Ref. 8 we have argued that a reasonable requirement is that the smooth cylindrical functions Cyl^∞ be in those domains. Let us adopt this requirement and cite from Ref. 8 a simple criterion for a measure μ to carry such a representation.

Proposition 3.1: Let a positive measure μ on $\overline{\mathcal{A}}$ be given. Then a necessary and sufficient condition for the existence of a representation of the $E_{S,f}$ on $\mathcal{H} = L^2(\overline{\mathcal{A}}, d\mu)$ by symmetric operators with domains containing Cyl^∞ is that the derivations $X_{S,f}$ map differentiable cylindrical functions that are zero almost everywhere with respect to μ to functions that are again zero μ-a.e. (condition (null) of Ref. 8), that they extend to Cyl^∞ as well-defined operators and that for each surface S and covector field f on S there exists a constant $C_{S,f}$ such that

$$\left| \Delta_{S,f}(c, \mathbf{1}) \right| \leq C_{S,f} \, \|c\|_{\mathcal{H}} \qquad \text{for all } c \in \text{Cyl}^\infty, \tag{5}$$

where the sesquilinear form $\Delta_{S,f}$ is given by

$$\Delta_{S,f}(c, c') \doteq \left\langle \pi\left(X_{S,f}[c]\right) \mathbf{1}, \, \pi\left(c'\right) \mathbf{1} \right\rangle_{\mathcal{H}} - \left\langle \pi\left(c\right) \mathbf{1}, \, \pi\left(X_{S,f}[c']\right) \mathbf{1} \right\rangle_{\mathcal{H}}, \qquad c, c' \in \text{Cyl}.$$

In the following, we will also denote by $\Delta_{S,f}(c)$ the antilinear form $\Delta_{S,f}(c, \mathbf{1})$.

Let us sketch how this result comes about. The Poisson brackets (4) suggest to represent the $E_{S,f}$ as $\pi(E_{S,f}) = i\hbar X_{S,f}$, since this obviously promotes these brackets to commutation relations. The problem is that despite the i in the definition of π suggested above, the $\pi(E_{S,f})$ will in general not be symmetric, since the measure can have a nonvanishing "divergence" with respect to the vector fields $X_{S,f}$, i.e., formally

$$i\hbar X_{S,f}[d\mu] \neq 0.$$

Certainly this equation does not make sense as it stands. However, the form $\Delta_{S,f}$ defined in Proposition 3.1 is the appropriate definition for this divergence. The condition on μ exhibited in Proposition 3.1 is simply the requirement that $\Delta_{S,f}$ be given by an L^2 function, $F_{S,f}$, say. If this is the case, we can represent $E_{S,f}$ as $\pi(E_{S,f}) = i\hbar X_{S,f} + \hbar F_{S,f}/2$, which, as can be easily checked, is symmetric.

Before we proceed, let us make two remarks: the first one is that *a priori* it is only necessary to require the "divergences" $\Delta_{S,f}$ to be operators on $\mathcal{H}$, i.e., they do not have to be square integrable. As soon as one requires Cyl^∞ to be part of the domain, they are automatically L^2. The second remark we want to make is that it was realized already in Ref. 5, that when considering more general measures then μ_{AL}, a divergence term will have to be added to the $X_{S,f}$ to make them symmetric. The requirement of *compatibility* between measure and vector-field used there, seems too restrictive, however, since it implies that the divergence is a cylindrical function.

Let us call a surface S *admissible* with respect to a positive measure μ, if the action of the derivations $X_{S,f}$ can be extended to equivalence classes under the measure μ, and if furthermore the $\Delta_{S,f}$ are in $L^2(\overline{\mathcal{A}}, d\mu)$ for all smooth covector fields f. Then the following is a simple corollary of Proposition 3.1.

Proposition 3.2: A surface S is admissible with respect to a positive measure μ, coming from a family $\{f_\gamma\}$ via (3) iff for all covector fields f there is a constant C_f such that

$$\|X_{S,f}[\ln f_\gamma]\|_{L^2(\mathcal{A}_\gamma, d\mu_\gamma)} \leq C_{S,f} \qquad \text{for all graphs } \gamma,$$

where the f_γ are the functions characterizing μ according to (3).

