On the general Kerr/CFT correspondence in arbitrary dimensions

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ABSTRACT: We study conformal symmetries on the horizon of a general stationary and axisymmetric black hole. We find that there exist physically reasonable boundary conditions that uniquely determine a set of symmetry generators, which form one copy of the Virasoro algebra. For extremal black holes, Cardy's formula reproduces exactly the Bekenstein-Hawking entropy.

Contents

1   Introduction

There is an intriguing possibility that quantum gravity on the horizon of a black hole could be dual to some 2D conformal filed theory (CFT) at finite temperatures. A key evidence is that, following [1], one can find appropriate boundary conditions which allow for asymptotic conformal symmetries on the horizon [2–5]. In the case of extremal black holes, it is possible to study the relevant conformal symmetries in a systematic fashion [6], based on the idea of the Kerr/CFT correspondence [5]. For non-extremal black holes, it appears more difficult to study (or identify) the relevant conformal symmetries directly. In [7] it has been proposed to use a probing scalar field to extract useful information about the possible hidden conformal symmetries in the Kerr background. The method has been further developed and applied to several other cases (see, e.g. [8] and references therein).

Such activity has also generated renewed interest in the earlier effort [3, 4, 9] that seeks to study the conformal symmetries on the horizons directly. In particular, it has been shown in [10, 11] that it is possible to use the "stretched horizon" method to re-derive some results known in the usual literature of Kerr/CFT correspondence [5]. What's more, Carlip [10] has shown that in the case when the boundary conditions are the same for both the extremal and non-extremal black holes, there is a Virasoro algebra which reproduces (via the Cardy formula) the full entropy for extremal black holes but only half the entropy for non-extremal ones. And the same paper also shows that an alternative way of stretching the horizon can enable one to obtain the entropy fully. In view of such ambiguities, it will be helpful to study boundary symmetries on the horizon without using an intermediate stretched horizon [10]. Our main purpose of this work is to present such a construction.

We find physically reasonable boundary conditions that uniquely determine a set of symmetry generators, which form one copy of the Virasoro algebra (one for each of the

azimuthal angles). Cardy's formula can then be used to calculate the black hole entropy. Our boundary conditions are directly imposed on the (inverse) metric elements on the horizon, in much the same spirit as Brown and Henneaux [1]. The construction is general and is valid for arbitrary stationary and axisymmetric black holes in arbitrary dimensions. For practical reasons, some of the calculation is only explicitly done for Einstein gravity plus a (possibly zero) cosmological constant. But with enough effort, a generalization to more complicated theories should still be possible.

The main result of the paper is the following:

In section 2, we briefly recall some general features of stationary and axisymmetric black holes, setting the stage for our discussion. In section 3, we explain our boundary conditions, solve for the boundary symmetry generators, showing that they constitute a copy of the Virasoro algebra. In section 4, we calculate the central charge and the Frolov-Thorne temperature for non-extremal black holes. We find that Cardy's formula gives exactly half the Bekenstein-Hawking entropy. In section 5, we do the same for extremal black holes. Here we find that Cardy's formula reproduces the Bekenstein-Hawking entropy fully. We end with a short summary in section 6.

Both the definition of black hole charges and the calculation of central charges are done by using the covariant phase space method, for which we collect some basic formulae in the appendix A.

2 Stationary and axisymmetric black holes

Stationary and axisymmetric black holes constitute the most important class of exact solutions in various gravitational theories. They are the objects that we want to focus on in this paper. For the convenience of later discussions and also to fix our notations, we briefly recall some general features of these black holes. Most result has already appeared in [13], but here we shall explain some of the points in more detail.

In general, a stationary and axisymmetric black hole is characterized by the presence of a time-like Killing vector ∂_t and one (or several) space-like Killing vector(s) ∂_ϕ, where ϕ is periodically identified. Although a general proof is not known, existing examples suggest that all the stationary and axisymmetric black holes share the following form of the metric,[1]

$$ds^2 = f\left[-\frac{\Delta}{v^2}dt^2 + \frac{dr^2}{\Delta} \right] + q_{ij}d\theta^i d\theta^j + g_{ab}(d\phi^a - w^a dt)(d\phi^b - w^b dt)\,, \qquad (2.1)$$

where the coordinates can be identified as t the asymptotic time, r the radial coordinate, θ^i the longitudinal angles and ϕ^a the azimuthal angles. All the functions in (2.1) depend

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[1]A careful check of the general metric against many existing examples can be found in [6]. (The metrics in [6] look slightly different, but it is easy to put them into the form of (2.1).) As a convention for our sub/superscripts, the beginning Latin letters $(a, b, \cdots)$ are only used for the azimuthal angles (e.g., ϕ^a), the middle Latin letters $(i, j, \cdots)$ are only used for the longitudinal angles (e.g., θ^i), and the Greek letters $(\mu, \nu, \cdots)$ are used for all the coordinates, $\mu, \nu, \cdots \in \{r, t, a, i\}$. What's more, it is often convenient to treat the time on the same footing as the azimuthal angles. So we define $w^t = \Omega^t \equiv 1$ and we use capital letter indices $(A, B, \cdots)$ to go over both the azimuthal angles and the time, i.e. $A, B, \cdots \in \{t, a\}$. (In the brackets $\{r, t, a, i\}$ and $\{t, a\}$, we use "a" to represent all indices for the azimuthal angles and "i" to represent all indices for the longitudinal angles.)

on r and θ^i, except for Δ which is only a function of r. The inverse of (2.1) is

$$(\partial_S)^2 = \frac{\Delta}{f}\partial_r^2 + q^{ij}\partial_i\partial_j + g^{ab}\partial_a\partial_b - \frac{v^2}{f\Delta}(\partial_t + w^a\partial_a)(\partial_t + w^b\partial_b)\,, \qquad (2.2)$$

where $\partial_i \equiv \partial_{\theta^i}$ and $\partial_a \equiv \partial_{\phi^a}$. Using $\tilde{g}_{\mu\nu}$ to denote elements of the full metric, we find

$$\tilde{g}_{rr} = \frac{f}{\Delta}\,, \qquad \tilde{g}_{ij} = q_{ij}\,, \qquad \tilde{g}_{ab} = g_{ab}\,, \qquad\qquad \tilde{g}_{at} = -w_a\,, \qquad \tilde{g}_{tt} = -N^2 + w^2\,,$$

$$\tilde{g}^{rr} = \frac{\Delta}{f}\,, \qquad \tilde{g}^{ij} = q^{ij}\,, \qquad \tilde{g}^{ab} = g^{ab} - \frac{w^a w^b}{N^2}\,, \qquad \tilde{g}^{at} = -\frac{w^a}{N^2}\,, \qquad \tilde{g}^{tt} = -\frac{1}{N^2}\,, \qquad (2.3)$$

where $N^2 = f\Delta/v^2$, $w_a = g_{ab}w^b$ and $w^2 = w_a w^a$. The determinant of the full metric is $\tilde{g} = -qgf^2/v^2$, where q is the determinant of q_{ij} and g is the determinant of g_{ab}.

