A Remark on the Sign Variation Method for Real Root Isolation*

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Abstract

For a polynomial $P(x) = \sum_{0 \le i \le n} a_i x^i \in \mathbb{R}[x]$, let $V(P)$ be the number of sign variations in its coefficient sequence, i.e., number of pairs i, j with $0 \le i < j \le n = \deg P$, $a_i a_j < 0$ and $a_{i+1} = \ldots = a_{j-1} = 0$. We show: If P has a single positive real root and all other roots of P lie in $\mathcal{C} := \left\{a + ib \; ; \; a \le 0 \text{ and } |b| \le |a|\sqrt{3}\right\} \subseteq \mathbb{C}$ then $V(P) = 1$. This strengthens previous partial converses of Descartes' rule of sign.

1 Statement of Results and Discussion

For a polynomial $P(x) = \sum_{0 \le i \le n} a_i x^i \in \mathbb{R}[x]$, let $V(P)$ be the number of sign variations in its coefficient sequence, i.e., number of pairs i, j with $0 \le i < j \le n = \deg P$, $a_i a_j < 0$ and $a_{i+1} = \ldots = a_{j-1} = 0$. Descartes' rule of sign states that $V(P) - pos(P)$ is an even non-negative integer, where $pos(P)$ is the number of positive real roots of P. In particular, if $V(P) = 0$, P has no positive real root, and if $V(P) = 1$, P has exactly one positive real root.

The purpose of this note is to prove the following partial converse.

Theorem 1 *Let $P \in \mathbb{R}[x]$ have the following properties:*

- *P has a single positive real root.*
- *All other roots of P lie in $\mathcal{C} := \left\{a + ib \; ; \; a \le 0 \text{ and } |b| \le |a|\sqrt{3}\right\}$. See Figure 1.*

Then $V(P) = 1$.

We give the proof in Section 2. In the remainder of this section, we relate the theorem above to results in the literature.

Descartes' rule of sign gives rise to a simple recursive algorithm for determining the number of real roots of a polynomial P in the open interval $(0, 1)$. We need the following additional notation.

Definition 1 *Let $P(x) = \sum_{0 \le i \le n} a_i x^i \in \mathbb{R}[x]$. Define polynomials RP, HP, and TP as follows:*

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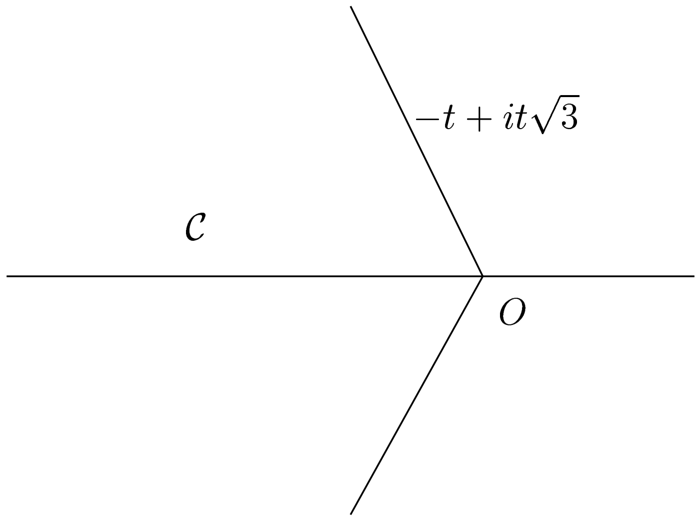

Figure 1: The cone $\mathcal{C}$ and its boundary rays $-t \pm it\sqrt{3}$, $t \geq 0$.

- $RP(x) = x^n P(1/x)$,
- $HP(x) = P(x/2)$,
- $TP(x) = P(x+1)$.