Proof: Assume S to be admissible with respect to μ, and let F be cylindrical on γ. Then there are constants $C_{S,f}$ such that

$$\left| \Delta_{S,f}(F) \right| = \left| \int \overline{F} X_{S,f}[\ln f_\gamma] f_\gamma \, d\mu_{\text{H}}^{|E(\gamma)|} \right| \leq C_{S,f} \, \|F\|_\mu = C_{S,f} \, \|F\|_{L^2(\mathcal{A}_\gamma, d\mu_\gamma)},$$

so $X_{S,f}[\ln f_\gamma]$ is in $L^2(\mathcal{A}_\gamma, d\mu_\gamma)$. This allows us to plug it into $\Delta_{S,f}$, yielding

$$\|X_{S,f}[\ln f_\gamma]\|_\mu^2 = |\Delta_{S,f}(X_{S,f}[\ln f_\gamma])| \le C_{S,f}\|X_{S,f}[\ln f_\gamma]\|_\mu,$$

whence $\|X_{S,f}[\ln f_\gamma]\|_{\mu_\gamma} \le C_{S,f}$, independently of γ.

Vice versa, assume that there are constants $C_{S,f}$, such that $\|X_{S,f}[\ln f_\gamma]\|_\mu \le C_{S,f}$, independently of γ. Then for F cylindrical on γ,

$$
\begin{aligned}
|\Delta_{S,f}(F)| &= \left| \int \overline{F} X_{S,f}[\ln f_\gamma] f_\gamma \, d\mu_{\mathrm{H}}^{|E(\gamma)|} \right| = |\langle F, \, X_{S,f}[\ln f_\gamma]\rangle_{L^2(\mathcal{A}_\gamma, d\mu_\gamma)}| \\
&\le \|F\|_{L^2(\mathcal{A}_\gamma, d\mu_\gamma)} \|X_{S,f}[\ln f_\gamma]\|_{L^2(\mathcal{A}_\gamma, d\mu_\gamma)} \\
&\le C_{S,f}\|F\|_{L^2(\overline{\mathcal{A}}, d\mu)},
\end{aligned}
$$

independently of γ. $\qquad\qquad\square$

IV. ADMISSIBILITY IN THE U(1) CASE

Up to now, our considerations applied either to a general compact connected gauge group, or at least to SU(2). From now on, we will turn our attention to the analogous but technically much simpler case $G = \mathrm{U}(1)$, i.e., the electromagnetic field. We caution the reader that from now on all measures are assumed to be obtained from families of functions $\{f_\gamma\}$ via (3), without explicitly stating it.

Let us start by introducing some notation. We will be very brief and refer to Ref. 10 for more thorough information on the U(1) theory. Let us parametrize U(1) as $g(\varphi) = \exp i\varphi$ where φ is in $[0, 2\pi]$. The Haar measure is then simply given by $d\varphi/2\pi$. The irreducible representations are labeled by $n \in \mathbb{Z}$ and are parametrized by $\pi_n(g(\varphi)) = \exp in\varphi$.

In the following, we will denote by $\overline{\mathcal{A}}_{\mathrm{U}(1)}$ the space of generalized U(1) connections. The *charge network functions* $T_{\gamma,\underline{n}}$ on $\overline{\mathcal{A}}_{\mathrm{U}(1)}$ are defined as

$$T_{\gamma,\underline{n}}(\varphi_1, \ldots, \varphi_{|E(\gamma)|}) = e^{in_1\varphi_1} \ldots e^{in_{|E(\gamma)|}\varphi_{|E(\gamma)|}}.$$

They form an orthonormal basis in $L^2(\overline{\mathcal{A}}_{\mathrm{U}(1)}, d\mu_{\mathrm{AL}})$. Let us also introduce their integrals with respect to a measure μ,

$$f_\gamma^{(n)} := \int \overline{T_{\gamma,\underline{n}}} \, d\mu.$$

Since $f_\gamma^{(n)}$ is nothing else then a specific Fourier coefficient of the function f_γ, we will call the family $\{f_\gamma^{(n)}\}$ the *Fourier coefficients of the measure* μ. The requirements (norm), (pos), (sub), and (add) have straightforward analogs in the family $\{f_\gamma^{(n)}\}$. We furthermore note that $\left\|T_{\gamma,\underline{n}}\right\|_\mu = 1$ for any normalized μ.

