On Local Routing of Two-Terminal Nets

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We examine the problem of finding edge-disjoint paths between pairs of vertices placed on inner as well as outer faces of a finite grid graph. For each path a global routing, which fixes the topology of the path relative to the nontrivial faces, is part of the input. We give necessary and sufficient conditions for the solvability of the problem and provide an algorithm to find a solution with running time quadratic in the size of the problem. © 1992 Academic Press, Inc.

I. INTRODUCTION

The planar rectangular grid consists of vertices $\{(x, y); x, y \in \mathbb{Z}\}$ and edges $\{((x, y), (x', y')); |x - x'| + |y - y'| = 1\}$. A *routing region* R is a finite subgraph of the planar rectangular grid.

In the sequel R always denotes a routing region. We call a bounded face F of R *trivial* if it has exactly four vertices on its boundary and nontrivial otherwise. We use M to denote the nontrivial bounded faces and $\bar{M}$ to denote M together with the unbounded face. Let B be the set of vertices of R with degree at most three. Note that a vertex $v \in B$ lies on the boundary of a face $F \in \bar{M}$.

A *local routing* is a path in the routing region R connecting two vertices of B. The endpoints of the path are called its *terminals*. Two local routings p and q are *elementarily equivalent* if there are paths p_1, p_2, q_2, p_3 such that $p = p_1 p_2 p_3$, $q = p_1 q_2 p_3$, and such that $p_2 q_2^{-1}$ (q_2^{-1} is the reverse of path q_2) is a boundary cycle of a trivial face. Two elementarily equivalent paths are hence the same except that they take two different routes around a single trivial face. Two local routings p and q are *equivalent* if there is a sequence $p_0, ..., p_k$, $k \geq 0$, of paths such that $p = p_0$, $q = p_k$, and p_i and p_{i+1} are elementarily equivalent for $0 \leq i < k$. Note that if p and q are equivalent then p and q have the same terminals.

We use $[p]$ to denote the equivalence class of local routing p, i.e., to denote the homotopy class of path p. A *global routing* or *net* is an equiva-

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lence class $[p]$; the terminals of the path p are also called the terminals of the net. We are now ready to state the *local routing problem* (LPR).

Input. A routing region R and a multi-set $\mathcal{N}$ of nets.

Output. A local routing $\mathrm{lr}(N)$ for each net $N \in \mathcal{N}$ such that

(1) $\mathrm{lr}(N) \in N$ for all $N \in \mathcal{N}$

(2) $\mathrm{lr}(N_1)$ and $\mathrm{lr}(N_2)$ are edge-disjoint for $N_1, N_2 \in \mathcal{N}$, $N_1 \neq N_2$

or an indication that there is no such set of local routings.

Figure 1 gives an example. We assume that each net $N \in \mathcal{N}$ is given by one of its representatives. We use r to denote the number of vertices of R and m to denote the total length of the representatives and call $n = r + m$ the size of the problem. A local routing problem is called *solvable*, if an appropriate set of local routings exists, and is called *unsolvable* otherwise. In this paper we will prove the following theorem.
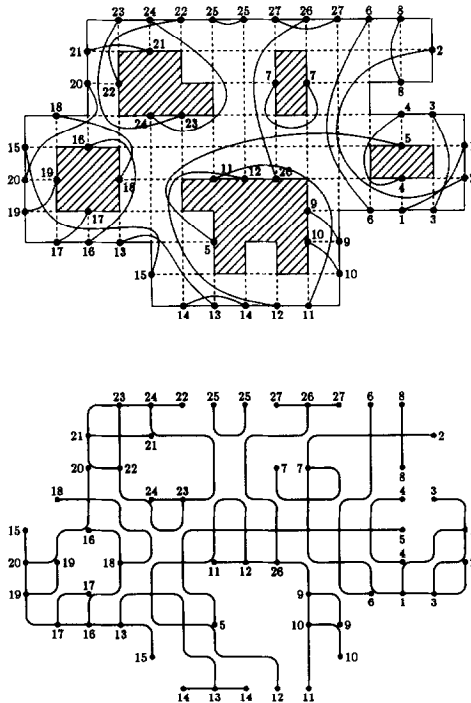

FIG. 1. A local routing problem and its solution. The global routings are shown as curves for added clarity.
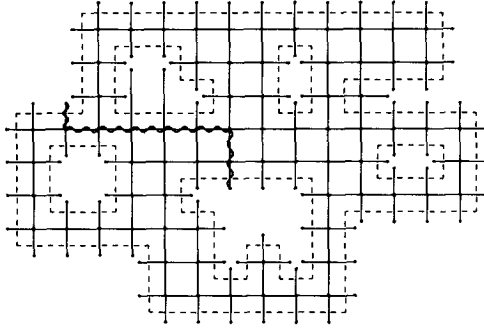
FIG. 2. The multiple-source dual $D(R)$ of the routing region of Fig. 1. A cut of capacity 8 is shown wiggled.

THEOREM 1. *Let $P = (R, \mathcal{N})$ be an even bounded LRP of size n.*

(a) *P is solvable if and only if the free capacity of every cut is nonnegative.*

(b) *In time $O(r^2 + m) = O(n^2)$ one can decide whether P is solvable and also construct a solution if it is.*

It remains to define the terms "even LRP," "cut," "free capacity of a cut," and "bounded." The *multiple-source dual $D(R)$* of routing region R is defined as follows (cf. Fig. 2). For every edge e of R there is a dual edge $d(e)$ with its endpoints lying in those faces of R which are separated by e. The endpoints of dual edges which lie in faces outside $\bar{M}$ are identified, the endpoints in faces in $\bar{M}$ are kept distinct and are called *sources* of the dual graph. A *cut* of R is a simple path in the dual graph connecting two sources. The *capacity* cap(C) of a cut C is its length (= number of edges of R intersected by the cut) (cf. Fig. 2). A cut can be viewed as a polygonal line $s_1, ..., s_k$, where each s_i is a straightline segment and s_i and s_{i+1} have different directions (one horizontal, one vertical). A cut is a 1-*bend cut* if $k \leqslant 2$ and a 0-*bend cut* if $k = 1$.

Let C be a cut and let p be a path. Then cross(p, C) is the number of edges e of p with $d(e)$ in C, i.e., the number of times p goes across C. For a global routing gr we define

$$\text{cross}(gr, C) = \min\{\text{cross}(p, C); p \in gr\}.$$

The *density* dens(C) of cut C is defined by

$$\text{dens}(C) = \sum_{N \in \mathcal{N}} \text{cross}(N, C)$$

and the *free capacity* fcap(C) is given by

$$\text{fcap}(C) = \text{cap}(C) - \text{dens}(C).$$

A cut C is *saturated* if fcap(C) = 0 and *oversaturated* if fcap(C) < 0.

An LRP is *even* if fcap(C) is even for every cut C and it is 1-*even* if fcap(C) is even for every 1-bend cut C.

For a vertex $v \in V$, deg(v) denotes the degree of v in the graph R and ter(v) denotes the number of nets having v as an endpoint. An LRP is *bounded*, if deg(v) + ter(v) ≤ 4 for all vertices v.

At this point all terms are defined and we can now put our work into perspective. We view the routing problem as the problem of finding edge-disjoint paths in a grid graph. This is usually called routing in *knock-knee mode*, since two solution paths may both bend in the same vertex. The mode where this is excluded is called Manhattan mode. Previous work on routing problems in knock-knee mode can be found in Preparata and Lipski [PL], Frank [F], Mehlhorn and Preparata [MP], Nishizeki, Saito and Suzuki [NSS], Kaufmann and Mehlhorn [KM], Becker and Mehlhorn [BM], Kramer and van Leeuwen [KvL], and Brady and Brown [BB]. The problems considered in the first five papers are special cases of the LRP considered here. They solve the routing problem for channels [PL], switchboxes [F, MP], convex generalized switchboxes [NSS], and generalized switchboxes [MK], respectively. In a generalized switchbox problem we have $M = \varnothing$, in a convex generalized switchbox problem we have in addition that every two boundary vertices are connected by a path with at most one bend, and in a switchbox problem R is a rectangle. The time bounds obtained in these papers are much better than the bound in the present paper, e.g., $O(n(\log n)^2)$ for the generalized switchbox problem. The paper by Becker and Mehlhorn is incomparable with the present paper; it is more restrictive in some ways, since all vertices in B have to lie on the boundary of the unbounded face, and it is more general in other ways, since the routing region R may be an arbitrary even (= all nodes not on the boundary of the unbounded face have even degree) planar graph, nets are simply pairs of points in B, and no homotopies are required in the input. Kramer and van Leeuwen prove that the global routing problem is NP-complete, i.e., if we drop the condition that lr(N) ∈ N for $N \in \mathcal{N}$ (in other words, a net is just a pair of vertices in B), then the problem becomes NP-complete. Finally, Brady and Brown treat the problem of layer assignment and show that any layout in knock-knee mode can be wired using four conducting layers. For channels three layers suffice [PL].

The present paper also has sources in graph theory, most notably the paper by Okamura and Seymour [OS]. They showed that the cut condition, i.e., fcap(X) ≥ 0 for all cuts X, is necessary and sufficient for the

solvability of even multi-commodity flow problems in planar graphs provided that all terminals lie on the same face. We drop this restriction and thus generalize their result; however, our solution *only* works for grid graphs. The generalization to general planer graphs remains a major challenge. A partial result was recently obtained by van Hoesel and Schrijver [HS]; they treat the case $|M| = 1$. An implementations of the Okamura and Seymour result can be found in Matsumoto, Nishizeki, and Saito [MNS] and in Becker and Mehlhorn.

We also want to mention the papers by Cole and Siegel [CS] and Leiserson and Maley [LM]. They prove the same result as we do but for river routing instead of routing in knock-knee mode, i.e., they require that solution paths are *vertex-disjoint*. Of course, nets cannot cross in their case.

Automatic VLSI design systems, e.g., CALCOS (Lauther [L]) and PI (Rivest [R]) for integrated circuits divide the routing problem into several stages:

(1) Determine a global routing for every net.

(2) Cut the routing region into regions of simple shape, e.g., channels.

(3) Determine for every net the exact positions where it crosses channel boundaries.

(4) Route each channel.

In some systems, e.g., CALCOS, stages (3) and (4) are combined into a single stage. Channels are routed one by one and the routings in the first i channels fix the positions of the nets which leave these channels. In all stages heuristic algorithms are usually used. The result of Kramer and van Leeuwen states that the general routing problem is NP-complete; our theorem states that the combination of stages (2) to (4) can be solved in polynomial time, at least for two-terminal nets and in knock-knee mode.

This paper is organized as follows. In Section II we give the algorithm which is then proved correct in Section III. Section IV, describes an implementation and its analysis. Section V is a short conclusion; it lists some open problems and recent extensions of the present work.

We close this section with a remark about notation. A path p is a sequence $e_1 e_2 \cdots e_k$ of edges where $e_i = (v_i, v_{i+1})$ for $1 \leqslant i \leqslant k$. We call v_1 the start vertex of p, v_{k+1} the end vertex of p and v_1 and v_{k+1} its terminals; i.e., we view a path as being oriented from v_1 to v_{k+1}. The reverse path $p^{-1} = e_k \cdots e_2 e_1$ is then oriented from v_{k+1} to v_1. A net $N = [p]$ is an equivalence class of paths and therefore also oriented. The start vertex of N is the start vertex of p. The reverse net N^{-1} is the equivalence class $[p^{-1}]$. Of course, a routing problem changes only in an inessential way, if

the orientation of some nets is changed. We will tacitly use this fact in the following sections.

II. The Algorithm

In this section we describe an algorithm for solving local routing problems. Let $P_0 = (V_0, E_0, \mathcal{N}_0)$ be an even local routing problem which satisfies the cut condition, i.e., $\mathrm{fcap}(C) \in 2 \cdot \mathbb{N}_0$ for every cut C. Here $2 \cdot \mathbb{N}_0$ denotes the set of even nonnegative integers. Our algorithm constructs a solution for P_0 iteratively by transforming P_0 into simpler and simpler routing problems. We will maintain the invariant that the *current routing problem* $P = (V, E, \mathcal{N})$ *is good with respect to* P_0.

DEFINITION 1.   A routing problem $P = (V, E, \mathcal{N})$ is *good* with respect to $P_0 = (V_0, E_0, \mathcal{N}_0)$ if

(a) P is a routing problem with $V \subseteq V_0$ and $E \subseteq E_0$.

(b) $\mathrm{fcap}(Y) \in 2 \cdot \mathbb{N}_0$ for all 1-bend cuts Y; i.e., the cut condition for 1-bend cuts is satisfied and the problem is 1-even.

(c) P is bounded.

(d)   If P is solvable then P_0 is solvable.