Observe that the roots of $HP(x)$ in $(0,1)$ (in $(1,2)$, respectively) correspond to the roots of $P(x)$ in $(0,1/2)$ (in $(1/2,1)$, respectively) and hence the roots of $THP(x)$ in $(0,1)$ correspond to the roots of $P(x)$ in $(1/2,1)$. Also observe that the roots of $TRP(x)$ in $(0,+\infty)$ correspond to the roots of $P(x)$ in $(0,1)$.

With these definitions the algorithm is easy to state. The recursive form is due to Akritas and Collins [CA76, Akr80]. The original form of Vincent [Vin36] simply applied Descartes' rule to sufficiently small intervals. Rouillier and Zimmermann [RZ01] discuss optimizations and implementations.

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int number_of_real_roots_in_zero_one(polynomial P) // P squarefree

1. Let $d = V(TRP)$ be the number of sign changes in the polynomial TRP.
2. If $d \leq 1$ return d.
3. Let $\delta = 1$ if $P(1/2) = 0$ and let $\delta = 0$ otherwise.
4. Let $Q_1(x) = HP(x)$ and let $Q_2(x) = THP(x)$. Return

$$\delta + \mathit{number\_of\_real\_roots\_in\_zero\_one}(Q_1) + \mathit{number\_of\_real\_roots\_in\_zero\_one}(Q_2)$$

It is easy to extend the procedure so that is also computes isolating intervals. The partial correctness of the procedure follows from Descartes' rule of sign. In order to prove termination, one needs partial converses. The following are known:

Theorem 2 (Vincent [Vin36]) *If $P(x)$ has no real or complex root in the open ball with center $1/2$ and radius $1/2$ then $V(TRP) = 0$, i.e., the transformed polynomial TRP has no sign variations in its coefficient sequence.*

Theorem 3 (Collins-Johnson [CJ89]) *If $P(x)$ has exactly one real root in the interval $(0,1)$ and no non-real root in the open disks of radius 1 centered at 0 or centered at 1, then $V(TRP) = 1$, i.e., the transformed polynomial TRP has a single sign variation in its coefficient sequence. The region is illustrated in Figure 2.*

Earlier, Collins and Loos [CL82] showed that if P has one real zero in $(0,1)$ and no other root in a disk of radius $\Omega(n^2)$ centered at the origin, then $V(TRP) = 1$. Our theorem strengthens these results. We have

Theorem 4 *If P has exactly one real root in $(0,1)$ and no non-real root inside the open disks centered at $1/2 \pm i/(2\sqrt{3})$ and passing through the origin (and hence radius $1/\sqrt{3}$), then $V(TRP) = 1$, i.e., the transformed polynomial TRP has a single sign variation in its coefficient sequence. The region is illustrated in Figure 2.*

Proof: Let

$$Q(x) = TRP(x) = (1+x)^n P(\frac{1}{1+x}) \ .$$

Then $Q(x)$ has a single positive real root and hence, by Theorem 1, $V(Q) = 1$ if all other roots of Q lie in $\mathcal{C}$. It remains to show that

$$x \in \mathcal{C} \quad \text{iff} \quad \frac{1}{1+x} \notin C_1 \cup C_2 \ .$$

We have $x \in \mathcal{C}$ iff $1 + x \in 1 + \mathcal{C}$. The boundaries of $1 + \mathcal{C}$ are given by the rays $1 - t \pm i\sqrt{3} \cdot t$, $t \geq 0$. The rays share the point 1.

What does the inversion map $x \mapsto 1/x$ do? It transforms lines not passing through the center of inversion into circles passing through the center of the inversion. Thus the line $\ell = 1 - t + i\sqrt{3} \cdot t$, $t \in \mathbb{R}$, is transformed into a circle passing through the origin and passing through the point 1 (since the latter point is mapped to itself by the inversion map). The center of the circle is therefore on the vertical line $a = 1/2$. By symmetry, the center of the circle must be on the line perpendicular to ℓ and passing through the origin. The line perpendicular to ℓ has parametric equation $t + it/\sqrt{3}$. The intersection point with the vertical line $a = 1/2$ has coordinates $1/2 + i/(2\sqrt{3}))$ $(t = 1/2)$. We conclude that ℓ is mapped onto the boundary of the disk C_1 and hence $x \in \mathcal{C}$ iff $1/(1+x) \notin C_1 \cup C_2$. ∎

