AVACS – Automatic Verification and Analysis of Complex Systems

REPORTS
of SFB/TR 14 AVACS

Editors: Board of SFB/TR 14 AVACS

System Description: H-PILoT Version 1.9

by

Carsten Ihlemann and Viorica Sofronie-Stokkermans

AVACS Technical Report No. 61
August 2010
ISSN: 1860-9821

Publisher: Sonderforschungsbereich/Transregio 14 AVACS
(Automatic Verification and Analysis of Complex Systems)
Editors: Bernd Becker, Werner Damm, Martin Fränzle, Ernst-Rüdiger Olderog,
Andreas Podelski, Reinhard Wilhelm
ATRs (AVACS Technical Reports) are freely downloadable from `www.avacs.org`

System Description: H-PILoT Version 1.9

Carsten Ihlemann and Viorica Sofronie-Stokkermans

Max-Planck-Institut für Informatik
Campus E1.4, Saarbrücken
e-mail: {ihlemann|sofronie}@mpi-inf.mpg.de

Abstract. This system description provides an overview of H-PILoT (Hierarchical Proving by Instantiation in Local Theory extensions), a program for hierarchical reasoning in extensions of logical theories. H-PILoT reduces deduction problems in the theory extension to deduction problems in the base theory. Specialized provers and standard SMT solvers can be used for testing the satisfiability of the formulae obtained after the reduction. For a certain type of theory extension (namely for *local theory extensions*) this hierarchical reduction is sound and complete and – if the formulae obtained this way belong to a fragment decidable in the base theory – H-PILoT provides a decision procedure for testing satisfiability of ground formulae, and can also be used for model generation.

Table of Contents

1 Introduction

H-PILoT (Hierarchical Proving by Instantiation in Local Theory extensions) is an implementation of the method for hierarchical reasoning in local theory extensions presented in [GSW04,GSW06,Sof05,Sof07]: it reduces the task of checking the satisfiability of a (ground) formula over the extension of a theory with additional function symbols subject to certain axioms (a set of clauses) to the task of checking the satisfiability of a formula over the base theory. The idea is to replace the set of clauses which axiomatize the properties of the extension functions by a finite set of instances thereof. This reduction is polynomial in the size of the initial set of clauses and is always sound. It is complete in the case of so-called *local extensions* [Sof05]; in this case, it provides a decision procedure for validity for the universal fragment of the theory extension (or alternatively for satisfiability of ground clauses w.r.t. the theory extension) if the clauses obtained by the hierarchical reduction belong to a fragment for which satisfiability is decidable in the base theory. The satisfiability of the reduced set of clauses is then checked with a specialized prover for the base theory.

State of the art SMT provers such as CVC3 [BT07], Yices [DdM06a,DdM06b] and Z3 [dMB08,BdM09] are very efficient for testing the satisfiability of *ground formulae* over standard theories, but use heuristics in the presence of *universally quantified* formulae, hence cannot detect *satisfiability* of such formulae. H-PILoT recognizes a class of local axiomatizations, performs the instantiation and hands in a ground problem to the SMT provers or other specialized provers, for which they are know to terminate with a yes/no answer, so it can be used as a tool for steering standard SMT provers, in order to provide decision procedures in the case of local theory extensions. H-PILoT can also be used for generating models of satisfiable formulae; and even more, it can be coupled to programs with graphic facilities to provide graphical representations of these models. Being a decision procedure for many theories important in verification, H-PILoT is extremely helpful for deciding truth or satisfiability in a large variety of verification problems.

This is an extended version of the description of H-PILoT presented at CADE 22 [ISS09].

2 Theoretical background

Many problems in mathematics and computer science can be reduced to proving the satisfiability of conjunctions of literals in a background theory (which can be the extension of a base theory with additional functions – e.g., free, monotone, or recursively defined – or a combination of theories). Considerable work has been dedicated to the task of identifying situations where reasoning in extensions and combinations of theories can be done efficiently and accurately. The most important issues which need to be addressed in this context are:

(i) finding possibilities of reducing the search space without losing completeness, and
(ii) making modular or hierarchical reasoning possible.

In [GM92,McA93,GM02], Givan and McAllester introduced and studied the so-called "local inference systems", for which validity of ground Horn clauses can be checked in polynomial time. A link between this proof theoretic notion of locality and algebraic arguments used for identifying classes of algebras with a word problem decidable in PTIME [Bur95] was established in [Gan01]. In [GSW04,GSW06,Sof05] these results were further extended to so-called *local extensions* of theories. Locality phenomena were also studied in the verification literature, mainly motivated by the necessity of devising methods for efficient reasoning in theories of pointer structures [MN05] and arrays [BMS06].

In [IJSS08] we showed that these results are instances of a general concept of locality of a theory extension – parameterized by a closure operator on ground terms. The main idea of locality and local extensions is to limit the search space for counterexamples (hence the name).

2.1 Local theory extensions

We consider the following setup. Let $\mathcal{T}_0$ be a theory in some signature Σ_0. We consider extensions $\mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}$ of $\mathcal{T}_0$ with function symbols in a set Σ_1 (extension functions) whose properties are axiomatized by a set $\mathcal{K}$ of (universally closed) $\Sigma_0 \cup \Sigma_1$-clauses. Let Σ_c be an additional set of constants.

Task. Let G be a set of ground $\Sigma_0 \cup \Sigma_1 \cup \Sigma_c$-clauses. We want to check whether or not G is satisfiable w.r.t. $\mathcal{T}_0 \cup \mathcal{K}$.

Method. Let $\mathcal{K}[G]$ be the set of those instances of $\mathcal{K}$ in which every subterm starting with an extension function is a ground subterm already appearing in $\mathcal{K}$ or G. If G is unsatisfiable w.r.t. $\mathcal{T}_0 \cup \mathcal{K}[G]$ then it is also unsatisfiable w.r.t. $\mathcal{T}_0 \cup \mathcal{K}$. The converse is not necessarily true.

Definition 1. *We say that the extension $\mathcal{T}_0 \cup \mathcal{K}$ of $\mathcal{T}_0$ is* local *if it satisfies the following condition*[1]*:*

(Loc) *For every set G of ground $\Sigma_0 \cup \Sigma_1 \cup \Sigma_c$-clauses it holds that $\mathcal{T}_0 \cup \mathcal{K} \cup G \models \perp$ if and only if $\mathcal{T}_0 \cup \mathcal{K}[G] \cup G \models \perp$.*

Thus, the method is sound and complete for *local theory extensions.*

Theorem 1 ([Sof05]). *Assume that the extension $\mathcal{T}_0 \subseteq \mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}$ is local and let G be a set of ground clauses. Let $\mathcal{K}^0 \cup G^0 \cup D$ be the purified form of $\mathcal{K} \cup G$ obtained by introducing fresh constants for the Σ_1-terms, adding their definitions $d \approx f(t)$ to D, and replacing $f(t)$ in G and $\mathcal{K}[G]$ by d. (Then Σ_1-functions occur only in D in unit clauses of the form $d \approx f(t)$.) The following are equivalent.*

1. *$\mathcal{T}_0 \cup \mathcal{K} \cup G$ has a (total) model.*
2. *$\mathcal{T}_0 \cup \mathcal{K}[G] \cup G$ has a partial model where all subterms of $\mathcal{K}$ and G and all Σ_0-functions are defined.*
3. *$\mathcal{T}_0 \cup \mathcal{K}^0 \cup G^0 \cup \mathsf{Con}^0$ has a total model, where $\mathsf{Con}^0 := \{ \bigwedge_{i=1}^{n} c_i \approx d_i \rightarrow c \approx d \mid f(c_1, ..., c_n) \approx c, f(d_1, ..., d_n) \approx d \in D\}$.*

A variant of this notion, namely Ψ-locality, was also studied, where the set of instances to be taken into account is $\mathcal{K}[\Psi(G)]$, where Ψ is a closure operator which may add a (finite) number of new terms to the subterms of G. We also analyzed a generalized version of locality, in which the clauses in $\mathcal{K}$ and the set G of ground clauses are allowed to contain first-order Σ_0-formulae.

2.2 Examples of local theory extensions

Among the theory extensions which we proved to be local or Ψ-local in previous work are:

- a fragment of the theory of pointers with stored scalar information in the nodes introduced in [MN05], further analyzed in [IJSS08,FIJS10];
- a fragment of the theory of arrays with integer indices, and elements in a given theory introduced in [BMS06], further analyzed in [IJSS08];
- theories of functions over an ordered domain or over a numerical domain satisfying monotonicity or boundedness conditions [Sof05,Sof06a,SI07a];
- various combinations of such extensions [Sof07,IJSS08].

We can also consider successive extensions of theories: $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \subseteq \cdots \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_n$. If every variable in $\mathcal{K}_i$ occurs below a function symbol in Σ_i, this reduction process can be iterated [IJSS08].

For local theory extensions, Theorem 1 allows us to reduce the original problem to a satisfiability problem over the base theory $\mathcal{T}_0$.

[1] It is easy to check that the formulation we give here and that in [Sof05] are equivalent.

3 Implementation

The software system H-PILoT (Hierarchical Proving by Instantiation in Local Theory extensions) for hierarchical reasoning in local theory extensions is implemented as follows: A given proof task (set of ground clauses), over the extension of a theory with functions axiomatized by a set of clauses, is reduced to an equi-satisfiable ground problem over the base theory in the manner of Theorem 1. After H-PILoT has carried out this reduction, it hands over the transformed problem to a dedicated prover for the base theory. This reduction is always sound. For local theory extensions the hierarchical reduction is sound and complete. If the formulas obtained in this way belong to a fragment decidable in the base theory, H-PILoT provides a decision procedure for testing satisfiability of ground formulas. If the reduced formulas are satisfiable (modulo the base theory), H-PILoT can be used for model generation, which is of great help in detecting and localizing errors.

3.1 Generalities

H-PILoT is implemented in Ocaml[2]. The system, together with a manual and examples, can be downloaded from `www.mpi-inf.mpg.de/~ihlemann/software/`. There is both a 32-bit and a 64-bit Linux version available.

To improve user-friendliness, a clausifier has also been integrated into H-PILoT (Sect. 12). H-PILoT recognizes a class of local axiomatizations; it has advanced abilities to handle the common data structures of *arrays* (Sect. 11.1) and *pointers* (Sect. 11.2)[3]. H-PILoT performs the instantiation and hands in a ground problem to the SMT provers or other specialized provers, for which they are known to terminate with a yes/no answer, so it can be used as a tool for steering standard SMT provers, in order to provide decision procedures in the case of local extensions. The provers integrated with H-PILoT are the general-purpose prover SPASS ([WDF+09]); the SMT-solvers Yices ([DdM06a,DdM06b]), CVC3 ([BT07]) and Z3 ([dMB08]); and the prover Redlog ([DS97]) for non-linear real problems. State-of-the-art SMT provers, such as the ones above, are very efficient for testing the satisfiability of ground formulas over standard theories, such as linear arithmetic (real, rational or integer), but use heuristics in the presence of universally quantified formulas, hence, cannot reliably detect satisfiability of such formulas. However, if SMT solvers are used for finding software bugs, being able to detect the actual satisfiability of satisfiable sets of formulas is crucial (cf. [dM09,dMB08,GdM09,BdM09]). For local theory extensions, H-PILoT offers the possibility of detecting satisfiability and of constructing models for satisfiable sets of clauses. On request, H-PILoT provides an extensive step-by-step trace of the reduction process, making its results verifiable (Sect. 15.2).

H-PILoT has been used in large case studies, where its ability (1) to handle *chains* of extensions, (2) to detect unsatisfiability *and* satisfiability, and (3) to construct models of satisfiable sets of clauses has been crucial.

[2] `http://caml.inria.fr/ocaml/index.en.html`

[3] H-PILoT automatically detects whether a given specification falls within the local fragment of these theories.

3.2 Structure of the program

The main algorithm which hierarchically reduces a decision problem in a theory extension to a decision problem in the base theory can be divided into a preprocessing part, the main loop and a post-processing part; see Figure 1.

Preprocessing. The input is read and parsed. If it is detected to be in SMT format, we set the options to "use arithmetic" (e.g., $+$, $-$,... are predefined). If the input is not in clause normal form (CNF), it is translated to CNF, then the input is flattened and linearized. The program then checks if the clauses in the axiomatization given are local extensions and sets the flag `-local` to true/false. This ends the preprocessing phase.

Main algorithm. The main loop proceeds as follows: We consider chains of extensions $\mathcal{T}_0 \subseteq \mathcal{T}_1 \subseteq \cdots \subseteq \mathcal{T}_n$, where $\mathcal{T}_i = \mathcal{T}_0 \cup \bigcup_{j=1}^{i} \mathcal{K}_j$ of $\mathcal{T}_0$ with function symbols in a set Σ_i (extension functions) whose properties are axiomatized by a set $\mathcal{K}_i$ of $(\Pi_0 \cup \bigcup_{j=1}^{i} \Sigma_j)$-clauses.

Let $i = n$. As long as the extension level i is greater than 0, we compute $\mathcal{K}_i[G]$ ($\mathcal{K}_i[\Psi]$ for arrays). If no separation of the extension symbols is required, we stop here (the result will be passed to an external prover that can reason about the extension of the theory $\mathcal{T}_0$ with free function symbols in Σ_n). Otherwise, we perform the hierarchical reduction by purifying $\mathcal{K}_i$ and G (to $\mathcal{K}_i^0$, G^0 respectively) and by adding corresponding instances of the congruence axioms Con_i. To prepare for the next iteration, we transform the clauses into the form $\forall \bar{x}.\Phi \vee \mathcal{K}_i$ (compute prenex form, skolemize). If $\mathcal{K}_i[G]$ $(i > 1)$ is not ground, we quit with a corresponding message. Otherwise we set $G' := \mathcal{K}_i^0 \wedge G^0 \wedge \mathsf{Con}_i$ and $\mathcal{T}' := \mathcal{T} \setminus \{\mathcal{K}_i\}$. We flatten and linearize $\mathcal{K}'$ and decrease i. If level $i = 0$ is reached G' is handed to an external prover.

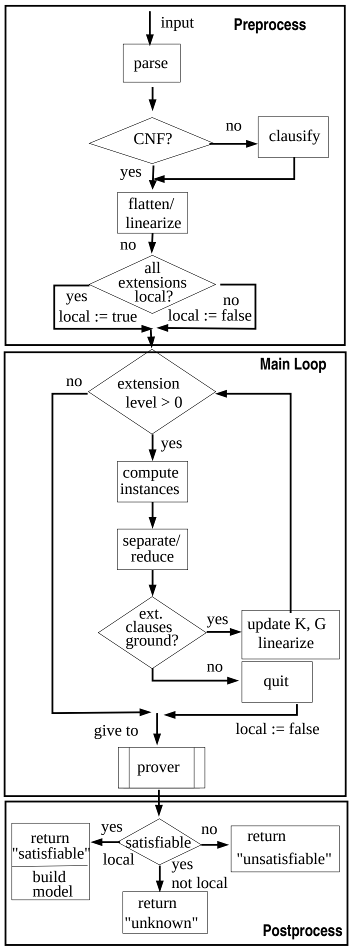


Fig. 1. H-PILoT Structure

Post-processing. If the answer is "unsatisfiable" then $G \models_{\mathcal{T}_n} \perp$. If the answer is "satisfiable" and all extensions were local, then G is satisfiable w.r.t. $\mathcal{T}_n$ and we know how to build a model. If the answer is satisfiable but we do not know that all extensions are local, or if the instantiated clauses (of level > 1) were not ground, we answer "unknown".

4 Modules of H-PILoT

We present the different parts of H-PILoT in more detail.

4.1 Preprocessing

H-PILoT receives as input a many-sorted specification of the signature; a specification of the hierarchy of local extensions to be considered; an axiomatization $\mathcal{K}$ of the theory extension(s); a set G of ground clauses containing possibly additional constants. H-PILoT allows the following preprocessing functionality.

Translation to clause form. H-PILoT provides a translator to clause normal form (CNF) for ease of use. First-order formulas can be given as input; H-PILoT translates them into CNF. In the present implementation, the CNF translator does not provide the full functionality of FLOTTER ([NW01]) – it has only restricted subformula renaming – but is powerful enough for most applications.

Flattening/linearization. Methods for recognizing local theory extensions usually require that the clauses in the set $\mathcal{K}$ extending the base theory are *flat* and *linear*, which does nothing to improve readability. If the flags -linearize and/or -flatten are used then the input is flattened and/or linearized (the general purpose flag -preprocess may also be used). H-PILoT allows the user to enter a more intelligible non-flattened version and will perform the flattening and linearization of $\mathcal{K}$.

Recognizing syntactic criteria which imply locality. Examples of local extensions include (fragments of) the theories of the common data structures: the theory of arrays (see Section 11.1) and the theory of pointers (see Section 11.2), respectively (and also iterations and combinations thereof). In the preprocessing phase H-PILoT analyzes the input clauses to check whether they are in one of these fragments.

– If the flag -array is on, H-PILoT checks if the input is in the "array property fragment".
– If the keyword "pointer" is detected, H-PILoT checks if the input is in the appropriate pointer fragment and adds missing "nullability" terms, i.e., it adds premises of the form "$t \neq \mathsf{null}$", in order to relieve the user of this clerical labor.

