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# The thickness of a minor-excluded class of graphs[1]

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**Abstract**

The thickness problem on graphs is $\mathscr{NP}$-hard and only few results concerning this graph invariant are known. Using a decomposition theorem of Truemper, we show that the thickness of the class of graphs without $G_{12}$ minors is less than or equal to two (and therefore, the same is true for the more well-known class of the graphs without $K_5$ minors). Consequently, the thickness of this class of graphs can be determined with a planarity testing algorithm in linear time.

*Keywords:* Thickness; Crossing number; Skewness; Graph minor; 1-sum; 2-sum; $\Delta$-sum

## 1. Introduction

The *thickness* $\theta(G)$ of a graph $G=(V,E)$ is the minimum number $k$ such that $G$ is the union of $k$ planar subgraphs (here, by 'union of $k$ planar subgraphs' we mean that the edge-set $E$ can be partitioned into $k$ sets so that the graph induced by each set is planar). Therefore, the thickness is one measure of the degree of nonplanarity of a graph.

Clearly, $\theta(G)=1$ if and only if $G$ is planar. The thickness problem, asking for the thickness of a given graph $G$, is $\mathscr{NP}$-hard [5], so there is little hope to find a polynomial-time algorithm for the thickness problem on general graphs. However, for some graph classes, the thickness can be determined in polynomial time. For example, the thickness is known for complete and complete bipartite graphs [1]. In some cases, there are (often relatively poor) bounds on the thickness of a graph [2,3].

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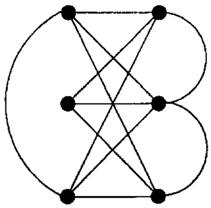

Fig. 1. Graph $G_{12}$.

The thickness problem has applications in VLSI design. In electronic circuits, components are joined by means of conducting strips. These may not cross, since this would lead to undesirable signals. In this case, an insulated wire must be used. For that reason, circuits with a large number of crossings are decomposed into several layers without crossings, which are then pasted together. The goal is to use as few layers as possible. In this application it would be desirable to know the thickness of a hypergraph whose nodes are cells to be placed and whose hyperedges correspond to the nets connecting the cells. If the thickness problem could be solved for graphs, it would be a useful engineering tool in the layout of electronic circuits.

We have restricted our attention to a minor-excluded class of graphs, the class of graphs without $G_{12}$ minors ($G_{12}$ is displayed in Fig. 1). Our method to determine the thickness of this class of graphs is based on a decomposition theorem of Truemper [6]. The paper is organized as follows. The concept of graph decomposition is introduced in Section 2. In Section 3 we prove the main result of this paper. Finally, in Section 4 we give negative results on using our approach for the two graph invariants crossing number and skewness.

## 2. Decomposition of graphs

In this section, we present the 1-, 2- and $\Delta$-sums of graphs. Furthermore, we describe a recursive construction process for graphs without $G_{12}$ minors, based on Truemper's decomposition theorem.

For that purpose, let $G=(V,E)$ be a connected graph. $G$ is called a *1-sum* of the graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$, denoted $G=G_1\bigoplus_1 G_2$, if the identification of an arbitrary node $v_1$ of $G_1$ with an arbitrary node $v_2$ of $G_2$ produces $G$. Analogously, $G$ is called a *2-sum* (respectively, *$\Delta$-sum*) of $G_1$ and $G_2$, denoted $G=G_1\bigoplus_2 G_2$ ($G=G_1\bigoplus_\Delta G_2$), if identification of an edge (resp., triangle) of $G_1$ with an edge (resp., triangle) of $G_2$ and subsequent deletion of this edge (resp. triangle) produces $G$ (see Fig. 2). Conversely, if $G=G_1\bigoplus_1 G_2$, $G=G_1\bigoplus_2 G_2$ or $G=G_1\bigoplus_\Delta G_2$, we say that $G_1$ and $G_2$ are a 1-, 2- or $\Delta$-sum decomposition of $G$. Let $\bigoplus\in\{\bigoplus_1,\bigoplus_2,\bigoplus_\Delta\}$. If, for $k\geqslant 2$, $G=(((G_1\bigoplus G_2)\bigoplus G_3)\bigoplus\cdots)\bigoplus G_k$, we call the graphs $G_i (1\leqslant i\leqslant k)$ *building blocks* of $G$.