For the U(1) theory, the covector-fields f in the definition of $E_{S,f}$ are just functions, so we can simplify notation by replacing them all by 1. The action of the vector-fields X on the charge network functions then read

$$X_S[T_{\gamma,\underline{n}}] = \frac{\kappa}{2}\left(\sum_{e\in E(\gamma)} I(S, e)n_e \right)T_{\gamma,\underline{n}},$$

where $I(S, e)$ is the *signed intersection number* of e and S, which we define as follows: call an intersection of an edge e and a surface S *proper* if it is not the start or endpoint of e and e is transversal to S at the intersection. Let $P_\pm$ be the number of proper intersections of S and e with positive/negative relative orientation of S and e at the intersection point and $I_\pm$ the number of intersections with positive/negative relative orientation that are not proper. Then $I(S, e) = P_+ - P_- + (I_+ - I_-)/2$. Finally, we note the following useful formula:

$$\Delta_S(T_{\gamma,\underline{n}}) = \frac{\kappa}{2}\left(\sum_{e\in E(\gamma)} I(S, e)n_e \right)f_\gamma^{(n)}. \qquad (6)$$

We will now examine more closely the admissibility of surfaces with respect to measures on $\overline{\mathcal{A}}_{U(1)}$. We will see that the reasons for surfaces not to be admissible can be manifold: first, note that since the representation labels n_e in (6) can be arbitrary large, the admissibility requires that the higher Fourier components of the measure have to be suitably damped. This is the reason why the r-Fock measures do not have admissible surfaces, as we will show below.

Another reason for nonadmissibility is that the vector fields X_S act on cylindrical functions as the sums of derivatives. The number of terms in this sum can be very large when the intersections of graph and surface become numerous. Therefore to allow for admissible surfaces, f_γ must contain sufficient information about the geometry of the edges contained in γ to tell how many times an edge can intersect with a given surface. This is not possible if the measure is required to be diffeomorphism invariant—we will prove below that, with the exception of μ_{AL}, such measures do not have any admissible surfaces.

Finally, for the same reason f_γ has to contain information about the positions of the edges of γ relative to each other. We will see below that this forces *factorizing measures*, i.e., measures for which the f_γ factorize into a product of functions just depending on single edges, to be extremely close to μ_{AL} in a certain sense, if they are to allow for admissible surfaces.

A. Diffeomorphism invariant measures

Of special importance for quantum gravity are measures that do not depend on any geometric background structures (such as a metric or a connection) on Σ. The requirement of background independence can be formalized as follows: analytic diffeomorphisms ϕ naturally act on the space of graphs by mapping a graph γ to its image $\phi(\gamma)$ which clearly is a graph again. Consequently, they also act on cylindrical functions by

$$F[A] \equiv F(h_{e_1}[A], \ldots, h_{e_n}[A]) \longmapsto F(h_{\phi(e_1)}[A], \ldots, h_{\phi(e_n)}[A]) =: \phi(F)[A].$$

A measure μ is called invariant under analytic diffeomorphisms, or shorter, *diffeomorphism invariant*, if

$$\int F \, d\mu = \int \phi(F) \, d\mu$$

for all $F \in \text{Cyl}$ and all analytic diffeomorphisms ϕ. A measure coming from a family $\{f_\gamma\}$ of functions in the sense of (3) is clearly diffeomorphism invariant iff $f_\gamma = f_{\phi(\gamma)}$ for all graphs and all analytic diffeomorphisms ϕ.