If the answer is "yes" then we know that the extensions we consider are local, i.e., that H-PILoT can be used as a decision procedure.

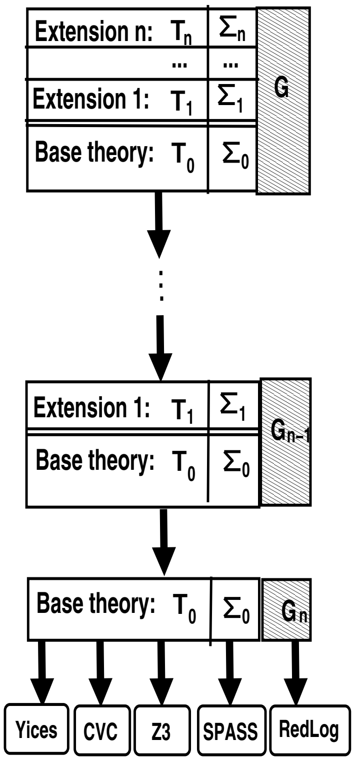
4.2 Main algorithm

The main algorithm hierarchically reduces a decision problem in a theory extension to a decision problem in the base theory.

Given a set of clauses $\mathcal{K}$ and a set of ground clauses G, the algorithm we use carries out a hierarchical reduction of G to a set of formulas in the base theory. It then hands over the new problem to a dedicated prover such as Yices, CVC3 or Z3. H-PILoT is also coupled with Redlog (for handling non-linear real arithmetic) and with SPASS[4].

Loop. For a chain of local extensions:

$$\mathcal{T}_0 \subseteq \mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}_1 \subseteq \mathcal{T}_2 = \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2 \\ \subseteq ... \subseteq \mathcal{T}_n = \mathcal{T}_0 \cup \mathcal{K}_1 \cup ... \cup \mathcal{K}_n.$$



a satisfiability check w.r.t. the last extension can be reduced (in n steps) to a satisfiability check w.r.t. $\mathcal{T}_0$. The only caveat is that at each step the reduced clauses $\mathcal{K}_i^0 \cup G^0 \cup \mathsf{Con}^0$ need to be ground. Groundness is assured if each variable in a clause appears at least once under an extension function. In that case, we know that at each reduction step the total clause size only grows polynomially in the size of G ([Sof05]). H-PILoT allows the user to specify a chain of extensions by tagging the extension functions with their place in the chain (e.g., if f belongs to $\mathcal{K}_3$ but not to $\mathcal{K}_1 \cup \mathcal{K}_2$ it is declared as level 3).

Let $i = n$. As long as the extension level i is larger than 0, we compute $\mathcal{K}_i[G]$ ($\mathcal{K}_i[\Psi(G)]$ in case of arrays). If no separation of the extension symbols is required, we stop here (the result will be passed to an external prover). Otherwise, we perform a hierarchical reduction by purifying $\mathcal{K}_i$ and G (to $\mathcal{K}_i^0, G^0$ respectively) and by adding corresponding instances of the congruence axioms Con_i. To prepare for the next iteration, we transform the clauses into the form $\forall x.\Phi \vee \mathcal{K}_i$ (compute prenex form, skolemize). If $\mathcal{K}_i[G]/\mathcal{K}_i^0$ is not ground, we quit with a corresponding message. Otherwise our new proof task G' becomes $G' := \mathcal{K}_i^0 \wedge G^0 \wedge \mathsf{Con}_i$, our new extension clauses are $\mathcal{K}' := \mathcal{K}_{i-1}$ and our new base theory becomes $\mathcal{T}' := \mathcal{T}_{i-1} \setminus \{\mathcal{K}_{i-1}\}$. We flatten and linearize $\mathcal{K}'$ and decrease i. If level $i = 0$ is reached, we exit the main loop and G' is handed to an external prover. Completeness is guaranteed if all extensions are known to be local and if each reduction step produces a set of ground clauses for the next step.

4.3 Post-processing

Depending on the answer of the external provers to the satisfiability problem G_n, we can infer whether the initial set G of clauses was satisfiable or not.

[4] H-PILoT only calls one of these solvers once.

- If G_n is unsatisfiable w.r.t. $\mathcal{T}_0$ then we know that G is unsatisfiable.
- If G_n is satisfiable, but H-PILoT failed to detect this, and the user did not assert the locality of the sets of clauses used in the axiomatization, its answer is "unknown".
- If G_n is satisfiable and H-PILoT detected the locality of the axiomatization, then the answer is "satisfiable". In this case, H-PILoT takes advantage of the ability of SMT-solvers to provide counterexamples for the satisfiable set G_n of ground clauses and specifies a counterexample of G by translating the basic SMT-model of the reduced proof task to a model of the original proof task. This improves readability greatly, especially when we have a chain of extensions. The counterexamples can be graphically displayed using Mathematica (cf. Section 15.3). This is currently done separately; an integration with Mathematica is planned for the future.

5 The input grammar

The input file consists of a *declaration* part (for function symbols in the base theory, for extension functions, for relations, and constants), specifications of the *types* and of the base theory, a part containing *axiomatizations* of base theory/extension functions; and a part containing the set of *ground clauses* whose satisfiability is being checked.

$\langle start \rangle$ ::= $\langle$*base_functions*$\rangle$ $\langle$*extension_functions*$\rangle$ $\langle$*relations*$\rangle$ $\langle$*constants*$\rangle$ $\langle$*interval*$\rangle$
$\langle$*baseTheory*$\rangle$ $\langle$*formulasOrClauses*$\rangle$ $\langle$*groundformulas*$\rangle$ $\langle$*query*$\rangle$

5.1 Declarations

Type declarations: We allow for declarations of standard types:

$\langle$*domain*$\rangle$ ::= `bool` | `int` | `real` | `pointer` | `pointer#` $\langle$*int*$\rangle$
| `scalar` | `free` | `free#` $\langle$*int*$\rangle$

Declarations of simple types such as intervals are also allowed:

$\langle$*interval*$\rangle$ ::= ϵ
| `Interval :=` $\langle$*int*$\rangle$ $\langle$*sm*$\rangle$ $\langle$*identifier*$\rangle$ `;`
| `Interval :=` $\langle$*identifier*$\rangle$ $\langle$*sm*$\rangle$ $\langle$*int*$\rangle$ `;`
| `Interval :=` $\langle$*int*$\rangle$ $\langle$*sm*$\rangle$ $\langle$*identifier*$\rangle$ $\langle$*sm*$\rangle$ $\langle$*int*$\rangle$ `;`

$\langle$*int*$\rangle$ ::= *any non-negative number.*

$\langle$*sm*$\rangle$ ::= `<=` | `<`

Further details are given in Section 10.

Function and relation declarations. The declaration part contains sort and type declarations for the function symbols in the base theory, for the extension function symbols, for the relation symbols and for the constants:

⟨*base_functions*⟩ ::= `Base_functions := {` ⟨*function_list*⟩ `}`
⟨*extension_functions*⟩ ::= `Extension_functions := {` ⟨*function_list*⟩ `}`
⟨*relations*⟩ ::= `Relations := {` ⟨*relation_list*⟩ `}`
⟨*constants*⟩ ::= ϵ | `Constants := {` constant_list `}`

⟨*constant_list*⟩ ::= ⟨*constant*⟩ ⟨*additional_constants*⟩
⟨*additional_constants*⟩ ::= ϵ | `,` ⟨*constant*⟩ ⟨*additional_constant*⟩

⟨*constant*⟩ ::= `(` ⟨*identifier*⟩ `, bool )`
  | `(` ⟨*identifier*⟩ `, int )`
  | `(` ⟨*identifier*⟩ `, real )`
  | `(` ⟨*identifier*⟩ `, scalar )`
  | `(` ⟨*identifier*⟩ `, pointer )`
  | `(` ⟨*identifier*⟩ `, pointer#` ⟨*int*⟩ `)`
  | `(` ⟨*identifier*⟩ `, free )`
  | `(` ⟨*identifier*⟩ `, free#` ⟨*int*⟩ `)`

⟨*function_list*⟩ ::= ϵ | ⟨*function*⟩ ⟨*additional_functions*⟩
⟨*additional_functions*⟩ ::= ϵ | `,` ⟨*function*⟩ ⟨*additional_functions*⟩
⟨*relation_list*⟩ ::= ϵ | ⟨*relation*⟩ ⟨*additional_relations*⟩
⟨*additional_relations*⟩ ::= ϵ | `,` ⟨*relation*⟩ ⟨*additional_relations*⟩

We allow for predefined relation declarations (e.g. for relations such as $\leq$ or $<$ as well as for new relation declarations. In each case we specify together with the relation symbols also their arity.

⟨*relation*⟩ ::= `(` ⟨*uneqs*⟩ `,` ⟨*int*⟩ `)` | `(` ⟨*identifier*⟩ `,` ⟨*int*⟩ `)`

We allow for several forms of function declaration: We can declare the number of arguments of the function ((1),(2)); the number of arguments and the level (for predefined arithmetical operations over the integers (3), or reals (4) or for uninterpreted functions (5)); the number of arguments, the level, and the sort of the domain and codomain (without repetitions if the domain and codomain are the same (6); separately specified if they are different (7)).

⟨*function*⟩ ::= `(` ⟨*identifier*⟩ `,` ⟨*int*⟩ `)` (1)
  | `(` ⟨*arithop*⟩ `,` ⟨*int*⟩ `)` (2)
  | `(` ⟨*arithop*⟩ `,` ⟨*int*⟩ `,` ⟨*int*⟩ `, int )` (3)
  | `(` ⟨*arithop*⟩ `,` ⟨*int*⟩ `,` ⟨*int*⟩ `, real )` (4)
  | `(` ⟨*identifier*⟩ `,` ⟨*int*⟩ `,` ⟨*int*⟩ `)` (5)
  | `(` ⟨*identifier*⟩ `,` ⟨*int*⟩ `,` ⟨*int*⟩ `,` ⟨*domain*⟩ `)` (6)
  | `(` ⟨*identifier*⟩ `,` ⟨*int*⟩ `,` ⟨*int*⟩ `,` ⟨*domain*⟩ `,` ⟨*domain*⟩ `)` (7)

At the moment declarations of functions which accept arguments of different sorts are not supported.

5.2 Axiomatizations

We support axiomatizations for the base theory:

$\langle \mathit{base\_theory} \rangle ::= \epsilon \mid \texttt{Base :=} \ \langle \mathit{clause\_list} \rangle$

axiomatizations of the properties of the extension functions:

$$\begin{aligned}\langle \mathit{formulasOrClauses} \rangle ::= {} & \epsilon \mid \langle \mathit{formulas} \rangle \mid \langle \mathit{clauses} \rangle \\ & \mid \langle \mathit{formulas} \rangle \langle \mathit{clauses} \rangle \mid \langle \mathit{clauses} \rangle \langle \mathit{formulas} \rangle\end{aligned}$$

as well as input of the ground formulae whose (un)satisfiability is being checked:

$\langle \mathit{formulas} \rangle ::= \texttt{Formulas :=} \ \langle \mathit{formula\_list} \rangle$

$\langle \mathit{formula\_list} \rangle ::= \langle \mathit{formula} \rangle \mid \langle \mathit{formula} \rangle \ \texttt{;} \ \langle \mathit{additional\_formulas} \rangle$

$\langle \mathit{additional\_formulas} \rangle ::= \epsilon \mid \langle \mathit{formula} \rangle \ \texttt{;} \ \langle \mathit{additional\_formulas} \rangle$

$$\begin{aligned}\langle \mathit{formula} \rangle ::= {} & \langle \mathit{atom} \rangle \\ & \mid \texttt{NOT (} \ \langle \mathit{formula} \rangle \ \texttt{)} \\ & \mid \texttt{OR (} \ \langle \mathit{formula} \rangle \ \langle \mathit{formula\_plus} \rangle \ \texttt{)} \\ & \mid \texttt{AND (} \ \langle \mathit{formula} \rangle \ \langle \mathit{formula\_plus} \rangle \ \texttt{)} \\ & \mid \texttt{(} \ \langle \mathit{formula} \rangle \ \texttt{-->} \ \langle \mathit{formula} \rangle \ \texttt{)} \\ & \mid \texttt{(} \ \langle \mathit{formula} \rangle \ \texttt{<-->} \ \langle \mathit{formula} \rangle \ \texttt{)} \\ & \mid \texttt{(FORALL} \ \langle \mathit{variables} \rangle \ \texttt{) .} \ \langle \mathit{formula} \rangle \\ & \mid \texttt{(EXISTS} \ \langle \mathit{variables} \rangle \ \texttt{) .} \ \langle \mathit{formula} \rangle\end{aligned}$$

$\langle \mathit{formula\_plus} \rangle ::= \texttt{,} \ \langle \mathit{formula} \rangle \ \langle \mathit{formula\_star} \rangle$

$\langle \mathit{formula\_star} \rangle ::= \epsilon \mid \texttt{,} \ \langle \mathit{formula} \rangle \ \langle \mathit{formula\_star} \rangle$

$\langle \mathit{ground\_formulas} \rangle ::= \epsilon \mid \texttt{Ground\_Formulas :=} \ \langle \mathit{formula\_list} \rangle$

$\langle \mathit{query} \rangle ::= \texttt{Query :=} \ \langle \mathit{ground\_clauses} \rangle$

$\langle \mathit{clauses} \rangle ::= \texttt{Clauses :=} \ \langle \mathit{clause\_list} \rangle$

$\langle \mathit{base\_clause\_list} \rangle ::= \epsilon \mid \langle \mathit{clause} \rangle \ \texttt{;} \ \langle \mathit{additional\_base\_clauses} \rangle$

$\langle \mathit{additional\_base\_clauses} \rangle ::= \epsilon \mid \langle \mathit{base\_clause} \rangle \ \texttt{;} \ \langle \mathit{additional\_base\_clauses} \rangle$

$\langle \mathit{base\_clause} \rangle ::= \langle \mathit{clausematrix} \rangle \mid \langle \mathit{universalQuantifier} \rangle \ \langle \mathit{clausematrix} \rangle$

$\langle \mathit{clause\_list} \rangle ::= \langle \mathit{clause} \rangle \mid \langle \mathit{clause} \rangle \ \texttt{;} \ \langle \mathit{additional\_clauses} \rangle$

$\langle \mathit{additional\_clauses} \rangle ::= \epsilon \mid \langle \mathit{clause} \rangle \ \texttt{;} \ \langle \mathit{additional\_clauses} \rangle$

$$\begin{aligned}\langle \mathit{clause} \rangle ::= {} & \langle \mathit{clausematrix} \rangle \\ & \mid \langle \mathit{universalQuantifier} \rangle \ \langle \mathit{clausematrix} \rangle \\ & \mid \{ \ \langle \mathit{formula} \rangle \ \} \ \texttt{OR} \ \langle \mathit{clausematrix} \rangle \\ & \mid \langle \mathit{universalQuantifier} \rangle \ \{ \ \langle \mathit{formula} \rangle \ \} \ \texttt{OR} \ \langle \mathit{clausematrix} \rangle \\ & \mid \{ \ \langle \mathit{formula} \rangle \ \} \ \texttt{-->} \ \langle \mathit{clausematrix} \rangle \\ & \mid \langle \mathit{universalQuantifier} \rangle \ \{ \ \langle \mathit{formula} \rangle \ \} \ \texttt{-->} \ \langle \mathit{clausematrix} \rangle\end{aligned}$$

```
⟨universalQuantifier⟩ ::= ( FORALL ⟨variables⟩ ) .
⟨variables⟩ ::= ⟨name⟩ ⟨additional_variable⟩
⟨additional_variables⟩ ::= ϵ | , ⟨name⟩ ⟨additional_variables⟩
⟨ground_clauses⟩ ::= ϵ | ⟨clausematrix⟩ ; ⟨ground_clauses⟩
⟨clausematrix⟩ ::= ⟨literal⟩ | ⟨disjunctive_clause⟩ | ⟨sorted_clause⟩

⟨disjunctive_clause⟩ ::= OR ( ⟨literal⟩ ⟨literal_plus⟩ )
⟨literal_plus⟩ ::= , ⟨literal⟩ ⟨literal_star⟩
⟨literal_star⟩ ::= ϵ | , ⟨literal⟩ ⟨literal_star⟩

⟨sorted_clause⟩ ::= ⟨atom_list⟩ --> ⟨atom_list⟩
⟨atom_list⟩ ::= ϵ | ⟨atom⟩ ⟨atom_star⟩
⟨atom_star⟩ ::= ϵ | , ⟨atom⟩ ⟨atom_star⟩

⟨literal⟩ ::= ⟨atom⟩ | NOT ( ⟨atom⟩ )
⟨atom⟩ ::= ⟨equality_atom⟩ | ⟨ineq_atom⟩ | ⟨predicate_atom⟩
⟨equality_atom⟩ ::= ⟨term⟩ = ⟨term⟩
⟨ineq_atom⟩ ::= ⟨term⟩ ⟨uneqs⟩ ⟨term⟩
⟨predicate_atom⟩ ::= ⟨identifier⟩ [ ⟨term⟩ ⟨additional_terms⟩ ]
⟨arguments⟩ ::= ⟨term⟩ ⟨additional_terms⟩
⟨additional_terms⟩ ::= ϵ | , ⟨term⟩ ⟨additional_terms⟩

⟨term⟩ ::= ⟨name⟩
        | ⟨operator⟩ ( ⟨arguments⟩ )
        | ⟨array⟩ ( ⟨arguments⟩ )
        | ⟨update⟩ ( ⟨arguments⟩ )
        | ⟨term_arith⟩ ⟨arithop⟩ ⟨term_arith⟩

⟨term_arith⟩ ::= ⟨name⟩
              | ⟨operator⟩ ( ⟨arguments⟩ )
              | ( ⟨term_arith⟩ ⟨arithop⟩ ⟨term_arith⟩ )

⟨arithop⟩ ::= + | - | * | /
⟨uneqs⟩ ::= <= | >= | < | >
⟨operator⟩ ::= ⟨identifier⟩

⟨array:⟩ ::= write ( ⟨identifier⟩ , ⟨term⟩ , ⟨term⟩ )
           | write ( ⟨array⟩ , ⟨term⟩ , ⟨term⟩ )

⟨update⟩ ::= update ( ⟨identifier⟩ , ⟨term⟩ , ⟨term⟩ )
           | update ( ⟨update⟩ , ⟨term⟩ , ⟨term⟩ )

⟨name⟩ ::= ⟨identifier⟩

⟨identifier⟩ ::= any string consisting of letters and numbers starting with a letter.
                 It may end with “’ ”.
```

6 Parameters of H-PILoT

H-PILoT has several input parameters controlling its behavior. They can be listed by calling `hpilot.opt -help`.