A decomposition theorem by Truemper [6] allows us to restrict our attention to certain building blocks for all 2-connected graphs without $G_{12}$ minors.

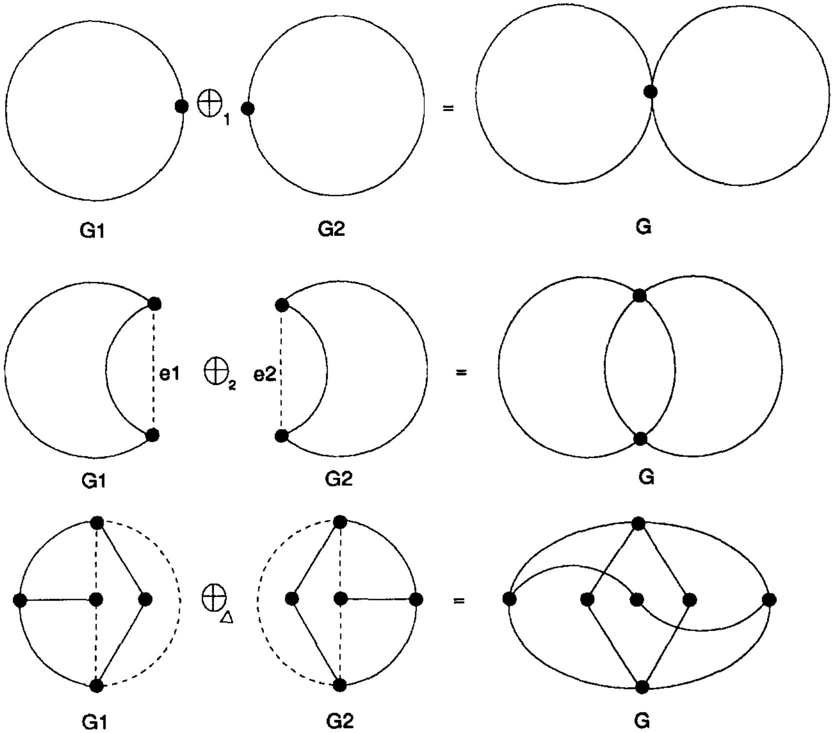


Fig. 2. 1-, 2- and Δ-sum.

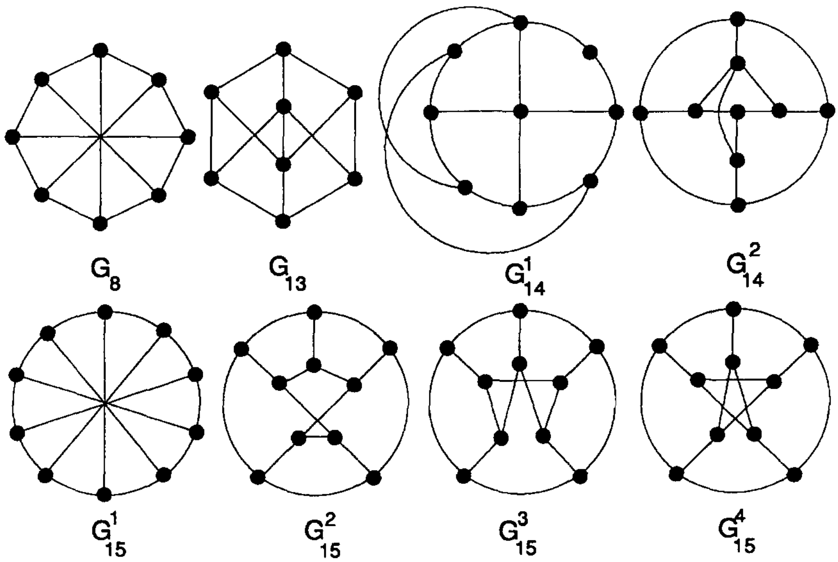


Fig. 3. Graphs of Theorem 2.1.