Remark.   For the problems P constructed by our algorithm the connection between a solution of P and a solution of P_0 is very simple. For every net $N \in \mathcal{N}_0$ there will be a set $\mathcal{N}(N) \subseteq \mathcal{N}$ of nets of the current problem P and a set $E(N) \subseteq E_0 - E$ of edges such that local routings for the nets in $\mathcal{N}(N)$ together with the edges in $E(N)$ form a local routing for N. In other words, the set $E(N)$ of edges has already been reserved for the net N and the fragments $\mathcal{N}(N)$ of net N still have to be routed. When P is trivial, i.e., $\mathcal{N} = \varnothing$, then a solution for P_0 was found.

We will mostly use the phrase P is good instead of P is good with respect to the basic problem P_0.

DEFINITION 2.   An LRP P is *reduced* if $\mathrm{ter}(v) < \deg(v)$ for all $v \in V$, and there is no cut with capacity one.

In each iteration of the algorithm (cf. Program 1) the routing problem is simplified in two phases (Procedures Simplify1 and Simplify2). Procedure Simplify1 eliminates all cuts with capacity one and all vertices v with $\deg(v) = \mathrm{ter}(v)$ and turns the routing problem into a reduced routing problem. Procedure Simplify2 works on reduced routing problems and

either discards some edges in the left upper corner of the routing region (if there is no saturated cut through the left upper corner) or it chooses a particular net to be routed through the left upper corner. In the second case a vertex v with $\deg(v) = \text{ter}(v)$ is created and hence Simplify1 applies in the next iteration. Thus if Simplify2 of one iteration does not remove an edge then Simplify1 of the next iteration will remove a vertex of the routing region and hence $O(n)$ iterations suffice to find a solution.

PROGRAM 1. ($* P_0 = (V_0, E_0, \mathcal{N}_0)$ is an even local routing problem satisfying the cut condition $*$).

$P = (V, E, \mathcal{N}) \leftarrow P_0$
while $E \neq \varnothing$
do($* P$ is good $*$)
  Simplify1;
  ($* P$ is good and reduced $*$)
  Simplify2;
  ($* P$ is good $*$)
od

LEMMA 1. *Let P be a 1-even local routing problem. Then $\deg(v) = \text{ter}(v)$ mod 2 for all v. In particular, if P is bounded then $\deg(v) = 3$ implies $\text{ter}(v) = 1$ and if P is reduced $\deg(v) = 2$ implies $\text{ter}(v) = 0$. Also there are no vertices of degree one.*

Proof.   Let v be any vertex. If $\deg(v) = 4$ then $v \notin B$ and hence $\text{ter}(v) = 0$ and the claim is true. So let us assume that $v \in B$. Then v lies on the boundary of c nontrivial faces, $1 \leqslant c \leqslant 3$. Thus there are c disjoint cuts $X_1, ..., X_c$ which separate v from the remainder of the graph. Note that $\deg(v) = \text{cap}(X_1) + \cdots + \text{cap}(X_c)$. Also $\text{ter}(v) = (\text{dens}(X_1) + \cdots + \text{dens}(X_c))$ mod 2, since exactly the paths having v as a terminal cross X an odd number of times. Thus $\deg(v) - \text{ter}(v) = (\text{fcap}(X_1) + \cdots + \text{fcap}(X_c))$ mod 2. It remains to be shown that $\text{fcap}(X_i) = 0$ mod 2 for all i, $1 \leqslant i \leqslant c$. If $c \geqslant 2$ or $\deg(v) \leqslant 2$ then the cuts X_i are 1-bend cuts and hence we have $\text{fcap}(X_i) = 0$ mod 2 for all i since P is 1-even. If $c = 1$ and $\deg(v) = 3$ then X_1 is a 2-bend cut. We will show $\text{fcap}(X_1) = 0$ mod 2 in Lemma 11c of Section III.2. Thus in either case we conclude $\deg(v) = \text{ter}(v)$ mod 2. In a reduced routing problem we also have $\text{ter}(v) < \deg(v)$ and in a bounded problem we have $\deg(v) + \text{ter}(v) = 4$ for the vertices v with $\deg(v) = 3$. This proves the second part of the claim. ∎

We will next describe procedures Simplify1 and Simplify2, cf. Programs 2 and 3. In Simplify1 we distinguish two cases, namely the existence of a cut X with $\text{cap}(X) = 1$ or the absence of such cuts and the existence of a vertex

v with $\deg(v) = \mathrm{ter}(v)$. Assume first that there is a cut X with $\mathrm{cap}(X) = 1$. Since P is good and hence is 1-even and satisfies the cut condition for X we must have $\mathrm{fcap}(X) = 0$. Thus there is a unique net N with $\mathrm{cross}(N, X) = 1$. Let $e \in E$ be the edge intersected by X and let $N = [p_1 e p_2]$ where $\mathrm{cross}(p_1 e p_2, X) = 1$. We remove edge e, reserve it for net N, and replace net N by the two nets $[p_1]$ and $[p_2]$. A solution for P is readily obtained from a solution for the modified problem which we denote by P'. We only have to combine the local routings for $[p_1]$ and $[p_2]$ with edge e and obtain a local routing for N.

PROGRAM 2.   Procedure Simplify1

begin ($*$ P is good $*$)
    while there is a cut X with $\mathrm{cap}(X) = 1$ or a vertex v with $\deg(v) = \mathrm{ter}(v)$
  do ($*$ P is good $*$)
    if there exists a cut X with $\mathrm{cap}(X) = 1$
    then let X be a cut with $\mathrm{cap}(X) = 1$ and let N be the unique net
          with $\mathrm{cross}(N, X) = 1$;
          let e be the edge intersected by cut X and let
          $N = [p_1 e p_2]$ where $\mathrm{cross}(p_1 e p_2, X) = 1$;
          remove edge e, reserve it for net N and replace net N by
          nets $[p_1]$ and $[p_2]$.
    else let v be a vertex with $\deg(v) = \mathrm{ter}(v)$;
          let e_i, $1 \leqslant i \leqslant 2$, be the edges incident to v and
          let N_i, $1 \leqslant i \leqslant 2$, be the nets incident to v where the edges
          are numbered as shown in Fig. 3 and N_1 is *right-of* N_2;
          let $N_1 = [e_1 p_1]$, where p_1 does not use edge e_1;
          remove edge e_1, reserve it for net N_1 and replace net N_1 by
          net $[p_1]$
    fi
    ($*$ P is good $*$)
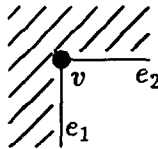 **od**
  ($*$ P is good and reduced $*$)
end

FIG. 3.   A vertex v with $\deg(v) = 2$ and the edges incident to it. The face in $\bar{M}$ is shown hatched.

LEMMA 2. *The problem P' defined above is good; i.e., the then-case of Procedure Simplify1 maintains the invariant.*

Proof. A proof will be given in Section III.4. ∎

Assume next that there is no cut with capcity one but that there is a vertex v with $\deg(v) = \text{ter}(v)$. Then $\deg(v) = 2$ and v is incident to exactly one nontrivial face. We will now define two orderings, one on the d edges incident to v and one on the d nets incident to v. We will then assign the ith edge in the ordering of edges to the ith net in the ordering of nets and simplify the routing problem in this way.

For a vertex v incident to exactly one non-trivial face an ordering on the edges incident to v is easy to define. Let e_i, $1 \leq i \leq \deg(v)$, be the edges incident to v ordered counterclockwise around v and numbered such that the faces following e_i, $1 \leq i \leq \deg(v)$, in counterclockwise order are trivial.

We turn to the ordering on nets next. We define this ordering in three steps. We first define an ordering on paths, then define the concept of a canonical representative of a net and then order nets via their canonical representatives.

DEFINITION 3. (a) Let p_1 and p_2 be local routings with a common start vertex v. Then p_1 is *right-of* p_2 if either
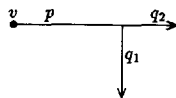
— $p_1 = p_2$ or

— p_1 is a proper prefix of p_2 and p_2 enters the endpoint t_1 of p_1 through edge e_i and leaves it through edge e_j where $i > j$ and $e_1, ..., e_d$, $d \in \{2, 3\}$ is the ordering of the edges incident to t_1 defined above or

— p_2 is a proper prefix of p_1 and p_1 enters the endpoint t_2 of p_2 through e_i and leaves it through edge e_j where $i < j$ and $e_1, ..., e_d$, $d \in \{2, 3\}$ is the ordering of the edges incident to t_2.

— p_1 and p_2 differ in their first edges, p_1 starts with edge e_i, p_2 starts with edge e_j and $1 \leq i < j \leq d$ or
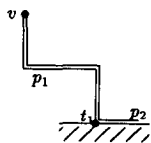
p_1 is a prefix of p_2 p_1 and p_2 have a common prefix p

p_1 and p_2 differ in their first edges

FIG. 4. The various cases of Definition 3(a). The local routing p_1 is *right-of* local routing p_2.
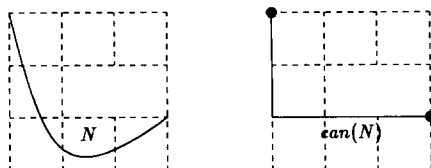
Fig. 5.   A net and its canonical representative.

— p_1 and p_2 have a maximal non-trivial common prefix p; i.e., $p_1 = pq_1$ and $p_2 = pq_2$ and the first edges of q_1 and q_2 are distinct, and the three edges "last edge of p," "first edge of q_1," "first edge of q_2" are ordered counterclockwise around their common endpoint.

Figure 4 illustrates this definition.

(b)   Let N be a net. A path $p \in N$ is the *canonical representative* of net N if p is a shortest path in N and if p is *right-of* all other shortest paths in N. We denote the canonical representative of net N by can(N), cf. Fig. 5.

(c)   Let N and N' be nets with a common start vertex terminal v. Then N is *right-of* N' if can(N) is *right-of* can(N').

Remark. Definition 3(a) is long but there is a simple idea behind it. Follow the paths p_1 and p_2 starting in their common start vertex v. When they separate one will proceed to the right of the other one. The special cases (one prefix of the other or no common nontrivial prefix) reduce to the general case if one artificially introduces a pseudo-edge at each vertex $w \in B$ entering w from the nontrivial face and extends all paths by pseudo-edges at both ends. The canonical representative of a net is unique since a net has only finitely many shortest representatives and since the relation *right-of* is a linear order for the paths with a common start vertex.

We are now ready for the else-case of Procedure Simplify1. Let e_i, $1 \leqslant i \leqslant 2$, be the edges incident to v in the ordering defined above and let N_i, $1 \leqslant i \leqslant 2$, be the nets incident to v, where N_1 is *right-of* N_2. (Here we implicitly assumed that the nets N_i incident to v have v as their start vertex. This might require changing the orientation of some nets. If there is a net N which has v as its start and end vertex then both orientations have to be considered, i.e., N is some N_i and N^{-1} is some N_j, $i \neq j$.) Write $N_1 = [e_1 p_1]$, where p_1 is a path not using edge e_1. N_1 can be written this way because e_1 can be replaced by an equivalent path of length three. Remove edge e_1 from the routing region, reserve it for net N_1 and replace net N_1 by net $[p_1]$. Call the new problem P', cf. Fig. 6. It is clear that a solution for P can be immediately derived from a solution for P'; we only
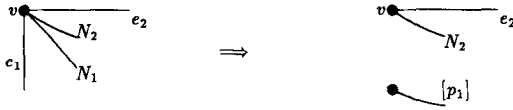
FIG. 6. The else-case of procedure Simplify1.

have to add edge e_1 to the local routing for $[p_1]$ and obtain a local routing for N_1.

LEMMA 3. *The problem P' defined above is good, i.e., the else-case of Procedure Simplify1 maintains the invariant.*

Proof. A proof will be given in Section II.4. ∎

Procedure Simplify1 leaves us with a reduced routing problem to which we apply Procedure Simplify2, cf. Program 3.

We need some additional concepts. A vertex $a = (x, y)$ of the routing region R is called the *left upper corner* of R if there is no vertex (x', y') of R with $y' > y$ or with $y' = y$ and $x' < x$. We use b to denote the vertex $b = (x, y - 1)$ and $e^* = (a, b)$ to denote the vertical edge incident to a. We consider 1-bend cuts X which go through edge e^* and distinguish two cases: either there exists a saturated 1-bend cut through edge e^* (then-case) or there exists no such cut (else-case). In the latter case we remove the four edges on the boundary cycle of the trivial face to the right of e^* (cf. Fig. 7).