-min	Use minimal locality. Currently, this is only relevant for the array property fragment.
-prClauses	Produce output: print all the clauses calculated and used.
-noProver	Do not hand over to prover, just produce output.
-arith	Use arithmetic. 'plus','+','-' etc. are predefined. Numerals (names for integers) must be used preceded by underscore _, e.g., '_3'.
-yices	Use Yices as background solver: 'plus', '+' etc. are predefined as are the order relations $\leq, \geq, <, >$. Numbers can also be given in the input. Numbers are integers by default (use '-real' for real numbers).
-cvc	Use CVC as background solver. Arithmetic is predefined as with '-yices'.
-z3	Use Z3 as background solver. Arithmetic is predefined as with '-yices'.
-flatten	Flatten clauses first.
-linearize	Linearize clauses first.
-flattenQuery	Flattens the proof task first.
-preprocess	Preprocess input: flatten/linearize clauses, flatten proof task. In array-context: split clauses which contain inequalities like $i \neq j$ into two clauses.
-noSeparation	Stop at calculating the instances $\mathcal{K}[G]$. Don't introduce names for extension terms and don't reduce to base theory.
-unPseudofy	Eliminate pseudo-quantifiers like $\forall i.i = 3$... before handing over to a prover.[5]
-noProcessing	No computation. Just translate into prover syntax and hand over. Overrides '-preprocess'. When using this flag one should provide the domains of functions too. When used in combination with CVC there may arise problems with boolean types.[6]
-clausification	Toggle clausification (true/false). Default is 'true'. 'false' implies '-noProcessing'.
-real	Use reals instead of integers as the default type.
-redlog	Call Redlog for base prover. Assumes '-real'.
-version	Print version number.

[5] This is automatically carried out if we have a multiple-step extension. This is because the next step can only be carried out if the current step resulted in ground clauses.

[6] This is because CVC only provides booleans as bit-vectors of length 1. The type 'BOOLEAN' is the type of formulas.

-freeType	Enables the use of an unspecified type 'free' in addition to 'real' and 'int'. Only CVC, Z3 and Yices accept free types. Yices is default.
-renameSubformulas	Toggles the renaming of subformulas during clausification (true/false). Subformula renaming avoids exponential growth. Default is "true".
-verbosity	Verbosity level (0,1,2). From taciturn to garrulous. To be used in conjunction with '-prClauses'. Default is 0
-arrays	Use settings for array. This combines '-preprocess', '-min' and '-arith'; It also splits clauses on negative equalities.
-model	Gives a counter-model for satisfiable proof tasks. Needs Yices or CVC (implies Yices by default).
-smt	Produce SMT-LIB output without calling a prover.
-isLocal	Use this flag (true/false) to tell the program whether all the extensions are local or not. This matters only if H-PILoT cannot derive a contradiction. In that case this means that there really is none only if the extensions are local. Default is false.
-help	Display list of options

7 Error handling

In case there is a parsing error one can use

`export OCAMLRUNPARAM='p'` (in *bash* syntax).

This produces a walk-through of the parsing process, which is of great help in localizing syntax errors. To turn it back off use

`export OCAMLRUNPARAM=''`.

8 Application areas

H-PILoT has applications in *mathematics*, *multiple-valued logics*, *data- structures* and *reasoning in complex systems.*

Mathematics. An important example of local extensions are extensions with *monotone functions* over partially ordered sets [SI07a,SI07b]. We will give an example of how to use H-PILoT on problems involving monotonicity in Section 9 below. Another example from mathematics is an extension with *free functions*, i.e., we have an empty set of extension clauses $\mathcal{K}$ but the proof task G contains new function symbols. Even in this simple case, local and hierarchical reasoning is useful, because expanding the signature might already derail a back-end prover. For instance, consider real arithmetic. Linear arithmetic is tractable and can be handled quite efficiently by state-of-the-art SMT-solvers. Non-linear arithmetic is another matter, however (cf. [FHT+07,BPT07]). Here the options are more limited. To handle non-linear real arithmetic, H-PILoT is integrated with the prover *Redlog* ([DS97]). Since Redlog uses quantifier elimination for real closed fields, it relies on a fixed signature. In a case like this, H-PILoT can be employed as a preprocessor that eliminates the new function symbols in the proof task, allowing the user to employ free function symbols together with (non-linear) real arithmetic. Another example from mathematics, taken from [Sof05], is that of a *Lipschitz function.* There it was shown that any extension of the real numbers with a Lipschitz function is local.

Multiple-valued logics. Another important application area of local reasoning and, by the same token, H-PILoT is reasoning in multiple-valued logics. These logics have more than two truth values, in fact they allow as truth values the whole real interval of $[0, 1]$. The semantics are often given algebraically. For example, the class $\mathcal{MV}$ of all MV-algebras is the quasi-variety generated by the real unit interval $[0, 1]$ with connectives $\{\vee, \wedge, \circ, \Rightarrow\}$ (cf. [SI07a,SI07b]). The connectives for these algebras can be defined in terms of real functions and relations. Hence, these connectives can be seen as the extension functions of a definitional extension over the reals, which is local. One may, therefore, reduce universal validity problems over the class of $\mathcal{MV}$-algebras, say, to a constraint satisfiability problem over the unit interval $[0, 1]$ (cf. [SI07a,SI07b]). This allows one to use solvers for the real numbers to discharge proof tasks over multiple-valued logics. We will give an example of this in Section 10.1.

Data structures. The ubiquitous data structures of arrays and lists satisfy locality conditions if we confine ourselves to appropriate fragments of their theories. This matters in particular if we have satisfiable problems. In order to have a full decision procedure - one that is also able to give the correct answer for satisfiable problems - one has to stay inside of these fragments. H-PILoT automates this task: it will check whether a given problem lies inside the appropriate fragment of the theory of arrays or pointers, respectively, and give the answer "satisfiable" only if this is the case. Otherwise, H-PILoT will give the answer "unknown" and warn the user that the problem did not fall inside the local fragment. (For unsatisfiable problems this never matters. If we can derive a contradiction from the local instances alone, we can derive one from the universal extension axioms a fortiori.)

Reasoning in complex systems. In order to be able to handle complex real life systems which mix many theories, a stratified approach is expedient: we consider *chains of local extensions* (cf. [JSS07]). This feature is supported in H-PILoT. The user simply has to tag extension functions with their respective level in the chain. A reduction is then carried out iteratively by the program. A full-fledged reduction is possible provided the reduced theory clauses are ground at each level of the extension chain. H-PILoT has been part of a vertically integrated tool chain, checking invariants of a transition system modeling a European train controller system (see Section 15.1). The correctness of the model was shown automatically [IJSS08]. The underlying track topology was complex and dynamic, making H-PILoT's ability to decide the pointer fragment essential. It was also a great help in practice, due to its ability to provide (readable) counterexamples in the cases where the problem together with the axiomatization was satisfiable. This aided the modeler in finding gaps in the specification.

9 Examples

We illustrate the way H-PILoT is implemented and can be used on two examples.

9.1 Monotone functions

We consider as base theory $\mathcal{T}_0$ the theory of a partial order, and its extension with two monotone functions f and g.
That is, our base theory $\mathcal{T}_0$ consists of the axioms for reflexivity, transitivity and anti-symmetry.

(1) $\forall x.\ x \leq x.$
(2) $\forall x, y.\ x \leq y \wedge y \leq x \rightarrow x = y.$
(3) $\forall x, y, z.\ x \leq y \wedge y \leq z \rightarrow x \leq z.$

The extension we consider consists of the two new function symbols together with the clauses $\mathcal{K}$ expressing their monotonicity.

(1) $\forall x, y.\ x \leq y \rightarrow f(x) \leq f(y).$
(2) $\forall x, y.\ x \leq y \rightarrow g(x) \leq g(y).$

We want to show that

$$\mathcal{T}_0 \cup \mathcal{K} \models c_0 \leq c_1 \leq d_1 \wedge c_2 \leq d_1 \wedge d_2 \leq c_3 \wedge d_2 \leq c_4 \wedge f(d_1) \leq g(d_2) \rightarrow f(c_0) \leq g(c_4).$$

Expressed as a satisfiability problem of the form "$\mathcal{T}_0 \cup \mathcal{K} \cup G \models \perp$?", where:

$$G = c_0 \leq c_1 \leq d_1 \wedge c_2 \leq d_1 \wedge d_2 \leq c_3 \wedge d_2 \leq c_4 \wedge f(d_1) \leq g(d_2) \wedge \neg f(c_0) \leq g(c_4).$$

As an input file for H-PILoT this looks as follows (we use "R" as order relation because $\leq$ is reserved).

```
% Two monotone functions over a poset.
% Status: unsatisfiable

Base_functions:={}
Extension_functions:={(f, 1), (g, 1)}
Relations:={(R, 2)}
```

```
% R is partial order
Base := (FORALL x).     R[x, x];
        (FORALL x,y,z). R[x, y], R[y, z] --> R[x, z];
        (FORALL x,y).   R[x, y], R[y, x] --> x = y;

Clauses := (FORALL x,y). R[x, y] --> R[f(x), f(y)];
           (FORALL x,y). R[x, y] --> R[g(x), g(y)];

Query := R[c0, c1];
         R[c1, d1];
         R[c2, d1];
         R[d2, c3];
         R[d2, c4];
         R[f(d1), g(d2)];
         NOT(R[f(c0), g(c4)]);
```

In this case we have no function symbols in the base theory and two functions symbols f and g of arity 1 in the extension clauses. This is expressed by:

```
Base_functions:={}
Extension_functions:={(f, 1), (g, 1)}
```

We have only one relation in our (base) clauses, namely 'R', with arity 2. This we express by:

`Relations:={(R, 2)}`.

For technical reasons, relations require square brackets for their arguments in H-PILoT as seen above. The symbols `<=`, `<`, `>=` and `>` are reserved for arithmetic over the integers or over the reals. They may be written infix and there are provers (e.g., Yices, CVC) that "understand" arithmetic and orderings. We wouldn't have needed to axiomatize '$\leq$' at all.
However, the above problem is more general. It concerns every partial order not only orderings of numbers. By default, H-PILoT calls SPASS. SPASS has no in-built understanding of orderings and, thus, `<=` would be just an arbitrary symbol. For clarity we used the letter 'R'.
As for the syntax of clauses, one should note that the syntax of H-PILoT requires that each clause must end with a semicolon, be in prenex normal form and all names meant to be (universal) variables must be explicitly quantified.

Every name which is not explicitly quantified will be considered a constant!

As we can see in our proof task.

```
Proof Task := R[c0, c1];
         R[c1, d1];
         R[c2, d1];
         R[d2, c3];
         R[d2, c4];
         R[f(d1), g(d2)];
         NOT(R[f(c0), g(c4)]);
```

Note further that because the background theory, extension theory and the proof task must all be clauses[7], we need to break up the conjunction in our original

[7] In fact, the extension clauses might be more general as we will see later.

proof task into a set of unit clauses. A non-unit clause may be written as above in a "sorted" manner $\varphi_1, ..., \varphi_n \rightarrow \psi_1, ..., \psi_k$ for $\varphi_1 \wedge ... \wedge \varphi_n \rightarrow \psi_1 \vee ... \vee \psi_k$, i.e., as an implication with the negated atoms of the clause in the antecedent and the (positive) atoms in the consequent (the operator `-->` is reserved for sorted clauses) or as an arbitrary disjunctions of literals.
The name of the input files for H-PILoT can be freely chosen, although it is customary to have them have the suffix ".loc". Suppose we have put the above problem in a file named `mono_for_poset.loc`, then we can run H-PILoT by calling

```
hpilot.opt mono_for_poset.loc
```

H-PILoT will parse the input file, carry out the reduction and then will hand over the reduced problem to SPASS (using the same name but with the suffix ".dfg"). SPASS terminates quickly with the result that a proof exists

```
SPASS beiseite: Proof found.
Problem: mono_for_poset.dfg
SPASS derived 35 clauses, backtracked 0 clauses and kept 41 clauses.
SPASS allocated 496 KBytes.
SPASS spent     0:00:02.32 on the problem.
                0:00:00.00 for the input.
                0:00:00.00 for the FLOTTER CNF translation.
                0:00:00.00 for inferences.
                0:00:00.00 for the backtracking.
                0:00:00.10 for the reduction.
```

meaning that the set of clauses is inconsistent, as we wanted to show.

One can see the full reduction process by using the option `-prClauses`.

9.2 Arrays

For a more complicated example, let us consider an algorithm for inserting an element x into a sorted array a with the bounds l and u. We want to check that the algorithm is correct, i.e., that the updated array a' remains sorted. This could be an invariant being checked in a verification task.
There are three different cases.

- x could be smaller than any element in a (equivalently, $x < a[l]$),
- x could be greater than any element of a ($x > a[u]$) or,
- there is a position p ($l < p \leq u$) such that $a[p-1] < x$ and $x \leq a[p]$.

In the first two cases we put x at the first respectively last position of the array. In the third case, we insert x at position p and shift the other elements to the right, i.e., $a'[i+1] = a[i]$ for $i > p$. We have to take care to cover also the cases where the array contains only 1 or 2 elements. As input it will look as follows.

```
Clauses :=
  % case 1
  (FORALL i). i = l, x <= a(i) --> a'(i) = x;
  (FORALL i). x <= a(l), l < i, i <= u + _1 --> a'(i) = a(i - _1);

  % case 2
  (FORALL i). i = u, a(i) <= x --> x <= a(l), a'(i + _1) = x;
  (FORALL i). a(u) <= x, l - _1 <= i, i < u
                    --> x <= a(l), a'(i + _1) = a(i + _1);

  % case 3
  (FORALL i). x < a(u), l <= i, i < u, a(i) < x, x <= a(i + _1)
               --> a'(i + _1) = x;
  (FORALL i). a(l) < x, x < a(u), l <= i, i < u, x <= a(i),
               x <= a(i + _1) --> a'(i + _1) = a(i);
  (FORALL i). a(l) < x, x < a(u), i = u + _1 --> a'(i) = a(i - _1);
  (FORALL i). a(l) < x, x < a(u), l - _1 <= i, i < u, a(i + _1) < x
               --> a'(i + _1) = a(i + _1);

  (FORALL i,j). l <= i, i <= j, j <= u --> a(i) <= a(j);
```

with the last clause saying that a was sorted at the beginning.

There are several things to note. Most importantly, we now have a two-step extension. First, an array can be simply seen as a partial function. This gives us the first extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$. $\mathcal{T}_0$ here is the theory of indices (integers, say) which we extend by the function a and the axiom for monotonicity of a. Now we update a, giving us a second extension $\mathcal{T}_2 \supseteq \mathcal{T}_1$ where our extension clauses $\mathcal{K}_2$ are given by the three cases above.

We need to make sure that the last extension is also local. This is easy to establish, because $\mathcal{K}_2$ is a *definitional extension* or case distinction (cf. [SI07a,SI07b]). A definitional extension is one where extension functions f only appear in the form $\varphi_i(\bar{x}) \to f(\bar{x}) = t_i(\bar{x})$ with t_i being a base theory term and the φ_i are mutually exclusive base theory clauses. This is the reason that we have written $\forall i. i = l, x \leq a(i) \to a'(i) = x$ instead of the shorter $x \leq a(l) \to a'(l) = x$: to ensure that the antecedents of the clauses are all mutually exclusive. Now we know that we are dealing with a definitional and therefore local extension. (Remember that when assessing whether $\mathcal{T}_2 \supseteq \mathcal{T}_1$ is a local extension, $\mathcal{T}_1$ is the base theory; that $\mathcal{T}_1$ is itself an extension is not important at the moment.)

We need to tell the program that we are dealing with a chain of extensions instead of a single one. We do this by declaring to which level of the chain an extension function belongs: $(f, arity, level)$.