Examples of diffeomorphism invariant measures are the Baez measures[7] and μ_{AL}. A bit surprisingly, it turns out that at least in the U(1) case considered here, μ_{AL} is the only such measure that has admissible surfaces. This shows again that μ_{AL} is a very special measure.

Proposition 4.1: Let μ be a diffeomorphism invariant normalized measure on $\overline{\mathcal{A}}_{U(1)}$ coming from a family $\{f_\gamma\}$ of functions in the sense of (3). If there exists an open, simply connected surface admissible with respect to μ, then $\mu = \mu_{AL}$.

Before proving this Proposition, we have to provide a rather technical lemma.

Lemma 4.2: Given a graph γ, an open, simply connected surface S and a vector $\underline{m} = (m_1, \ldots, m_{|E(\gamma)|}) \in \mathbb{Z}^{|E(\gamma)|}$, there is an analytic diffeomorphism $\phi_{\underline{m}}^A$ of Σ such that

$$I(S, \phi_{\underline{m}}^A(e_1)) = m_1, \quad \ldots \quad I(S, \phi_{\underline{m}}^A(e_{|E(\gamma)|})) = m_{|E(\gamma)|},$$

where $e_1, \ldots, e_{|E(\gamma)|}$ are the edges of γ.

Proof: Let us start by considering a single edge e. The first observation we make is that for arbitrary $m \in \mathbb{Z}$ there is a smooth diffeomorphism ϕ_m of Σ such that $I(S, \phi_m(e)) = m$. For example, this diffeomorphism might "drag out" some part of e to create the desired intersections but it might be the identity far away from e (see Fig. 3). Note that such a diffeomorphism exists, whether e is a
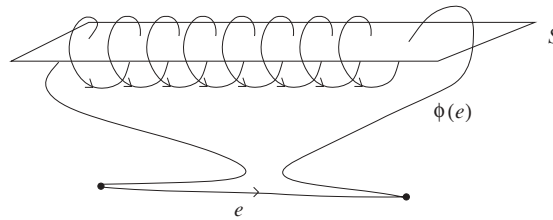
FIG. 3. An example for e and $\phi(e)$.

loop or not, that its existence depends however on just admitting surfaces that are open and simply connected.

The problem is that ϕ_m will in general not be analytic. Therefore we have to establish now that there is even an analytic diffeomorphism ϕ_m^A which does the same job, i.e., for which $I(S, \phi_m^A(e)) = m$. We will do this by using powerful mathematical results on real analytic manifolds and analytic approximations to diffeomorphisms.

Note first that since S does not contain its boundary, all intersections of $\phi_m(e)$ with S happen in the inside of S. Therefore there is a tube T containing $\phi_m(e)$ such that every other curve starting at the starting point and ending at the endpoint of $\phi_m(e)$ and lying entirely inside T will also have the signed intersection number m with S (see Fig. 4). Therefore, if we manage to map e into T such that the endpoints of the image are close to that of $\phi_m(e)$, we are done.

Let us first recall the famous result by Grauert[17] that every real analytic paracompact manifold can be analytically embedded into $\mathbb{R}^n$ for some large enough n. Therefore we will, without loss of generality, regard Σ as embedded into $\mathbb{R}^n$ in this way.

Now R^n comes with a globally defined metric $\rho(p, p') = \| p - p' \|$ and induces a metric tensor on Σ that locally on Σ gives rise to a metric ρ'. As Σ is embedded, the topology induced by ρ on Σ and that given by ρ' coincide. That said, we turn to a powerful result by Huebsch and Morse[18] on approximations of diffeomorphisms of manifolds analytically embedded in $\mathbb{R}^n$: Theorem 1.4 of Ref. 18, slightly specialized to our needs, reads the following.

