Published for SISSA by Springer

Received: *August 9, 2012*
Accepted: *September 28, 2012*
Published: *October 22, 2012*

The limit of $N = (2,2)$ superconformal minimal models

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Abstract: The limit of families of two-dimensional conformal field theories has recently attracted attention in the context of AdS/CFT dualities. In our work we analyse the limit of $N = (2,2)$ superconformal minimal models when the central charge approaches $c = 3$. The limiting theory is a non-rational $N = (2,2)$ superconformal theory, in which there is a continuum of chiral primary fields. We determine the spectrum of the theory, the three-point functions on the sphere, and the disc one-point functions.

Keywords: Field Theories in Lower Dimensions, Conformal and W Symmetry, Conformal Field Models in String Theory

ArXiv ePrint: 1204.0446

doi:10.1007/JHEP10(2012)141

JHEP10(2012)141

Contents

1 Introduction

The analysis and construction of non-rational conformal field theories (CFTs) in two dimensions is a highly non-trivial task. On the other hand, rational CFTs are well investigated and understood. Some non-rational theories can be constructed as limits of rational theories, and their properties can be inferred from our knowledge about rational models. The first example of such a limiting theory is the Runkel-Watts theory [1] at central charge $c = 1$ that arises as the limit of Virasoro minimal models. Similar constructions have been considered for W_n minimal models [2], and supersymmetric $N = 1$ minimal models [3]. A different approach towards taking limits of conformal field theories is discussed in [4].

It is an obvious question whether such a construction is also possible for the $N = (2, 2)$ supersymmetric minimal models. For several reasons this is far from being a straightforward generalisation of the known cases. Firstly, all other examples are constructed as diagonal coset models of the form $\frac{\mathfrak{g}_k \oplus \mathfrak{g}_\ell}{\mathfrak{g}_{k+\ell}}$ where the level k is sent to infinity. On the other hand, the $N = (2, 2)$ Grassmannian Kazama-Suzuki models [5], of which the $N = (2, 2)$ minimal models are the simplest example, have a coset description as $su(n)/u(n-1)$, so their structure is different. Secondly, for the diagonal cosets it is known [6–8] that there are renormalisation group (RG) flows that connect theories with different levels k (triggered by the $(1, 1; \text{Adjoint})$ field). These flows become short[1] for large levels k (being accessible to conformal perturbation theory), and the models come closer in the space of theories, therefore one would intuitively expect a "convergence" to a limiting theory. For the $N = (2, 2)$ minimal models there are also RG flows connecting different models, but they are not accessible to conformal perturbation theory [10, 11] and are not short with respect to the Zamolodchikov metric, so that the theories do not seem to approach a limiting point in theory space. From this point of view, one might even doubt that a limit of $N = (2, 2)$ minimal models for large levels can be defined.

In this article we are investigating precisely this question. We analyse the spectrum of the $N = (2, 2)$ minimal models in the limit of large levels k for which the central charge approaches $c = 3$. In the Neveu-Schwarz sector we find primary fields $\Phi_{q,n}$ with a continuous charge $0 < |q| < 1$ and a discrete label $n = 0, 1, \ldots$. The fields with label $n = 0$ are chiral or anti-chiral primaries. By taking the limit of the known three-point functions of minimal models, we show that the fields in the limit theory have well-defined and non-trivial three-point functions. We also can define two classes of boundary conditions, a discrete one labelled by an integer M and a continuous one labelled — similarly to the fields — by a continuous parameter Q and a discrete parameter N, and we determine the disc one-point functions. One might still wonder whether the limit theory is fully consistent, but the results so far indicate that it is well behaved. It would be interesting to check that the resulting theory satisfies crossing symmetry.