In our example that would be

```
Extension_functions:={(a', 1, 2), (a, 1, 1)}
```

The program will now automatically determine the level of each extension clause. In our example, an extension clause will have level 2 if and only if a' occurs in it and level 1 otherwise (level 0 refers to a clause in the base theory).

Also recall from the explanations in Section 6 that numerals (names for integers) must be preceded by an underscore, and that + and − may be written infix for readability; $(=, +, -, *, /)$ are the only functions for which this is allowed[8]. Our declarations, therefore should look like this.

```
Base_functions:={(+,2), (-, 2)}
Extension_functions:={(a', 1, 2), (a, 1, 1)}
Relations:={(<=, 2), (<, 2)}
```

All that is left to do now is add the proof task – the negation of

$$\forall i, j.\ (l \leq i \leq j \leq u + 1 \rightarrow a'(i) \leq a'(j))$$

namely:

$$1 \leq m \wedge m \leq n \wedge n \leq u + 1 \wedge \neg(a'(m) \leq a'(m))$$

to the file and hand it over to H-PILoT . The file looks like this.

```
Base_functions:={(+,2), (-, 2)}
Extension_functions:={(a', 1, 2), (a, 1, 1)}
Relations:={(<=, 2), (<, 2)}

% K
Clauses :=
  % case 1
  (FORALL i). i = l, x <= a(i) --> a'(i) = x;
  (FORALL i). x <= a(l), l < i, i <= u + _1 --> a'(i) = a(i - _1);

  % case 2
  (FORALL i). i = u, a(i) <= x --> x <= a(l), a'(i + _1) = x;
  (FORALL i). a(u) <= x, l - _1 <= i, i < u
                     --> x <= a(l), a'(i + _1) = a(i + _1);

  % case 3
  (FORALL i). x < a(u), l <= i, i < u, a(i) < x, x <= a(i + _1)
              --> a'(i + _1) = x;
  (FORALL i). a(l) < x, x < a(u), l <= i, i < u, x <= a(i),
              x <= a(i + _1) --> a'(i + _1) = a(i);
  (FORALL i). a(l) < x, x < a(u), i = u + _1 --> a'(i) = a(i - _1);
  (FORALL i). a(l) < x, x < a(u), l - _1 <= i, i < u, a(i + _1) < x
              --> a'(i + _1) = a(i + _1);

  (FORALL i,j). l <= i, i <= j, j <= u --> a(i) <= a(j);

Query := l <= m;
         m <= n;
         n <= u + _1;
         NOT( a'(m) <= a'(n) );
```

We do not need to declare a base theory here because we will be using Yices and Yices already "knows" integer arithmetic. We call H-PILoT thus:

```
hpilot.opt -yices -preprocess ai.loc
```

[8] When using SPASS, they may also be written infix but nevertheless they are just free functions for SPASS.

H-PILoT will produce a reduction, put it in a file called *ai.ys* and pass it over to Yices which will say `unsat` or `sat`. A note on the flag `-preprocess`: Establishing that some extension is local presupposes that the extension clauses in $\mathcal{K}$ are *flat* and *linear*. Flatness means that the clauses contain no nesting of extension functions. Linearity means that:

- no variable occurrs twice in any extension term and
- if any variable occurs in two extension terms, the terms are the same.

In this example, we have non-flat clauses such as

```
(FORALL i). i = u, a(i) <= x --> x <= a(l), a'(i + _1) = x;
```

We rectify matters by a *flattening* operation - rewriting the above clause to

```
(FORALL i,j). j = i + _1, i = u, a(i) <= x -->
                                    x <= a(l), a'(j) = x;
```

This will not affect consistency of any proof task w.r.t. $\mathcal{K}$. However, it does not improve readability. Therefore the program will perform flattening/linearization only if the option `-preprocess` is chosen.

10 Example: Specifying the type information

10.1 Global constraints[9]

Sometimes we want to restrict the domain of the problem, e.g., we want to consider natural numbers instead of integers or we are interested in a real interval $[a, b]$ only. Yices and CVC support the definition of subtypes. When using one of these it is possible to state a global constraint on the domain of the models in the preamble as follows:

```
Interval := 0 <= x <= 1;
```

This will restrict the domain of the models of the theory to the unit interval $[0, 1]$. It is equivalent to adding the antecedent $0 \leq x \wedge x \leq 1$, for every variable x, to each formula in the clauses and the proof task.
The bounds of the interval can also be exclusive or mixed as in

```
Interval := 0 < x <= 1;
```

or one-sided as in

```
Interval := 2 <= x;
```

[9] This feature is not supported for Z3.

Consider the following example, taken from [SI07a,SI07b], concerning multiple-valued logic. The class $\mathcal{MV}$ of all MV-algebras is the quasi-variety generated by the real unit interval $[0,1]$ with the Łukasiewicz connectives $\{\vee, \wedge, \circ, \Rightarrow\}$, i.e., the algebra $[0,1]_L = ([0,1], \vee, \wedge, \circ, \Rightarrow)$. The Łukasiewicz connectives can be defined in terms of the real functions '+','−' and the relation '≤', giving us a local extension over the real unit interval.
Therefore, the following are equivalent:

(1) $\mathcal{MV} \models \forall \overline{x} \bigwedge_{i=1}^{n} s_i(\overline{x}) = t_i(\overline{x}) \rightarrow s(\overline{x}) = t(\overline{x})$
(2) $[0,1]_L \models \forall \overline{x} \bigwedge_{i=1}^{n} s_i(\overline{x}) = t_i(\overline{x}) \rightarrow s(\overline{x}) = t(\overline{x})$
(3) $\mathcal{T}_0 \cup \mathsf{Def}_L \wedge \bigwedge_{i=1}^{n} s_i(\overline{c}) = t_i(\overline{c}) \wedge s(\overline{c}) \neq t(\overline{c}) \models \perp$,

where $\mathcal{T}_0$ consists of the real unit interval $[0,1]$ with the operations $+, -$ and predicate symbol $\leq$.

For instance, we might want to establish whether linearity $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$ follows from the axioms. As an input file for H-PILoT it looks like this.

```
Base_functions:={(+, 2), (-, 2)}
Extension_functions:={(V, 1), (M, 1), (o, 1), (r, 1)}
Relations:={(<=, 2), (<, 2), (>, 2), (>=, 2)}

Interval := 0 <= x <= 1;

% K
Clauses := % definition of \/
           (FORALL x,y). x <= y --> V(x, y) = y;
           (FORALL x,y). x > y  --> V(x, y) = x;

           % definition of /\
           (FORALL x,y). x <= y --> M(x, y) = x;
           (FORALL x,y). x > y  --> M(x, y) = y;

           % definition of o
           (FORALL x,y). x + y <  _1 --> o(x, y) = _0;
           (FORALL x,y). x + y >= _1 --> o(x, y) = (x + y) - _1;

           % definition of =>
           (FORALL x,y). x <= y --> r(x, y) = _1;
           (FORALL x,y). x > y  --> r(x, y) = (_1 - x) + y;

Query := % linearity: x => y . \/ . y => x = 1
         NOT(V(r(a, b), r(b, a)) = _1);
```

10.2 Using standard types

In default mode using SPASS, H-PILoT hands over a set of general first-order formulas without types. However, H-PILoT also provides support for the standard types `int, real, bool` and for free types. When using CVC or Yices the default type is `int`, for Redlog it is `real`. The default type does not need to be specified in the input file. One can also use the `-real` flag to set the default type to real for Yices and CVC.
Free types are specified as `free#`i, $i = 1, 2, \ldots$ or simply as `free` if there is only one free type. When using free types the flag `-freeType` must be set. Only

Yices and CVC are able to handle free types (Yices is default when the flag is set). When using mixed type in one input file, the types of the functions and the constants need to be declared. If the domain of a function is the same as the range it is enough to specify the domain as in

$(foo, arity, level, domainType)$

if they differ say

$(foo, arity, level, domainType, rangeType)$.

Constants are simply declared as

$(name, type)$.

The following example is taken from [Sof06b].

```
% Pointers
% status unsatisfiable
Base_functions:={(+,2), (-, 2)}
Extension_functions:={(next, 1, 1, free#1), (prev, 1, 1, free#1),
                         (priority, 1, 1, free#1, real),
                            (state, 1, 1, free#1, free#2)}
Relations:={(>=, 2)}
Constants:={(null, free#1), (eps, real), (a, free#1), (b, free#1),
               (RUN, free#2), (WAIT, free#2), (IDLE, free#2)}

% K
Clauses :=
  (FORALL x). OR(state(x) = RUN, state(x) = WAIT, state(x) = IDLE);
  % prev and next are inverse
  (FORALL p). OR(p = null, prev(next(p)) = null, prev(next(p)) = p);
  (FORALL p). --> p = null, next(prev(p)) = null, next(prev(p)) = p;
  (FORALL p, q). next(q) = next(p) --> p = null, q = null, p = q;
  (FORALL p, q). prev(q) = prev(p) --> p = null, q = null, p = q;
  (FORALL p). --> p = null, next(p) = null, state(p) = IDLE,
                  state(next(p)) = IDLE, state(p) = state(next(p));
  (FORALL p). OR(p = null, next(p) = null, NOT(state(p) = RUN),
                    priority(p) >= priority(next(p)));


Query := NOT(eps = _5);
         NOT(eps = _6);
         priority(a) = _5;
         priority(b) = _6;
         a = prev(b);
         state(a) = RUN;
         NOT(next(a) = null);
         NOT(a = null);
         NOT(b = null);
```

11 Example: Handling data structures

11.1 Arrays

We consider the local fragment of the theory of arrays in more detail and show how it can be dealt with by H-PILoT. For the *array property fragment* ([BMS06]) the following syntactical restrictions are imposed. Let A be a set of function symbols used for denoting arrays.

1. An *index guard* is a positive Boolean combination of atoms of the form $t \leq u$ or $t = u$ where t and u are either a variable or a ground term of linear arithmetic.
2. A *value restriction* is a formula $\varphi_V(\overline{c}, \overline{x})$ containing constants among those in $\overline{c} = c_1, \dots, c_k$ and free variables among those in $\overline{x} = x_1, \dots, x_n$, with the property that:
   (1) all occurrences of the variables are shielded by function symbols in A;
   (2) no nested array reads are allowed
   i.e., the free variables x_i occur in φ_V only in direct array reads $a[x_i]$.
3. A universal formula of the form $(\forall \bar{x})(\varphi_I(\bar{x}) \rightarrow \varphi_V(\bar{x}))$ is an *array property* if it is flat, φ_I is an index guard and ϕ_V a value restriction.

In this section we only consider extensions by clauses of the above form. Our base theory is the disjoint, many-sorted combination of linear integer arithmetic (Presburger) with a theory of elements. The extension functions are in this case the function symbols in A, used for denoting arrays. In order to be able to handle this fragment we have to use a particular type of locality, namely minimal locality. To use this feature we call H-PILoT with parameter `-arrays`:

```
hpilot.opt -arrays k.loc
```

Consider the example of inserting a new element into a sorted array a. Arrays are modeled as free functions and array updates are dealt with by introducing new array names. In this fashion, let d be identical to a except for position k at which it takes value w and let e be identical to d except possibly at position l where we have written x and similarly for c, b and a. The set $\mathcal{K}$ of extension clauses we consider is:

$$\left.\begin{array}{ll}(\forall i,j)(0 \leq i \leq j \leq n-1 \rightarrow c[i] \leq c[j]) & (1)\\ (\forall i,j)(0 \leq i \leq j \leq n-1 \rightarrow e[i] \leq e[j]) & (2)\\ (\forall i)(i \neq l \rightarrow b[i] = c[i]) & (3)\\ (\forall i)(i \neq k \rightarrow a[i] = b[i]) & (4)\\ (\forall i)(i \neq l \rightarrow d[i] = e[i]) & (5)\\ (\forall i)(i \neq k \rightarrow a[i] = d[i]). & (6)\end{array}\right\} \quad \mathcal{K}$$

Our proof task (with additional constraints) is

$$\left.\begin{array}{l}w < x < y < z\\ 0 < k < l < n\\ k+3 < l\\ c[l] = x\\ b[k] = w\\ e[l] = z\\ d[k] = y.\end{array}\right\} \quad G$$

The input file looks as follows. (The operators are written prefix here which requires the names `plus` and `minus`, because `+` and `-` are reserved for infix.)

```
% Arrays for minimal locality
Base_functions:={(plus,2), (minus, 2)}
Extension_functions:={(a, 1), (b, 1), (c, 1), (d, 1), (e, 1)}
Relations:={(<=, 2)}

% K
Clauses := (FORALL i,j). _0 <= i, i <= j,
                             j <= minus(n, _1) --> c(i) <= c(j);
           (FORALL i,j). _0 <= i, i <= j,
                             j <= minus(n, _1) --> e(i) <= e(j);
           (FORALL i).  --> i=l, b(i) =  c(i);
           (FORALL i).  --> i=k, a(i) =  b(i);
           (FORALL i).  --> i=l, d(i) =  e(i);
           (FORALL i).  --> i=k, a(i) =  d(i);


Query := plus(w, _1)  <= x;
         plus(x, _1)  <= y;
         plus(y, _1)  <= z;
         plus(_0, _1) <= k;
         plus(k, _1)  <= l;
         plus(l, _1)  <= n;
         plus(k, _3)  <= l;
         c(l) = x;
         b(k) = w;
         e(l) = z;
         d(k) = y;
```

$\mathcal{K}$ does not yet fulfil the syntactic requirements (index guards must be positive!). We rewrite $\mathcal{K}$ as follows: We change an expression $i \neq l$ where i is the (universally quantified) variable to $i \leq l - 1 \vee l + 1 \leq i$. We rewrite it like this because the universally quantified variable i must appear unshielded in the index guard. This gives us the following set of clauses.

$$\left.\begin{array}{lr}
(\forall i, j)(0 \leq i \leq j \leq n - 1 \rightarrow c[i] \leq c[j]) & (1)\\
(\forall i, j)(0 \leq i \leq j \leq n - 1 \rightarrow e[i] \leq e[j]) & (2)\\
(\forall i)(i \leq l - 1 \rightarrow b[i] = c[i]) & (3)\\
(\forall i)(l + 1 \leq i \rightarrow b[i] = c[i]) & (4)\\
(\forall i)(i \leq k - 1 \rightarrow a[i] = b[i]) & (5)\\
(\forall i)(k + 1 \leq i \rightarrow a[i] = b[i]) & (6)\\
(\forall i)(i \leq l - 1 \rightarrow d[i] = e[i]) & (7)\\
(\forall i)(l + 1 \leq i \rightarrow d[i] = e[i]) & (8)\\
(\forall i)(i \leq k - 1 \rightarrow a[i] = d[i]) & (9)\\
(\forall i)(k + 1 \leq i \rightarrow a[i] = d[i]). & (10)
\end{array}\right\} \quad \mathcal{K}'$$

H-PILoT performs this and the following rewrite steps automatically to spare the user this tedious labor. Also, $\mathcal{K}$ is not linear, this must also be taken care of. H-PILoT carries out all necessary rewrite steps for the user, who can simply input the above file to the system.

Instead of using free functions to specify array updates, H-PILoT allows the user to *model array updates directly* by using a “write” function – for example, write(a, i, x) denotes a new array which is identical to a except (possibly) at position i where the value of the new array is set to x. In this way we can specify our problem above as:

```
% Arrays for minimal locality with 'write'function.
Base_functions:={(+, 2), (-, 2)}
Extension_functions:={(a, 1)}
Relations:={(<=, 2)}

% K
Clauses :=
 (FORALL i,j). _0 <= i, i <= j, j <= n - _1 -->
    write(write(a,k,w), l, x)(i) <= write(write(a,k,w), l, x)(j);

 (FORALL i,j). _0 <= i, i <= j, j <= n - _1 -->
    write(write(a,k,y), l, z)(i) <= write(write(a,k,y), l, z)(j);

Query := w + _1  <= x;
         x + _1  <= y;
         y + _1  <= z;
         _0 + _1 <= k;
         k + _1  <= l;
         l + _1  <= n;
         k + _3  <= l;
```

As above, H-PILoT will also automatically split on disequations in the antecedent. Note also that since we assume that indices of arrays are integers, it makes no difference whether we write `w + _1` or `plus(w, _1)` in the input file. Linear integer arithmetic will be used (Yices is default).

11.2 Pointers

The local fragment of the theory of pointers (cf. [MN05,IJSS08,FIJS10]) is also implemented in H-PILoT. We consider pointer problems over a two-sorted language, containing one sort `pointer` and another scalar sort. The scalar sort can be concrete e.g. `real`, or is kept abstract in which case it is written as `scalar`. There are two function types involving pointers, namely `pointer` $\to$ `pointer` and `pointer` $\to$ *scalar*, where *scalar* is either a concrete scalar sort (e.g. `real`) or the abstract sort "scalar".[10] The axioms we consider are all of the form

$$\forall \bar{p}.\ \ \mathcal{E} \vee \mathcal{C}$$

where $\bar{p}$ is a set of pointer variables containing all the pointer variables occurring in $\mathcal{E} \vee \mathcal{C}$, $\mathcal{E}$ contains disjunctions of pointer equalities and $\mathcal{C}$ is a disjunction of scalar constraints (i.e., literals of scalar type). $\mathcal{E} \vee \mathcal{C}$ may also contain free variables of scalar type or, equivalently, free scalar constants.