PROGRAM 3. Procedure Simplify2.

begin (∗ P is good and reduced ∗)
    let $e^* = (a, b)$ be the vertical edge incident to the left upper corner a;
    if ∃ saturated 1-bend cut through e^*
    then let X be the leftmost saturated 1-bend cut through e^*
        let $([p_1], [p_2])$ be the rightmost decomposition with respect to X.
        replace net $[p_1 p_2]$ by nets $[p_1]$ and $[p_2]$.
    else remove the four edges on the boundary of the trivial
        face to the right of e^*
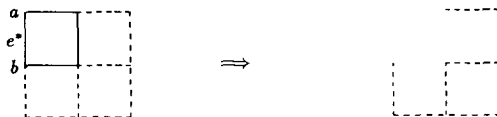   **fi**
    (∗ P is good ∗)
end

FIG. 7. The else-case of procedure Simplify2.

LEMMA 4. *The else-case of procedure Simplify2 maintains the invariant.*

Proof. A proof will be given in Section III.4. ∎

Let us consider the then-case next. We observe first that a saturated 1-bend cut through edge e^* must consist of two segments s_1 and s_2 and that the second segment runs downwards. This can be seen as follows: A horizontal cut through edge e^* cannot be saturated since each of the vertices of degree two in the top row are not terminal of any net and each of the l vertices of degree three is the terminal of exactly one net. Hence the capacity exceeds the density by at least two. A similar argument shows that s_2 cannot bend upwards. So s_2 must run downwards. Among the saturated 1-bend cuts through edge e^* we choose the one with the maximal segment s_1. We call this cut the *leftmost saturated 1-bend cut through edge e^**.

Next consider any net N. A *decomposition of N with respect to vertex a* is a pair $([p_1], [p_2])$, where p_1 ends in vertex a, p_2 starts in vertex a and $N = [p_1 p_2]$. Let $([p_1], [p_2])$ be a decomposition of net N and let $([q_1], [q_2])$ be a decomposition of N' with respect to a. The decomposition $([p_1], [p_2])$ is *right-of* decomposition $([q_1], [q_2])$ if $[p_2]$ is *right-of* $[q_2]$.

We are now ready for the description of the then-case. Let X be the leftmost saturated 1-bend cut through edge e^*. Let

$$D = \{([q_1], [q_2]); ([q_1], [q_2]) \text{ is the decomposition with respect to } a$$
$$\text{of some net } N \text{ where either } N \in \mathcal{N} \text{ or } N^{-1} \in \mathcal{N}$$
$$\text{and cross}([q_1 q_2], X) = \text{cross}([q_1], X) + \text{cross}([q_2], X)\}.$$

Intuitively, in D we collect those decompositions with respect to a of the nets which do not provide additional crossings of cut X. Let $([p_1], [p_2]) \in D$ be *right-of* every other element of D. We call $([p_1], [p_2])$ the *rightmost decomposition with respect to X*.
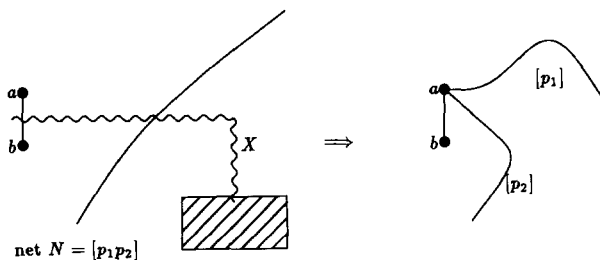

FIG. 8. The then-case of procedure Simplify2. X is the leftmost saturated 1-bend cut through edge e^* and $([p_1], [p_2])$ is the rightmost decomposition with respect to X.

Remark. The set D is not-empty sinced the cut X is saturated. Let $N \in \mathcal{N}$ be a net which goes across X. We can write $N = [q_1 e^* q_2]^\varepsilon$ with $\varepsilon \in \{-1, +1\}$. Then $([q_1], [e^* q_2])$ is an element of D.

In the then-case we replace the net $[p_1 p_2]$ (or $[p_1 p_2]^{-1}$, whichever is in $\mathcal{N}$) by the nets $[p_1]$ and $[p_2]$, cf. Fig. 8. Call the modified problem P'. It is clear that a solution for the modified problem P' directly yields a solution for P. We also have

LEMMA 5. *The problem P' defined above is good, i.e., the then-case of procedure Simplify2 maintains the invariant.*

Proof. A proof will be given in Section III.4. ∎

THEOREM 2. *Let P_0 be a solvable even local routing problem. Then our algorithm constructs a solution in $O(n)$ iterations of the main loop.*

Proof. If P_0 is solvable then P_0 is good (with respect to P_0). We infer from Lemmas 2 to 5 that all intermediate problems are good, and hence the final problem is good. The set of edges in the final problem is empty and hence our algorithm finds a solution for the routing problem. The bound on the number of iterations follows from the remark made immediately before Definition 2. ∎

Theorem 2 implies part (a) of Theorem 1. Clearly, if P_0 is solvable then the cut condition holds. Conversely, if the cut condition holds then the cut condition for 1-bend cuts holds and P_0 is good. Thus our algorithm constructs a solution.

III. CORRECTNESS

We prove the correctness of our algorithm in this section. The section is divided into four subsections. The last subsection contains the proof of Lemmas 2 to 5 of Section II; the first three subsections contain preliminary lemmas. Section III.1 is on cuts and reduced representatives, Section III.2 gives an alternative definition of the ordering *right-of* on nets which avoids the use of canonical representatives and is more amenable to counting densities, and Section III.3 shows that the cut condition for a small subset of the 1-bend cuts implies the cut condition for all cuts.

III.1. *Cuts and Reduced Paths*

Let C be a cut and let N be a net. By definition of cross(N, C) there is a path $p \in N$ with cross(N, C) = cross(p, C). Figure 9 shows that for a net N and cuts C_1 and C_2 there is not necessarily a single representative $p \in N$
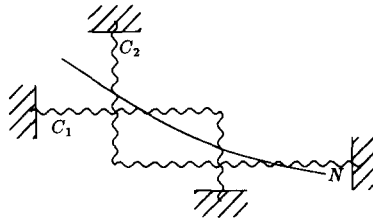
FIG. 9. Two cuts C_1 and C_2 and a net N which does not have a common reduced representative for both cuts.

with $\mathrm{cross}(N, C_i) = \mathrm{cross}(p, C_i)$, $i = 1, 2$. We call a representation $p \in N$ *reduced* with respect to a set of cuts if $\mathrm{cross}(p, C) = \mathrm{cross}(N, C)$ for every cut in the set. In this section we derive sufficient conditions for the existence of such a common "reduced" representative.

Let C be a cut and let $F_1, ..., F_m$ be the sequence of trivial faces through which C goes. Let $e_1^{(i)} p_1^{(i)} e_2^{(i)} p_2^{(i)}$ be the boundary cycle of face F_i in clockwise order where $e_1^{(i)}$ and $e_2^{(i)}$ are the edges intersected by C, $p_1^{(i)}$ and $p_2^{(i)}$ are paths, and $e_2^{(i)}$ and $e_1^{(i+1)}$ are the same edge but in opposite orientation. Then $p_1 = p_1^{(1)} p_1^{(2)} \cdots p_1^{(m)}$ and $p_2 = p_2^{(1)} p_2^{(2)} \cdots p_2^{(m)}$ are called the two *paths along cut C*, cf. Fig. 10. If x and y are vertices on path p_1 (or p_2) then we call the subpath of p_1 (or p_2) connecting x and y the path along C connecting x and y.

LEMMA 6. *Let C be a cut and let p be a path. If* $\mathrm{cross}(p, C) > \mathrm{cross}([p], C)$ *then p can be written* $p = p_1 e_1 p_2 e_2 p_3$, *where* $e_1 = (c, d)$ *and* $e_2 = (e, f)$ *go across C in opposite directions, p_2 does not intersect C,* $\mathrm{cross}(p_1 p_4 p_3, C) = \mathrm{cross}(p, C) - 2$, *and* $p_4 \in [e_1 p_2 e_2]$, *where p_4 is the path from c to f along C.*

Proof. The claim is intuitively obvious, cf. Fig. 11. A proof goes as follows. Let $q \in [p]$ be such that $m := \mathrm{cross}(q, C) = \mathrm{cross}([p], C)$ and let $q_0, ..., q_k$ be a sequence of paths such that $q = q_0$, $q_k = p$, and q_i and q_{i+1} are elementarily equivalent, $0 \leqslant i < k$. We show by induction on i that the intersections between q_i and C can be labelled by the labels "proper" and
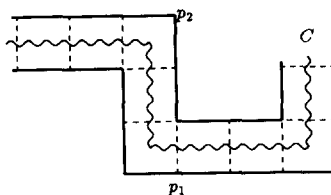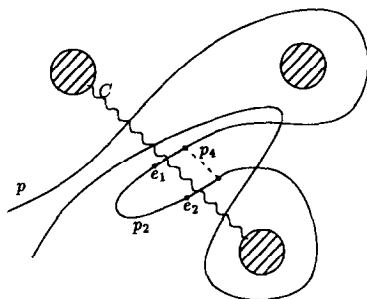

FIG. 10. A cut C and the two paths along C.

Fig. 11. An illustration of Lemma 6.

"improper" such that the improper intersections can be paired and such that

(1) no pair encloses a proper intersection,

(2) the pairs form a set of properly nested parentheses, and

(3) each pair induces a cycle (consisting of the subpath of p connecting the two elements of the pair and the path along C connecting it) which is homotopic to 0 (the faces in M are, of course, the holes in the plane). Also at most m intersections are labelled proper.

For $i = 0$ all intersections are labelled proper. Let us go from i to $i + 1$ next. We have $q_i = q_i^1 q_i^2 q_i^3$, $q_{i+1} = q_i^1 q_i^4 q_i^3$ and $q_i^2 (q_i^4)^{-1}$ is a boundary cycle of a trivial face. Clearly $\mathrm{cross}(q_i^2, C) = \mathrm{cross}(q_i^4, C) \mod 2$ and $\mathrm{cross}(q_i^2, C) + \mathrm{cross}(q_i^4, C) \leqslant 2$, since C is a simple dual path. We thus have to distinguish four cases. If $\mathrm{cross}(q_i^2, C) = \mathrm{cross}(q_i^4, C) = 0$ then there is nothing to show. If $\mathrm{cross}(q_i^2, C) = \mathrm{cross}(q_i^4, C) = 1$ then the labelling and the pairing is easily modified. If $\mathrm{cross}(q_i^4, C) = 2$ then we pair the two additional points of intersection and label them improper. This leaves the case $\mathrm{cross}(q_i^2, C) = 2$. If both intersections are labelled proper or if both are labelled improper and are paired, then there is nothing to show.

If one is labelled proper and one improper (call the improper intersection x) then we relabel the partner of the improper intersection (call it y) proper. Note that y cannot be enclosed by a pair because this pair would also contain x (pairs were nested properly in q_i) and hence the intersection labelled proper in q_i because this intersection is an immediate neighbor of x. Thus the nesting property is again satisfied. Finally, if both intersections are improper then we pair their partners. We show that the new pairing satisfies conditions (1) and (2). Note first that if one of the partners is contained in a parenthesis then one of the disappearing intersections was and hence the other one was also, since the disappearing intersections are immediate neighbors. Thus the other partner is also contained and hence

property (2) is again satisfied. For property (1) we only observe that any proper intersection contained in the newly formed parenthesis must been contained in one of the two disappearing parentheses. We leave it to the reader to check the claim about 0-homotopy.

The labelling and pairing with respect to q_k immediately yields the desired edges e_1 and e_2 and path p_2. We only have to consider a pair of improper intersections which contains no further pair. ∎

LEMMA 7. *Let C be a cut and let p be a path. Then* $\text{cross}(p, C) = \text{cross}([p], C) \pmod 2$.

Proof. By Lemma 6 we can reduce $\text{cross}(p, C)$ to $\text{cross}([p], C)$ by repeatedly removing pairs of intersections. ∎

Let C and D be two cuts and let p_1 and p_2 be the paths along C and q_1 and q_2 be the paths along D. We say that C and D do *not interfere* if p_i intersects D at most once and q_i intersects C at most once, $i = 1, 2$. A set of cuts is *interferencefree* if any two cuts in S do not interfere.

LEMMA 8. (a) *Let C and D be 1-bend cuts. C and D do not interfere if either C and D have at most one vertex v in common, C uses the dual edges d_1, d_2 incident to v and D uses d_3 and d_4 incident to v such that d_1, d_3, d_2, and d_4 are distinct and lie in that order around v or if $C = EF$ and $D = EG$, where F and G are vertex-disjoint except for their common endpoint.*

(b) *Let S be an interferencefree set of cuts and let N be a net. Then there is a path $p \in N$ which is reduced with respect to all cuts in S, i.e., $\text{cross}(p, C) = \text{cross}(N, C)$ for all $C \in S$.*

Proof. (a) Obvious.