The (outer) black hole horizon r_0 is located at the (largest) root of $\Delta(r_0) = 0$. Near the black hole horizon, $f, v^2, (q_{ij})$ and (g_{ab}) are all positive definite. The fact that black holes are intrinsically regular on the horizon puts extra constraints on the functions,

$$v(r, \theta^i) = v_0(r) + v_1(r, \theta^i)\Delta + \mathcal{O}(\Delta^2)\,,$$
$$w^a(r, \theta^i) = w_0^a(r) + w_1^a(r, \theta^i)\Delta + \mathcal{O}(\Delta^2)\,, \qquad (2.4)$$

which means that any dependence of v and w^a on θ^i can only begin at the order Δ. With these conditions, it is then possible to completely remove the divergence at $\Delta \to 0$ from the metric,

$$-\frac{\Delta}{v^2}dt^2 + \frac{dr^2}{\Delta} = \left(\frac{\Delta}{v_0^2} - \frac{\Delta}{v^2}\right)dt^2 - \frac{\Delta}{v_0^2}du_+ du_-\,,$$
$$d\phi^a - w^a dt = d\phi_\pm^a - w^a du_\pm \pm (w^a - w_0^a)\frac{v_0}{\Delta}dr\,, \qquad (2.5)$$

where

$$du_\pm = dt \pm \frac{v_0}{\Delta}dr\,, \quad d\phi_\pm^a = d\phi^a \pm w_0^a\frac{v_0}{\Delta}dr\,. \qquad (2.6)$$

The constraints that w_0^a and v_0 depend only on r comes from the required integrability of (2.6). In the presence of matter fields, similar constraints should also apply. For example, if there is a U(1) gauge field, then it must be of the form

$$A = A_a(r, \theta^i)(d\phi^a - w^a dt) + \left[A_t(r) + \mathcal{O}(\Delta)\right]dt\,. \qquad (2.7)$$

Finally, one can choose the coordinate system to be non-rotating at spatial infinity, which means

$$w^a(r, \theta^i) \longrightarrow 0 \quad \text{as} \quad r \to +\infty\,. \qquad (2.8)$$

In this case, $\Omega^a = w_0^a(r_0)$ is the angular velocity of the horizon along ϕ^a. Using the null Killing vector on the horizon $\partial_t + \Omega^a\partial_a$, one can find that the black hole temperature is

$$T = \frac{\kappa}{2\pi} = \frac{\Delta'}{4\pi v}\Big|_{r=r_0} = \frac{\Delta'(r_0)}{4\pi v_0(r_0)}\,, \qquad (2.9)$$

where κ is the surface gravity on the horizon. For extremal black holes, $\Delta'(r_0) = 0$ and so the temperature vanishes.

Charges of the black hole can be calculated by using (A.18) in the appendix,

$$\delta E = \int_{\text{horizon}} \delta \mathbf{Q}_{(\partial_t)} - i_{(\partial_t)} \mathbf{\Theta}_\delta \,, \tag{2.10}$$

$$\delta J_a = - \int_{\text{horizon}} \delta \mathbf{Q}_{(\partial_a)} - i_{(\partial_a)} \mathbf{\Theta}_\delta = - \int_{\text{horizon}} \delta \mathbf{Q}_{(\partial_a)} \,. \tag{2.11}$$

In these definitions, the charges will not be well defined unless the corresponding defining equations are δ-integrable. For J_a, the δ-integrability of (2.11) is obvious and one has

$$J_a = - \int_{\text{horizon}} \mathbf{Q}_{(\partial_a)} \,. \tag{2.12}$$

For E, one can use (2.1) to explicitly check [13] that in the context of Einstein gravity plus a (possibly zero) cosmological constant,[2]

$$\bar{\delta} E = T \bar{\delta} S + \Omega^a \bar{\delta} J_a \,, \quad S = \frac{\mathcal{A}_{\text{rea}}}{4} \,, \tag{2.13}$$

where S is the Bekenstein-Hawking entropy and $\mathcal{A}_{\text{rea}} \equiv \int_{\text{horizon}} (d^{D-2}x)_{tr} 2\sqrt{qg}$ is the area of the horizon. It is obvious that the $\bar{\delta}$-integrability of (2.10) is intimately related to the presence of the first law of thermodynamics (2.13) for the black holes. In fact, the authors of [16] have noticed integrating the first law of thermodynamics as a practical method for calculating the mass of black holes, especially for ones that do not have an easier alternative.

3 Boundary conditions and the Virasoro algebra

As is obvious from the last section, all thermodynamical quantities of a black hole can be calculated purely by using data from the neighborhood of the horizon. From this perspective, two black holes are *intrinsically* the same if they approach each other fast enough as one takes the limit to the horizon, while the exterior of the black holes may be rather different due to matter fields living outside the horizon. This is our most important reason for choosing to impose boundary conditions on the horizon. A loosely related and interesting idea can be found in [8, 17, 18].

The fluctuations over a given background $\tilde{g}_{\mu\nu}$ should satisfy the linearized equations of motion, which we denote as

$$\tilde{E}_{\mu\nu} = 0 \,, \quad \Longrightarrow \quad \delta \tilde{E}_{\mu\nu} = 0 \,. \tag{3.1}$$

A particular class of solutions are generated by a Lie derivative,

$$\delta \tilde{g}_{\mu\nu} = \pounds_\xi \tilde{g}_{\mu\nu} \,, \quad \Longrightarrow \quad \delta \tilde{E}_{\mu\nu} = \pounds_\xi \tilde{E}_{\mu\nu} = 0 \,. \tag{3.2}$$

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[2]The operator $\bar{\delta}$ is defined to perturb only the free parameters (such as mass and angular momenta) in a given solution. This is what's usually needed to test the first law of thermodynamics for a given black hole solution. For all other types of perturbations, one can consult [14, 15].

This may not be much a surprise, because $\delta \tilde{g}_{\mu\nu} = \pounds_\xi \tilde{g}_{\mu\nu}$ (when paired with $\delta x^\mu = -\xi^\mu$) is nothing but the general diffeomorphism. However, the new metric $\tilde{g}'_{\mu\nu} = \tilde{g}_{\mu\nu} + \pounds_\xi \tilde{g}_{\mu\nu}$ does represent a new configuration if the coordinate system is held fixed. For the corresponding physical meaning of ξ, note a real physical coordinate system is nothing but a lattice of observers (clocks and rulers). The observed fluctuation in the metric can also be interpreted as that, the metric is held fixed, but it is the lattice of observers oscillate involuntarily, driven by quantum fluctuations of the spacetime. In order for the observers to see a metric fluctuation $\delta \tilde{g}_{\mu\nu} = \pounds_\xi \tilde{g}_{\mu\nu}$, the involuntary motion of the observers must be $\delta x^\mu = -\xi^\mu$. As such, we immediately have the following constraints on ξ:

- First of all, since ξ^μ corresponds to the involuntary motion of observers, it should be finite;

- Secondly, $\xi^r(r_0) \neq 0$ means that the horizon either expands or shrinks, and this in general leads to a different black hole. So we must require $\xi^r(r_0) = 0$;

- Similarly, $\xi^i(r_0) \neq 0$ in general changes the metric on the horizon, and which should also be forbidden if we want to talk about the same black hole.

With these considerations, one can expand ξ^μ near the black hole horizon as

$$\xi^\mu = \sum_{k=0}^\infty \xi^\mu_{(k)} (r - r_0)^k, \quad \xi^r_{(0)} = \xi^i_{(0)} = 0, \tag{3.3}$$

where all the functions $\xi^\mu_{(k)}$ depend only on θ^i, ϕ^a and t.