**Theorem 2.1** (Truemper [6]). *Any 2-connected graph without $G_{12}$ minors is planar, or isomorphic to $K_5$, $K_{3,3}$, $G_8$, $G_{13}$, $G_{14}^1$, $G_{14}^2$, $G_{15}^1$, $G_{15}^2$, $G_{15}^3$, $G_{15}^4$, or may be constructed recursively by 2-sums and Δ-sums. The building blocks of such a construction are as follows*:

*2-sums*: *planar graphs, and graphs isomorphic to $K_5$, $K_{3,3}$, $G_8$, $G_{13}$, $G_{14}^1$, $G_{14}^2$, $G_{15}^1$, $G_{15}^2$, $G_{15}^3$, or $G_{15}^4$.*

*Δ-sums*: *planar graphs, and graphs isomorphic to $K_5$.*

The building blocks of Theorem 2.1 can be seen in Fig. 3. All graphs are not planar, but obviously their thickness equals 2.

## 3. Thickness theorem

Before we state the main result of this paper, we prove several lemmas. For notational convenience, we denote the planar graphs demonstrating thickness 2 for a given graph $G$ as *planar decomposition graphs* of $G$.

**Lemma 3.1.** *Any* 1- *or* 2-*sum of two planar graphs is planar.*

**Proof.** The sum operations cannot introduce $K_{3,3}$ or $K_5$-minors, hence must preserve planarity. □

**Lemma 3.2.** *Any* 1- *or* 2-*sum* $G_3 = G_1 \bigoplus_1 G_2$ *or* $G_3 = G_1 \bigoplus_2 G_2$, *where graph* $G_1$ *has thickness* 2, *and graph* $G_2$ *is planar has thickness* 2.

**Proof.** Let $G_1'$ and $G_1''$ be planar decomposition graphs of $G_1$. The 1-sum of $G_1'$ and $G_2$ is planar by Lemma 3.1. Clearly, the obtained 1-sum and $G_1''$ are planar decomposition graphs of the 1-sum of $G_1$ and $G_2$.

We can assume without loss of generality that the edge $e$ to be identified in the 2-sum is embedded in $G_1'$. Then the 2-sum of $G_1'$ and $G_2$ is planar by Lemma 3.1, and hence the obtained 2-sum and $G_1''$ are planar decomposition graphs of the 2-sum of $G_1$ and $G_2$. □

**Lemma 3.3.** *Let* $G_1$ *and* $G_2$ *be two graphs with thickness* 2, *say with planar decomposition graphs* $G_1'$, $G_1''$ *and* $G_2'$, $G_2''$, *respectively. Suppose* $G_2'$ *contains the edge* $e$ *to be identified in a* 2-*sum together with all edges incident with* $e$. *Then the* 2-*sum* $G_3 = G_1 \bigoplus_2 G_2$ *has thickness* 2.

**Proof.** Again, we can assume, without loss of generality, that edge $e$ is embedded in $G_1'$. Then the 2-sum of $G_1'$ and $G_2'$, and the union of $G_1''$ and $G_2''$ are planar decomposition graphs of $G_3$. Note that there are no edges between $G_1''$ and $G_2''$. □

**Lemma 3.4.** *Any Δ-sum* $G_3 = G_1 \bigoplus_{\Delta} G_2$ *of a graph* $G_1$ *with thickness at most* 2 *and of a planar graph* $G_2$ *has thickness at most* 2.

**Proof.** Let $e=(u,v)$ be one of the edges of the triangle and let $w$ be the vertex of the triangle that is not an endpoint of $e$. Since $G_2$ is planar, we can decompose $G_2$ into a graph $G_2'$, containing $e$ together with all edges incident to $u$ or $v$, and a graph $G_2''$ consisting of all edges incident to $w$ that do not go to any endpoint of $e$. The remaining edges can be distributed arbitrarily to $G_2'$ or $G_2''$.

If $G_1$ has thickness 2, we have two planar decomposition graphs for $G_1$, say $G_1'$ and $G_1''$. Without loss of generality $G_1'$ may contain $e$. Define $G_3'$ to be the 2-sum of $G_1'$ and $G_2'$, and $G_3''$ to be the 1-sum of $G_1''$ and $G_2''$. Due to Lemma 3.1, $G_3'$ and $G_3''$ are planar decomposition graphs for $G_3$. Note that after the sum operations, the remaining edges of the triangle, which connect $u$ with $w$ as well as $v$ with $w$, are deleted.