Theorem 4.3 (Theorem 1.4 of Ref. 18): *Let Σ be analytically embedded in $\mathbb{R}^n$, ϕ a smooth diffeomorphism of Σ, and $\delta > 0$. Then there is an analytic diffeomorphism ϕ^A of Σ that satisfies*
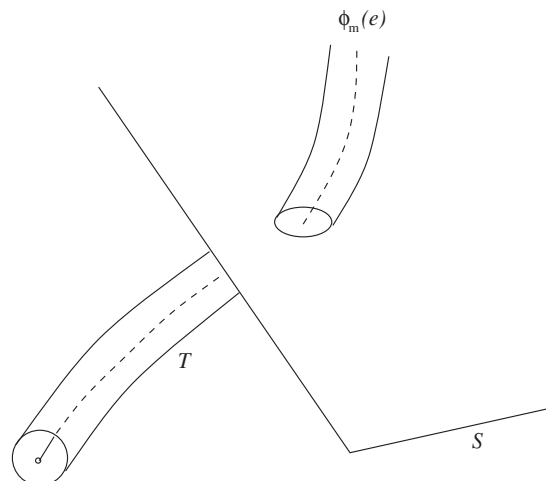
$$\rho(\phi(p), \phi^A(p)) \leq \delta.$$

FIG. 4. The tube T.

That is, we can approximate ϕ_m by an analytic diffeomorphism, the quality of the approximation being measured by ρ. What remains to be shown is that this also gives an approximation controlled by ρ'. To this end, let us parameterize e by $t \in [0, 1]$ and let ϵ be chosen such that the ϵ-balls $U_\epsilon(e(t))$ with respect to ρ' in Σ lie within the tube T for all parameter-values t. Since the topology coming from ρ' and that induced by ρ on Σ coincide, we can find $\delta(t) > 0$ such that the intersection of the δ-ball $U_{\delta(t)}(e(t))$ in $\mathbb{R}^n$ with Σ lies within $U_\epsilon(e(t))$. Since the embedding of Σ and the metric tensors are all continuous, we can choose $\delta(t)$ continuous. Letting δ be the (nonzero) minimum of $\delta(t)$, we can invoke the above theorem to find ϕ_m^A which indeed sends all the points of e into T, the endpoints of the image being close to those of $\phi_m(e)$. Therefore, $I(S, \phi_m^A(e)) = m$ and we have proven the theorem for the case of a single edge.

A similar reasoning shows the existence of an analytic diffeomorphism $\phi_{\underline{m}}^A$, mapping a given graph γ to one such that $I(S, \phi_{\underline{m}}^A(e_1)) = m_1$, $I(S, \phi_{\underline{m}}^A(e_2)) = m_2, \ldots$: again it is not hard to see that there is a smooth diffeomorphism doing the job and that it can be suitably approximated by an analytic diffeomorphism. Since no new idea but just a lot more notation is needed in this case, we refrain from giving the details. $\qquad\square$

With the above lemma at hand, the proof of the proposition is now straightforward.

Proof of Proposition 4.1: Let μ be a diffeomorphism invariant normalized measure on $\overline{\mathcal{A}}$ and S be a surface which is admissible with respect to μ. Pick an arbitrary graph γ and a vector $\underline{m} \in \mathbb{Z}^{|E(\gamma)|}$ and denote the analytic diffeomorphism provided by Lemma 4.2 by $\phi_{\underline{m}}$. One computes

$$\left| \Delta_S(T_{\underline{n}, \phi_{\underline{m}}(\gamma)}) \right| = \hbar \frac{\kappa}{2} \left| \sum_I n_I m_I \right| \left| f_{\phi_{\underline{m}}(\gamma)}^{(\underline{n})} \right| = \hbar \frac{\kappa}{2} \left| \sum_I n_I m_I \right| \left| f_\gamma^{(\underline{n})} \right|,$$

where the last equality is due to the diffeomorphism invariance of the measure. Since S is assumed to be admissible, this has to be bounded independently of $\underline{n}$. But as $\underline{m}$ is arbitrary, $|\sum_I n_I m_I|$ can be made arbitrarily large whenever $\underline{n} \neq 0$. So in this case $f_\gamma^{(\underline{n})}$ has to be zero. $f_\gamma^{(0)}$ has to be 1 due to normalization. Since γ was arbitrary, we have shown that indeed $\mu = \mu_{\mathrm{AL}}$. $\qquad\square$