(b) Let $p \in N$ be such that $k := \sum_{E \in S} \text{cross}(p, E)$ is minimal. Assume that p is not reduced with respect to some cut $C \in S$. Then $p = p_1 e_1 p_2 e_2 p_3$, where $e_1 = (c, d)$ and $e_2 = (e, f)$ go across C, p_2 does not go across C, $\text{cross}(p_1 p_4 p_3, C) = \text{cross}(p, C) - 2$, and $p_4 \in [e_1 p_2 e_2]$, where p_4 is the path from c to f along C; cf. Lemma 6. Let $D \in S$ be arbitrary. Then $\text{cross}(p_4, D) \leqslant 1$, since C and D do not interfere and hence $\text{cross}(p_1 p_4 p_3, D) \leqslant \text{cross}(p_4, D) + 1$. Since $p_1 p_4 p_3 \in [p]$ we conclude $\text{cross}(p_1 p_4 p_3, D) \leqslant \text{cross}(p, D)$ from Lemma 7 and hence $\sum_{E \in S} \text{cross}(p_1 p_4 p_3, E) < k$, a contradiction. ∎

III.2. *Slicings and the Ordering of Nets*

The goal of this section is a more topological characterization of the ordering *right-of* of nets and of the density of a cut. Whilst the notion of canonical representative gives us a concise definition of the ordering *right-of* on nets it is of very limited use in density arguments because
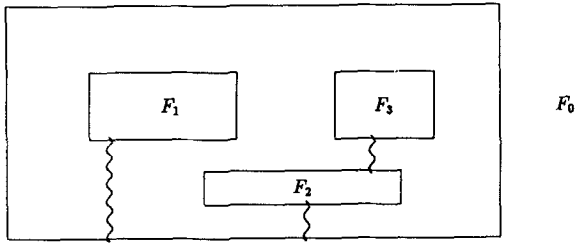
FIG. 12. A slicing S. We have parent$(F_1) = F_0$, parent$(F_3) = F_2$, and parent$(F_2) = F_0$, where F_0 is the unbounded face.

canonical representatives are by definition unique and therefore we cannot assume them to be reduced with respect to certain cuts. The alternative definition of the ordering given in this section, albeit longer, can be used in density and ordering arguments. Throughout this section we assume that there is no cut of capacity one.

DEFINITION 4. A slicing S of the routing region R is a set $\{C(F); F \in M\}$ of cuts and a function *parent*: $M \to \bar{M}$ such that

    (1) $C(F)$ has one endpoint in face F and the other endpoint in face *parent*(F) for all $F \in M$;

    (2) the function *parent* defines a tree on $\bar{M}$ with the infinite face being the root;

    (3) each cut $C(F)$, $F \in M$, is a 0-bend cut and two cuts $C(F)$ and $C(G)$ are vertex-disjoint for $F \neq G$.

Figure 12 illustrates this definition.

If S is a slicing then the removal of all edges of R which are intersected by a cut in S turns the routing region R into a routing region $R(S)$ which has only one nontrivial face, the unbounded face, cf. Fig. 13. This follows from the observation that for every cut $C(F)$ of the slicing the paths along $C(F)$ are also paths in $R(S)$, since the cuts of S are pairwise vertex-disjoint.
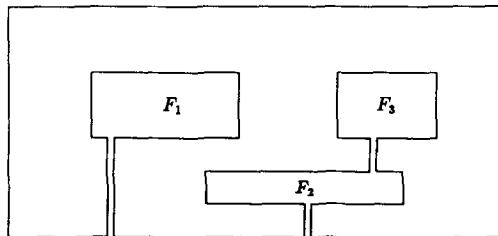

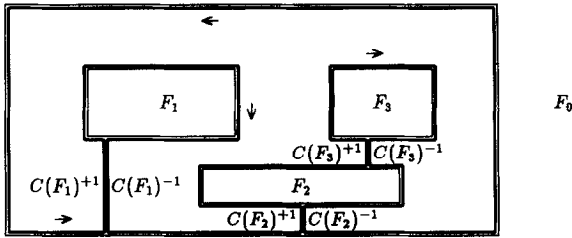FIG. 13. The routing region $R(S)$, where R and S are as in Fig. 12.

FIG. 14. The ordering on $B \cup C$ for the example of Fig. 12 and 13. The cyclic ordering is indicated by a heavy line with arrows.

For a cut $C(F)$ let $C(F)^{-1}$ and $C(F)^{+1}$ be two new symbols. We use the new symbols to "represent" the two sides of the cut $C(F)$. Let $C = \{ C(F)^{-1}, C(F)^{+1}; F \in M \}$ and let B be the set of vertices of R of degree three or less as defined in the introduction. We define a cyclic ordering on the set $B \cup C$ by a clockwise traversal of the boundary of the unbounded face of $R(S)$, where $C(F)^{+1}$ represents the sequence of vertices and edges used to reach face F from face parent(F) and $C(F)^{-1}$ represents the sequence of vertices and edges used to reach face parent(F) from face F. Figure 14 illustrates this definition. Note that this ordering is well defined, since no node in B can belong to the boundary of two faces in $\bar{M}$. Otherwise, there would be a cut of capacity one.

We can use slicings to decompose paths (and then nets) into *elementary pieces*. Let p be an oriented path. We can write $p = p_1 e_1 p_2 e_2 \cdots p_m e_m p_{m+1}$, where e_i goes across a cut in S, say $C(F_{j_i})$ and the p_i's do not go across cuts in S. Assume further that e_i crosses the cut $C(F_{j_i})$ in the direction from $C(F_{j_i})^{+d_i}$ to $C(F_{j_i})^{-d_i}$, where $d_i \in \{ -1, +1 \}$. Then

$$(s, C(F_{j_1})^{d_1}), (C(F_{j_1})^{-d_1}, C(F_{j_2})^{d_2}), \ldots, (C(F_{j_m})^{d_m}, t)$$

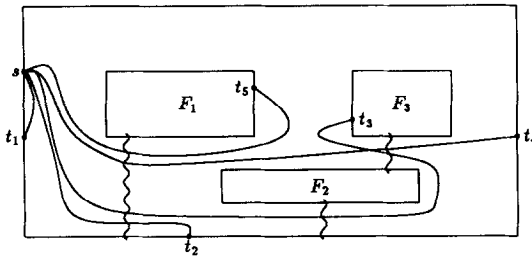FIG. 15. Net N_i connects s and t_i. The elementary pieces of N_5 are $(s, C(F_1)^{+1})$, $(C(F_1)^{-1}, t_5)$ and the elementary pieces of N_4 are $(s, C(F_1)^{+1})$, $(C(F_1)^{-1}, C(F_3)^{+1})$, $(C(F_3)^{-1}, t_4)$. Since $C(F_1)^{-1}, C(F_3)^{+1}, t_5$ occur in that order in the cyclic ordering of $B \cup C$ (cf. Fig. 14), N_4 is "right-of" N_5. In general, N_i is "right-of" N_j for $i < j$.

is the *decomposition* of path p into elementary pieces; here s and t denote the start- and end-vertex of p. An elementary piece is an element of $(B \cup C)^2$. This definition is illustrated by Fig. 15.

LEMMA 9. *Let p and q be equivalent paths and let p and q be reduced with respect to all cuts in slicing S. Then p and q yield the same sequence of elementary pieces.*

Proof. Obvious. ∎

Lemma 9 allows us to extend the decomposition into elementary pieces from paths to nets. Let N be a net and let $p \in N$ be reduced with respect to S. Then a decomposition of N is defined as a decomposition of p.

We are now ready for the alternative definition of the ordering *right-of* on nets with a common start vertex. Let N_i be a net with start-vertex s and let $\alpha_{i1}, ..., \alpha_{ik_i}$ be the decomposition of N_i into elementary pieces, $i = 1, 2$. Then N_1 is "right-of" N_2 iff there is a j such that $\alpha_{1l} = \alpha_{2l}$ for $l < j$, $\alpha_{1j} = (u, v)$, $\alpha_{2j} = (u, w)$, where $u, v, w \in B \cup C$, $v \neq w$, and u, v, and w occur in *that* order in the cyclic ordering of $B \cup C$ (cf. Fig. 15) or if $N_1 = N_2$.

Since the equivalence of this definition with our old definition of *right-of* still needs to be verified, we used "right-of" instead of *right-of* in this definition. Also note that the definition of "right-of" is with respect to a particular slicing S. We will next show that the orderings *right-of* and "right-of" are the same; this also implies that the ordering "right-of" is independent of the particular slicing used in its definition.

LEMMA 10. *Let N_1 and N_2 be nets with common start-vertex s. Then N_1 is "right-of" N_2 iff N_1 is right-of N_2.*

Proof. The claim is obvious if $N_1 = N_2$. So let us assume that $N_1 \neq N_2$ and that N_1 is *right-of* N_2. Since both orderings are linear it suffices to show that N_1 is "right-of" N_2.

Let $\alpha_{i1}, ..., \alpha_{ik_i}$ be the decomposition of N_i into elementary pieces. Since $N_1 \neq N_2$ there is a j such that $\alpha_{1l} = \alpha_{2l}$ for $l < j$ and $\alpha_{1j} = (u, v)$, $\alpha_{2j} = (u, w)$ with $v \neq w$.

Let $\text{can}(N_i)$ be the canonical representative of net N_i. Since canonical representatives are shortest representatives they are reduced with respect to all 0-bend cuts and hence with respect to all cuts in the slicing. We can therefore write

$$\text{can}(N_i) = p_{i1} e_{i1} p_{i2} e_{i2} \cdots e_{ik_i - 1} p_{ik_i}$$

where e_{il} goes across some cut in S and the p_{il}'s do not go across cuts in S. Also, if e_{il} crosses the cut $C(F_{il})$ in the direction from $C(F_{il})^{d_{il}}$

to $C(F_{il})^{-d_{il}}$ then $\alpha_{il} = (C(F_{il-1})^{d_{il}-1}, C(F_{il})^{-d_{il}})$ for $2 \leqslant l < k_i$, $\alpha_{i1} = (s, C(F_{i1})^{d_{i1}})$, and $\alpha_{ik_i} = (C(F_{ik_i-1})^{d_{ik_i}-1}, t_i)$. Thus $F_{1l} = F_{2l}$ and $d_{1l} = d_{2l}$ for $l < j$.

Let h be minimal such that $p_{1h} e_{1h} \neq p_{2h} e_{2h}$. Then clearly $h \leqslant j$. Also, we can write $p_{1h} e_{1h} = q r_{1h}$ and $p_{2h} e_{2h} = q r_{2h}$, where q is maximal with this property. Since N_1 is *right-of* N_2 the path r_{1h} is *right-of* r_{2h}.

CLAIM 1. *r_{1h} and r_{2h} are vertex-disjoint except for their common start vertex and p_{1l} and p_{2l} are vertex-disjoint for $h < l \leqslant j$.*

Proof. This follows immediately from the fact that canonical representatives are shortest rightmost representatives and the observation that a point of intersection would induce a cycle which is homotopic to 0. ∎

Let s_{il} be the start vertex of p_{il} and let t_{il} be the end vertex of p_{il}. Then $s_{1h} = s_{2h}$ and $\alpha_{il} = (t_{il}, s_{il+1})$. Also $s_{1l} \neq s_{2l}$ for $h < l \leqslant j$ and $t_{1l} \neq t_{2l}$ for $h \leqslant l \leqslant j$, by Claim 1. Observe next that the vertices s_{1h}, t_{1h}, t_{2h} appear in that order in a counterclockwise traversal of the boundary of region $R(S)$. This follows from Claim 1 and the assumption that p_{1h} is *right-of* p_{2h}.

CLAIM 2. *The vertices s_{2l}, s_{1l}, t_{1l}, t_{2l} appear in that order in a counterclockwise traversal of the boundary of region $R(S)$ for $h < l \leqslant j$.*

Proof. We use induction on l. Let l, $h < l \leqslant j$ be arbitrary. Then path p_{il} connects s_{il} and t_{il} and by Claim 1 the two paths p_{il} and p_{2l} do not intersect, i.e., p_{1l} and p_{2l} are two non-intersecting chords of region $R(S)$. We conclude that only the four orderings s_{2l}, s_{1l}, t_{1l} and s_{2l}, t_{1l}, s_{1l}, t_{2l} and s_{2l}, t_{2l}, s_{1l}, t_{1l} and s_{2l}, t_{2l}, t_{1l}, s_{1l} are possible. It remains to argue that s_{1l} directly follows s_{2l} in a counterclockwise traversal because this will leave only the first possibiility.