Given this expansion, there are some further constraints that need to be satisfied,

- Firstly, we need to determine under what conditions can a perturbed configuration be regarded as the *same* to the original black hole. The most straightforward choice seems to be that the induced metric on the horizon should remain fixed. But since the time t is in many sense on the same footing as the azimuthal angles ϕ^a, we impose a stronger condition that the induced metric on the $r = r_0$ hypersurface should remain fixed, which means[3]

$$\delta \tilde{g}_{ij} \approx \delta \tilde{g}_{iA} \approx \delta \tilde{g}_{AB} \approx 0, \quad \forall \ i, j, A, B. \tag{3.4}$$

- Similarly, we also require that the volume density of the full spacetime remains the same on the horizon,[4]

$$\delta \sqrt{-\tilde{g}} \approx 0. \tag{3.5}$$

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[3]Throughout the paper, we use "$\approx$" to relate quantities that are equal in the limit $r \to r_0$.

[4]Steve Carlip told me that he had known the significance of this condition for sometime, but he had been reluctant to use it for lack of a good justification. Here we choose to use this condition because (1) it is intuitively consistent with the requirement that the black hole should remain the same on the horizon and (2) it is technically very helpful.

- Lastly, we require that there is no mixing between θ^i and any other directions,

$$\delta\tilde{g}_{ir} \approx \delta\tilde{g}^{ir} \approx \delta\tilde{g}^{ij} \approx \delta\tilde{g}^{iA} \approx 0, \quad \forall\; i, A. \tag{3.6}$$

This is motivated by the desire to preserve θ^i as longitudinal angles. (Otherwise one cannot say for sure that r is the radial direction and $r = r_0$ is the horizon.) But beyond that, there is not a very good reason for why one must do this. So we will treat (3.6) as a hand-put-in condition.

To study the consequences of the boundary conditions (3.4), (3.5) and (3.6), let's use (3.3) and write down the corresponding perturbation over the background (2.1) explicitly. The results are

$$\pounds_\xi \tilde{g}_{rr} \approx \partial_r \left(\frac{f}{\Delta'} \xi^r_{(1)} \right) + \frac{\xi^i_{(1)} \partial_i f}{\Delta'} + \frac{f}{\Delta} \xi^r_{(1)}, \tag{3.7}$$

$$\pounds_\xi \tilde{g}_{rA} \approx \frac{f}{\Delta'} \partial_A \xi^r_{(1)} + \tilde{g}_{AB} \xi^B_{(1)}, \tag{3.8}$$

$$\pounds_\xi \tilde{g}_{ri} \approx \frac{f}{\Delta'} \partial_i \xi^r_{(1)} + q_{ij} \xi^j_{(1)}, \quad \pounds_\xi \tilde{g}^{ri} \approx 0, \tag{3.9}$$

$$\pounds_\xi \tilde{g}_{ij} \approx \pounds_\xi \tilde{g}^{ij} \approx 0, \tag{3.10}$$

$$\pounds_\xi \tilde{g}_{ia} \approx g_{ab} D_i \xi^b_{(0)}, \tag{3.11}$$

$$\pounds_\xi \tilde{g}_{it} \approx -w_a D_i \xi^a_{(0)}, \tag{3.12}$$

$$\pounds_\xi \tilde{g}^{iA} \approx \frac{v^2}{f\Delta'} w^A w^B \partial_B \xi^i_{(1)} - q^{ij} \partial_j \xi^A_{(0)}, \tag{3.13}$$

$$\pounds_\xi \tilde{g}_{ab} \approx g_{ac} D_b \xi^c_{(0)} + g_{bc} D_a \xi^c_{(0)}, \tag{3.14}$$

$$\pounds_\xi \tilde{g}_{at} \approx g_{ab} D_t \xi^b_{(0)} - w_b D_a \xi^b_{(0)}, \tag{3.15}$$

$$\pounds_\xi \tilde{g}_{tt} \approx -2w_a D_t \xi^a_{(0)}, \tag{3.16}$$

$$\pounds_\xi \sqrt{-\tilde{g}} \approx \sqrt{-\tilde{g}} \left(\xi^r_{(1)} + \partial_A \xi^A_{(0)} \right). \tag{3.17}$$

where $D_\mu \xi^a_{(0)} \equiv \partial_\mu \xi^a_{(0)} - w^a \partial_\mu \xi^t_{(0)}$. Comparing these results with (3.4), (3.5) and (3.6), we find that

$$D_i \xi^a_{(0)} \approx D_a \xi^a_{(0)} \approx D_t \xi^a_{(0)} \approx 0, \quad \xi^r_{(1)} = -\partial_A \xi^A_{(0)},$$
$$\xi^i_{(1)} = -q^{ij} \frac{f}{\Delta'} \partial_j \xi^r_{(1)}, \qquad \partial_i \xi^A_{(0)} = q_{ij} \frac{v^2}{f\Delta'} w^A w^B \partial_B \xi^j_{(1)}. \tag{3.18}$$

To solve these equations, note $D_\mu \xi^a_{(0)} \approx 0$ are easily solved with

$$\xi^a_{(0)} = \Omega^a \xi^t_{(0)}, \quad \Longrightarrow \quad D_\mu \xi^a_{(0)} = (\Omega^a - w^a) \partial_\mu \xi^t_{(0)} \approx 0, \tag{3.19}$$

where Ω^a is defined below (2.8). The other equations are then uniquely solved by

$$\xi^r_{(1)} = -\partial_A \xi^A_{(0)} = -\Omega^A \partial_A \xi^t_{(0)}, \quad \partial_i \xi^t_{(0)} = 0, \quad \xi^i_{(1)} = 0. \tag{3.20}$$

With these results, the only non-vanishing variation of the metric elements are

$$\pounds_\xi \tilde{g}_{rr} \approx \mathcal{O}\left(\frac{1}{\Delta} \right), \quad \pounds_\xi \tilde{g}_{rA} \approx \mathcal{O}\left(\frac{1}{\Delta'} \right). \tag{3.21}$$

Note the variation of $\tilde{g}_{rr}$ is of the same order as $\tilde{g}_{rr}$ itself. Something similar also exists in the usual Kerr/CFT correspondence [5].

Now at the leading order ξ^μ depends only on $\xi^t_{(0)}$, which is a free function of ϕ^a and t. One can expand $\xi^t_{(0)}$ using the Fourier modes $e^{-im(\phi^{\bar{a}}-\tilde{\Omega}^{\bar{a}}t)}$, where $\phi^{\bar{a}}$ is one of the azimuthal angles, m is an integer, and $\tilde{\Omega}^{\bar{a}}$ is a constant to be determined. When appropriately normalized, we find ($\rho \equiv r - r_0$)

$$\bar{a}_m \equiv \xi^\mu \partial_\mu = -e^{-im(\phi^{\bar{a}}-\tilde{\Omega}^{\bar{a}}t)}\left\{\left[i\,m\,\rho+\mathcal{O}(\rho^2)\right]\partial_r+\mathcal{O}(\rho^2)\partial_i+\left[\frac{\Omega^A}{\Omega^{\bar{a}}-\tilde{\Omega}^{\bar{a}}}+\mathcal{O}(\rho)\right]\partial_A\right\}. \quad (3.22)$$