We require that pointer terms appearing below a function should not be `null` in order to rule out null pointer errors. That is, for all terms $f_1(f_2(\dots f_n(p)))$,

[10] In [FIJS10] we use an extension of H-PILoT which allows several pointer sorts, as well as functions of sort $p_1, \dots p_n \to p$ (where p_i, p are pointer sorts) and $p_1, \dots, p_m \to$ *scalar*. This aspect is discussed at the end of this section.

$i = 1, .., n$, occurring in the axiom, the axiom also contains the disjunction $p = \mathsf{null} \vee f_n(p) = \mathsf{null} \vee \cdots \vee f_2(\ldots f_n(p)) = \mathsf{null}$.
Pointer/scalar formulas complying with this restriction are called *nullable*. The locality result in [IJSS08] allows the integration of pointer reasoning with the above features into H-PILoT. We now present an example given in [MN05], which looks like this as input for H-PILoT. (We have added an appropriate proof task.)

```
Base_functions:={(+,2), (-, 2)}
Extension_functions:={(next, 1, 1, pointer),
                      (prev, 1, 1, pointer),
                      (priority, 1, 1, pointer, real)}
Relations:={(>=, 2)}
Constants:={(a, pointer), (b, pointer)}

Clauses :=
     (FORALL p). prev(next(p)) = p;
     (FORALL p). --> next(prev(p)) = p;
     (FORALL p). --> q = null, priority(p) >= priority(next(p));

Query := priority(a) = _5;
         priority(b) = _6;
         a = prev(b);
         NOT(a = null);
         NOT(b = null);
```

H-PILoT can be called without any parameters because the keyword `pointer` is present. This will trigger H-PILoT's pointer mode so that it will add all the nullable terms to the axioms and use the specific (stable) locality required.

Because the scalar type is concrete here (`real`), H-PILoT will use Yices as the back-end prover (its default for arithmetic). If we want to leave the scalar type abstract we could write something like

```
%psiPointers.scalar.loc
Base_functions:={}
Extension_functions:={(next, 1, 1, pointer),
                      (prev, 1, 1, pointer),
                      (priority, 1, 1, pointer, scalar)}
Relations:={}
Constants:={(a, pointer), (b, pointer), (c5, scalar), (c6, scalar)}

Clauses := (FORALL p). prev(next(p)) = p;
           (FORALL p). next(prev(p)) = p;
           (FORALL p). NOT(priority(p) = priority(next(p)));


Query := priority(a) = c5;
         priority(b) = c6;
         a = prev(b);
         c5 = c6;
         NOT(a = null);
         NOT(b = null);
```

We again can simply type

```
hpilot.opt -preprocess psiPointers.scalar.loc
```

without any parameters. H-PILoT will recognize this as a pointer problem and use Yices as default, this time because of the free type `scalar`. There can also be more than one pointer type and pointer extensions can be fused with other types of extensions in a hierarchy. However, due to the different types of locality that need to be employed, the user must specify which levels are pointer extensions. He does this by using the keyword `Stable`.
For example, the header of a more complicated verification task which mixes different pointer types might look like this.

```
Base_functions:={(-, 2), (+, 2)}
Extension_functions:=
   { % level 4
     (next',1,4, pointer#2,pointer#2), (pos',1,4,pointer#2,real)
     % level 3
     (next,1,3,pointer#2,pointer#2), (pos,1,3,pointer#2,real),
     (spd,1,3,pointer#2,real), (segm,1,3,pointer#2,pointer#1),
     % level 2
     (bd,1,2,real,real),
     % level 1
     (lmax,1,1,pointer#1,real), (length,1,1,pointer#1,real),
     (nexts,1,1,pointer#1,pointer#1), (alloc,1,1,pointer#1,int)}

Relations :={(<=, 2), (>=, 2), (>, 2), (<, 2)}

Constants:= {(t3,pointer#2), (t2,pointer#2), (t1,pointer#2),
              (d,real), (State0,int), (s,pointer#1), (State1,int)}

Stable := 1, 3;
```

Note that the type `pointer#2` must be declared with a higher level than `pointer#1` because `pointer#2` refers to `pointer#1` but not vice versa.

12 Example: Using the built-in CNF translator

H-PILoT also provides a clausifier for ease of use. First-order formulas can be given as input and H-PILoT translates them into clausal normal form (CNF). The CNF-translator does not provide the full functionality of FLOTTER. It uses structural formula renaming ([PG86]) and standard Skolemization, not the more exotic variants thereof (cf. [NW01]). Nevertheless, it is powerful enough for most applications. As a simple example consider the following.

```
% cnf.fol
Base_functions:={(delta, 2), (abs, 1), (-, 2)}
Extension_functions:={(f, 1)}
Relations:={}


Formulas :=
    (FORALL eps, a, x). (_0 < eps -->
            AND( _0 < delta(eps, a),
                    (abs(x - a) < delta(eps, a)
                            --> abs(f(x) - f(a)) < eps)));

Query :=
```

H-PILoT translates `Formulas` to clause normal form. To see the output, we use

```
hpilot.opt -preprocess -prClauses cnf.fol
```

We obtain the following output file:

```
!- Adding formula:
   (FORALL eps, a, x).
       (_0 < eps -->
          AND( _0 < delta(eps, a), (abs(-(x, a)) < delta(eps, a)
            --> abs(-(f(x), f(a))) < eps)))
!- add_formulas
!- We have 1 levels.
!- done
!- Our base theory is:
!- empty.
!- Clausifying formulas...
!- (FORALL z_1, z_3). OR( _0 < delta(z_1, z_3), NOT(_0 < z_1))
!- (FORALL z_1, z_2, z_3).
                OR( NOT(abs(-(z_2, z_3)) < delta(z_1, z_3)),
                    abs(-(f(z_2), f(z_3))) < z_1, NOT(_0 < z_1))
!- Yielding 2 new clauses:
!- [z_1, z_2, z_3] abs(-(z_2, z_3)) < delta(z_1, z_3), _0 < z_1
                        ---> abs(-(f(z_2), f(z_3))) < z_1
!- [z_1, z_3] _0 < z_1 ---> _0 < delta(z_1, z_3)
!- After rewriting we have as clauses K:
!- [z_1, z_2, z_3] abs(-(z_2, z_3)) < delta(z_1, z_3), _0 < z_1
                        ---> abs(-(f(z_2), f(z_3))) < z_1
!- [z_1, z_3] _0 < z_1 ---> _0 < delta(z_1, z_3)
```

telling us that the above formula resulted in two new clauses (in addition to those given outright under `Clauses`), viz.

$$\forall z_1, z_3.\ \ 0 < delta(z_1, z_3) \vee \neg(0 < z_1)$$

and

$$\forall z_1, z_2, z_3.\ \ \neg(abs(z_2 - z_3) < delta(z_1, z_3)) \vee abs(f(z_2) - f(z_3)) < z_1 \vee \neg(0 < z_1).$$

In this case no ground clause resulted and H-PILoT stops.

13 Extended locality

For some applications we would like to allow more complicated extension clauses, say we want them to be inductive ($\forall\exists$) instead of universal. H-PILoT is also able to handle extensions with augmented clauses, i.e., formulas of the form $\forall x.\Phi(x) \vee C(x)$, where Φ is an arbitrary formula *which does not contain extension functions* and C is a clause which does (cf. [IJSS08]). Consider the following example taken from [IJSS08].

Suppose there is a parametric number m of processes. The priorities associated with the processes (non-negative real numbers) are stored in an array p. The states of the process – enabled (1) or disabled (0) – are stored in an array a. At each step only the process with maximal priority is enabled, its priority is set to x and the priorities of the waiting processes are increased by y. This can be expressed by the following set of axioms which we denote as $\mathsf{Update}(p, p', a, a')$.

$$\begin{array}{l}
\forall i(1 \leq i \leq m \wedge \quad (\forall j(1 \leq j \leq m \wedge j{\neq}i \rightarrow p(i){>}p(j))) \rightarrow a'(i) = 1)\\
\forall i(1 \leq i \leq m \wedge \quad (\forall j(1 \leq j \leq m \wedge j{\neq}i \rightarrow p(i){>}p(j))) \rightarrow p'(i){=}x)\\
\forall i(1 \leq i \leq m \wedge \neg(\forall j(1 \leq j \leq m \wedge j{\neq}i \rightarrow p(i){>}p(j))) \rightarrow a'(i){=}0)\\
\forall i(1 \leq i \leq m \wedge \neg(\forall j(1 \leq j \leq m \wedge j{\neq}i \rightarrow p(i){>}p(j))) \rightarrow p'(i){=}p(i){+}y),
\end{array}$$

where x and y are parameters. We may need to check whether, given that at the beginning the priority list is injective, i.e., formula $(\mathsf{Inj})(p)$ holds:

$$(\mathsf{Inj})(\mathsf{p}) \quad \forall i, j(1 \leq i \leq m \wedge 1 \leq j \leq m \wedge i \neq j \rightarrow p(i) \neq p(j)),$$

then it remains injective after the update, i.e., check whether

$$(\mathsf{Inj})(p) \wedge \mathsf{Update}(p, p', a, a') \wedge (1 \leq c \leq m \wedge 1 \leq d \leq m \wedge c \neq d \wedge p'(c) = p'(d)) \models \perp.$$

We need to deal with alternations of quantifiers in the extension. The extension formulas $\mathcal{K}$ are augmented clauses of the form

$$\forall x_1, ..., x_n. \ (\Phi(x_1, \ldots, x_n) \vee C(x_1, \ldots, x_n)),$$

where $\Phi(x_1, \ldots, x_n)$ is an *arbitrary first-order formula* in the base signature with free variables $x_1, \ldots, x_n$ and $C(x_1, \ldots, x_n)$ is a *clause* in the extended signature. As input for H-PILoT, extended clauses may be either written as $\forall \bar{x}. \ (\Phi(\bar{x}) \vee C(\bar{x}))$ or as $\forall \bar{x}. \ (\Phi(\bar{x}) \rightarrow C'(\bar{x}))$. The input file for H-PILoT looks as follows.

```
% Updating of priorities of processes
% File update_AE.loc
Base_functions:={(+,2), (-, 2)}
Extension_functions:={(a', 1, 2, bool), (a, 1, 1, bool), (p', 1, 2, real), (p, 1, 1, real)}
Relations:={(<=, 2), (<, 2), (>, 2)}
Constants:={(x, real), (y, real)}

% K
Clauses :=
 (FORALL i). _1 <= i, i <= m --> _0 <= p(i);
 (FORALL i). { AND(_1 <= i, i <= m,
    (FORALL j). (AND(_1 <= j, j <= m, NOT(j = i))
                                --> p(i) > p(j)))}
                                    --> a'(i) = _1;

 (FORALL i). { AND(_1 <= i, i <= m,
    (FORALL j). (AND(_1 <= j, j <= m, NOT(j = i))
                                --> p(i) > p(j)))}
                                    --> p'(i) = x;
 (FORALL i). { AND(_1 <= i, i <= m,
    NOT((FORALL j,i).(AND(_1 <= j, j <= m, NOT(j = i))
                                --> p(i) > p(j))))}
                                    --> a'(i) = _0;
 (FORALL i). { AND(_1 <= i, i <= m,
    NOT((FORALL j).(AND(_1 <= j, j <= m, NOT(j = i))
                                --> p(i) > p(j))))}
                                    --> p'(i) = p(i) + y;

 (FORALL i,j). _1 <= i, i <= m, _1 <= j, j <= m, p(i) = p(j)
                                    --> i = j;

Query := _1 <= c;
         c <= m;
         _1 <= d;
         d <= m;
         x <= _0;
         y > _0;
         NOT(c=d);
         p'(c) = p'(d);
```

The curly braces '{', '}' are required to mark the beginning and the end of the base formula Φ.

14 System evaluation

We have used H-PILoT on a variety of local extensions and on chains of local extensions. An overview of the tests we made is given below. For these tests, we have used Yices as the back-end solver for H-PILoT. We distinguish between satisfiable and unsatisfiable problems.

Unsatisfiable Problems. For simple unsatisfiable problems, there hardly is any difference in run-time whether one uses H-PILoT or an SMT-solver directly. This is due to the fact that a good SMT-solver uses the heuristic of trying out all the occurring ground terms as instantiations of universal quantifiers. For local extensions this is always sufficient to derive a contradiction.
When we consider chains of extensions the picture changes dramatically. On one test example – the array insertion of Section 9 which used a chain of two local extensions – Yices performed considerably slower than H-PILoT: The original problem took Yices over 5 minutes to solve. The hierarchical reduction yielded 113 clauses of the background theory (integers) which were proved to be unsatisfiable by Yices in a mere 0.07s.

Satisfiable Problems. For satisfiable problems over local theory extensions, H-PILoT always provides the right answer. In local extensions, H-PILoT is a decision procedure whereas completeness of other SMT-solvers is not guaranteed. In the test runs, Yices either ran out of memory or took more than 6 hours when given any of the unreduced problems. This even was the case for small problems, e.g., problems over the reals with less than ten clauses. With H-PILoT as a front end, Yices solved all the satisfiable problems in less than a second with the single exception of monotone functions over posets/distributive lattices.

14.1 Test runs for H-PILoT

We analyzed the following examples. The satisfiable variant of a problem carries the suffix ".sat".

array insert. Insertion of an element into a sorted integer array. This is the example from Section 9. Arrays are definitional extensions here.
array insert (∃). Insertion of an element into a sorted integer array. Arrays are definitional extensions here. Alternate version with (implicit) existential quantifier.
array insert (linear). Linear version of array insert.
array insert real. Like array insert but with an array of reals.
array insert real (linear). Linear version of array insert real.
update process priorities (∀∃). Updating of priorities of processes. This is the example from Section 13. We have an ∀∃-clause.

list1. Made-up example of integer lists. Some arithmetic is required

chain1. Simple test for chains of extensions (plus transitivity).

chain2. Simple test for chains of extensions (plus transitivity and arithmetic).

double array insert. A sorted array is updated twice. This is the example from Section 11.1. It is inside the array property fragment.

mono. Two monotone functions over integers/reals for SMT solver.

mono for distributive lattices.R. Two monotone functions over a distributive lattice. The axioms for a distributive lattice are stated together with the definition of a relation R: $R(x, y) :\Leftrightarrow x \wedge y = x$. Monotonicity of f (respectively of g) is given in terms of R: $R(x, y) \rightarrow R(f(x), f(y))$. Flag `-freeType` must be used.

mono for distributive lattices. Same as mono for distributive lattices.R except that no relation R is defined. Monotonicity of the two functions f, g is directly given: $x \wedge y = x \rightarrow f(x) \wedge f(y) = f(x)$. Flag `-freeType` must be used.

mono for poset. Two monotone functions over a poset with poset axioms as in Section 9. Same as mono, except the order is modeled by a relation R.

mono for total order. Same as mono except linearity is an axiom. This makes no difference unless SPASS is used.

own. Simple test for monotone function.

mvLogic/mv1. The example for MV-algebras from Section 10.1. The Łukasiewicz connectives can be defined in terms of the (real) operations $+, -, \leq$. Linearity is deducible from axioms.

mvLogic/mv2. Example for MV-algebras. The Łukasiewicz connectives can be defined in terms of $+, -, \leq$.

mvLogic/bl1. Example for MV-algebras with BL axiom (redundantly) included. The Łukasiewicz connectives can be defined in terms of $+, -, \leq$.

mvLogic/example_6.1. Example for MV-algebras with monotone and bounded function. The Łukasiewicz connectives can be defined in terms of $+, -, \leq$.

RBC_simple. Example with train controller.

RBC_variable2. Example with train controller.

14.2 Test results

The running times are given in User + sys times (in s). Run on an Intel Xeon 3 GHz, 512 kB cache; median of 100 runs (entries marked with ^{1}: 10 runs; marked with ^{2}: 3 runs). The third column lists the number of clauses produced; "unknown" means Yices answer was `unknown`, "out. mem." means out of memory and time out was set at 6h. Yices version 1.0.19 was used.

The answer "unknown*" for the satisfiable examples with monotone functions over distributive lattices/posets (H-PILoT followed by Yices) is due to the fact that Yices cannot handle the universal axioms of distributive lattices/posets. A translation of such problems to SAT provides a decision procedure (cf. [Sof05] and also [Sof03]).