If $G_1$ is planar, let $G_1'$ have all edges of $G_1$, and $G_1''$ consist just of the nodes of $G_1$. Then define the planar decomposition graphs as above. □

We are now prepared to prove the main result of this paper.

**Theorem 3.5.** *If $G$ is a graph without $G_{12}$ minors, then $\theta(G)\leqslant 2$.*

**Proof.** According to Theorem 2.1, every 2-connected graph without $G_{12}$-minors can be obtained by a sequence of 2- (respectively, $\Delta$-)sums with special building blocks. The above lemmas show that the thickness stays at 2 under sum operations with these building blocks. All these graphs can be decomposed in such a way that one of their two planar decomposition graphs contains the edge to be identified together with all the edges incident with that edge.

In the case of a $\Delta$-sum with a planar graph, Lemma 3.4 applies directly. In the case of a $\Delta$-sum with $K_5$, we can decompose $K_5$ into a graph $G_2'$ containing one edge $e$ of the triangle together with all edges incident to both endpoints of $e$ and a graph $G_2''$ consisting of the node $w$ involved in the $\Delta$-sum, which is not an endpoint of $e$, together with the edges incident at $w$ that do not go to any endpoint of $e$. Clearly, $G_2'$ and $G_2''$ are planar and hence we can define the same sum graphs as in Lemma 3.4.

Therefore, the theorem is proved for 2-connected graphs. If $G$ is not 2-connected, the decomposition theorem applies for every 2-connected block of the graph and hence for the whole graph. □

As a corollary, we obtain that the thickness problem in the class of graphs without $G_{12}$-minors is solvable in linear time.

**Corollary 3.6.** *The thickness of a graph $G$ without $G_{12}$ minors can be determined in linear time in the number of nodes of $G$.*

**Proof.** Apply a linear time planarity testing algorithm [4] to $G$. If $G$ is planar, then $\theta(G)=1$, otherwise $\theta(G)=2$. □

Since $G_{12}$ contains a $K_5$ minor, the class of graphs without $G_{12}$ minors contains the class of graphs without $K_5$ minors, and hence, we have proved the result for the more well-known class of graphs without $K_5$ minors as well. Wagner [7] produced for these graphs a decomposition theorem that has become a prototype for a number of decomposition results, including Theorem 2.1 used here.

## 4. Other invariants

One may think that applying certain sum operations might also be applicable to control other topological invariants of graphs, such as the *crossing number* $\nu(G)$ or the *skewness* $\mu(G)$ of a graph $G$. The crossing-number $\nu(G)$ of a given graph $G$ is the minimum number of pairwise intersections of edges when $G$ is drawn in the plane. The skewness is the minimum number of edges which have to be deleted from graph $G$ to make it planar.

Unfortunately, such a transfer is not possible, since by a 2-sum there is neither additivity of the crossing number resp. skewness of the building blocks nor a fixed value as for the thickness. We prove this by giving counterexamples.

**Theorem 4.1.** *For each $n \in \mathbb{N}$ there exist graphs $G_1$ and $G_2$ such that, for any graph $G = G_1 \bigoplus_2 G_2$, the following holds:*

$$\nu(G) > \nu(G_1) + \nu(G_2) + n.$$

**Proof.** For $n \in \mathbb{N}$, denote by $M_{n+4}$ the planar graph shown in Fig. 4 with $n+4$ vertices and $2n+5$ edges. Start with the graph $K_{3,3}$ and take successively 2-sums with seven edges of the $K_{3,3}$ and $M_{n+4}$ as shown in Fig. 5. The resulting graph $H$ has crossing-number one. Take a further 2-sum of $H$ and $M_{n+4}$ by identifying the edges $e$ and $f_1$.

In every drawing of the graph, the edge $f_2$ crosses a complete subgraph $M_{n+4} - e$ and therefore at least $n+2$ edges. Therefore, we have $\nu(H \bigoplus_2 M_{n+4}) = n+2 > \nu(H) + \nu(M_{n+4}) + n$. □

An example of the nonadditivity of the skewness can be obtained by a slight modification of the proof of Theorem 4.1.