By induction hypothesis the vertices s_{2l-1}, s_{1l-1}, t_{1l-1}, t_{2l-1} occur in that order in a counterclockwise traversal. Also $F_{1l-1} = F_{2l-1}$ and hence the vertices t_{1l-1}, t_{2l-1}, s_{2l}, s_{1l} occur in that order in a counterclockwise traversal of the boundary of $R(S)$. Finally, since s_{1l} and s_{2l} lie on the same side of the same cut and since the paths can(N_i) are reduced, the vertices t_{2l} and t_{1l} must follow s_{1l} in the counterclockwise travesal. This proves Claim 2. ∎

We are now ready to complete the proof of Lemma 10. The vertices s_{2j}, s_{1j}, t_{1j}, t_{2j} occur in that order in a counterclockwise traversal of the boundary of $R(S)$, s_{2j} and s_{1j} lie on the same side of the same cut and t_{1j} and t_{2j} do not. Hence $\alpha_{1j} = (u, v)$, $\alpha_{2j} = (u, w)$, $v \neq w$ and u, v, w occur in the same order in the cyclic ordering of $B \cup C$ as s_{1j}, t_{1j}, t_{2j} occur in a counterclock-
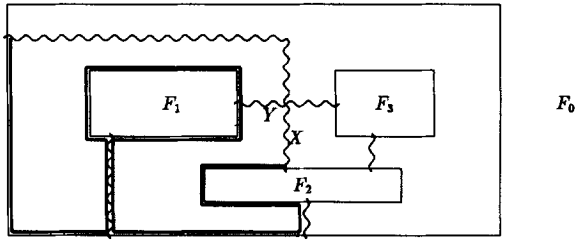
FIG. 16. The slicing S of Fig. 12 and two cuts X and Y. S, X, and Y are strongly inter-ferencefree. The set $L(X)$ is indicated by a heavy line.

wise traversal of the boundary of $R(S)$. Thus N_1 is "right-of" N_2 and the proof is complete. ∎

Lemma 10 justifies that we write *right-of* instead of "right-of" from now on. We will next show that slicings are a very convenient tool for counting densities.

Let X and Y be interferencefree cuts and let S be a slicing. We say that S, X, and Y are *strongly interferencefree* if every cut $C \in S$ is vertex-disjoint from X and Y, cf. Fig. 16.

Assume now that S, X, and Y are strongly interference-free. Then cut X splits the region $R(S)$ into two parts and in this way induces a partition $(L(X), R(X))$ of $B \cup C$. The partition $(L(Y), R(Y))$ induced by the cut Y of $B \cup C$ is defined analogously. For sets $D_1, D_2 \subseteq B \cup C$ define $\overline{\text{dens}}(D_1, D_2)$ as the sum over all nets $N \in \mathcal{N}$ of the number of elementary pieces (x, y) of net N, when $|D_1 \cap \{x, y\}| = 1$ and $|D_2 \cap \{x, y\}| = 1$. For $\overline{\text{dens}}(D_1, (B \cup C) - D_1)$ we write simply $\overline{\text{dens}}(D_1)$.

LEMMA 11. *Let S be a slicing and let X and Y be 1-bend cuts such that S, X, and Y are strongly interferencefree. Let $(L(X), R(X))$ and $(L(Y), R(Y))$ be the partitions of $B \cup C$ induced by X and Y. Then*

(a) $\text{dens}(X) = \overline{\text{dens}}(L(X))$ *and* $\text{dens}(Y) = \overline{\text{dens}}(L(Y))$

(b) $\overline{\text{dens}}(L(X)) + \overline{\text{dens}}(L(Y))$
$\qquad = \overline{\text{dens}}(L(X) \cap L(Y)) + \overline{\text{dens}}(L(X) \cup L(Y))$
$\qquad + 2\,\overline{\text{dens}}(L(X) - L(Y), L(Y) - L(X))$

(c)   *Let P be a 1-even local routing problem, let v be a vertex of degree 3 and let Z be a 2-bend cut of capacity 3 which separates v from the remainder of the graph. Then $\text{fcap}(Z) = 0$ mod 2.*

Proof. (a)   We show $\text{dens}(X) = \overline{\text{dens}}(L(X))$. Let Q be any net and let $q \in Q$ be reduced with respect to $S \cup \{X\}$. Write $q = q_1 e_1 q_2 e_2 \cdots q_{m-1} e_{m-1} q_m$, where the e_i's cross cuts in S and the q_i's do
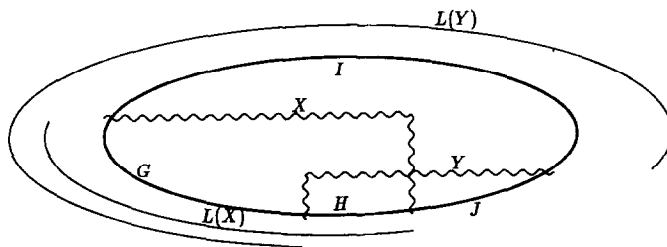
FIG. 17. Cuts X and Y and the partition (G, H, I, J) of $B \cup C$.

not. Then the q_i's correspond to the elementary pieces of Q. Clearly $\text{cross}(Q, X)$ is the number of q_i which go across X. Moreover, q_i goes across X iff the corresponding elementary piece has exactly one end point in $L(X)$.

(b) Let $G = L(X) \cap L(Y)$, $H = L(X) - L(Y)$, $I = L(Y) - L(X)$, and $J = (B \cup C) - (L(X) \cup (L(Y))$, cf. Fig. 17. Then

$$\overline{\text{dens}}(L(X)) + \overline{\text{dens}}(L(Y))$$

$$= \overline{\text{dens}}(G \cup H, I \cup J) + \overline{\text{dens}}(G \cup I, H \cup J)$$

$$= \overline{\text{dens}}(G, I) + \overline{\text{dens}}(G, J) + \overline{\text{dens}}(H, I) + \overline{\text{dens}}(H, J)$$

$$\quad + \overline{\text{dens}}(G, H) + \overline{\text{dens}}(G, J) + \overline{\text{dens}}(I, H) + \overline{\text{dens}}(I, J)$$

$$= \overline{\text{dens}}(G, I \cup J \cup H) + \overline{\text{dens}}(G \cup H \cup I, J) + 2 \overline{\text{dens}}(H, I)$$

$$= \overline{\text{dens}}(L(X) \cap L(Y)) + \overline{\text{dens}}(L(X) \cup L(Y))$$

$$\quad + 2 \overline{\text{dens}}(L(X) - L(Y), L(Y) - L(X)).$$

(c) The cut Z consists of three segments s_1, s_2, s_3 of length one each. Consider the cuts $X = s_1 s_2'$ and $Y = s_3 s_2''$, where s_2' results from s_2 by extending it beyond s_3 until it hits a boundary and s_2'' results from s_2 by extending it beyond s_1 until it hits a boundary. Define the partitions $(L(X), R(X))$ and $L((Y), R(Y))$ such that $\{v\} = L(X) \cap L(Y)$ and let S be a slicing such that S, X, and Y are interferencefree. Then

$$\overline{\text{dens}}(L(X) \cap L(Y))$$

$$= \overline{\text{dens}}(L(X)) + \overline{\text{dens}}(L(Y)) - \overline{\text{dens}}(L(X) \cup L(Y))$$

$$\quad - 2 \overline{\text{dens}}(L(X) - L(Y), L(Y) - L(X)).$$

Next note that $(L(X) \cup L(Y), R(X) \cap R(Y))$ is a partition which corresponds to a 0-bend cut, say U, and that $\text{cap}(U) = 1 + a + b$,

$\mathrm{cap}(X) = 2 + b$, and $\mathrm{cap}(Y) = 2 + a$ for some integers a and b; a is the length of s_2' minus 1 and b is the length of s_2'' minus 1. Then

$$\mathrm{fcap}(Z) = 3 - \overline{\mathrm{dens}}(L(X) \cap L(Y))$$

$$= (2 + b - \overline{\mathrm{dens}}(L(X))) + (2 + a - \overline{\mathrm{dens}}(L(Y)))$$

$$\quad - (1 + a + b - \overline{\mathrm{dens}}(L(X) \cup L(Y)))$$

$$\quad + 2\,\overline{\mathrm{dens}}(L(X) - L(Y), L(Y) - L(X))$$

$$= \mathrm{fcap}(X) + \mathrm{fcap}(Y) - \mathrm{fcap}(U) + 2\,\overline{\mathrm{dens}}(L(X) - L(Y), L(Y) - L(X))$$

$$= 0 \bmod 2. \quad \blacksquare$$

Remark. In part (c) of Lemma 11 we prove the evenness of a 2-bend cut by expressing it as the intersection of two 1-bend cuts. In a similar way one can express a k-bend cut as the intersection of two $(k-1)$-bend cuts. This suggests that one can prove by induction on the number of bends that in a 1-even routing problem all cuts have even free capacity, i.e., 1-evenness implies evenness. Such an inductive proof is not possible, however, because the slicing S in part (c) of the lemma does not necessarily exist if Z is an arbitrary cut. We want to mention, that arguments similar to the ones used in the proof of Lemma 12 can be used to prove that 1-evenness implies evenness. We do not need that claim in the present paper and therefore do not include that proof.

Lemma 11 will play a major role in Section III.4.

III.3. *On the Form of Cuts*

This section is concerned with the form of cuts. We will first (Lemma 12(a)) show that the cut condition for 1-bend cuts implies the cut condition for all cuts. We will then show (Lemma 12(b)) that only 0-bend cuts and 1-bend cuts connecting concave corners (cf. Definition 5 below) need to be considered and we will finally show that the leftmost saturated cut through edge e^* (the reader finds the definition of edge e^* in Section II immediately before Lemma 4) is a 1-bend cut (strictly speaking, we defined leftmost cut only with respect to 1-bend cuts; Lemma 12(c) justifies this).

DEFINITION 5. (a) A pair $((v', v), (v, v''))$ of boundary edges of some non-trivial face is a *concave corner* if $\deg(v) = 4$.

(b) A 1-bend cut $X = s_1$, s_2 connects two concave corners $((v', v), (v, v''))$ and $((w', w), (w, w''))$, if X intersects one edge of each pair, say (v', v) and (w, w'), X is not a 0-bend cut, $Y := s_1'$, s_2', where s_1' intersects (v, v'') and has the same length and direction as s_2, and s_2' intersects (w, w'') and has the same length and direction as s_1, is a cut, and the rectangle
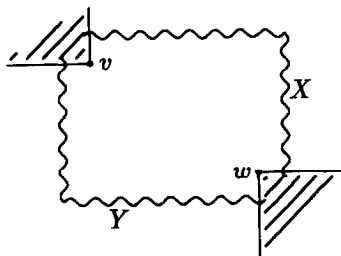
Fig. 18.   A 1-bend cut X connecting two concave corners v and w.

formed by X and Y contains only vertices of degree 4. Figure 18 illustrates this definition. ∎

Remark. $Y = s'_1, s'_2$ is a cut if neither of the two line segments s'_1 and s'_2 passes through a non-trivial face. Furthermore, the rectangle formed by X and Y contains only vertices of degree 4 iff v and w are the only vertices on the boundary of a non-trivial face which lie inside the rectangle.

LEMMA 12.   *Let P be a 1-even bounded LRP.*

(a)   *If there is an oversaturated cut then there is an oversaturated 1-bend cut.*

(b)   *If there is an oversaturated cut then there is an oversturated 0-bend cut or an oversaturated 1-bend cut connecting two concave corners.*

(c)   *If there is a saturated cut $X = s_1, ..., s_k$, $k \geqslant 3$, through edge e^* then there either exists an oversaturated 1-bend cut (not necessarily through e^*) or a saturated 1-bend cut $X = s'_1, s'_2$ through e^* with s'_1 longer than s_1.*

Proof. (a)   Let us assume the existence of an oversaturated cut. Among the oversaturated cuts let $X = s_1, s_2, ..., s_k$ be such that k is minimal and such that among the cuts with minimal k the length of s_1 is maximal and among the cuts with minimal k and maximal s_1 the total length of X is minimal. Let us assume for the sake of a contradiction that X is not a 1-bend cut, i.e., $k \geqslant 3$.

CLAIM 1.   *There is no segment s_i, $2 \leqslant i < k$, such that s_{i-1} and s_{i+1} lie on the same side of the line L_i supporting s_i.*

Proof.   Let $C = \{i; s_{i-1}$ and s_{i+1} lie on the same side of the line L_i supporting $s_i\}$. For $i \in C$ call the side of L_i on which s_{i-1} and s_{i+1} lie the crucial side of s_i. We show first that there is an $i \in C$ such that there is no segment s_j which intersects a square one unit away from s_i and on the crucial side of s_i.

Assume otherwise. Let $i^* \in C$ be such that s_{i^*} has the minimal length of

any segment in C. We may assume w.l.o.g. that s_{i*} runs vertically and that the left side of s_{i*} is crucial. By the choice of s_{i*} only the segments s_1 and s_k can intersect a square one unit to the left of s_{i*}. Let us assume w.l.o.g. that s_1 does. Then s_2 must extend to the left as seen from the common endpoint of s_1 and s_2. Hence there must be $j, l \in C$ such that s_j runs vertically and the right side of s_j is crucial and s_l runs horizontally. Let j^*, $l^* \in C$ be such that s_{j*} runs vertically, its right side is crucial and s_{j*} has minimal length and s_{l*} runs horizontally and has minimal length. Then either s_{j*} or s_{l*} must have the property claimed.