Name	**status**	**#cl.**	**H-PILoT + yices**	**H-PILoT + yices** stop at $\mathcal{K}[G]$	**yices**
array insert (implicit ∃)	Unsat	310	0.29	0.06	0.36
array insert (implicit ∃).sat	Sat	196	0.13	0.04	time out
array insert	Unsat	113	0.07	0.03	318.22[1]
array insert (linear version)	Unsat	113	0.07	0.03	7970.53[2]
array insert.sat	Sat	111	0.07	0.03	time out
array insert real	Unsat	113	0.07	0.03	360.00[1]
array insert real (linear)	Unsat	113	0.07	0.03	7930.00[2]
array insert real.sat	Sat	111	0.07	0.03	time out
update process priorities	Unsat	45	0.02	0.02	0.03
update process priorities.sat	Sat	37	0.02	0.02	unknown
list1	Unsat	18	0.02	0.01	0.02
list1.sat	Sat	16	0.02	0.01	unknown
chain1	Unsat	22	0.01	0.01	0.02
chain2	Unsat	46	0.02	0.02	0.02
mono	Unsat	20	0.01	0.01	0.01
mono.sat	Sat	20	0.01	0.01	unknown
mono for distributive lattices.R	Unsat	27	0.22	0.06	0.03
mono for distributive lattices.R.sat	Sat	26	unknown*	unknown*	unknown
mono for distributive lattices	Unsat	17	0.01	0.01	0.02
mono for distributive lattices.sat	Sat	17	0.01	0.01	unknown
mono for poset	Unsat	20	0.02	0.02	0.02
mono for poset.sat	Sat	19	unknown*	unknown*	unknown
mono for total order	Unsat	20	0.02	0.02	0.02
own	Unsat	16	0.01	0.01	0.01
mvLogic/mv1	Unsat	10	0.01	0.01	0.02
mvLogic/mv1.sat	Sat	8	0.01	0.01	unknown
mvLogic/mv2	Unsat	8	0.01	0.01	0.06
mvLogic/bl1	Unsat	22	0.02	0.01	0.03
mvLogic/example_6.1	Unsat	10	0.01	0.01	0.03
mvLogic/example_6.1.sat	Sat	10	0.01	0.01	unknown
RBC_simple	Unsat	42	0.03	0.02	0.03
double array insert	Unsat	791	1.16	0.20	0.07
double array insert	Sat	790	1.10	0.20	unknown
RBC_simple.sat	Sat	40	0.03	0.02	out. mem.
RBC_variable2	Unsat	137	0.08	0.04	0.04
RBC_variable2.sat	Sat	136	0.08	0.04	out. mem.

15 Examples

15.1 A case study

In [IJSS08] we used H-PILoT for verifying the correctness of the controller of a system of trains moving on a linear track. In [FIJS10], H-PILoT's ability to decide the pointer fragment of Section 11.2 has been used in the verification

of real-time systems which exhibit rich and dynamic data structures. There H-PILoT was part of a tool chain employed for the verification of a case study from the European Train Control System standard, describing the controller of a system of trains moving on a rail track with complex topology – modeled using two-sorted pointer structures. The tool chain received as input a high level specification and a formula (a safety property), and generated proof obligations, which were automatically verified using H-PILoT (with Yices).
The verification problem we considered are expressed as satisfiability problems for universally quantified formulas, hence cannot be solved by SMT-solvers alone. The experimental results show H-PILoT to be a very efficient tool for the discharging of all the proof tasks of the case study. The full type system implemented in H-PILoT increased the efficiency considerably by blocking unnecessary instantiations. The tool chain used in the case study range from a specification language for real-time systems called COD to the translation of such a specification via phase-event automata (Syspect/PEA) to transition constraint systems (TCS) which can then be exported to H-PILoT. The invariant for every transition in the TCS was checked.
Since the invariant was too complex to be handled by the clausifier of H-PILoT we checked the invariant for every transition in two parts yielding 92 proof obligations. Further, we performed tests to ensure that the specifications are consistent. The time to compute the TCS from the specification was insignificant. The overall time to verify all transition updates with Yices and H-PILoT in the unsatisfiable case (when the invariant is correct) does not differ much. There is one example – the speed update – on which H-PILoT was 5 times faster than Yices alone.
We also made tests with the verification of a set of conditions which was not inductive over all transitions. Here, H-PILoT was able to provide a model after 8s whereas Yices detected unsatisfiability for 17 problems, returned "unknown" for 28, and timed out once. For the consistency check H-PILoT was able to provide a model after 3s, whereas Yices answered "unknown".
During the development of the case study H-PILoT helped us finding the correct transition invariants by providing models for satisfiable sets of clauses (occurring when the safety formulae were not invariant under transitions).

15.2 A run example of H-PILoT

We consider an example taken from [BM07] (Example 11.10). The input file looks as follows.

```
% arrays_from_book.loc
Base_functions:={(+, 2), (-, 2)}
Extension_functions:={(a, 1, 1, int, int), (b, 1, 1, int, int)}
Relations:={(<=, 2)}

% K
Formulas :=
 AND( (FORALL i). (AND(l <= i, i <= u) --> a(i) = b(i)),
       NOT((FORALL i). (AND(l <= i, i <= u + _1) -->
                        write(a, u + _1, b(u +_1))(i) = b(i))));
```

The arrays a and b are considered to be equal between the constants l and u. We prove that if we update a at $u+1$ to $b(u+1)$ then a and b should be equal between l and $u+1$. The formula above denies this and should therefore be inconsistent. We call H-PILoT with

```
hpilot.opt -preprocess -prClauses arrays_from_book.loc
```

`-preprocess` is needed as usual; we use `-prClauses` to get a trace of the program. (Because the array keyword `write` appears in the input we don't have to use the flag `-arrays`: it is implicit.) The trace looks as follows (to improve readability we aligned the level labels and often left out the listing of the extension ground terms due to space constraints).
First, H-PILoT reads the input and clausifies the formula.

```
 ********************************************* Starting hpilot*********************************************
 arrays_from_book.loc
 Adding formula:
 AND( (FORALL i). (AND( l <= i, i <= u) --> a(i) = b(i)),
      NOT((FORALL i). (AND( l <= i, i <= +(u, _1)) --> read(write(a, +(u, _1),b(+(u, _1))), i) = b(i))))
 done.
 Clausifying formulas...
 (FORALL z_1). OR( NOT(l <= z_1), NOT(z_1 <= u), a(z_1) = b(z_1))
 l <= sk_1
 sk_1 <= +(u, _1)
 NOT(read(write(a, +(u, _1), b(+(u, _1))), sk_1) = b(sk_1))
 Yielding 4 new clauses:
 read(write(a, +(u, _1), b(+(u, _1))), sk_1) = b(sk_1) --->    L: 0; Extension ground terms: b(sk_1), b(+(u, _1))
  ---> sk_1 <= +(u, _1)                                        L: 0; Extension ground terms:
  ---> l <= sk_1                                               L: 0; Extension ground terms:
 [z_1] l <= z_1, z_1 <= u ---> a(z_1) = b(z_1)                 L: 0; Extension ground terms:
```

H-PILoT then replaces array writes by introducing new arrays:
write$(a, u+1, b(u+1))$ is replaced by: $\forall i.i \neq u+1 \rightarrow a_{w1}(i) = a(i)$ and $a_{w1}(u+1) = b(u+1)$. To remain in the decidable fragment, H-PILoT replaces $\forall i.i \neq u+1 \rightarrow a_{w1}(i) = a(i)$ with $\forall i.i \leq u+1-1 \rightarrow a_{w1}(i) = a(i)$ and $\forall i.u+1+1 \leq i \rightarrow a_{w1}(i) = a(i)$.

```
 Replacing writes...
 We have 1 levels.
 Our base theory is:
 empty.
 Splitting clause [i]  ---> i = +(u, _1), a_w1(i) = a(i)                       L: 1;
terms:  on eq i = +(u, _1)
 Checking APF for clause [i] i <= -(+(u, _1), _1) ---> a_w1(i) = a(i)         L: 0;
Extension ground terms: ---> true
 Checking APF for clause [i] +(+(u, _1), _1) <= i ---> a_w1(i) = a(i)         L: 0;
Extension ground terms: ---> true
 Checking APF for clause [z_1] l <= z_1, z_1 <= u ---> a(z_1) = b(z_1)        L: 1;
Extension ground terms: ---> true
 Recalculating all levels.
```

H-PILoT then flattens and linearizes the result.

```
After rewriting we have as clauses K:
 [i, x_1] x_1 = i, i <= -(+(u, _1), _1) ---> a_w1(i) = a(x_1)  L: 1; Extension ground terms:
 [i, x_1] x_1 = i, +(+(u, _1), _1) <= i ---> a_w1(i) = a(x_1)  L: 1; Extension ground terms:
 [z_1, x_1] x_1 = z_1, l <= z_1, z_1 <= u ---> a(z_1) = b(x_1) L: 1; Extension ground terms:
 and as query:
  ---> l <= sk_1                                               L: 0; Extension ground terms:
  ---> sk_1 <= +(u, _1)                                        L: 0; Extension ground terms:
 a_w1(sk_1) = b(sk_1) --->                L: 1; Extension ground terms: a_w1(sk_1), b(sk_1)
  ---> a_w1(+(u, _1)) = b(+(u, _1))  L: 1; Extension ground terms: a_w1(+(u, _1)),b(+(u, _1))
 Our query G is :
  ---> l <= sk_1                                               L: 0; Extension ground terms:
  ---> sk_1 <= +(u, _1)                                        L: 0; Extension ground terms:
 a_w1(sk_1) = b(sk_1) --->                L: 1; Extension ground terms: a_w1(sk_1), b(sk_1)
  ---> a_w1(+(u, _1)) = b(+(u, _1))  L: 1; Extension ground terms: a_w1(+(u, _1)),b(+(u, _1))
 xxxxxxxxxxx End preprocessing.
```

H-PILoT then calculates the set of instances $\mathcal{K}[\Psi(G)]$ and simplifies the terms to avoid redundant instances of clauses (e.g. $u+1-1$ is replaced by u).

```
We have 5 index terms for minimal locality l, sk_1, u, +(u, _1), +(u, _2)
K has 3 members.
[i, x_1] x_1 = i, i <= -(+(u, _1), _1) ---> a_w1(i) = a(x_1)                          L: 1;
[i, x_1] x_1 = i, +(+(u, _1), _1) <= i ---> a_w1(i) = a(x_1)                          L: 1;
[z_1, x_1] x_1 = z_1, l <= z_1, z_1 <= u ---> a(z_1) = b(x_1)                         L: 1;
Computing K<G>...
K<G> looks as follows:
K_G has 75 members.
[] l = l, l <= -(+(u, _1), _1) ---> a_w1(l) = a(l)                                    L: 0;
[] l = sk_1, sk_1 <= -(+(u, _1), _1) ---> a_w1(sk_1) = a(l)                           L: 0;
[] l = u, u <= -(+(u, _1), _1) ---> a_w1(u) = a(l)                                    L: 0;
[] l = +(u, _1), +(u, _1) <= -(+(u, _1), _1) ---> a_w1(+(u, _1)) = a(l)               L: 0;
[] l = +(u, _2), +(u, _2) <= -(+(u, _1), _1) ---> a_w1(+(u, _2)) = a(l)               L: 0;
[] sk_1 = l, l <= -(+(u, _1), _1) ---> a_w1(l) = a(sk_1)                              L: 0;
[] sk_1 = sk_1, sk_1 <= -(+(u, _1), _1) ---> a_w1(sk_1) = a(sk_1)                     L: 0;
[] sk_1 = u, u <= -(+(u, _1), _1) ---> a_w1(u) = a(sk_1)                              L: 0;
[] sk_1 = +(u, _1), +(u, _1) <= -(+(u, _1), _1) ---> a_w1(+(u, _1)) = a(sk_1)         L: 0;
[] sk_1 = +(u, _2), +(u, _2) <= -(+(u, _1), _1) ---> a_w1(+(u, _2)) = a(sk_1)         L: 0;
[] u = l, l <= -(+(u, _1), _1) ---> a_w1(l) = a(u)                                    L: 0;
[] u = sk_1, sk_1 <= -(+(u, _1), _1) ---> a_w1(sk_1) = a(u)                           L: 0;
[] u = u, u <= -(+(u, _1), _1) ---> a_w1(u) = a(u)                                    L: 0;
[] u = +(u, _1), +(u, _1) <= -(+(u, _1), _1) ---> a_w1(+(u, _1)) = a(u)               L: 0;
[] u = +(u, _2), +(u, _2) <= -(+(u, _1), _1) ---> a_w1(+(u, _2)) = a(u)               L: 0;
[] +(u, _1) = l, l <= -(+(u, _1), _1) ---> a_w1(l) = a(+(u, _1))                      L: 0;
[] +(u, _1) = sk_1, sk_1 <= -(+(u, _1), _1) ---> a_w1(sk_1) = a(+(u, _1))             L: 0;
[] +(u, _1) = u, u <= -(+(u, _1), _1) ---> a_w1(u) = a(+(u, _1))                      L: 0;
[] +(u, _1) = +(u, _1), +(u, _1) <= -(+(u, _1), _1) ---> a_w1(+(u, _1)) = a(+(u,_1))  L: 0;
[] +(u, _1) = +(u, _2), +(u, _2) <= -(+(u, _1), _1) ---> a_w1(+(u, _2)) = a(+(u,_1))  L: 0;
[] +(u, _2) = l, l <= -(+(u, _1), _1) ---> a_w1(l) = a(+(u, _2))                      L: 0;
[] +(u, _2) = sk_1, sk_1 <= -(+(u, _1), _1) ---> a_w1(sk_1) = a(+(u, _2))             L: 0;
[] +(u, _2) = u, u <= -(+(u, _1), _1) ---> a_w1(u) = a(+(u, _2))                      L: 0;
[] +(u, _2) = +(u, _1), +(u, _1) <= -(+(u, _1), _1) ---> a_w1(+(u, _1)) = a(+(u,_2))  L: 0;
[] +(u, _2) = +(u, _2), +(u, _2) <= -(+(u, _1), _1) ---> a_w1(+(u, _2)) = a(+(u,_2))  L: 0;
[] l = l, +(+(u, _1), _1) <= l ---> a_w1(l) = a(l)                                    L: 0;
[] l = sk_1, +(+(u, _1), _1) <= sk_1 ---> a_w1(sk_1) = a(l)                           L: 0;
[] l = u, +(+(u, _1), _1) <= u ---> a_w1(u) = a(l)                                    L: 0;
[] l = +(u, _1), +(+(u, _1), _1) <= +(u, _1) ---> a_w1(+(u, _1)) = a(l)               L: 0;
[] l = +(u, _2), +(+(u, _1), _1) <= +(u, _2) ---> a_w1(+(u, _2)) = a(l)               L: 0;
[] sk_1 = l, +(+(u, _1), _1) <= l ---> a_w1(l) = a(sk_1)                              L: 0;
[] sk_1 = sk_1, +(+(u, _1), _1) <= sk_1 ---> a_w1(sk_1) = a(sk_1)                     L: 0;
[] sk_1 = u, +(+(u, _1), _1) <= u ---> a_w1(u) = a(sk_1)                              L: 0;
[] sk_1 = +(u, _1), +(+(u, _1), _1) <= +(u, _1) ---> a_w1(+(u, _1)) = a(sk_1)         L: 0;
[] sk_1 = +(u, _2), +(+(u, _1), _1) <= +(u, _2) ---> a_w1(+(u, _2)) = a(sk_1)         L: 0;
[] u = l, +(+(u, _1), _1) <= l ---> a_w1(l) = a(u)                                    L: 0;
[] u = sk_1, +(+(u, _1), _1) <= sk_1 ---> a_w1(sk_1) = a(u)                           L: 0;
[] u = u, +(+(u, _1), _1) <= u ---> a_w1(u) = a(u)                                    L: 0;
[] u = +(u, _1), +(+(u, _1), _1) <= +(u, _1) ---> a_w1(+(u, _1)) = a(u)               L: 0;
[] u = +(u, _2), +(+(u, _1), _1) <= +(u, _2) ---> a_w1(+(u, _2)) = a(u)               L: 0;
[] +(u, _1) = l, +(+(u, _1), _1) <= l ---> a_w1(l) = a(+(u, _1))                      L: 0;
[] +(u, _1) = sk_1, +(+(u, _1), _1) <= sk_1 ---> a_w1(sk_1) = a(+(u, _1))             L: 0;
[] +(u, _1) = u, +(+(u, _1), _1) <= u ---> a_w1(u) = a(+(u, _1))                      L: 0;
[] +(u, _1) = +(u, _1), +(+(u, _1), _1) <= +(u, _1) ---> a_w1(+(u, _1)) = a(+(u,_1))  L: 0;
[] +(u, _1) = +(u, _2), +(+(u, _1), _1) <= +(u, _2) ---> a_w1(+(u, _2)) = a(+(u,_1))  L: 0;
[] +(u, _2) = l, +(+(u, _1), _1) <= l ---> a_w1(l) = a(+(u, _2))                      L: 0;
[] +(u, _2) = sk_1, +(+(u, _1), _1) <= sk_1 ---> a_w1(sk_1) = a(+(u, _2))             L: 0;
[] +(u, _2) = u, +(+(u, _1), _1) <= u ---> a_w1(u) = a(+(u, _2))                      L: 0;
[] +(u, _2) = +(u, _1), +(+(u, _1), _1) <= +(u, _1) ---> a_w1(+(u, _1)) = a(+(u,_2))  L: 0;
[] +(u, _2) = +(u, _2), +(+(u, _1), _1) <= +(u, _2) ---> a_w1(+(u, _2)) = a(+(u,_2))  L: 0;
[] l = l, l <= l, l <= u ---> a(l) = b(l)                                             L: 0;
[] sk_1 = l, l <= l, l <= u ---> a(l) = b(sk_1)                                       L: 0;
[] u = l, l <= l, l <= u ---> a(l) = b(u)                                             L: 0;
[] +(u, _1) = l, l <= l, l <= u ---> a(l) = b(+(u, _1))                               L: 0;
[] +(u, _2) = l, l <= l, l <= u ---> a(l) = b(+(u, _2))                               L: 0;
[] l = sk_1, l <= sk_1, sk_1 <= u ---> a(sk_1) = b(l)                                 L: 0;
[] sk_1 = sk_1, l <= sk_1, sk_1 <= u ---> a(sk_1) = b(sk_1)                           L: 0;
[] u = sk_1, l <= sk_1, sk_1 <= u ---> a(sk_1) = b(u)                                 L: 0;
[] +(u, _1) = sk_1, l <= sk_1, sk_1 <= u ---> a(sk_1) = b(+(u, _1))                   L: 0;
[] +(u, _2) = sk_1, l <= sk_1, sk_1 <= u ---> a(sk_1) = b(+(u, _2))                   L: 0;
[] l = u, l <= u, u <= u ---> a(u) = b(l)                                             L: 0;
[] sk_1 = u, l <= u, u <= u ---> a(u) = b(sk_1)                                       L: 0;
[] u = u, l <= u, u <= u ---> a(u) = b(u)                                             L: 0;
[] +(u, _1) = u, l <= u, u <= u ---> a(u) = b(+(u, _1))                               L: 0;
[] +(u, _2) = u, l <= u, u <= u ---> a(u) = b(+(u, _2))                               L: 0;
[] l = +(u, _1), l <= +(u, _1), +(u, _1) <= u ---> a(+(u, _1)) = b(l)                 L: 0;
[] sk_1 = +(u, _1), l <= +(u, _1), +(u, _1) <= u ---> a(+(u, _1)) = b(sk_1)           L: 0;
[] u = +(u, _1), l <= +(u, _1), +(u, _1) <= u ---> a(+(u, _1)) = b(u)                 L: 0;
[] +(u, _1) = +(u, _1), l <= +(u, _1), +(u, _1) <= u ---> a(+(u, _1)) = b(+(u,_1))    L: 0;
[] +(u, _2) = +(u, _1), l <= +(u, _1), +(u, _1) <= u ---> a(+(u, _1)) = b(+(u,_2))    L: 0;
[] l = +(u, _2), l <= +(u, _2), +(u, _2) <= u ---> a(+(u, _2)) = b(l)                 L: 0;
[] sk_1 = +(u, _2), l <= +(u, _2), +(u, _2) <= u ---> a(+(u, _2)) = b(sk_1)           L: 0;
[] u = +(u, _2), l <= +(u, _2), +(u, _2) <= u ---> a(+(u, _2)) = b(u)                 L: 0;
[] +(u, _1) = +(u, _2), l <= +(u, _2), +(u, _2) <= u ---> a(+(u, _2)) = b(+(u,_1))    L: 0;
[] +(u, _2) = +(u, _2), l <= +(u, _2), +(u, _2) <= u ---> a(+(u, _2)) = b(+(u,_2))    L: 0;
```