As a comparison, the generators found in [10, 11] are

$$\bar{a}_m = -e^{-im(\phi^{\bar{a}}-\Omega^{\bar{a}}t)}\left[i\,m\,\rho\,\partial_r+\partial_{\bar{a}}+\text{subleading terms}\right]. \quad (3.23)$$

At the leading order, the generators (3.22) satisfy the (centerless) Virasoro algebra,

$$i[\bar{a}_m\,,\,\bar{a}_n] = (m-n)\bar{a}_{m+n}\,. \quad (3.24)$$

Like in (2.10) and (2.11), one can define the charge corresponding to $\bar{a}_m$ by using (A.18),

$$\delta L^{\bar{a}}_m = -\int_{\text{horizon}}\left(\delta\mathbf{Q}_{(\bar{a}_m)}-i_{(\bar{a}_m)}\mathbf{\Theta}_\delta\right). \quad (3.25)$$

For $m \neq 0$, we find because of the factor $e^{-im\phi^{\bar{a}}}$,

$$\bar{\delta} L^{\bar{a}}_m = 0\,, \quad m = \pm 1, \pm 2, \cdots. \quad (3.26)$$

So $L^{\bar{a}}_m\,,\,\forall\,m \neq 0$ are independent on the background metric (2.1). For $m = 0$,

$$\bar{a}_0 = -\frac{\check{\xi}}{\Omega^{\bar{a}}-\tilde{\Omega}^{\bar{a}}} + \mathcal{O}(\rho)\,, \quad \check{\xi} \equiv \Omega^A\partial_A = \partial_t + \Omega^a\partial_a\,. \quad (3.27)$$

In the case when $\bar{\delta}(\Omega^{\bar{a}}-\tilde{\Omega}^{\bar{a}}) = 0$, we find using (2.10) and (2.11)

$$\bar{\delta} L^{\bar{a}}_0 = -\int_{\text{horizon}}\bar{\delta}\mathbf{Q}_{(\bar{a}_0)}-i_{(\bar{a}_0)}\mathbf{\Theta}_\delta = \bar{\delta}\left(\frac{E-\Omega^a J_a}{\Omega^{\bar{a}}-\tilde{\Omega}^{\bar{a}}}\right),$$

$$\implies \quad L^{\bar{a}}_0 = \frac{E-\Omega^a J_a}{\Omega^{\bar{a}}-\tilde{\Omega}^{\bar{a}}} + (\text{background independent constant})\,. \quad (3.28)$$

4 The non-extremal case

Given the charges (3.25), there is a central extension to (3.24), just as described in (A.9) and (A.21) in the appendix. The central charge can be found through (A.24) and (A.25). To find the result explicitly, let's firstly look at the quantity defined in (A.20) and (A.32). The relevant component is (note $\pounds_m \equiv \pounds_{\bar{a}_m}$)

$$K^{tr}(\pounds_n, \pounds_m) = \frac{1}{16\pi}\left\{-\frac{\tilde{h}}{2}\bar{a}^{tr}_m+\tilde{h}^{t\rho}\tilde{\nabla}_\rho\bar{a}^r_m-\tilde{h}^{r\rho}\tilde{\nabla}_\rho\bar{a}^t_m-(\tilde{\nabla}^t\tilde{h}^{r\rho}-\tilde{\nabla}^r\tilde{h}^{t\rho})\bar{a}_{m\rho}\right.$$

$$\left.+\bar{a}^t_m(\tilde{\nabla}_\rho\tilde{h}^{r\rho}-\tilde{\nabla}^r\tilde{h})-\bar{a}^r_m(\tilde{\nabla}_\rho\tilde{h}^{t\rho}-\tilde{\nabla}^t\tilde{h})\right\}. \quad (4.1)$$

where $\tilde{h}_{\mu\nu} \equiv \pounds_n \tilde{g}_{\mu\nu}$. For later convenience, we shall calculate the result for the more general Virasoro generators,

$$\bar{a}_m = -e^{-im(\phi^{\bar{a}} - \hat{\Omega}^{\bar{a}} t)} \left\{ \left[i\, m\, \rho + \mathcal{O}(\rho^2) \right] \partial_r + \mathcal{O}(\rho^2) \partial_i + \left[\chi^A + \mathcal{O}(\rho) \right] \partial_A \right\}, \qquad (4.2)$$

where $\hat{\Omega}^{\bar{a}}$ and χ^A are arbitrary constants, except for

$$\chi^{\bar{a}} = 1 + \hat{\Omega}^{\bar{a}} \chi^t. \qquad (4.3)$$

Note in (3.22), we have

$$\hat{\Omega}^{\bar{a}} = \tilde{\Omega}^{\bar{a}}, \quad \chi^{\bar{a}} = \frac{\Omega^{\bar{a}}}{\Omega^{\bar{a}} - \tilde{\Omega}^{\bar{a}}}, \quad \chi^t = \frac{1}{\Omega^{\bar{a}} - \tilde{\Omega}^{\bar{a}}}, \qquad (4.4)$$

which obviously satisfy (4.3).

Using (4.3) and assuming $T \propto \Delta'(r_0) \neq 0$, we find

$$K^{tr}(\pounds_{-m}, \pounds_m) \approx -\frac{4im^3}{16\pi} \left[\chi^t (\Omega^{\bar{a}} - \hat{\Omega}^{\bar{a}}) - \frac{1}{2} \right] \frac{v^2(\Omega^{\bar{a}} - \hat{\Omega}^{\bar{a}})}{f\Delta'} + \cdots, \qquad (4.5)$$

where we have only kept terms that are third order polynomials in m. The omitted terms are all finite and are linear in m. Note again that "$\approx$" means equal on the horizon. The subleading terms (those of $\mathcal{O}(\rho^2)$ for ∂_r and ∂_i, and those of $\mathcal{O}(\rho)$ for ∂_A) are not constrained in (3.22). So it is reassuring to note that the corresponding subleading terms in (4.2) have *no* contribution to (4.5) either.

Now using (A.25), the central charge is

$$\begin{aligned} c^{\bar{a}} &= \frac{3}{\pi} \int_{\text{horizon}} (d^{D-2}x)_{tr} 2\sqrt{-\tilde{g}} \left[\chi^t (\Omega^{\bar{a}} - \hat{\Omega}^{\bar{a}}) - \frac{1}{2} \right] \frac{v^2(\Omega^{\bar{a}} - \hat{\Omega}^{\bar{a}})}{f\Delta'} \\ &= \frac{3}{\pi} \int_{\text{horizon}} (d^{D-2}x)_{tr} 2\sqrt{qg} \left[\chi^t (\Omega^{\bar{a}} - \hat{\Omega}^{\bar{a}}) - \frac{1}{2} \right] \frac{v_0(r_0)(\Omega^{\bar{a}} - \hat{\Omega}^{\bar{a}})}{\Delta'(r_0)} \\ &= \frac{3}{\pi^2} \left[\chi^t (\Omega^{\bar{a}} - \hat{\Omega}^{\bar{a}}) - \frac{1}{2} \right] \cdot \frac{\Omega^{\bar{a}} - \hat{\Omega}^{\bar{a}}}{T} \cdot S, \end{aligned} \qquad (4.6)$$

where T is the temperature (2.9) and S is the entropy (2.13). For the generators (3.22), we can use (4.4) to further reduce the result to

$$c^{\bar{a}} = \frac{3}{2\pi^2} \cdot \frac{\Omega^{\bar{a}} - \tilde{\Omega}^{\bar{a}}}{T} \cdot S. \qquad (4.7)$$