2 Proof of Theorem 1

A polynomial $R(x) = \sum_{0 \leq i \leq n} a_i x^i$ is said to have the *ratio-decreasing property* if $a_i \geq 0$ for all i and

$$\frac{a_n}{a_{n+1}} \geq \cdots \geq \frac{a_k}{a_{k+1}} \geq \cdots \geq \frac{a_0}{a_1} \geq \frac{a_{-1}}{a_0} \ ,$$

where $a_{n+1} = a_{-1} = 0$, $0/0 = 0$ and $x/0 = \infty$ for $x > 0$. Collins and Loos observed the relevance of polynomials with the ratio-decreasing property.

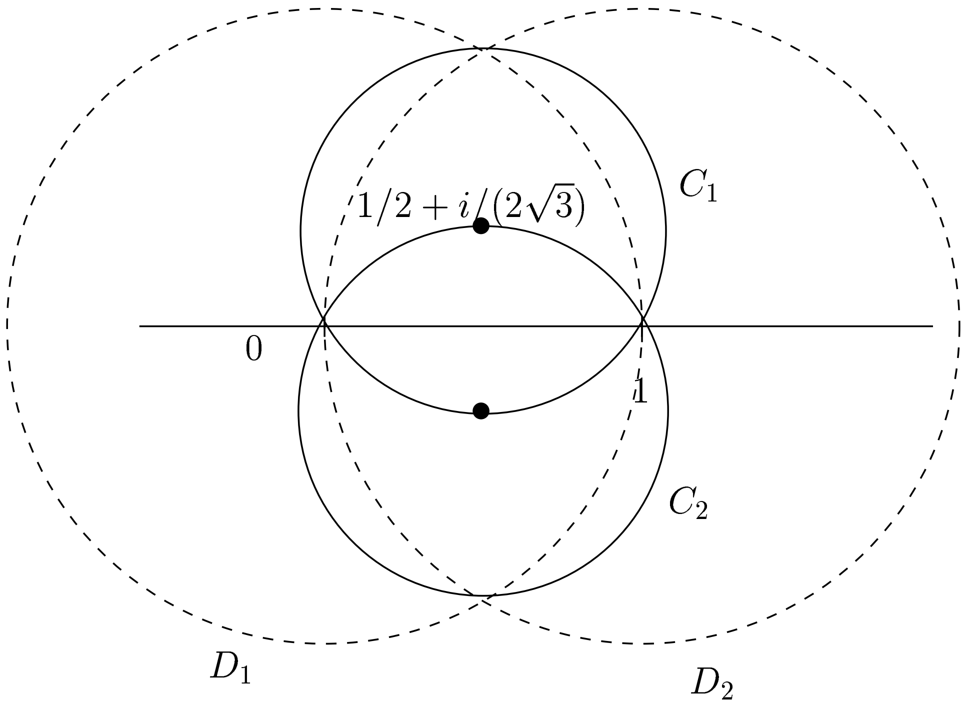

Figure 2: The theorem by Collins and Johnson states that if $P(x)$ has a single real root in $(0, 1)$ and no non-real root in $D_1 \cup D_2$, then $V(TRP) = 1$. Our theorem states that if $P(x)$ has a single real root in $(0, 1)$ and no non-real root in $C_1 \cup C_2$, then $V(TRP) = 1$. Observe that $C_1 \cup C_2 \subseteq D_1 \cup D_2$. Disks C_1 and C_2 have centers $1/2 \pm i/(2\sqrt{3})$ and pass through the origin.