B. Factorizing measures

Let $\{f_\gamma\}_\gamma$ be a family defining a positive normalized measure on $\overline{\mathcal{A}}_{\mathrm{U}(1)}$. We will call this measure *factorizing* if

$$f_\gamma(g_1, \ldots, g_{|E(\gamma)|}) = \prod_{i=1}^{|E(\gamma)|} f_{e_i}(g_i) \qquad where \qquad E(\gamma) = \{e_1, \ldots, e_{|E(\gamma)|}\}.$$

Examples for factorizing measures are the heat kernel measures and μ_{AL}. Factorizing measures are particularly easy to deal with, and regarding admissibility, we find the following.

Proposition 4.4: If μ is a positive, normalized, factorizing measure on $\overline{\mathcal{A}}_{\mathrm{U}(1)}$ (defined by a family of functions $\{f_e\}$) and S is an admissible surface with respect to μ. Then, of all the edges e intersecting S once and with a given relative orientation, at most countably infinitely many can have $f_e \neq 1$.

This result is a bit technical but it has interesting consequences, for example, for Euclidean invariant measures: if $\Sigma = \mathbb{R}^3$, we call a measure μ *Euclidean invariant* if for all cylindrical functions F and all Euclidean transformations T,

$$\int F \, d\mu = \int T(F) \, d\mu.$$
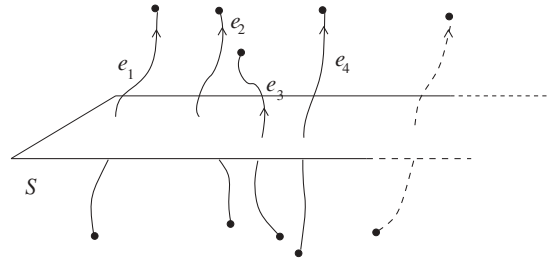
A consequence of Proposition 4.4 is

FIG. 5. Surface and graph considered in the proof of Proposition 4.4.

Corollary 4.5: Let μ be an Euclidean invariant positive, normalized, and factorizing measure on $\overline{\mathcal{A}}_{\mathrm{U}(1)}$, possessing an admissible surface. Then $\mu = \mu_{AL}$.

Let us first prove the proposition.

Proof of Proposition 4.4: Consider a surface S and a graph γ with N edges such that all edges are intersecting S exactly once and with the same relative orientation (see Fig. 5). An easy computation shows that

$$f_\gamma^{-1}(\underline{\varphi})|X_S[f_\gamma]|^2 = \sum_{I=1}^N \left| \frac{f'_{e_I}(\varphi_{e_I})}{f_{e_I}(\varphi_{e_I})} \right|^2 \prod_{K=1}^N f_{e_K}(\varphi_{e_K}) + \sum_{I \neq J} \frac{\overline{f'_{e_I}(\varphi_{e_I})}}{\overline{f_{e_I}(\varphi_{e_I})}} \frac{f'_{e_J}(\varphi_{e_J})}{f_{e_J}(\varphi_{e_J})} \prod_{K=1}^N f_{e_K}(\varphi_{e_K}).$$

Using (norm) and the symmetry properties of the derivatives with respect to the Haar measure, we see that the integral over the second term vanishes; therefore,

$$\int_{\mathrm{U}(1)^N} f_\gamma^{-1} |X_S[f_\gamma]|^2 \, d\mu_{\mathrm{H}}^N = \sum_{I=1}^N \int_0^{2\pi} \frac{1}{f_{e_I}} |\partial_\varphi f_{e_I}(\varphi)|^2 \, d\varphi. \tag{7}$$