Note in order for $c^{\bar{a}}$ to be non-negative, we need $\Omega^{\bar{a}} \geq \tilde{\Omega}^{\bar{a}}$. On the other hand, the Frolov-Thorne temperature for the modes $e^{-im(\phi^{\bar{a}} - \tilde{\Omega}^{\bar{a}} t)}$ is [5, 10]

$$T^{\bar{a}} = \frac{T}{\tilde{\Omega}^{\bar{a}} - \Omega^{\bar{a}}}, \qquad (4.8)$$

which is negative for $\Omega^{\bar{a}} > \tilde{\Omega}^{\bar{a}}$. If we want both $c^{\bar{a}}$ and $T^{\bar{a}}$ to be non-negative, then the only choice is $\tilde{\Omega}^{\bar{a}} = \Omega^{\bar{a}}$.[5] In this case, $c^{\bar{a}}$ vanishes and $T^{\bar{a}}$ diverges. The generators (3.22)

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[5]There is another way to see why one should take $\tilde{\Omega}^{\bar{a}} = \Omega^{\bar{a}}$. This is related to the fact that the horizon of (2.1) is a *frozen* surface. That is, for an observer comoving with the horizon, there should not be any propagating signals in the θ^i or ϕ^a directions on the horizon (because such signals are always space-like, i.e., superluminal). So on the horizon, all perturbations can only be a function of $\phi^a - \Omega^a t$. As a consequence, the modes $e^{-im(\phi^{\bar{a}} - \Omega^{\bar{a}} t)}$ should in fact be understood as $e^{-im(\phi^{\bar{a}} - \Omega^{\bar{a}} t)} = e^{-im(\phi^{\bar{a}} - \Omega^{\bar{a}} t)} \prod_{a \neq \bar{a}} e^{-i0(\phi^a - \Omega^a t)}$.

also diverge. As suggested in [10], the origin of such singular behaviors could be physical. Using the canonical version of Cardy's formula, we find

$$S^{\bar{a}} = \frac{\pi^2}{3} c^{\bar{a}} T^{\bar{a}} = \frac{S}{2}. \tag{4.9}$$

This result resembles that in section 2 of [10] in a remarkable way. Note Cardy's formula only gives half the Bekenstein-Hawking entropy. We will discuss more about this issue when we conclude.

Note the above result does not depend on which azimuthal angle is used. This is similar to what happens in the case of extremal Kerr/CFT correspondence [20]. Although we have the same number of Virasoro algebras as that of the azimuthal angles, they are not independent in terms of counting the degrees of freedom for the black hole.

5 The extremal case

As mentioned in the previous section, the subleading terms in (4.2) have no contribution to the central term (4.5). Unfortunately, this is not true for extremal black holes. In order to talk sensibly about the central charges for extremal black holes, one must further require that (4.2) obey the Virasoro algebra up to the sub-leading order. We then find

$$\bar{a}_m = -e^{-im(\phi^{\bar{a}} - \hat{\Omega}^{\bar{a}} t)} \Bigg(\left\{ i\,m\,\rho + \left[mu^r + \frac{im^2}{2}(u^{\bar{a}} - \Omega^{\bar{a}} u^t) \right] \rho^2 + \mathcal{O}(\rho^3) \right\} \partial_r$$
$$+ \left[mu^i \rho^2 + \mathcal{O}(\rho^3) \right] \partial_i + \left[\chi^A + mu^A \rho + \mathcal{O}(\rho^2) \right] \partial_A \Bigg), \quad (5.1)$$

where u^r, u^i and u^A are free functions of θ^i. With (5.1), we find that the contribution of the subleading terms to the central charge vanishes again. The following result is valid for both extremal and non-extremal black holes,

$$K^{tr}(\pounds_{-m}, \pounds_m) \approx -m^3 \left(\frac{\Delta'}{\Delta^2} Z_1 + \frac{Z_2}{\Delta} \right) + \cdots, \tag{5.2}$$

where "$\cdots$" denotes terms linear in m and

$$Z_1 = -\frac{2iv_0^2(r_0)(\Omega^{\bar{a}} - \hat{\Omega}^{\bar{a}})}{f(r_0, \theta^i)} \rho^2 + \left[-\frac{2iv_0^2(r_0)w'^{\bar{a}}(r_0)}{f(r_0, \theta^i)} + (\Omega^{\bar{a}} - \hat{\Omega}^{\bar{a}})G_1(r_0, \theta^i) \right] \rho^3 + \mathcal{O}(\rho^4),$$

$$Z_2 = \frac{4iv_0^2(r_0)(\Omega^{\bar{a}} - \hat{\Omega}^{\bar{a}})^2 \chi^t}{f(r_0, \theta^i)} \rho + \left[\frac{2iv_0^2(r_0)w'^{\bar{a}}(r_0)}{f(r_0, \theta^i)} + (\Omega^{\bar{a}} - \hat{\Omega}^{\bar{a}})G_2(r_0, \theta^i) \right] \rho^2 + \mathcal{O}(\rho^3). \tag{5.3}$$

The detail of the two functions G_1 and G_2 will not be important for us. We only need to know that they are both of order $\mathcal{O}(1)$. Note we have preserved the dependence of (5.2) and (5.3) on r only through Δ, Δ' and $\rho(= r - r_0)$.

One can check that (5.2) reduces to (4.5) in the non-extremal case, $\Delta'(r_0) \neq 0$. In the extremal case $\Delta'(r_0) = 0$,

$$K^{tr}(\pounds_{-m}, \pounds_m) \approx \frac{4im^3 v_0^2(r_0) w_0'^{\bar{a}}(r_0)}{16\pi \Delta''(r_0) f(r_0, \theta^i)}(1 + G) + \cdots,$$

$$G = \frac{\Omega^{\bar{a}} - \hat{\Omega}^{\bar{a}}}{w_0'^{\bar{a}}(r_0)} \left\{ \frac{2}{\rho} \left[1 - \chi^t(\Omega^{\bar{a}} - \hat{\Omega}^{\bar{a}}) \right] + \frac{2\Delta'''(r_0)}{3\Delta''(r_0)} \left[\chi^t(\Omega^{\bar{a}} - \hat{\Omega}^{\bar{a}}) - \frac{1}{2} \right] \right.$$
$$\left. - \frac{f(r_0, \theta^i)}{2iv_0^2(r_0)}(2G_1 + G_2) \right\}, \tag{5.4}$$

The first term in G diverges as $\rho \to 0$. But this divergence will go away once we use (4.4). In fact, as was discussed in the previous section, $\tilde{\Omega}^{\bar{a}} = \Omega^{\bar{a}}$. So $G = 0$ upon using (4.4). From (A.25), the central charge for the extremal case is

$$c^{\bar{a}} = -\frac{3}{\pi} \int_{\text{horizon}} (d^{D-2}x)_{tr} 2\sqrt{-\tilde{g}} \, \frac{v_0^2(r_0) w_0'^{\bar{a}}(r_0)}{\Delta''(r_0) f(r_0, \theta^i)} = \frac{3}{\pi^2} \frac{S}{\tilde{T}^{\bar{a}}}, \tag{5.5}$$

where $\tilde{T}^{\bar{a}}$ is the Frolov-Thorne temperature related to $\phi^{\bar{a}}$ from the usual Kerr/CFT calculation [6],

$$\tilde{T}^{\bar{a}} = -\frac{\Delta''(r_0)}{4\pi v_0(r_0) w'^{\bar{a}}(r_0)}. \tag{5.6}$$

In the present context, (4.8) is valid for both the extremal and non-extremal cases. But $T^{\bar{a}}$ is now indefinite because $T = \Omega^{\bar{a}} - \tilde{\Omega}^{\bar{a}} = 0$. As suggested in [10], one can identify $T^{\bar{a}}$ with $\tilde{T}^{\bar{a}}$. In this case, Cardy's formula gives

$$S^{\bar{a}} = \frac{\pi^2}{3} c^{\bar{a}} T^{\bar{a}} = S, \tag{5.7}$$

which is exactly the Bekenstein-Hawking entropy.