The result is then purified:

```
computing defs ...
 We have the following definitions:
  ---> e_1 = a(l)                                                        L: 0; Extension ground terms: a(l)
  ---> e_2 = a(sk_1)                                                     L: 0; Extension ground terms: a(sk_1)
  ---> e_3 = a(u)                                                        L: 0; Extension ground terms: a(u)
  ---> e_4 = a(+(u, _1))                                                 L: 0; Extension ground terms: a(+(u, _1))
  ---> e_5 = a(+(u, _2))                                                 L: 0; Extension ground terms: a(+(u, _2))
  ---> e_6 = a_w1(l)                                                     L: 0; Extension ground terms: a_w1(l)
  ---> e_7 = a_w1(sk_1)                                                  L: 0; Extension ground terms: a_w1(sk_1)
  ---> e_8 = a_w1(u)                                                     L: 0; Extension ground terms: a_w1(u)
  ---> e_9 = a_w1(+(u, _1))                                              L: 0; Extension ground terms: a_w1(+(u, _1))
  ---> e_10 = a_w1(+(u, _2))                                             L: 0; Extension ground terms: a_w1(+(u, _2))
  ---> e_11 = b(l)                                                       L: 0; Extension ground terms: b(l)
  ---> e_12 = b(sk_1)                                                    L: 0; Extension ground terms: b(sk_1)
  ---> e_13 = b(u)                                                       L: 0; Extension ground terms: b(u)
  ---> e_14 = b(+(u, _1))                                                L: 0; Extension ground terms: b(+(u, _1))
  ---> e_15 = b(+(u, _2))                                                L: 0; Extension ground terms: b(+(u, _2))
 Purified:
 K_G has 75 members.
 [] l = l, l <= -(+(u, _1), _1) ---> e_6 = e_1                           L: 0; Extension ground terms:
 [] l = sk_1, sk_1 <= -(+(u, _1), _1) ---> e_7 = e_1                     L: 0; Extension ground terms:
 [] l = u, u <= -(+(u, _1), _1) ---> e_8 = e_1                           L: 0; Extension ground terms:
 [] l = +(u, _1), +(u, _1) <= -(+(u, _1), _1) ---> e_9 = e_1             L: 0; Extension ground terms:
 [] l = +(u, _2), +(u, _2) <= -(+(u, _1), _1) ---> e_10 = e_1            L: 0; Extension ground terms:
 [] sk_1 = l, l <= -(+(u, _1), _1) ---> e_6 = e_2                        L: 0; Extension ground terms:
 [] sk_1 = sk_1, sk_1 <= -(+(u, _1), _1) ---> e_7 = e_2                  L: 0; Extension ground terms:
 [] sk_1 = u, u <= -(+(u, _1), _1) ---> e_8 = e_2                        L: 0; Extension ground terms:
 [] sk_1 = +(u, _1), +(u, _1) <= -(+(u, _1), _1) ---> e_9 = e_2          L: 0; Extension ground terms:
 [] sk_1 = +(u, _2), +(u, _2) <= -(+(u, _1), _1) ---> e_10 = e_2         L: 0; Extension ground terms:
 [] u = l, l <= -(+(u, _1), _1) ---> e_6 = e_3                           L: 0; Extension ground terms:
 [] u = sk_1, sk_1 <= -(+(u, _1), _1) ---> e_7 = e_3                     L: 0; Extension ground terms:
 [] u = u, u <= -(+(u, _1), _1) ---> e_8 = e_3                           L: 0; Extension ground terms:
 [] u = +(u, _1), +(u, _1) <= -(+(u, _1), _1) ---> e_9 = e_3             L: 0; Extension ground terms:
 [] u = +(u, _2), +(u, _2) <= -(+(u, _1), _1) ---> e_10 = e_3            L: 0; Extension ground terms:
 [] +(u, _1) = l, l <= -(+(u, _1), _1) ---> e_6 = e_4                    L: 0; Extension ground terms:
 [] +(u, _1) = sk_1, sk_1 <= -(+(u, _1), _1) ---> e_7 = e_4              L: 0; Extension ground terms:
 [] +(u, _1) = u, u <= -(+(u, _1), _1) ---> e_8 = e_4                    L: 0; Extension ground terms:
 [] +(u, _1) = +(u, _1), +(u, _1) <= -(+(u, _1), _1) ---> e_9 = e_4      L: 0; Extension ground terms:
 [] +(u, _1) = +(u, _2), +(u, _2) <= -(+(u, _1), _1) ---> e_10 = e_4     L: 0; Extension ground terms:
 [] +(u, _2) = l, l <= -(+(u, _1), _1) ---> e_6 = e_5                    L: 0; Extension ground terms:
 [] +(u, _2) = sk_1, sk_1 <= -(+(u, _1), _1) ---> e_7 = e_5              L: 0; Extension ground terms:
 [] +(u, _2) = u, u <= -(+(u, _1), _1) ---> e_8 = e_5                    L: 0; Extension ground terms:
 [] +(u, _2) = +(u, _1), +(u, _1) <= -(+(u, _1), _1) ---> e_9 = e_5      L: 0; Extension ground terms:
 [] +(u, _2) = +(u, _2), +(u, _2) <= -(+(u, _1), _1) ---> e_10 = e_5     L: 0; Extension ground terms:
 [] l = l, +(+(u, _1), _1) <= l ---> e_6 = e_1                           L: 0; Extension ground terms:
 [] l = sk_1, +(+(u, _1), _1) <= sk_1 ---> e_7 = e_1                     L: 0; Extension ground terms:
 [] l = u, +(+(u, _1), _1) <= u ---> e_8 = e_1                           L: 0; Extension ground terms:
 [] l = +(u, _1), +(+(u, _1), _1) <= +(u, _1) ---> e_9 = e_1             L: 0; Extension ground terms:
 [] l = +(u, _2), +(+(u, _1), _1) <= +(u, _2) ---> e_10 = e_1            L: 0; Extension ground terms:
 [] sk_1 = l, +(+(u, _1), _1) <= l ---> e_6 = e_2                        L: 0; Extension ground terms:
 [] sk_1 = sk_1, +(+(u, _1), _1) <= sk_1 ---> e_7 = e_2                  L: 0; Extension ground terms:
 [] sk_1 = u, +(+(u, _1), _1) <= u ---> e_8 = e_2                        L: 0; Extension ground terms:
 [] sk_1 = +(u, _1), +(+(u, _1), _1) <= +(u, _1) ---> e_9 = e_2          L: 0; Extension ground terms:
 [] sk_1 = +(u, _2), +(+(u, _1), _1) <= +(u, _2) ---> e_10 = e_2         L: 0; Extension ground terms:
 [] u = l, +(+(u, _1), _1) <= l ---> e_6 = e_3                           L: 0; Extension ground terms:
 [] u = sk_1, +(+(u, _1), _1) <= sk_1 ---> e_7 = e_3                     L: 0; Extension ground terms:
 [] u = u, +(+(u, _1), _1) <= u ---> e_8 = e_3                           L: 0; Extension ground terms:
 [] u = +(u, _1), +(+(u, _1), _1) <= +(u, _1) ---> e_9 = e_3             L: 0; Extension ground terms:
 [] u = +(u, _2), +(+(u, _1), _1) <= +(u, _2) ---> e_10 = e_3            L: 0; Extension ground terms:
 [] +(u, _1) = l, +(+(u, _1), _1) <= l ---> e_6 = e_4                    L: 0; Extension ground terms:
 [] +(u, _1) = sk_1, +(+(u, _1), _1) <= sk_1 ---> e_7 = e_4              L: 0; Extension ground terms:
 [] +(u, _1) = u, +(+(u, _1), _1) <= u ---> e_8 = e_4                    L: 0; Extension ground terms:
 [] +(u, _1) = +(u, _1), +(+(u, _1), _1) <= +(u, _1) ---> e_9 = e_4      L: 0; Extension ground terms:
 [] +(u, _1) = +(u, _2), +(+(u, _1), _1) <= +(u, _2) ---> e_10 = e_4     L: 0; Extension ground terms:
 [] +(u, _2) = l, +(+(u, _1), _1) <= l ---> e_6 = e_5                    L: 0; Extension ground terms:
 [] +(u, _2) = sk_1, +(+(u, _1), _1) <= sk_1 ---> e_7 = e_5              L: 0; Extension ground terms:
 [] +(u, _2) = u, +(+(u, _1), _1) <= u ---> e_8 = e_5                    L: 0; Extension ground terms:
 [] +(u, _2) = +(u, _1), +(+(u, _1), _1) <= +(u, _1) ---> e_9 = e_5      L: 0; Extension ground terms:
 [] +(u, _2) = +(u, _2), +(+(u, _1), _1) <= +(u, _2) ---> e_10 = e_5     L: 0; Extension ground terms:
 [] l = l, l <= l, l <= u ---> e_1 = e_11                                L: 0; Extension ground terms:
 [] sk_1 = l, l <= l, l <= u ---> e_1 = e_12                             L: 0; Extension ground terms:
 [] u = l, l <= l, l <= u ---> e_1 = e_13                                L: 0; Extension ground terms:
 [] +(u, _1) = l, l <= l, l <= u ---> e_1 = e_14                         L: 0; Extension ground terms:
 [] +(u, _2) = l, l <= l, l <= u ---> e_1 = e_15                         L: 0; Extension ground terms:
 [] l = sk_1, l <= sk_1, sk_1 <= u ---> e_2 = e_11                       L: 0; Extension ground terms:
 [] sk_1 = sk_1, l <= sk_1, sk_1 <= u ---> e_2 = e_12                    L: 0; Extension ground terms:
 [] u = sk_1, l <= sk_1, sk_1 <= u ---> e_2 = e_13                       L: 0; Extension ground terms:
 [] +(u, _1) = sk_1, l <= sk_1, sk_1 <= u ---> e_2 = e_14                L: 0; Extension ground terms:
 [] +(u, _2) = sk_1, l <= sk_1, sk_1 <= u ---> e_2 = e_15                L: 0; Extension ground terms:
 [] l = u, l <= u, u <= u ---> e_3 = e_11                                L: 0; Extension ground terms:
 [] sk_1 = u, l <= u, u <= u ---> e_3 = e_12                             L: 0; Extension ground terms:
 [] u = u, l <= u, u <= u ---> e_3 = e_13                                L: 0; Extension ground terms:
 [] +(u, _1) = u, l <= u, u <= u ---> e_3 = e_14                         L: 0; Extension ground terms:
 [] +(u, _2) = u, l <= u, u <= u ---> e_3 = e_15                         L: 0; Extension ground terms:
 [] l = +(u, _1), l <= +(u, _1), +(u, _1) <= u ---> e_4 = e_11           L: 0; Extension ground terms:
 [] sk_1 = +(u, _1), l <= +(u, _1), +(u, _1) <= u ---> e_4 = e_12        L: 0; Extension ground terms:
 [] u = +(u, _1), l <= +(u, _1), +(u, _1) <= u ---> e_4 = e_13           L: 0; Extension ground terms:
 [] +(u, _1) = +(u, _1), l <= +(u, _1), +(u, _1) <= u ---> e_4 = e_14    L: 0; Extension ground terms:
 [] +(u, _2) = +(u, _1), l <= +(u, _1), +(u, _1) <= u ---> e_4 = e_15    L: 0; Extension ground terms:
 [] l = +(u, _2), l <= +(u, _2), +(u, _2) <= u ---> e_5 = e_11           L: 0; Extension ground terms:
 [] sk_1 = +(u, _2), l <= +(u, _2), +(u, _2) <= u ---> e_5 = e_12        L: 0; Extension ground terms:
 [] u = +(u, _2), l <= +(u, _2), +(u, _2) <= u ---> e_5 = e_13           L: 0; Extension ground terms:
 [] +(u, _1) = +(u, _2), l <= +(u, _2), +(u, _2) <= u ---> e_5 = e_14    L: 0; Extension ground terms:
  [] +(u, _2) = +(u, _2), l <= +(u, _2), +(u, _2) <= u ---> e_5 = e_15    L: 0; Extension ground terms:
  ---> l <= sk_1                                                          L: 0; Extension ground terms:
  ---> sk_1 <= +(u, _1)                                                   L: 0; Extension ground terms:
 e_7 = e_12 --->                                                          L: 0; Extension ground terms:
  ---> e_9 = e_14                                                         L: 0; Extension ground terms:
```

The introduced definitions are replaced by the corresponding congruence axioms.

```
Replacing D by N0:
 This yields 30 clauses.
 +(u, _1) = +(u, _2) ---> e_14 = e_15                                    L: 0; Extension ground terms:
 u = +(u, _2) ---> e_13 = e_15                                           L: 0; Extension ground terms:
 u = +(u, _1) ---> e_13 = e_14                                           L: 0; Extension ground terms:
 sk_1 = +(u, _2) ---> e_12 = e_15                                        L: 0; Extension ground terms:
 sk_1 = +(u, _1) ---> e_12 = e_14                                        L: 0; Extension ground terms:
 sk_1 = u ---> e_12 = e_13                                               L: 0; Extension ground terms:
 l = +(u, _2) ---> e_11 = e_15                                           L: 0; Extension ground terms:
 l = +(u, _1) ---> e_11 = e_14                                           L: 0; Extension ground terms:
 l = u ---> e_11 = e_13                                                  L: 0; Extension ground terms:
 l = sk_1 ---> e_11 = e_12                                               L: 0; Extension ground terms:
 +(u, _1) = +(u, _2) ---> e_9 = e_10                                     L: 0; Extension ground terms:
 u = +(u, _2) ---> e_8 = e_10                                            L: 0; Extension ground terms:
 u = +(u, _1) ---> e_8 = e_9                                             L: 0; Extension ground terms:
 sk_1 = +(u, _2) ---> e_7 = e_10                                         L: 0; Extension ground terms:
 sk_1 = +(u, _1) ---> e_7 = e_9                                          L: 0; Extension ground terms:
 sk_1 = u ---> e_7 = e_8                                                 L: 0; Extension ground terms:
 l = +(u, _2) ---> e_6 = e_10                                            L: 0; Extension ground terms:
 l = +(u, _1) ---> e_6 = e_9                                             L: 0; Extension ground terms:
 l = u ---> e_6 = e_8                                                    L: 0; Extension ground terms:
 l = sk_1 ---> e_6 = e_7                                                 L: 0; Extension ground terms:
 +(u, _1) = +(u, _2) ---> e_4 = e_5                                      L: 0; Extension ground terms:
 u = +(u, _2) ---> e_3 = e_5                                             L: 0; Extension ground terms:
 u = +(u, _1) ---> e_3 = e_4                                             L: 0; Extension ground terms:
 sk_1 = +(u, _2) ---> e_2 = e_5                                          L: 0; Extension ground terms:
 sk_1 = +(u, _1) ---> e_2 = e_4                                          L: 0; Extension ground terms:
 sk_1 = u ---> e_2 = e_3                                                 L: 0; Extension ground terms:
 l = +(u, _2) ---> e_1 = e_5                                             L: 0; Extension ground terms:
 l = +(u, _1) ---> e_1 = e_4                                             L: 0; Extension ground terms:
 l = u ---> e_1 = e_3                                                    L: 0; Extension ground terms:
 l = sk_1 ---> e_1 = e_2                                                 L: 0; Extension ground terms:
```

Finally we hand over to a prover (Yices is default here). The program checked earlier that the problem was in a decidable fragment of the theory of arrays (APF), which is Ψ-local. Hence Yices' answer can always be trusted irrespective of whether the answer is 'satisfiable' or 'unsatisfiable'.

```
The problem is in APF
Handing over to Yices:
Total number of clauses: 109.
    unsat
unsat
H-PILoT spent                     0.161975s on the problem.
Of which clausification took      0.006998s.
The prover needed                 0.021996s for the problem.
Total running time:               0.183971s.
```

15.3 Model generation and visualization

We illustrate the method for model generation we use on two examples.