Because of (pos), $1/f_{e_I}$ is strictly positive, so

$$\int_0^{2\pi} \frac{1}{f_{e_I}} |\partial_\varphi f_{e_I}(\varphi)|^2 \, d\varphi \geq 0,$$

with equality iff $f_{e_I}(\varphi) = const$. Normedness fixes the constant to be 1. On the other hand, because S is assumed to be admissible, Proposition 3.2 requires the right hand side of (7) to be bounded independently of γ. If there would be a more than countably infinite number of edges e intersecting S once and with the chosen relative orientation, for which f_e is nonconstant, there would be subsequences $e_1, e_2, \ldots$ among these edges such that with $\gamma_N := \cup_{I=1}^N e_N$, $\int f_\gamma^{-1} |X_S[f_\gamma]|^2 \, d\mu_{\mathrm{H}}^N$ would get arbitrarily large for large N. So at most a countably infinite number of these edges can have f_e nonconstant. $\qquad\square$

Let us now prove the corollary.

Proof of Corollary 4.5: Let μ be an Euclidean invariant, positive, normalized, factorizing measure and S be a surface admissible with respect to μ. With $\{f_e\}$, we denote the family of functions on $\mathrm{U}(1)$ defining it. Consider an arbitrary edge e. By Euclidean moves, it can always be mapped to an edge e' which intersects S at least once. Subdivide e' into edges $e'_1, e'_2, \ldots$ such that each of them intersects S *precisely* once. Consider one of those, e'_I. By moving it around by Euclidean moves, one can obtain an uncountable family of edges intersecting S once. Apply Proposition 4.4 and use Euclidean invariance to conclude $f_{e'_I} = 1$. Do this for all the $e'_1, e'_2, \ldots$. Then (sub) shows that $f_{e'} = 1$. Use Euclidean invariance to finally conclude $f_e = 1$. $\qquad\square$

C. Varadarajan measures

In Ref. 10 Varadarajan made the remarkable observation that one can "import" measures defined on the space of tempered distributions on $\mathbb{R}^3$ to $\overline{\mathcal{A}}_{U(1)}$. In this subsection, we will consider the properties of such imported measures with respect to admissibility of surfaces. Let for this purpose $\Sigma = \mathbb{R}^3$, equipped with the Euclidean metric. Furthermore denote with $\mathcal{S}$ the Schwarz test function space on $\mathbb{R}^3$.

We start with the observation that

$$h_e[A] = \exp i \int_e A \, ds = \exp i \int_{\mathbb{R}^3} F_e^{(0)}(x) A(x) \, d^3x,$$

where

$$F_e^{(0)i}(x) = \int_0^1 \dot{e}^i(t) \delta(x - e(t)) \, dt$$

is the "distributional formfactor" of the edge e. Using this notation, the Fourier transforms of a measure μ can be written as

$$f_\gamma^{(n)} = \int \exp -i \int_{\mathbb{R}^3} A(x) \left(\sum_I n_I F_{e_I}^{(0)} \right)(x) \, d^3x \, d\mu[A]. \tag{8}$$

An important observation of Varadarajan was that the right hand side of (8) formally has the same structure as the Fourier transform (or generating functional),

$$\mathcal{F}(F) = \int \exp -i \left(\int_{\mathbb{R}^3} A(x) F(x) d^3x \right) DA, \qquad F \in \mathcal{S}^3 \tag{9}$$

for a positive regular Borel measure DA on the space $(\mathcal{S}^3)'$ of tempered distributions. Such measures were extensively studied in a quantum field theory. The analogy between (8) and (9) is *a priori* only formal, because the $F_{e_I}^{(0)}$ of (8) are certainly not in $\mathcal{S}^3$. But Varadarajan realized that one can *define* measures on $\overline{\mathcal{A}}$ by setting

$$f_\gamma^{(n)} =: \int \exp -i \int_{\mathbb{R}^3} A(x) \left(\sum_I n_I F_{e_I} \right)(x) \, d^3x \, DA, \tag{10}$$

where now

$$F_e^i(x) := \int_0^1 \dot{e}^i(t) F(x - e(t)) \, dt \tag{11}$$

for a fixed positive $F \in \mathcal{S}^3$. Consistency of the $\{f_\gamma^{(n)}\}$ follows from the behavior of (11) under composition, and positivity and normedness of the resulting measure on $\overline{\mathcal{A}}$ from the corresponding properties of the measure DA on $(\mathcal{S}^3)'$.