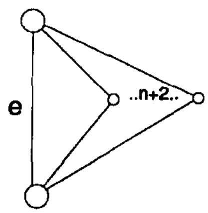


Fig. 4. Graph $M_{n+4}$.

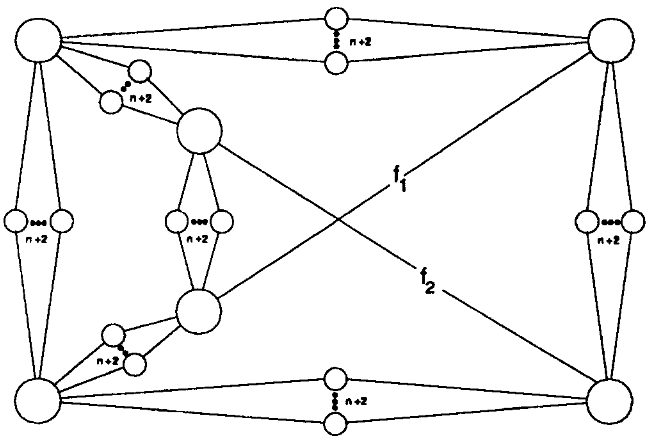


Fig. 5. Graph $H$.

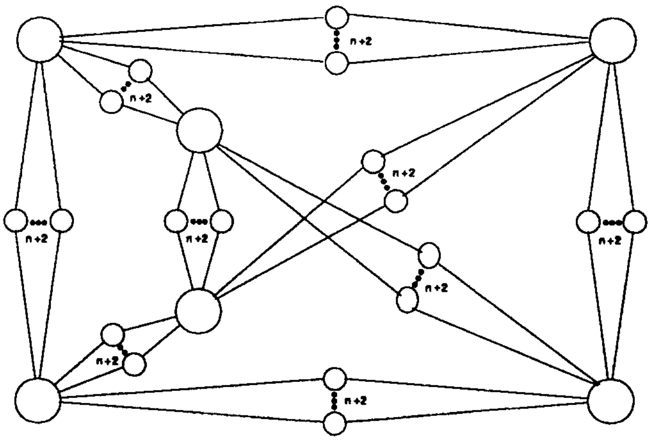


Fig. 6. Graph $F$.

**Theorem 4.2.** *For each* $n \in \mathbb{N}$ *there exist graphs* $G_1$ *and* $G_2$ *such that the following holds for the graph* $G = G_1 \bigoplus_2 G_2$:

$$\mu(G) > \mu(G_1) + \mu(G_2) + n.$$

**Proof.** Take 2-sums of eight edges of $K_{3,3}$ with $M_{n+4}$. The skewness of the resulting graph equals one. A further 2-sum of the remaining edge of $K_{3,3}$ with $M_{n+4}$ gives the graph $F$ of Fig. 6. In order to achieve planarity, a graph $M_{n+4} - e$ must be removed, i.e., the skewness is $n+2$. □

Since we only used building blocks according to Theorem 2.1, the above theorems are valid even if we restrict ourselves to graphs without $G_{12}$ minors.

## Acknowledgements

In an earlier version of this paper we had obtained our main result for the class of graphs without $K_5$ minors. We are grateful to Klaus Truemper who pointed out the generalization to the class of graphs without $G_{12}$ minors and for valuable suggestions for simplifying the presentation.

## References

[1] L.W. Beineke, R.J. Wilson (Eds.), Selected Topics in Graph Theory, Academic Press, London, 1978.
[2] A.M. Dean, J.P. Hutchinson, E.R. Scheinerman, On the thickness and arboricity of a graph, J. Combin. Theory B 52 (1991) 147–151.
[3] J. Halton, On the thickness of graphs of given degree, Inform. Sci. 54 (1991) 219–238.
[4] J. Hopcroft, R.E. Tarjan, Efficient planarity testing, J. ACM 21 (1974) 549–568.
[5] A. Mansfield, Determining the thickness of graphs is NP-hard, Math. Proc. Camb. Philos. Soc. 9 (1983) 9–23.
[6] K. Truemper, Matroid Decomposition, Academic Press, Boston, 1992.
[7] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, Math. Ann. 114 (1937) 570–590.