Example 1: Theory of pointers

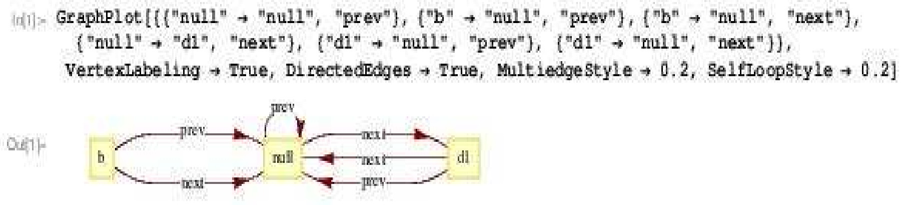
Consider the following problem, specified in H-PILoT input syntax:

```
Base_functions:={(+,2), (-, 2)}
Extension_functions:={(next, 1, 1, pointer), (prev, 1, 1, pointer),
                      (priority, 1, 1, pointer, real)}
Relations:={(>=, 2)}
Constants:={(null, pointer), (a, pointer), (b, pointer)}

% K
Clauses := (FORALL p). prev(next(p)) = p;
           (FORALL p). --> next(prev(p)) = p;
           (FORALL p). priority(p) >= priority(next(p));

Query := priority(a) = _5; priority(b) = _6; a = prev(b);
        %NOT(a = null); % pivotal for satisfiability
         NOT(b = null);
```

We used CVC3 to generate a model for the set of clauses obtained after the hierarchical reduction. A partial model is given below (after preprocessing/deleting repetitions):

```
null = a
prev(b) = a
prev(a) = d_5
next(prev(prev(b))) = e_5
next(prev(b)) = d_1
next(b) = a
prev(prev(b)) = d_5
prev(next(prev(b))) = d_5
prev(next(b)) = d_5
prev(next(a)) = d_5
next(a) = d_1
priority(next(a)) = 0
priority(next(prev(b))) = 0
priority(prev(b)) = 5
priority(b) = 6
priority(a) = 5
```

We can make this model total by defining next(x) := null and prev(y) := null whenever next(x) resp. prev(y) are undefined (cf. Section 11.2). The obtained model can be visualized using Mathematica (this last step is currently performed separately; it is not yet integrated into H-PILoT, but an integration is planned for the near future.). The result is presented below.



```
In[1]:= GraphPlot[{{"null" → "null", "prev"}, {"b" → "null", "prev"}, {"b" → "null", "next"},
          {"null" → "d1", "next"}, {"d1" → "null", "prev"}, {"d1" → "null", "next"}},
         VertexLabeling → True, DirectedEdges → True, MultiedgeStyle → 0.2, SelfLoopStyle → 0.2]

Out[1]=
```

Example 2: Theory of functions over the real numbers

Consider now the following example: decide whether

$$\mathsf{Mon}_f \cup \mathsf{Mon}_g \models_{\mathbb{R}} \forall x, y, z, u, v (x \leq y \wedge z \leq y \wedge f(y) \leq g(u) \wedge u \leq v \wedge u \leq w \rightarrow f(x) \leq g(v)).$$

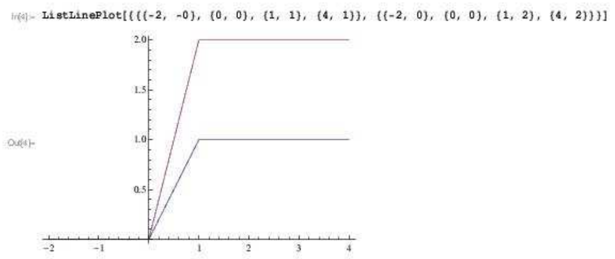
We formulate a satisfiable version, by replacing the argument x in the conclusion with a new variable x_0. The problem obtained this way can be formulated as follows in the input format of H-PILoT.

```
Base_functions:={}
Extension_functions:={(f, 1), (g, 1)}
Relations:={(<=, 2)}

Clauses := (FORALL x,y). x <= y --> f(x) <= f(y);
           (FORALL x,y). x <= y --> g(x) <= g(y);

Query := c1 <= d1; c2 <= d1; d2 <= c3; d2 <= c4;
         f(d1) <= g(d2); NOT(f(c0) <= g(c4));
```

CVC3 can be used to generate the following model of the proof task:

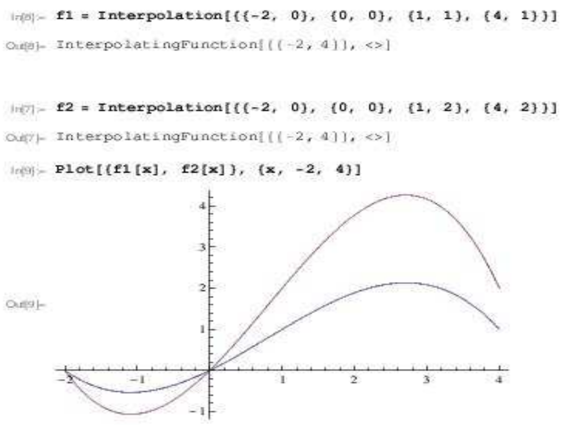
```
model:
c0 = 1
c1 = 0
d1 = 0
c2 = 0
c3 = 0
c4 = 1
d2 = 0
g(d2) = 0
f(d1) = 0
g(c4) = 1
f(c0) = 2
```

From this partial model, a total model can be constructed as explained in [Sof08]. This model can then be visualized as follows in Mathematica:



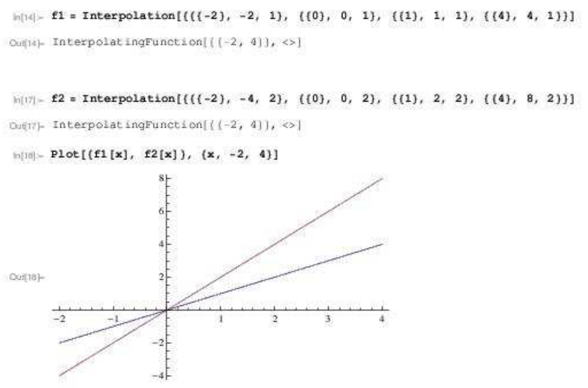
```
In[4]:= ListLinePlot[{{{-2, -0}, {0, 0}, {1, 1}, {4, 1}}, {{-2, 0}, {0, 0}, {1, 2}, {4, 2}}}]
```

Note that if we require differentiability of f and g then – with the completion described in the previous example – monotonicity of the extensions may not be preserved:



```
In[6]:= f1 = Interpolation[{{-2, 0}, {0, 0}, {1, 1}, {4, 1}}]
Out[6]= InterpolatingFunction[{{-2, 4}}, <>]

In[7]:= f2 = Interpolation[{{-2, 0}, {0, 0}, {1, 2}, {4, 2}}]
Out[7]= InterpolatingFunction[{{-2, 4}}, <>]
In[9]:= Plot[{f1[x], f2[x]}, {x, -2, 4}]
Out[9]=
```

In the example above we can enforce the functions to be linear. A general study of the properties which can be guaranteed when building the models is work in progress.



```
In[14]:= f1 = Interpolation[{{{-2}, -2, 1}, {{0}, 0, 1}, {{1}, 1, 1}, {{4}, 4, 1}}]
Out[14]= InterpolatingFunction[{{-2, 4}}, <>]

In[17]:= f2 = Interpolation[{{{-2}, -4, 2}, {{0}, 0, 2}, {{1}, 2, 2}, {{4}, 8, 2}}]
Out[17]= InterpolatingFunction[{{-2, 4}}, <>]
In[18]:= Plot[{f1[x], f2[x]}, {x, -2, 4}]
Out[18]=
```

Acknowledgments. This work was partly supported by the German Research Council (DFG) as part of the Transregional Collaborative Research Center "Automatic Verification and Analysis of Complex Systems" (SFB/TR 14 AVACS, `www.avacs.org`).

References

[BdM09] N. Bjørner and L. M. de Moura. Z3[10]: Applications, enablers, challenges and directions. In *Proceedings of the Sixth International Workshop on Constraints in Formal Verification, CFV'09*, 2009. `http://research.microsoft.com/en-us/um/people/leonardo/cfv09.pdf`.

[BT07] C. Barrett and C. Tinelli. CVC3. In W. Damm and H. Hermanns, editors, *Proceedings of CAV'07, LNCS 4590*, pages 298–302. Springer, 2007.

[BG01] D. Basin and H. Ganzinger. Automated complexity analysis based on ordered resolution. *Journal of the ACM* 48(1), 70–109, 2001.

[BM07] A. R. Bradley and Z. Manna. *The Calculus of Computation*. Springer, 1st edition, 2007.

[BMS06] A. R. Bradley, Z. Manna and H. B. Sipma. What's decidable about arrays? In E.A. Emerson and K. S. Namjoshi, editors, *Proceedings of VMCAI'06, LNCS 3855*, pages 427–442. Springer, 2006.

[BPT07] A. Bauer, M. Pister, and M. Tautschnig. Tool-support for the analysis of hybrid systems and models. In R. Lauwereins and J. Madsen, editors, *Design, Automation, and Test in Europe, DATE'07*, pages 924–929. ACM, 2007.

[Bur95] S. Burris. Polynomial time uniform word problems. *Mathematical Logic Quarterly* 41, 173–182, 1995.

[DdM06a] B. Dutertre and L. M. de Moura. A fast linear-arithmetic solver for DPLL(T). In T. Ball and R. B. Jones, editors, *Proceedings CAV'06, LNCS 4144*, pages 81–94. Springer, 2006.

[DdM06b] B. Dutertre and L. M. de Moura. Integrating Simplex with DPLL(T). *CSL Technical Report, SRI-CSL-06-01*, 2006.

[dM09] L. M. de Moura. SMT@Microsoft. Talk given at the Max-Planck-Institut für Informatik, Saarbrücken, Germany, 2009. Slides available at `http://research.microsoft.com/en-us/um/people/leonardo/mpi2009.pdf`.

[dMB08] L. M. de Moura and N. Bjørner. Z3: An efficient SMT solver. In C. R. Ramakrishnan and J. Rehof, editors, *Proceedings of TACAS'08, LNCS 4963*, pages 337–340. Springer, 2008.

[DS97] A. Dolzmann and T. Sturm. Redlog: Computer algebra meets computer logic. *ACM SIGSAM Bulletin* 31(2), pp. 2–9, 1997.

[FIJS10] J. Faber, C. Ihlemann, S. Jacobs, V. Sofronie-Stokkermans. Automatic Verification of Parametric Specifications with Complex Topologies. In D. Méry and S.Merz editors, *Proceedings of IFM'10, LNCS 6396*, pages 152–167, Springer, 2010.

[FHT+07] M. Fränzle, C. Herde, T. Teige, S. Ratschan, and T. Schubert. Efficient solving of large non-linear arithmetic constraint systems with complex boolean structure. *Journal on Satisfiability, Boolean Modeling and Computation*, 1(3-4):209–236, 2007.

[Gan01] H. Ganzinger: Relating semantic and proof-theoretic concepts for polynomial time decidability of uniform word problems. In: *16th IEEE Symposion on Logic in Computer Science*, pages 81–92. IEEE press, New York, 2001.

[GSW04] H. Ganzinger, V. Sofronie-Stokkermans and U. Waldmann. Modular proof systems for partial functions with weak equality. In D. Basin and M. Rusinowitch, editors, *Proceedings of IJCAR'04, LNAI 3097*, pages 168–182, 2004.

[GSW06] H. Ganzinger, V. Sofronie-Stokkermans and U. Waldmann. Modular proof systems for partial functions with Evans equality. *Information and Computation* 204 (10): 1453-1492, 2006.

[GdM09] Y. Ge and L. M. de Moura. Complete instantiation for quantified formulas in satisfiabiliby modulo theories. In A. Bouajjani and O. Maler, editors, *Proceedings of CAV'09, LNCS 5643*, pages 306–320. Springer, 2009.

[GM92] R. Givan and D. A. McAllester. New results on local inference relations. In B. Nebel, C. Rich, W.R. Swartout, editors, *Knowledge Representation and Reasoning, KR'92* 403–412, Morgan Kaufmann, 1992.

[GM02] R. Givan and D. A. McAllester. Polynomial-time computation via local inference relations. *ACM Transactions on Comp. Logic*, 3(4), 521–541, 2002.

[ISS09] C. Ihlemann and V. Sofronie-Stokkermans. System description: H-PILoT. In R. A. Schmidt, editor, *Proceedings of CADE-22, LNCS(LNAI) 5663*, pages 131–139. Springer, 2009.

[IJSS08] C. Ihlemann, S. Jacobs, and V. Sofronie-Stokkermans. On local reasoning in verification. In C. R. Ramakrishnan and J. Rehof, editors, *Proceedings of TACAS'08, LNCS 4963*, pages 265–281. Springer, 2008.

[JSS07] S. Jacobs and V. Sofronie-Stokkermans. Applications of hierarchical reasoning in the verification of complex systems. *Electronic Notes in Theoretical Computer Science*, 174(8):39–54, 2007.

[McA93] D. A. McAllester. Automatic recognition of tractability in inference relations. *Journal of the ACM*, 40(2):284–303, 1993.

[MN05] S. McPeak and G. C. Necula. Data structure specifications via local equality axioms. In K. Etessami and S. K. Rajamani, editors, *Proceedings of CAV'05, LNCS 3576*, pages 476–490. Springer, 2005.

[NW01] A. Nonnengart and C. Weidenbach. Computing Small Clause Normal Forms. In: A. Robinson and A. Voronkov, editors, *Handbook of Automated Reasoning* pp. 335 – 367, Elsevier, Amsterdam, 2001.

[PG86] D. A. Plaisted and S. Greenbaum. A structure-preserving clause form translation. *Journal of Symbolic Computation*, 2(3):293–304, 1986.

[Sof03] V. Sofronie-Stokkermans. Resolution-based decision procedures for the universal theory of some classes of distributive lattices with operators. *Journal of Symbolic Computation*, 36(6):891–924, 2003.

[Sof05] V. Sofronie-Stokkermans. Hierarchic reasoning in local theory extensions. In: R. Nieuwenhuis, editor *Proceedings of CADE-20, LNCS(LNAI) 3632*, pp. 219–234. Springer, 2005.

[Sof06a] V. Sofronie-Stokkermans. Interpolation in local theory extensions. In U. Furbach and N. Shankar, editors, *Proceedings of IJCAR'06, LNCS(LNAI) 4130*, pp. 235–250. Springer, 2006.

[Sof06b] V. Sofronie-Stokkermans. Local reasoning in verification. In S. Autexier and H. Mantel, editors, *Verification Workshop, VERIFY'06*, 2006.

[Sof07] V. Sofronie-Stokkermans. Hierarchical and modular reasoning in complex theories: The case of local theory extensions. In: B. Konev and F. Wolter, editors, *Proceedings of FroCos'07, LNCS 4720*, pages 47–71. Springer, 2007.

[Sof08] V. Sofronie-Stokkermans. Efficient Hierarchical Reasoning about Functions over Numerical Domains. In *Proceedings of KI'08, LNAI 5243*, pages 135–143. Springer, 2008.

[SI07a] V. Sofronie-Stokkermans and C. Ihlemann. Automated reasoning in some local extensions of ordered structures. In *Proceedings of ISMVL'07*, IEEE Press, paper 1, 2007.

[SI07b] V. Sofronie-Stokkermans and C. Ihlemann. Automated reasoning in some local extensions of ordered structures. *J. Multiple-Valued Logics and Soft Computing* 13(4–6), 397–414, 2007.

[WDF+09] C. Weidenbach, D. Dimova, A. Fietzke, R. Kumar, M. Suda, and P. Wischnewski. SPASS version 3.5. In R. A. Schmidt, editor, *Proceedings of CADE-22, LNCS(LNAI) 5663*, pages 140–145. Springer, 2009.