At this point we have the existence of an $i \in C$ such that no segment s_j intersects a square one unit away from s_i and on the crucial side of s_i. Assume w.l.o.g. that s_i extends vertically and that the left side of s_i is crucial.

We now move s_i to the left (note that each move by one unit decreases the capacity without changing the density and hence leaves us with an oversaturated cut) until either s_{i-1} or s_{i+1} becomes empty (contradiction to the minimality of k) or there are boundary points immediately to the left of s_i (this might be the case already initially). We now have to distinguish four cases, according to the lengths of the segments s_{i-1} and s_{i+1}. We only treat the case that s_{i-1} and s_{i+1} have both length at least two and leave the other cases to the reader.

The boundary vertices to the left of s_i lie in $h \geq 1$ segments as shown in Fig. 19. We consider cuts $X_1, ..., X_{h+1}$ shown in the right part of Fig. 19. Then cuts $X_2, ..., X_h$ are 1-bend cuts and X_1 and X_{h+1} have less bends than X. Thus $\mathrm{fcap}(X_i) \geq 0$ for $1 \leq i \leq h+1$. Let l_i be the number of vertices in the segment between a_i and b_i inclusive, $1 \leq i \leq h$. Note that $\deg(a_i) = \deg(b_i) = 4$ and hence $\mathrm{ter}(a_i) = \mathrm{ter}(b_i) = 0$. We have

$$\mathrm{cap}(X) = \mathrm{cap}(X_1) + \cdots + \mathrm{cap}(X_{h+1}) + \sum_{i=1}^{h} (l_i - 2) + 2$$

and

$$\mathrm{dens}(X) \leq \mathrm{dens}(X_1) + \cdots + \mathrm{dens}(X_{h+1}) + \sum_{i=1}^{h} (l_i - 2)$$
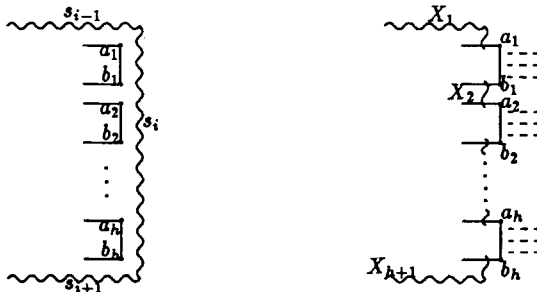
FIGURE 19
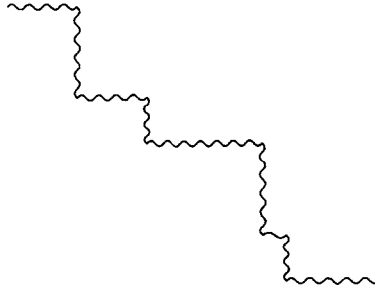
FIGURE 20

and hence

$$\mathrm{fcap}(X) > \mathrm{fcap}(X_1) + \cdots + \mathrm{fcap}(X_h) + \mathrm{fcap}(X_{h+1})$$
$$\geqslant 0,$$

a contradiction. ∎

We have now shown that an oversaturated cut with a minimal number of bends has the form of a staircase (cf. Fig. 20). We may assume that s_1 is horizontal, s_2 is vertical, starts at the right end of s_1 and extends downwards; s_3 extends to the right.

Assume first that s_2 intersects no horizontal edge whose right endpoint lies on the boundary of a face in $\bar{M}$. We can then move s_2 one unit to the right and obtain a cut X' with $\mathrm{fcap}(X') = \mathrm{fcap}(X)$ and a longer initial segment, a contradiction. Thus there are boundary points immediately to the right of s_2. Let w be the lowest such boundary point. Then either the edge e_1 directed upwards from w or the edge e_2 directed downwards from w is a boundary edge.
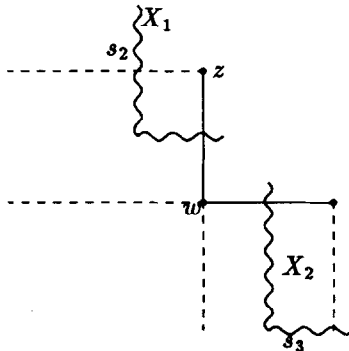
FIGURE 21
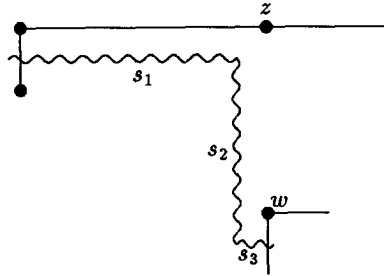
FIGURE 22

Case 1. Edge e_2 is not a boundary edge. Then $e_1 = (w, z)$ must be a boundary edge. We consider the two cuts shown in Fig. 21. Note that X_2 exists, since w was chosen as the lowest boundary point to the right of s_2. We have

$$\text{cap}(X) = \text{cap}(X_1) + \text{cap}(X_2)$$

and

$$\text{dens}(X) \leqslant \text{dens}(X_1) + \text{dens}(X_2),$$

since vertex w has degree 4, and hence $\text{ter}(w) = 0$. Also X_1 has no more bends than X, has the same initial segment, and is shorter than X_1, and X_2 has one bend less than X. Thus $\text{fcap}(X_i) \geqslant 0$ for $i = 1, 2$, and hence

$$\text{fcap}(X) \geqslant \text{fcap}(X_1) + \text{fcap}(X_2)$$
$$\geqslant 0,$$

a contradiction.

Case 2. The edge e_2 is a boundary edge and hence s_3 cuts only one edge. If edge e_1 is also a boundary edge then we can certainly shorten X

FIGURE 23

FIGURE 24

and still have an oversaturated cut, a contradiction. So let us assume that edge e_1 is not a boundary edge. Let z be the boundary point which lies above w and is closest to w. Then z either lies above the horizontal line supporting s_1 or it does not.

Case 2.1. z lies in the top row of R, cf. Fig. 22. Consider cut Y as shown in Fig. 23. Then $\text{dens}(X) = \text{dens}(Y)$, since $\text{ter}(w) = 0$ (note that $\deg(w) = 4$) and $\text{cap}(X) = \text{cap}(Y)$. Thus Y is oversaturated and a 1-bend cut, a contradiction.

Case 2.2. z does not lie above the horizontal line supporting s_1 (cf. Fig. 24). Consider cuts X_1 and X_2 as indicated in Fig. 25. We have $\text{cap}(X) = \text{cap}(X_1) + \text{cap}(X_2)$ and $\text{dens}(X) \leqslant \text{dens}(X_1) + \text{dens}(X_2)$. Also X_2 is 1-bend and X_1 has as many bends as X, the same initial segment and is shorter. Thus $\text{fcap}(X_i) \geqslant 0$ for $i = 1, 2$, and hence $\text{fcap}(X) \geqslant 0$, a contradiction.

FIGURE 25

(a) The case $v = w$ (b) Cuts X_1 and X_2

FIG. 26.   Cuts X_1 and X_2.

(b)   Let us assume that there is no saturated 0-bend cut or over-saturated 1-bend cut connecting two concave corners but there is an over-saturated 1-bend cut. Let $X = s_1, s_2$ be an oversaturated 1-ben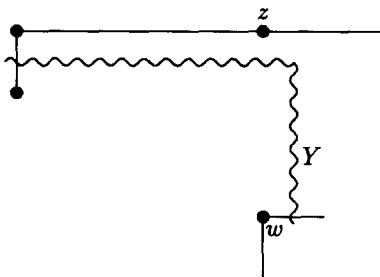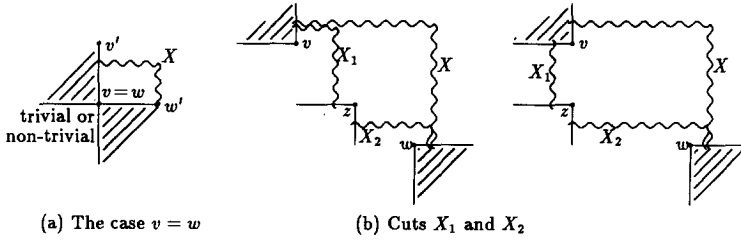d cut of minimal length. We may assume w.l.o.g. that s_1 is horizontal, s_2 is vertical, and the upper endpoint of s_2 and the right endpoint of s_1 coincide. Let (v, v') be the vertical boundary edge intersected by s_1 with v below v' and let (w, w') be the horizontal boundary edge intersected by s_2 with w left of w'.

Assume first that $v = w$. Then $cap(X) = 2$ and $dens(X) \geq 3$. Since $\deg(v) = 4$ ($\deg(v) = 2$ implies $dens(X) \leq ter(v) \leq \deg(v)$ and thus is impossible; $\deg(v) = 3$ implies the existence of an 0-bend cut of capacity 1, and hence density 1, and thus is impossible) and since X is not a 0-bend, the vertex v must lie on the boundary of at least two non-trivial faces, cf. Fig. 26a. Note that these faces are not necessarily distinct. Form $Y = s_1', s_2'$ as described in Definition 5. If Y is a cut, i.e., v lies on the boundary of exactly two non-trivial faces, then all conditions of Definition 5 are satisfied and X connects two concave corners. If Y is not a cut, i.e., v lies on the boundary of three non-trivial faces, then consider the two 0-bend cuts $Y_1 = s_1'$ and $Y_2 = s_2'$. We have $dens(Y_1) + dens(Y_2) = dens(X) \geq 3$, since $ter(v) = 0$ and hence the existence of an oversaturated 0-bend cut. This completes the discussion of the case $v = w$.

Assume next that $v \neq w$. The cut X starts in some non-trivial face, say F. Let (v, v'') be the other boundary edge of F incident to v. Define edge (w, w'') analogously. We observe first $((v', v), (v, v''))$ and $((w', w), (w, w''))$ are concave corners. Otherwise we could obtain shorter saturated cuts by arguments as in Claims 1 and 2 of part (a). Let $Y = s_1', s_2'$ be defined as in Definition 5. If Y is not a cut or if Y is a cut and the rectangle formed by X and Y contains a vertex of degree less than four then the rectangle formed by X and Y must contain a boundary point different from v and w. Let z be such a boundary point with maximal y-coordinate and, among such points, with maximal x-coordinate. Then $\deg(z) = 4$ and hence $ter(z) = 0$. Consider the cuts X_1 and X_2 shown in Fig. 26b (the case that the y-coordinate of z lies strictly between the y-coordinates of v and w is shown

in that figure. The case that z has the same x-coordinate as v and the same y-coordinate as w is similar). Both of them are 1-bend and shorter than X. Hence $\mathrm{fcap}(X_i) \geqslant 0$ for $i = 1, 2$. Also, $\mathrm{fcap}(X) \geqslant \mathrm{fcap}(X_1) + \mathrm{fcap}(X_2) \geqslant 0$, a contradiction. This proves part (b).

(c) Among the saturated cuts through e^* let $X = s_1, ..., s_k$ be such that k is minimal and such that among the cuts with minimal k the length of s_1 is maximal and among the cuts with minimal k and maximal s_1 the total length of X is minimal, cf. part (a). Let us assume also that there is no oversaturated cut. We will now argue almost as in part (a). The differences are as follows. In Claim 1, from the fact that $0 > \mathrm{fcap}(X) > \mathrm{fcap}(X_1) + \cdots + \mathrm{fcap}(X_{n+1})$ we conclude the existence of an oversaturated cut. In Claim 2, Case 1, we infer that X_1 is saturated, contradicting the choice of X. In Case 2.1, we conclude that Y is saturated and has the desired form, and in Case 2.2, we conclude that X_1 is saturated, contradicting the choice of X. ∎

III.4. *Proofs of Lemmas 2 to 5*

In this section we will finally prove the correctness of our algorithm by filling in the proofs of Lemmas 2 to 5. Throughout this section we will use the following convention on notation. P, P', and P'' are local routing problems. If X is a cut then $\mathrm{dens}(X)$, $\mathrm{dens}'(X)$, and $\mathrm{dens}''(X)$ denote the density of X with respect to P, P', and P'', respectively. Similar conventions are used for the capacity and free capacity.