Limits of conformal field theories also appear in the context of AdS/CFT dualities for higher spin gravity theories. Starting from the observation that the asymptotic symmetry of a higher spin gravity theory on AdS_3 is given by a W-algebra [12, 13], Gaberdiel and Gopakumar proposed a certain limit of W_n minimal models as the corresponding CFT dual [14]. In this limit, both the level k and the label n are sent to infinity such that the 't Hooft coupling $\lambda = \frac{n}{k+n}$ is kept fixed. This proposal was generalised to $N = (2, 2)$ superconformal theories in [15, 16]. In this context, our approach to send k to infinity in a given theory (with fixed n) is related to the situation where the 't Hooft coupling is zero.

[1] With respect to the Zamolodchikov metric [9].

The paper is organised as follows. In section 2 we consider the behaviour of the spectrum of minimal models in the limit. We define fields in the limit theory and show that they have sensible two-point functions. In section 3 we compute the limit of the three-point function, the necessary technical and computational details are collected in three appendices. Section 4 discusses boundary conditions and disc one-point functions. In section 5 we deal with the question whether we can define further fields of charge zero in the limit theory, and we conclude in section 6.

2 The spectrum

In this section we will analyse the spectrum of the limit theory. We start by reviewing some facts about minimal models, and then study their spectrum for large levels and define the corresponding fields in the limit theory.

2.1 Minimal models

The $N = (2,2)$ superconformal minimal models[2] come in a family parameterised by a positive integer k with central charges

$$c = 3\frac{k}{k+2} \ . \tag{2.1}$$

They possess a discrete spectrum. The unitary representations of the bosonic subalgebra of the $N = 2$ superconformal algebra are labelled by three integers (l, m, s), where

$$0 \leq l \leq k \quad , \quad m \equiv m + 2k + 4 \quad , \quad s \equiv s + 4 \ . \tag{2.2}$$

Only those triples (l, m, s) are allowed for which $l+m+s$ is even, and triples are identified according to the relation

$$(l, m, s) \equiv (k - l, m + k + 2, s + 2) \ . \tag{2.3}$$

The conformal weight and the $U(1)$ charge of the vectors in a representation $\mathcal{H}_{(l,m,s)}$ are given by

$$h \in h_{l,m,s} + \mathbb{N} \qquad\qquad h_{l,m,s} = \frac{l(l+2) - m^2}{4(k+2)} + \frac{s^2}{8} \tag{2.4}$$

$$q \in q_{m,s} + 2\mathbb{Z} \qquad\qquad q_{m,s} = -\frac{m}{k+2} + \frac{s}{2} \ . \tag{2.5}$$

We consider models with a diagonal spectrum, i.e. with equal left- and right-moving weights, $\bar{h} = h$, and charges, $\bar{q} = q$, of the ground states. The conformal weight and the $U(1)$ charge of the ground states of $\mathcal{H}_{(l,m,s)}$ are exactly given by $h_{l,m,s}$ and $q_{m,s}$ (without integer shifts) if the labels satisfy

$$|m - s| \leq l \, , \tag{2.6}$$

which is sometimes called the standard range. Contrary to some claims in the literature, the identification rule (2.3) does not allow one in general to map a given triple into the

[2] For an introduction see e.g. the textbooks [17, 18].

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standard range. Exceptions are provided by superdescendants of chiral primary or Ramond ground states (e.g. $(0,0,2) \equiv (k,k+2,0)$ cannot be mapped to the standard range).

Representations with even s belong to the Neveu-Schwarz sector. The direct sum $\mathcal{H}_{(l,m,0)} \oplus \mathcal{H}_{(l,m,2)}$ constitutes a representation of the full superconformal algebra. The primary fields $\phi_{l,m}$ with respect to the superconformal algebra are then labelled by a pair of integers (l,m), where

$$0 \leq l \leq k \quad , \quad |m| \leq k \quad , \quad l+m \text{ even} . \tag{2.7}$$

Their conformal weights and $U(1)$-charges are given by

$$h_{l,m} = h_{l,m,0} = \frac{l(l+2)-m^2}{4(k+2)} \tag{2.8}$$

$$q_{m,0} = -\frac{m}{k+2} . \tag{2.9}$$

The chiral primary fields are those with $m=-l$ obeying $h_{l,-l} = q_{l,-l}/2$, the anti-chiral primary states have $m=l$.