As a side remark, note the contribution to the central charge (5.2) comes from different terms for extremal and non-extremal cases. So in terms of the central charge, one cannot expect a smooth transition from the non-extremal case to the extremal case. This is another indication of the discontinuity that arises in taking the extremal limit for non-extremal black holes (see, e.g. [19]).

6 Summary

To summarize, we have studied conformal symmetries one the horizon of a general stationary and axisymmetric black hole. We find that consistent and physically reasonable boundary conditions exist, which uniquely determine a set of symmetry generators that form a copy of the Virasoro algebra. The construction is designed for black holes in arbitrary spacetime dimensions and in arbitrary theories. But for practical reasons, explicit calculation is only done for Einstein gravity plus a (possibly zero) cosmological constant. We find that one can deduce the full Bekenstein-Hawking entropy by using Cardy's formula for extremal black holes. For non-extremal black holes, Cardy's formula only gives half the Bekenstein-Hawking entropy.

As a possible explanation to the failure in the non-extremal case, we note that our boundary conditions (3.4), (3.5) and (3.6) are very stringent, and it uniquely allows for only one copy of the Virasoro algebra. In contrast, such as indicated in [7], the dual CFT for a non-extremal black hole could be non-chiral and there might be two copies of Virasoro algebras. This means that there might be a second Virasoro algebra but which is filtered out by (3.4), (3.5) and (3.6). However, so far we have not been able to find the more general boundary conditions that allow for two copies of the Virasoro algebras.

On the other hand, it is also possible that the problem is due to something else. One indication is that, as mentioned before, the horizon is a *frozen* surface, and all "fluctuations" on the horizon can only be a function of $\phi^a - \Omega^a t$. From this perspective, it seems very unlikely that there could be a second independent copy of the Virasoro algebra, because we do not have a second independent coordinate to work with.[6] So it is possible that (3.22) is all we have on a black hole horizon. For extremal black holes, it is already with some luck that Cardy's formula does reproduce the full Bekenstein-Hawking entropy. And in fact no one knows why this must work. For non-extremal black holes, it is then conceivable that one may have to go beyond simply applying Cardy's formula to deduce the full black hole entropy.

Another issue with non-extremal black holes is that the central charge vanishes while the Frolov-Thorne temperature diverges. However, here one is not sure if the singular behavior is intrinsic to the problem, or if one can find a better alternative construction that does not have such singular behaviors. We wish to understand this issue better in the future.

Despite such problems with the non-extremal case, we note Cardy's formula does give qualitatively correct result for the entropy. More importantly, it is remarkable that although our boundary conditions (3.4), (3.5) and (3.6) are as stringent as one can imagine, they still allow for non-trivial physical results. What's more, our boundary conditions are imposed on the metric elements on the horizon directly, and we do not need to introduce an intermediate stretched horizon. So if the boundary conditions (3.4), (3.5) and (3.6) are truly physically relevant, then quantum fluctuations near the horizon must be generated by (3.22), making it more convincing that quantum gravity on a black hole horizon is dual to some 2D conformal field theory.

Acknowledgments

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[6]This is to be contrasted with the asymptotic symmetries at spatial infinity of a BTZ black holes, where it is natural to define two independent coordinates like $\phi \pm t/\ell$ (ℓ is the AdS radius), and which are both periodic in ϕ.

A The covariant phase space method

In this section we compile all the necessary tools that are needed in the bulk discussion of the paper. Although we do re-drive some of the formulae that are particularly important for us, there is nothing essentially new here. For original works on the covariant phase space method, one can consult [14, 15, 21–25]. For earlier works on using the method to calculate central charges for black holes, one can see [4, 26, 27]. Part of our description follow [14, 15, 28] closely.

As a motivating example, we start with the case of one-dimensional motion in classical mechanics. The lagrangian is $L = L(q, \dot{q})$, with $q = q(t)$. Under a general operation $\hat{\delta}$ on q,[7]

$$\hat{\delta}L = \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\hat{\delta}q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\hat{\delta}q\right) = E\hat{\delta}q + \frac{d}{dt}\Theta_{\hat{\delta}}\,,$$

$$E = \frac{\partial L}{\partial q} - \dot{p}\,, \qquad p = \frac{\partial L}{\partial \dot{q}}\,, \qquad \Theta_{\hat{\delta}} = p\hat{\delta}q\,. \tag{A.1}$$

In the canonical phase space, p and q are treated as independent "coordinates", and one may denote them as $z^1 = q$ and $z^2 = p$. The Poisson bracket of any two functions $f = f(q, p)$ and $g = g(q, p)$ can be written as

$$\left\{f,\,g\right\}_{P.B.} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} = \Omega^{mn}\frac{\partial f}{\partial z^m}\frac{\partial g}{\partial z^n}\,, \quad m, n = 1, 2\,,$$

$$(\Omega^{mn}) = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\,, \quad \Longrightarrow \quad (\Omega_{mn}) = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\,. \tag{A.2}$$

For two arbitrary operations (say, $\hat{\delta}_1$ and $\hat{\delta}_2$), one can also define the presymplectic potential as

$$\Omega(\hat{\delta}_1, \hat{\delta}_2) \equiv \hat{\delta}_1\Theta_{\hat{\delta}_2} - \hat{\delta}_2\Theta_{\hat{\delta}_1} = \hat{\delta}_1 p\hat{\delta}_2 q - \hat{\delta}_2 p\hat{\delta}_1 q = \Omega_{mn}\hat{\delta}_1 z^m\hat{\delta}_2 z^n\,. \tag{A.3}$$

An interesting way to define the Hamiltonian is to take $\hat{\delta}_1$ to be a particular variation δ, which takes one solution to a nearby one, and to take $\hat{\delta}_2 = \frac{d}{dt}$,

$$\delta H = \Omega\left(\delta, \frac{d}{dt}\right) = \delta\Theta_{\left(\frac{d}{dt}\right)} - \frac{d}{dt}\Theta_\delta = \delta p\dot{q} - \dot{p}\delta q\,. \tag{A.4}$$

The Hamilton-Jacobi equations then follow in a straightforward manor,

$$\dot{q} = \frac{\delta H}{\delta p}\,, \quad \dot{p} = -\frac{\delta H}{\delta q}\,. \tag{A.5}$$