LEMMA 2. *The then-case of procedure Simplify1 turns a good LRP into a good LRP.*

Proof. Let P be a good LRP and let X be the cut of capacity one which was chosen in the then-case. Let P' be the modified problem. Then $\mathrm{cap}'(Y) = \mathrm{cap}(Y)$ and $\mathrm{dens}'(Y) = \mathrm{dens}(Y)$ for all cuts $Y \neq X$. This shows that P' is 1-even and satisfies the cut condition for all 1-bend cuts. Also a solution to P' directly yields a solution for P as argued in Section II and P' is bounded. Thus P' is good. ∎

LEMMA 3. *The else-case of procedure Simplify1 turns a good LRP into a good LRP.*

Proof. Let P be a good LRP having no cut with capacity one and let v be a vertex with $\deg(v) = \mathrm{ter}(v) = 2$. Let e_i, $1 \leqslant i \leqslant 2$, be the edges incident to v and let N_i, $1 \leqslant i \leqslant 2$, be the nets incident to v. The numbering is as defined in Section II. Let $N_1 = [e_1 p_1]$, where p_1 does not use edge e_1. It was argued in Section II that such a representation of net N_1 exists. We obtain problem P' be removing edge e_1 and replacing net N_1 by net $[p_1]$.
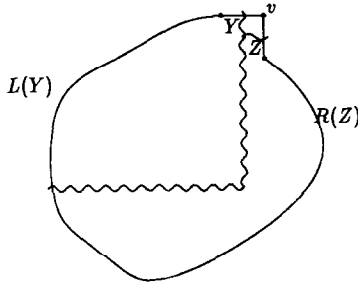
FIGURE 27

In order to show that P' is good we consider an intermediate problem P''. P'' is obtained from problem P by replacing net N_1 by nets $[e_1]$ and $[p_1]$ and leaving the routing region unchanged. It is clear that P' is good if P'' is good, since P' results from P'' by removing net $[e_1]$ and edge e_1. It therefore suffices to show that P'' is 1-even and satisfies the cut condition for 1-bend cuts.

Let Y be any 1-bend cut. If Y does not go through edge e_1 then $\mathrm{fcap}''(Y) = \mathrm{fcap}(Y) \in 2 \cdot \mathbb{N}_0$ and we are done. So let us assume that Y goes through edge e_1. Then $\mathrm{cross}([p_1], Y) = \mathrm{cross}(N_1, Y) + \varepsilon$ with $\varepsilon \in \{+1, -1\}$ and hence $\mathrm{fcap}''(Y) = \mathrm{fcap}(Y) - \varepsilon - 1$. We conclude that P'' is 1-even and $\mathrm{fcap}''(Y) \in 2 \cdot \mathbb{N}_0$ if either $\varepsilon = -1$ or $\mathrm{fcap}(Y) > 0$. This leaves the case $\mathrm{fcap}(Y) = 0$ and $\mathrm{cross}([p_1], Y) = \mathrm{cross}(N_1, Y) + 1$. Let Z be the cut obtained from Y by replacing the dual edge $d(e_1)$ by the dual edge $d(e_2)$, cf. Fig. 27. Then $\mathrm{cap}(Z) = \mathrm{cap}(Y)$.

We claim that $\mathrm{dens}(Z) = \mathrm{dens}(Y) + 2$ and hence $\mathrm{fcap}(Z) = \mathrm{fcap}(Y) - 2 = -2$. Thus Z is oversaturated and hence the problem P was not good by Lemma 12(a). It remains to show that $\mathrm{dens}(Z) = \mathrm{dens}(Y) + 2$. Let S be any slicing such that S, Y, and Z is strongly interferencefree and let $\alpha_{i1}, ..., \alpha_{ik_i}$ be a decomposition of N_i with respect to S, $1 \leqslant i \leqslant 2$. Define partitions $(L(Y), R(Y))$ and $(L(Z), R(Z))$ of $B \cup C$ such that $v \in R(Y) \cap L(Z)$. Then the cuts Y and Z partition $B \cup C$ into the three sets $\{v\}$, $L(Y)$, $R(Z)$, cf. Fig. 27. Also, $\alpha_{11} = (v, w_1)$, where $w_1 \in R(Z)$, since $\mathrm{cross}([p_1], Y) = \mathrm{cross}(N_1, Y) + 1$. Next observe that N_1 is right-of N_2 and hence $\alpha_{21} = (v, w_2)$, where $w_2 \in R(Z)$. This implies $\mathrm{dens}(Z) = \mathrm{dens}(Y) + 2$. ∎

LEMMA 4. *The else-case of procedure Simplify2 maintains the invariant.*

Proof. Let P be a reduced good routing problem and let $e^* = (a, b)$ be as defined in Section II. In the else-case there is no saturated cut through edge e^*. We obtain problem P' by deleting the boundary cycle of the trivial face to the right of e^* (cf. Fig. 7). We need to show that P' is good. Let Y

be any 1-bend cut. If Y does not start in any of the new dual sources then $\text{fcap}'(Y) = \text{fcap}(Y)$ and we are done. If Y does start in one of the new dual sources then Y can be extended to a cut Z through e^* and to a 1-bend cut Z' in problem P with $\text{cap}(Z) = \text{cap}(Z') = \text{cap}'(Y) + 2$ and $\text{dens}(Z) = \text{dens}(Z') = \text{dens}'(Y)$. Thus $\text{fcap}'(Y) = \text{fcap}(Z) - 2 = \text{fcap}(Z') - 2$. Since P is 1-even, $\text{fcap}(Z')$ and hence $\text{fcap}'(Y)$ are even. Also, by Lemma 12(c) there is no saturated cut through edge e^* and hence $\text{fcap}(Z) > 0$. This implies $\text{fcap}'(Y) \geqslant -1$ and since $\text{fcap}'(Y)$ is even, also $\text{fcap}'(Y) \geqslant 0$. Thus P' is good. ∎

LEMMA 5.   *The then-case of procedure Simplify2 maintains the invariant.*

Proof. Let P be a reduced good routing problem and let $e^* = (a, b)$ be as defined in Section II. Let X be the leftmost saturated 1-bend cut through edge e^* and let $([p_1], [p_2])$ be the rightmost decomposition of any net with repsect to X. We obtain the problem P' from the problem P by replacing the net $N = [p_1 p_2]$ by the nets $[p_1]$ and $[p_2]$. It is clear (cf. Section II) that a solution for P can be obtained from a solution for P'. It remains to show that P' is 1-even and satisfies the cut condition for 1-bend cuts. Note that since P is reduced we have $\text{ter}(a) = \text{ter}(c) = 0$ and $\text{ter}(v) = 1$ for all other vertices v between a and c in the top row. While proceeding from P to P' we change the ter-value only for vertex a, i.e., $\text{ter}'(a) = 2$, $\text{ter}'(c) = 0$, and $\text{ter}'(v) = 1$ for all other vertices v in the top row.

Let Y be any 1-bend cut. Let $q_i \in [p_i]$ be such that $\text{cross}(q_i, Y) = \text{cross}([p_i], Y)$. Then $q_1 q_2 \in [p_1 p_2]$ and hence $\text{cross}(q_1, Y) + \text{cross}(q_2, Y) = \text{cross}(q_1 q_2, Y) \bmod 2 = \text{cross}([p_1 p_2], Y) \bmod 2$ by Lemma 7. This shows that the problem P' is 1-even.

We turn to the cut condition next. Let us assume for the sake of a contradiction that the 1-bend cut Y is oversaturated, i.e., $\text{fcap}'(Y) < 0$. We will next show in a series of lemmas that there is also an oversaturated cut of a very restricted form.

LEMMA 13.   *There is an oversaturated cut (in P') which is either 0-bend or connects two concave corners. Furthermore the cut does not go through edge e', where $e' = (c, d)$ is vertical and c has maximal x-coordinate among all vertices conected to a by horizontal edges; cf. Fig. 28.*

Proof. The first claim follows immediately from Lemma 12(b) and our assumption that there is an oversaturated cut. So let us assume that the
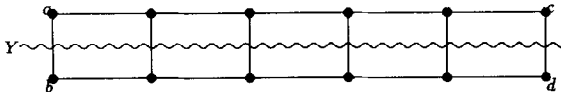

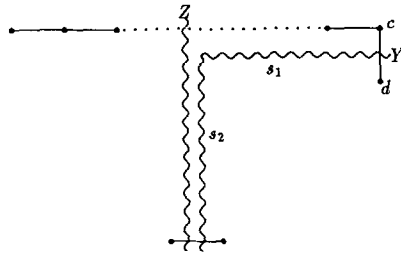FIG. 28.   Vertices a, b, c, d and a 0-bend cut Y.

FIG. 29.   The cuts Y and Z.

oversaturated cut Y is either a 0-bend cut or a 1-bend cut which connects two concave corners and that Y goes through edge e'. In the former case, we have ter'$(a) = 2$, ter'$(c) = 0$, and ter'$(v) = 1$ for all other vertices v between a and c in the top row. Hence fcap'$(Y) \geqslant 0$, cf. Fig. 28. In the latter case Y consists of a horizontal segment s_1 through edge e' and a vertical segment s_2 directed downwards. We claim that the 0-bend cut Z obtained from s_2 by extending s_2 by one edge at its top is also oversaturated, cf. Fig. 29. Let l be the length of segment s_1. Then dens'$(Z) \geqslant $ dens'$(Y) - (l-1)$, since all vertices v above s_1 except c have ter$(v) = 1$ and ter$(c) = 0$. Also cap$(Z) = $ cap$(Y) - l + 1$ and hence fcap'$(Z) = $ fcap'$(Y) < 0$. This shows that we have a cut of the desired form which does not go through edge e'. ∎

By Lemma 13 we may assume that the oversaturated cut Y does not go through edge e' and is either a 0-bend cut or a 1-bend cut which connects two concave corners. In the latter case the cut Y cannot intersect the top row, since no edge of the top row participates in a concave corner.

LEMMA 14.   *X and Y do not interfere.*

Proof. We show that X and Y satisfy the hypothesis of Lemma 8(a). If Y has a vertex in common with the horizontal segment of X then Y must be either 0-bend or must go through edge e^* (note that Y does not go through e' and is 0-bend if it intersects the top row.). In either case the hypothesis of Lemma 8(a) is clearly satisfied. If Y does no have a vertex in common with the horizontal part of X then the hypothesis is also satisfied. So X and Y do not interfere. ∎

Let $q \in [p_1 p_2]$ be reduced with respect to X and Y. Write $q = e_1 e_1 q_2$, where q_1 is the maximal prefix of q such that cross$(q_1, X) = $ cross$([p_1], X)$. Then the edge $e_1 = (v, w)$ crosses the cut X. Let r_1 be the path from v to a which runs along X and let r_2 be the path from b to w which runs along X, cf. Fig. 30. Then $[p_1] = [q_1 r_1]$ and $[p_2] = [e^* r_2 q_2]$. Since X and Y do

FIGURE 30

not interfere, the path r_i, $i = 1, 2$, can cross Y at most once. Thus $\text{cross}([p_1], Y) + \text{cross}([p_2], Y) \leqslant \text{cross}(q_1 r_1 e^* r_2 q_2, Y) \leqslant \text{cross}(q_1 e_1 q_2, Y) + 3$ $= \text{cross}(q, Y) + 3$ and, since $q_1 r_1 e^* r_2 q_2$ and q are equivalent, we even have $\text{cross}(q_1 r_1 e^* r_2 q_2, Y) \leqslant \text{cross}(q, Y) + 2$ by Lemma 7. This proves $\text{fcap}'(Y) \geqslant \text{fcap}(Y) - 2$ and hence $\text{fcap}(Y) = 0$, $\text{fcap}'(Y) = -2$. Recall that we assumed $\text{fcap}'(Y) < 0$. Also the cuts X and Y cannot be vertex-disjoint because then $\text{cross}(q_1 r_1 e^* r_2 q_2, Y) = \text{cross}(q_1 e_1 q_2, Y)$ and hence $\text{fcap}'(Y) = \text{fcap}(Y) \geqslant 0$.

LEMMA 15. *X and Y have exactly one vertex in common.*

Proof. Let us assume that X and Y have more than one vertex in common. Assume first that Y does not go through e^*. Since X and Y do not infer, we conclude that the vertical segments of X and Y have a common tail. Also at most one of the paths r_1 and r_2 can intersect Y and none of them does if e_1 does not intersect Y. Hence $\text{cross}(q_1 r_1 e^* r_2 q_2, Y) \leqslant \text{cross}(q_1 e_1 q_2, Y)$ and hence $\text{fcap}'(Y) \geqslant \text{fcap}(Y)$, a contradiction.

Let us assume next that Y goes through edge e^* and that the horizontal segment of Y is not shorter than the horizontal segment of X. Then either

FIG. 31. The paths r_1, r_2, and r_3.