Lemma 1 (Collins-Loos [CL82]) *If $R(x)$ has the ratio decreasing property, α is a positive real, and $P(x) = (x - \alpha)R(x)$ then $V(P(x)) = 1$.*

Proof: We have $P(x) = (x - \alpha)R(x) = \sum_{-1 \leq i \leq n}(a_i - \alpha a_{i+1})x^{i+1} = \sum_{0 \leq i \leq n+1} b_i x^i$. Since $R(x)$ has the ratio-decreasing property and $\alpha > 0$, there is a k such that

$$\frac{a_n}{a_{n+1}} \geq \ldots \geq \frac{a_k}{a_{k+1}} \geq \alpha \geq \frac{a_{k-1}}{a_k} \ldots \geq \frac{a_0}{a_1} \geq \frac{a_{-1}}{a_0}$$

and $n > k \geq 0$. Thus $b_{n+1} > 0$, $b_0 < 0$, $b_i \geq 0$ for $i \geq k$ and $b_i \leq 0$ for $i < k$ and hence $V(P(x)) = 1$. ∎

The Lemma above is, in a certain sense, best possible.

Lemma 2 *If $R(x)$ does not have the ratio-decreasing property, there is an $\alpha > 0$ such that $V(P(x)) \geq 2$, where $P(x) = (x - \alpha)R(x)$.*

Proof: As above, let $P(x) = \sum_{0 \leq i \leq n+1} b_i x^i$, where $b_i = a_{i-1} - \alpha a_i$. We distinguish cases. If $V(R) \geq 2$ then $V(P) \geq 2$ for small enough positive α; simply choose α such that for all i, a_{i-1} and $a_{i-1} - \alpha a_i$ have the same sign whenever $a_{i-1} \neq 0$. If $V(R) = 1$, R has a positive real root and hence P has at least two positive real roots. Thus $V(P) \geq 2$. Finally, assume

$V(R) = 0$. We may assume $a_n > 0$ and $a_0 > 0$. Then $b_{n+1} = a_n > 0$. Since R does not have the ratio decreasing property, there is a k such that $a_k/a_{k+1} < a_{k-1}/a_k$. Choose α such that $a_k/a_{k+1} < \alpha < a_{k-1}/a_k$. Then $b_{k+1} = a_k - \alpha a_{k+1} < 0$, $b_k = a_{k-1} - \alpha a_k > 0$ and $b_0 < 0$. Thus $V(P) \geq 3$. ∎

We will next use induction on the degree to show that if a real polynomial has all its roots in $\mathcal{C}$, it has the ratio-decreasing property. This strengthens a result of Collins and Loos [CL82] who showed that if a polynomial has all its roots in a disk of radius $O(1/n^2)$ centered at -1, it has the ratio-decreasing property. The base cases are trivial.

Lemma 3 *If $\alpha \in \mathbb{R}$ and $\alpha \leq 0$, the coefficient vector of $x - \alpha$ is ratio-decreasing.*

Lemma 4 *Let $a, b \in \mathbb{R}$ with $a \leq 0$ and $|b| \leq \sqrt{3} \cdot |a|$. The polynomial R with roots $a + ib$ and $a - ib$ has the ratio decreasing property.*

Proof: The coefficient vector of $R(x)$ is $(1, -2a, a^2 + b^2)$ and the sequence of fractions to be considered is

$$\frac{1}{0}, \quad \frac{-2a}{1}, \quad \frac{a^2+b^2}{-2a}, \quad \frac{0}{a^2+b^2} \ .$$

If $a = 0$ and hence $b = 0$, $R(x)$ is ratio-decreasing. So assume $a < 0$. We need $4a^2 \geq a^2 + b^2$ or $b^2 \leq 3a^2$. This is true by assumption. ∎

The three Lemmas above give a complete characterization of all degree two polynomials $R(x) = ax^2 + bx + c \in \mathbb{R}[x]$ such that $V((x - \alpha)R(x)) = 1$ for all positive real α. R has this property if and only if its roots lie in $\mathcal{C}$.