In parallel, one can do the same for a general system. Denoting the generalized coordinates of the canonical phase space as z^m, $m = 1, 2, \cdots$, one defines the presymplectic potential as

$$\Omega(\hat{\delta}_1, \hat{\delta}_2) = \Omega_{mn}\hat{\delta}_1 z^m\hat{\delta}_2 z^n\,. \tag{A.6}$$

For any symmetric transformations $\hat{\delta}_\xi$, the variation of the corresponding charge H_ξ is

$$\delta H_\xi = \Omega(\delta, \hat{\delta}_\xi) = \Omega_{mn}\delta z^m\hat{\delta}_\xi z^n\,, \tag{A.7}$$

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[7]Note $\hat{\delta}$ can either be a usual variation of q, or of some other types such as a time derivative on q.

where δ is again the particular variation that takes one solution to a nearby one. The Poisson bracket (or more generally the Dirac bracket) of any two such charges is

$$\left\{ H_\xi \, , H_\zeta \right\} = \Omega^{mn} \frac{\delta H_\xi}{\delta z^m} \frac{\delta H_\zeta}{\delta z^n} = -\Omega(\hat{\delta}_\xi, \hat{\delta}_\zeta) = \Omega(\hat{\delta}_\zeta, \hat{\delta}_\xi) \,. \tag{A.8}$$

Using the Jacobi identity and an arbitrary charge Q, one can further derive that

$$\{Q, H_{[\xi,\zeta]}\} = \hat{\delta}_{[\xi,\zeta]}Q = (\hat{\delta}_\xi\hat{\delta}_\zeta - \hat{\delta}_\zeta\hat{\delta}_\xi)Q = \hat{\delta}_\xi\{Q, H_\zeta\} - \hat{\delta}_\zeta\{Q, H_\xi\}$$
$$= \{\{Q, H_\zeta\}, H_\xi\} - \{\{Q, H_\xi\}, H_\zeta\} = \{\{H_\xi, H_\zeta\}, Q\} \,,$$
$$\implies \quad \{H_\xi, H_\zeta\} = -H_{[\xi,\zeta]} + K_{[\xi,\zeta]} \,, \quad \text{where} \quad \{K_{[\xi,\zeta]} \, , Q\} = 0 \,, \quad \forall \, Q \,. \tag{A.9}$$

Hence, simply because of the Jacobi identity, the algebra of the Poisson (Dirac) bracket is isomorphic (up to a possible central extension $K_{[\xi,\zeta]}$) to the Lie algebra $\hat{\delta}_\xi\hat{\delta}_\zeta - \hat{\delta}_\zeta\hat{\delta}_\xi = \hat{\delta}_{[\xi,\zeta]}$.

Now consider the general action,

$$S = \int_{\mathcal{M}} \mathbf{L} \,, \quad \mathbf{L} = \mathcal{L}(\Phi, \partial_\mu\Phi, \partial_\mu\partial_\nu\Phi, \cdots) * \mathbf{1} \,, \tag{A.10}$$

where Φ denotes all possible fields collectively. From now on, a bold faced letter such as $\mathbf{L}$ always stands for a differential form. Under an arbitrary operation $\hat{\delta}$ on the fields,

$$\hat{\delta}\mathbf{L} = (\hat{\delta}\Phi)E_\Phi * \mathbf{1} + d\boldsymbol{\Theta}_{\hat{\delta}} \,, \tag{A.11}$$

where all terms involving a derivative on $\hat{\delta}\Phi$ have been moved into $d\boldsymbol{\Theta}_{\hat{\delta}}$. The Euler-Lagrange equations are just $E_\Phi = 0$. In the special case when $\hat{\delta}$ is identified with a Lie derivative $\pounds_\xi = d \cdot i_\xi + i_\xi \cdot d$,

$$\pounds_\xi\mathbf{L} = d(i_\xi\mathbf{L}) = (\pounds_\xi\Phi)E_\Phi * \mathbf{1} + d\boldsymbol{\Theta}_\xi \,, \quad \mathbf{J}_\xi = \boldsymbol{\Theta}_\xi - i_\xi\mathbf{L} \,,$$
$$\implies \quad d\mathbf{J}_\xi = -(\pounds_\xi\Phi)E_\Phi * \mathbf{1} \cong 0 \,, \quad \implies \quad \mathbf{J}_\xi \cong d\mathbf{Q}_\xi \,, \tag{A.12}$$

where "$\cong$" means equal after using the equations of motion $E_\Phi = 0$. For a Killing vector ξ, one may call $\mathbf{J}_\xi$ the corresponding Noether current. Now consider the variation $\hat{\delta} = \delta$ that takes a classical solution to a nearby one,

$$\delta\mathbf{J}_\xi = \delta\boldsymbol{\Theta}_\xi - \delta(i_\xi\mathbf{L}) = \delta\boldsymbol{\Theta}_\xi - i_\xi \cdot d\boldsymbol{\Theta}_\delta = \mathbf{w}(\delta, \pounds_\xi) + d(i_\xi\boldsymbol{\Theta}_\delta) \,, \tag{A.13}$$

where $\mathbf{w}(\delta, \pounds_\xi) \equiv \delta\boldsymbol{\Theta}_\xi - \pounds_\xi\boldsymbol{\Theta}_\delta$. Since δ takes one solution to a nearby one, $\mathbf{J}_\xi$ stays exact as in (A.12). As a result,

$$\delta\mathbf{J}_\xi = d\delta\mathbf{Q}_\xi \,, \quad \implies \quad \mathbf{w}(\delta, \pounds_\xi) = d\mathbf{k}(\delta, \pounds_\xi) \,, \quad \mathbf{k}(\delta, \pounds_\xi) \equiv \delta\mathbf{Q}_\xi - i_\xi\boldsymbol{\Theta}_\delta \,. \tag{A.14}$$

In the case when ξ is a Killing vector,

$$\pounds_\xi = 0 \quad \implies \quad \mathbf{w}(\delta, \pounds_\xi) = 0 \,, \quad \implies \quad 0 = \int_V \mathbf{w}(\delta, \pounds_\xi) = \oint_{\partial V} \mathbf{k}(\delta, \pounds_\xi) \,, \tag{A.15}$$

where V is a Cauchy surface. We are particularly interested in the spacetime of a stationary black hole, where one can take V to be the space outside the horizon. As a result, ∂V has two disconnect pieces: one at spatial infinity and one at the horizon,

$$\oint_{\partial V} = \int_{+\infty} - \int_{\text{Horizon}} \,. \tag{A.16}$$

Usually one defines the charge corresponding to $\pounds_\xi$ through an integral at spatial infinity,

$$\delta H_\xi = \int_{+\infty} \mathbf{k}(\delta, \pounds_\xi) = \int_{+\infty} \delta\mathbf{Q}_\xi - i_\xi \mathbf{\Theta}_\delta \,. \tag{A.17}$$

But because of (A.15) and (A.16), this is equivalent to defining

$$\delta H_\xi = \int_{\text{horizon}} \mathbf{k}(\delta, \pounds_\xi) = \int_{\text{horizon}} \delta\mathbf{Q}_\xi - i_\xi \mathbf{\Theta}_\delta \,. \tag{A.18}$$

It is the second definition that we want to us in this paper. It is also straightforward to generalize such a definition to charges of boundary symmetries.