A natural question to ask is whether the resulting measures have admissible surfaces. The answer is in the negative, as the following proposition shows.

Proposition 4.6: Any measure obtained from a regular normalized Borel measure on $(\mathcal{S}^3)'$ by Varadarajan's method as described above has no admissible surfaces.

Proof: Let μ be a measure obtained from a regular Borel measure DA on $(\mathcal{S}^3)'$ as described above, let S be any surface and e be an edge intersecting S precisely once. For convenience, let us choose a parametrization of e such that $e \cap S = e(1/2)$.

As a preparation we note that

$$|\Delta_S(T_{e,n})| = \hbar \frac{\kappa}{2} |n| \left| f_e^{(n)} \right| = \hbar \kappa \, |n| \, |\mathcal{F}(nF_e)| \,,$$

where $\mathcal{F}$ is the Fourier transform of DA and F_e is the form factor of e as defined above. Now we define the edge e_ϵ for $0 < \epsilon < 1$,

$$e_\epsilon(t) := e((1 - \epsilon)/2 + \epsilon t), \qquad t \in [0, 1].$$

Thus e_ϵ becomes shorter and shorter with vanishing ϵ but still intersects S. Moreover, it is easy to check that $F_{e_\epsilon} \longrightarrow 0$ for $\epsilon \longrightarrow 0$ in the topology of $\mathcal{S}^3$.

Now we appeal to the Bochner–Minlos theorem (see, for example, Ref. 19) which states that the Fourier transform $\mathcal{F}$ of DA is continuous (in the topology of $\mathcal{S}^3$) and that $\mathcal{F}(0) = 1$. Therefore, by making ϵ small, we can bring $\mathcal{F}(n F_{e_\epsilon})$ as close to 1 as we wish. As, on the other hand, $|n|$ can be arbitrarily large, we see that $|\Delta_S(T_{e_\epsilon, n})|$ can be made arbitrarily large. Hence it can not be bounded by a constant independent of e_ϵ, n and therefore S can not be admissible. Since S was completely arbitrary, this completes the proof. □

V. DISCUSSION

In the present paper, we have investigated under which circumstances, certain "flux-like" variables can (not) be represented on measure spaces over the space of generalized connections. This investigation was motivated by recent work on the semiclassical sector of loop quantum gravity. However, many of the results we have obtained concern only the simpler case of a U(1) gauge theory. Thus, an immediate task would be to generalize the results obtained in this work to other gauge groups, most notably to SU(2). However, in view of the results obtained above, we have to acknowledge already now that the task of finding interesting measures supporting the flux operators is more difficult then on might at first think. There are two ways to interpret these difficulties: one is to say that the background dependent measures lead to a different phase of the theory whose ultraviolet behavior is simply not suited for this kind of observables, so they cease to exist. A situation vaguely similar is encountered in quantum field theory on curved spacetimes. There, in some representations of the field algebra, the operator quantizing the stress energy tensor of the field is well defined. In other representations whose energy content is less well behaved, this operator is not well defined anymore (elementary discussion and further references in Ref. 20) The other way to interpret the difficulties is to maintain that we have simply not gathered enough experience with background dependent representations yet, to see how such measures with admissible surfaces have to be constructed.

At this point, we are quite dissatisfied with the ratio of mathematical to physical considerations we produced in the present work. Our guess from the experiences with the admissibility condition is that if measures other than μ_{AL} exist which have a sufficiently large number of admissible surfaces, then we are not likely to find them by chance or by changing the measures that have already been constructed just a little bit. Rather, one would need some idea from physics on how to construct such measures. If on the other hand, such measures do not exist, we would like to better understand why this is so from the point of view of a physicist. The works[10, 19, 21] among other things, take first steps in this direction. In any case, many interesting questions are still to be investigated and we hope to come back to some of them in future publications.

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