Lemma 5 *If all zeros of a polynomial lie in $\mathcal{C}$, it has the ratio-decreasing property.*

Proof: We use induction on the degree. The cases $n = 1$ and $n = 2$ have already been dealt with. For the induction step, consider a polynomial R with the ratio decreasing property. We show that $(x - \alpha)R(x)$ and $(cx^2 + dx + e)R(x)$ have the ratio-decreasing property where α is a non-positive real and (c, d, e) is ratio-decreasing, i.e., $c > 0$, $d \geq 0$, $e \geq 0$, and $d^2 \geq ce$. If $e = 0$, $cx^2 + dx = x(cx + d)$ and we are really adding two non-positive real roots. We may therefore assume $e > 0$. Then also $d > 0$.

We first consider $(x + \beta)R(x)$ where $\beta > 0$. The generic coefficient of our new polynomial is $b_k = a_k + \beta a_{k+1}$. We need to show $b_k^2 \geq b_{k+1}b_{k-1}$. This is equivalent to

$$\begin{aligned} a_k^2 + 2\beta a_k a_{k+1} + \beta^2 a_{k+1}^2 &\geq (a_{k+1} + \beta a_{k+2})(a_{k-1} + \beta a_k) \\ &= a_{k+1}a_{k-1} + \beta(a_{k+1}a_k + a_{k+2}a_{k-1}) + \beta^2 a_{k+2}a_k \ . \end{aligned}$$

The inequality holds since $a_k^2 \geq a_{k+1}a_{k-1}$, $a_k a_{k+1} \geq a_{k+2}a_{k-1}$ (since $a_{k+1}/a_{k+2} \geq a_{k-1}/a_k$), and $a_{k+1}^2 \geq a_{k+2}a_k$.

We next consider $(cx^2 + dx + e)R(x)$ where c, d, and e are positive and $d^2 \geq ce$. The generic coefficient of our new polynomial is $b_k = ca_k + da_{k+1} + ea_{k+2}$. We need to show $b_k^2 \geq b_{k+1}b_{k-1}$, i.e.,

$$(ca_k + da_{k+1} + ea_{k+2})^2 \geq (ca_{k+1} + da_{k+2} + ea_{k+3})(ca_{k-1} + da_k + ea_{k+1}) \ .$$

We compare the terms involving c^2, e^2, cd, de, and either d^2 or ce.

The terms involving c^2 are a_k^2 and $a_{k+1}a_{k-1}$, respectively, and we have our desired inequality.

The terms involving e^2 are a_{k+2}^2 and $a_{k+3}a_{k+1}$, respectively, and we have our desired inequality.

The terms involving cd are $2a_k a_{k+1}$ and $a_k a_{k+1} + a_{k+2}a_{k-1}$, respectively, and we have our desired inequality (since $a_{k+1}/a_{k+2} \geq a_{k-1}/a_k$).

The terms involving de are $2a_{k+1}a_{k+2}$ and $a_{k+1}a_{k+2} + a_{k+3}a_k$, respectively, and we have our desired inequality (since $a_{k+2}/a_{k+3} \geq a_k/a_{k+1}$).

The terms involving d^2 or ce are $d^2 a_{k+1}^2 + ce2a_k a_{k+2}$ and $d^2 a_k a_{k+2} + ce(a_{k+1}^2 + a_{k+3}a_{k-1})$, respectively. We need to show that the former is at least as large as the latter. Since $d^2 \geq ce$ and $a_{k+1}^2 \geq a_k a_{k+2}$ it suffices to show

$$ce(a_{k+1}^2 + 2a_k a_{k+2}) \geq ce(a_k a_{k+2} + a_{k+1}^2 + a_{k+3}a_{k-1}) \quad \text{or} \quad a_k a_{k+2} \geq a_{k+3}a_{k-1} \ .$$

The inequality holds since $a_{k+2}/a_{k+3} \geq a_{k-1}/a_k$. ∎

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