$X = Y$ and hence $\text{fcap}'(Y) = \text{fcap}(Y)$, a contradiction, or X is not the leftmost saturated cut, a contradiction. This leaves the case that Y goes through e^* and has a shorter horizontal segment than X. Then $\text{cross}(q_1 r_1 e^* r_2 q_2, Y) = \text{cross}(q, Y) + 2$ is only possible if the edge $e_1 = (v, w)$ does not cross Y and hence the path r_2 does. Consider the path r_3 shown in Fig. 31. It connects a and w and does not cross Y. Also $[p_1] = [q_1 r_1]$ and $[p_2] = [r_3 q_2]$ and hence $\text{cross}([p_1], Y) + \text{cross}([p_2], Y) \leqslant \text{cross}(q_1 r_1 r_3 q_2, Y) = \text{cross}(q, Y)$. Thus $\text{fcap}'(Y) = \text{fcap}(Y)$, a contradiction. ∎

At this point we severely restricted the shape of the cut Y: Y has exactly one vertex v in common with X and X and Y do not interfere.

Let S be a slicing such that S, X, and Y are strongly interferencefree. Note that S exists because Y is either 0-bend or 1-bend connecting two concave corners, and hence every face can be connected by a sequence of 0-bend cuts with the unbounded face. We split X at vertex v into pieces X_1 and X_2 and Y at vertex v into pieces Y_1 and Y_2, where X_1 goes through edge e^* and X_1, Y_1, X_2, Y_2 occur clockwise around v. Let $Z_1 = X_1 Y_1$ and $Z_2 = X_2 Y_2$ and let $(L(X), R(X))$ and $(L(Y), R(Y))$ be the partitions of $B \cup C$ induced by the cuts X and Y, where $a \in L(X) \cap L(Y)$, cf. Fig. 32. Then

$$
\begin{aligned}
\text{dens}(X) + \text{dens}(Y) &= \overline{\text{dens}}(L(X)) + \overline{\text{dens}}(L(Y)) \\
&= \overline{\text{dens}}(L(X) \cap L(Y)) + \overline{\text{dens}}(L(X) \cup L(Y)) \\
&\quad + 2\,\overline{\text{dens}}(L(X) - L(Y), L(Y) - L(X)) \\
&= \text{dens}(Z_1) + \text{dens}(Z_2) \\
&\quad + 2\,\overline{\text{dens}}(L(X) - L(Y), L(Y) - L(X)).
\end{aligned}
$$

Suppose now that $\overline{\text{dens}}(L(X) - L(Y), L(Y) - L(X)) = 0$. Then $\text{fcap}(Z_1) + \text{fcap}(Z_2) = \text{fcap}(X) + \text{fcap}(Y) = 0$ and thus either $\text{fcap}(Z_1) \leqslant 0$ or
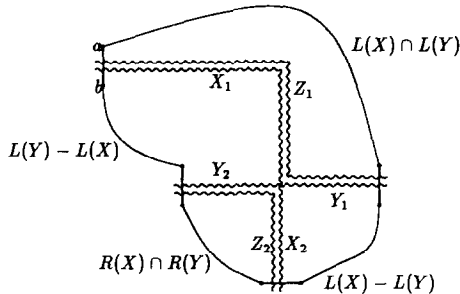

FIG. 32. The cuts X, Y, Z_1, and Z_2.

$fcap(Z_2) < 0$. In the former case there is a saturated 1-bend cut through e^* with a longer horizontal segment than X by Lemma 12(c), a contradiction to the choice of X; and in the second case we have a contradiction to the invariant by Lemma 12(a).

It remains to show that $\overline{\text{dens}}(L(X) - L(Y), L(Y) - L(X)) = 0$. Assume otherwise. Let (x, y) be an elementary piece of some net Q with $x \in L(X) - L(Y)$ and $y \in L(Y) - L(X)$. We can decompose Q with respect to vertex a into $([q_1], [q_2])$, where (a, y) is the first elementary piece of $[q_2]$. Let (a, z) be the first, elementary piece of $[p_2]$. Since r_1 and r_2 intersect Y (recall that $fcap'(Y) = -2$) we must have $z \notin L(Y) - L(X)$. Thus decomposition $([q_1], [q_2])$ of Q is *right-of* decomposition $([p_1], [p_2])$ of N, a contradiction to the choice of N. This completes the proof of Lemma 5. ∎

IV. IMPLEMENTATION

The input to our routing algorithm is a routing region R and a set $\mathcal{N}$ of nets. The routing region is given by its set of vertices and edges and their embedding into the plane and each net $N \in \mathcal{N}$ is given by some representative $rep(N) \in N$. We use r to denote the number of vertices of R and m to denote the total length of the representatives $rep(N)$. Then $r + m = n$.

In the implementation we represent nets by their decomposition with respect to a particular slicing S which we define next. For each face $F \in M$, $C(F)$ is a 0-bend vertical cut extending downwards from F. This choice ensures that S and X are vertex-disjoint for any 1-bend cut X through edge e^*.

The implementation is a fairly direct realization of the algorithm of Section II. The major question left open there is the choice of the data structures used to represent nets. We use the following data structures:

(1)   Each net is represented in both its orientations by the linear list of its elementary pieces. The two occurrences of any elementary piece in its two orientations are linked.

(2)   For each element $v \in B \cup C$ we have a linked list of all occurrences of elementary pieces (v, w).

LEMMA 16.   (a)   *The data structures can be constructed in time $O(n)$ from the input. Moreover, if the problem is solvable then their space requirement is $O(r)$.* (b) *The cyclic ordering on $B \cup C$ can be computed in time $O(r)$.*

Proof.   (a)   Color the edges of R which are intersected by some cut in S red and color all other edges black. Then trace the representatives

rep(N), $N \in \mathcal{N}$, and decompose them into elementary pieces. Finally, reduce the decomposition by the operation:

Replace(v, $C(F)^d$), ($C(F)^{-d}$, $C(F)^{-d}$), ($C(F)^d$, w) by (v, w);

i.e., eliminate unnecessary crossings with slicing cuts. All of this takes time $O(r + m) = O(n)$. For net $N \in \mathcal{N}$ let $l(N)$ be the number of elementary pieces in the decomposition of N. Clearly, every path $p \in N$ has length at least $l(N)$ and hence $\sum \{l(N); N \in \mathcal{N}\} = O(r)$ if the problem is solvable.

(b)   In order to compute the cyclic ordering on $B \cup C$ we remove all edges intersected by a cut in S (the red edge of part (a)) and then perform a clockwise traversal of the boundary of the unbounded face. All of this clearly takes time $O(r)$. ∎

We argued already that there are at most $O(r)$ calls to procedures Simplify1 and Simplify2. We will show below that a call Simplify1 has cost $O((1 + k)r)$, where k is the number of edges of R which are removed by the call and that a call of Simplify2 has cost $O(r)$. We will also show that a call of Simplify1 increases the number of elementary pieces by at most k and a call of Simplify2 increases the number of elementary pieces by at most one. Thus the number of elementary pieces is always $O(r)$. Note that it is $O(r)$ initially by Lemma 16, if the problem is solvable. Thus total running time is $O(n + r^2) = O(n^2)$ and Theorem 1(b) is shown.

LEMMA 17.   *A call of Simplify1 has cost $O((1 + k)r)$, where k is the number of edges of R which are removed by the call. Also the number of elementary pieces is increased by at most k.*

Proof.   We first check whether there is a cut of capacity one (by inspecting all dual edges) or a vertex v with $\deg(v) = \text{ter}(v)$ (by inspecting all vertices). This takes clearly time $O(r)$. Assume first that there is a cut, say X, of capacity one. Let us assume further that X is not a slicing cut; the case that X is a slicing cut is simpler and therefore left to the reader. We show first how to find in time $O(r)$ the elementary piece which goes across X. The cut X divides $B \cup C$ into parts $L(X)$ and $R(X)$. In time $O(r)$ we can clearly label the elements of $L(X)$ red and the elements of $R(X)$ black. We then only have to run through all elementary pieces and check the color of the endpoints until a bicolored piece is found. We next split this elementary piece at the edge intersected by X and remove this edge, say f, from the routing region. The edge f separated two faces F and G. If $F = G$ then we are done. If $F \neq G$ then we may assume w.l.o.g. that F is not an ancestor of G with respect to the function *parent* (cf. Definition 4 in Section III.2.). We have to remove the cut $C(F)$ from our set of slicing cuts and hence to change the decomposition into elementary pieces of all nets. This can be

done easily in time $O(r)$. We only have to replace all pairs $(x, C(F)^d)$, $(C(F)^{-d}, y)$ of elementary pieces by the single elementary piece (x, y).

Assume next that there is no cut of capacity one but a vertex v with $\deg(v) = \text{ter}(v)$. We first number the elements of $B \cup C$ in counterclockwise order starting at some arbitrary vertex. Using this numbering one can decide the cyclic ordering of three elements u, v, w of $B \cup C$ in time $O(1)$. It is now easy to find the rightmost net starting in v by tracing the nets and always selecting the rightmost possible continuation. Again this clearly takes time $O(r)$. ∎

LEMMA 18. *A call of Simplify2 takes time $O(r)$. Also the number of elementary pieces is increased by at most one.*

Proof. Let X_i be the 1-bend cut through edge e^* where the horizontal segment has length exactly i. We can clearly compute $\text{cap}(X_i)$ for all i in total time $O(r)$. We will next show how to compute $\text{dens}(X_i)$ for all i in time $O(r)$. Number $B \cup C$ in counterclockwise order starting at vertex b. Let (L_i, R_i) be the partition of $B \cup C$ induced by X_i with $b \in L_i$. Then $L_i \subseteq L_{i+1}$ for all i and hence

$$
\begin{aligned}
\text{dens}(X_{i+1}) &= \overline{\text{dens}}(L_{i+1}, R_{i+1}) \\
&= \overline{\text{dens}}(L_i \cup (L_{i+1} - L_i), R_{i+1}) \\
&= \overline{\text{dens}}(L_i, R_{i+1}) + \overline{\text{dens}}(L_{i+1} - L_i, R_{i+1}) \\
&= \overline{\text{dens}}(L_i, R_i - (L_{i+1} - L_i)) + \overline{\text{dens}}(L_{i+1} - L_i, R_{i+1}) \\
&= \text{dens}(X_i) - \overline{\text{dens}}(L_i, L_{i+1} - L_i) + \overline{\text{dens}}(L_{i+1} - L_i, R_{i+1}).
\end{aligned}
$$

Thus $\text{dens}(X_{i+1})$ can be computed from $\text{dens}(X_i)$ in time proportional to the number of elementary pieces incident to $L_{i+1} - L_i$. Note that using the numbering of $B \cup C$ one can decide in time $O(1)$ whether an element of $B \cup C$ belongs to L_i or R_{i+1}. Thus $\text{dens}(X_2)$, $\text{dens}(X_3)$, ... can be computed in time $O(r)$ once $\text{dens}(X_1)$ is known. Finally $\text{dens}(X_1)$ can be computed in time $O(r)$ by checking all elementary pieces.

We have now computed $\text{fcap}(X_i)$ for all i in time $O(r)$. If no X_i is saturated then we are almost done. We only have to remove the four edges on the boundary cycle of the trivial face to the right of e^*. Also if this merges two nontrivial faces then we have to remove a slicing cut as described in the proof of Lemma 17. Otherwise let X be the leftmost saturated cut through e^*. We first find all α's going across X in time $O(r)$ and then trace nets starting with these pieces in order to find the rightmost decomposition. This is done as follows: Let (v_1, w_1), (v_2, w_2), ... be the pieces which go across X with $v_j \in R(X)$, $w_j \in L(X)$. These pieces can be found in time $O(r)$ by first coloring the elements of $L(X)$ red and the

elements of $R(X)$ black and then checking all elementary pieces. Each such piece (v_i, w_i) defines a decomposition (p_i, q_i) of some net with respect to the cut X. For two decompositions (p_i, q_i) and (p_j, q_j) we can clearly determine which one is right-of the other one in time $\min(l(q_i), l(q_j))$ by simply tracing q_i and q_j. Here, $l(q_i)$ is the number of elementary pieces q_i consists of. Thus the rightmost decomposition can be computed in time $O(\sum l(q_i)) = O(r)$. ∎

V. CONCLUSION

We showed that the local routing problem for two-terminal nets is solvable in quadratic time. Recently, an alternative and considerably more complex algorithm with linear running time $O(n)$ has been found; cf. the first author's Ph.D. thesis [K] and [KM2]. In [K] it has also been shown that the technique described above can be applied almost directly to other grids like the hexagonal grid and the octo-square grid, which are based on the square grid extended by diagonals in one or both directions. There are three major open problems:

(1) Allow more general routing regions than grid graphs. In particular, prove a similar result for arbitrary planar graphs.

(2) Extend the result to nets with more than two terminals. It is known [S], that multiterminal net routing is NP-complete. An approach for the multiterminal net routing might be to partition each multiterminal net into a collection of two-terminal nets and then to use the result in this paper as a heuristic. To find a good approximation algorithm is still open.

(3) Construct layouts which are provably 3-layer wirable. Using Brady and Brown's result we can only guarantee a 4-layer wiring. But for very simple routing regions (channels) [PL] we know how to construct layouts which are 3-layer wirable.

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