Comparing (A.18) with (A.7), we see that in the present discussion, $\hat{\delta}_\xi = \pounds_\xi$ and

$$\Omega(\delta, \pounds_\xi) = \int_{\text{horizon}} \mathbf{k}(\delta, \pounds_\xi) \,. \tag{A.19}$$

But this result is not enough for us to recover the full presymplectic potential $\Omega(\pounds_{\xi_1}, \pounds_{\xi_2})$, which is needed to define the Poisson/Dirac bracket (A.8) for the corresponding charges. On the other hand, one can still define the following quantity,

$$K[\pounds_\zeta, \pounds_\xi] \equiv \Omega(\delta, \pounds_\xi)|_{\delta \to \pounds_\zeta} = \int_{\text{horizon}} \mathbf{k}(\delta, \pounds_\xi)|_{\delta \to \pounds_\zeta} \,. \tag{A.20}$$

And one can write

$$\left\{ H_\xi \,, H_\zeta \right\} = \Omega(\pounds_\zeta, \pounds_\xi) = -H[\pounds_\zeta, \pounds_\xi] + K[\pounds_\zeta, \pounds_\xi] \,, \tag{A.21}$$

$$H[\pounds_\zeta, \pounds_\xi] = K[\pounds_\zeta, \pounds_\xi] - \Omega(\pounds_\zeta, \pounds_\xi) \,. \tag{A.22}$$

Although (A.21) looks like (A.9) very much, it is not guaranteed that

$$H_{[\xi,\zeta]} = H[\pounds_\zeta, \pounds_\xi] \,, \qquad K_{[\xi,\zeta]} = K[\pounds_\zeta, \pounds_\xi] \,. \tag{A.23}$$

But rigorous treatment (see, e.g. [24, 25]) does show that $K[\pounds_\zeta, \pounds_\xi]$ contains all the information about $K_{[\xi,\zeta]}$. This is good enough for our purpose here. Now if the Virasoro algebra

$$\left[L_m \,, L_n \right] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n} \,, \tag{A.24}$$

is realized from (A.21) through the canonical quantization $i\{\cdot, \cdot\} \to [\cdot, \cdot]$, then one can read off the central charge from the coefficient of m^3 in the term $K[\pounds_{-m}, \pounds_m]$,

$$c = 12i\Big(\text{the coefficient of } m^3 \text{ in } K[\pounds_{-m}, \pounds_m]\Big) \,. \tag{A.25}$$

In the following, let's calculate $\mathbf{k}(\delta, \pounds_\xi)$ for Einstein gravity plus a (possibly zero) cosmological constant. To simplify notations, we will drop the "tilde" from the full metric in (2.3) from now on. For differential forms, we use the notation

$$(d^{D-p}x)_{\mu_1 \cdots \mu_p} \equiv \frac{1}{p!(D-p)!} \varepsilon_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_{D-p}} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{D-p}} \,, \quad |\varepsilon_{...}| = 1 \,, \tag{A.26}$$

with which the Hodge-$*$ dual of a p-form $\mathbf{w}_p = \frac{1}{p!} w_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$ can be written as

$$* \mathbf{w}_p = \sqrt{|g|} \, (d^{D-p}x)_{\mu_1 \cdots \mu_p} w^{\mu_1 \cdots \mu_p} \,, \quad \implies \quad *\mathbf{1} = \sqrt{|g|} \, d^D x \,. \tag{A.27}$$

For the exterior and interior products, one then has

$$d * \mathbf{w}_p = \sqrt{|g|} \, (d^{D-p+1}x)_{\mu_1 \cdots \mu_{p-1}} \nabla_{\mu_p} w^{\mu_1 \cdots \mu_p} \,,$$
$$i_\xi (d^{D-p}x)_{\mu_1 \cdots \mu_p} = (d^{D-p-1}x)_{\mu_1 \cdots \mu_p \mu} (p+1) \xi^\mu \,. \tag{A.28}$$

Now the action is

$$\mathbf{L} = \left(\frac{R - 2\Lambda}{16\pi} \right) * \mathbf{1} \,. \tag{A.29}$$

Under an arbitrary operation $\hat{\delta}$ on the fields,

$$\hat{\delta} \mathbf{L} = \frac{1}{16\pi} \left\{ \frac{\hat{h}}{2}(R - 2\Lambda) + (-R^{\mu\nu} + \nabla^\mu \nabla^\nu - \nabla^2 g^{\mu\nu}) \hat{h}_{\mu\nu} \right\} * \mathbf{1} \,,$$
$$\implies \quad E^{\mu\nu} = \frac{1}{16\pi} \left[\frac{1}{2} g^{\mu\nu}(R - 2\Lambda) - R^{\mu\nu} \right] = 0 \,,$$
$$\mathbf{\Theta}_{\hat{\delta}} = \sqrt{-g} \, (d^{D-1}x)_\mu \left(\frac{\nabla_\nu \hat{h}^{\mu\nu} - \nabla^\mu \hat{h}}{16\pi} \right) \,, \tag{A.30}$$

where $\hat{h}_{\mu\nu} \equiv \hat{\delta} g_{\mu\nu}$. In the case when $\hat{\delta} = \pounds_\xi$,

$$\mathbf{J}_\xi = \mathbf{\Theta}_\xi - i_\xi \mathbf{L} = \sqrt{-g} \, (d^{D-1}x)_\mu \left\{ \frac{-\nabla_\nu \xi^{\mu\nu} + 2R^{\mu\nu} \xi_\nu}{16\pi} - \left(\frac{R - 2\Lambda}{16\pi} \right) \xi^\mu \right\}$$
$$= \sqrt{-g} \, (d^{D-1}x)_\mu \left(\frac{-\nabla_\nu \xi^{\mu\nu}}{16\pi} \right) = d\mathbf{Q}_\xi \,,$$
$$\implies \quad \mathbf{Q}_\xi = \sqrt{-g} \, (d^{D-2}x)_{\mu\nu} \left(\frac{-\xi^{\mu\nu}}{16\pi} \right) \,, \quad \xi^{\mu\nu} = \nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu \,. \tag{A.31}$$

One can further derive that for $\mathbf{k}(\delta, \pounds_\xi) = \delta \mathbf{Q}_\xi - i_\xi \mathbf{\Theta}_\delta = \sqrt{-g} \, (d^{n-2}x)_{\mu\nu} K^{\mu\nu}(\delta, \pounds_\xi)$,

$$K^{\mu\nu}(\delta, \pounds_\xi) = \frac{1}{16\pi} \left\{ -\frac{\delta(\sqrt{-g} \, \xi^{\mu\nu})}{\sqrt{-g}} + \xi^\mu (\nabla_\rho h^{\nu\rho} - \nabla^\nu h) - \xi^\nu (\nabla_\rho h^{\mu\rho} - \nabla^\mu h) \right\} \,,$$
$$= \frac{1}{16\pi} \left\{ -\frac{h}{2} \xi^{\mu\nu} + h^{\mu\rho} \nabla_\rho \xi^\nu - h^{\nu\rho} \nabla_\rho \xi^\mu - (\nabla^\mu h^{\nu\rho} - \nabla^\nu h^{\mu\rho}) \xi_\rho \right.$$
$$\left. + \xi^\mu (\nabla_\rho h^{\nu\rho} - \nabla^\nu h) - \xi^\nu (\nabla_\rho h^{\mu\rho} - \nabla^\mu h) \right\} \,. \tag{A.32}$$

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