Representations with odd s belong to the Ramond sector. The Ramond ground states have labels $(l,l+1,1)$ with weight and charge given by

$$h_{l,l+1,1} = \frac{1}{8} - \frac{1}{4(k+2)} \tag{2.10}$$

$$q_{l+1,1} = \frac{1}{2} - \frac{l+1}{k+2} , \tag{2.11}$$

the corresponding field will be denoted by ψ^0_l. The full representation of the superconformal algebra built on such Ramond ground states is then $\mathcal{H}_{(l,l+1,1)} \oplus \mathcal{H}_{(l,l+1,-1)}$. The other Ramond representations of the superconformal algebra are given by the sum $\mathcal{H}_{(l,m,1)} \oplus \mathcal{H}_{(l,m,-1)}$ with $|m| \leq l-1$. The ground states in the two summands have the same conformal weight and differ by 1 in the $U(1)$ charge,

$$h_{l,m,\pm 1} = \frac{l(l+2)-m^2}{4(k+2)} + \frac{1}{8} \tag{2.12}$$

$$q_{m,\pm 1} = -\frac{m}{k+2} \pm \frac{1}{2} . \tag{2.13}$$

We denote the corresponding two fields by $\psi^{\pm}_{l,m}$.

2.2 Taking the limit

We want to take the limit $k \to \infty$, and analyse what happens to the spectrum. Let us first consider the primary states in the Neveu-Schwarz sector. When the level k becomes large, the spectrum of $U(1)$-charges becomes continuous in the range $-1 < q < 1$. We want to keep the $U(1)$ charge and the conformal weight fixed in the limit. For a fixed charge q we have to scale m with k such that

$$m \approx -q(k+2) . \tag{2.14}$$

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On the other hand, the label l is determined by $h_{l,m,0}$ and $q_{m,0}$ by

$$l = \sqrt{(k+2)^2 q_{m,0}^2 + 4(k+2)h_{l,m,0} + 1} - 1 \ . \tag{2.15}$$

Keeping $q_{m,0} \approx q \neq 0$ and $h_{l,m,0} \approx h$ fixed, the label l scales as

$$l = |m| + 2\frac{h}{|q|} - 1 + \mathcal{O}(1/k) \ . \tag{2.16}$$

The label l thus differs from the linearly growing $|m|$ only by a fixed finite number, which has to be an even integer (see (2.7)),

$$l = |m| + 2n \quad , \quad n = 0, 1, 2, \dots \ . \tag{2.17}$$

Whereas $|q|$ can take any value between 0 and 1, we see by comparing (2.16) and (2.17) that the ratio $h/|q|$ can only take discrete values,

$$h_n(q) = (2n+1)|q|/2 \, , \tag{2.18}$$

and $n = 0$ corresponds to chiral primary and anti-chiral primary fields.

In the h-q-plane, the Neveu-Schwarz spectrum is thus concentrated on lines going through the origin (see figure 1), and the fields $\Phi_{q,n}$ are labelled by their continuous $U(1)$-charge q and a discrete label n.

By a similar analysis we find in the Ramond sector on the one hand the Ramond ground states leading to fields Ψ^0_q with $h = \frac{1}{8}$ and $-\frac{1}{2} < q < \frac{1}{2}$ built from fields ψ^0_l with $l \approx (k+2)(\frac{1}{2} - q)$. In addition there are the fields $\Psi^{\pm}_{q,n}$ with $-\frac{1}{2} < \pm q < \frac{3}{2}$ and

$$h^{\pm}_n(q) = \frac{1}{8} + n\left| q \mp \frac{1}{2} \right| \ . \tag{2.19}$$

They are obtained from fields $\psi^{\pm}_{l,m}$ with $l = |m| + 2n - 1$ and $m \approx -(k+2)\left(q \mp \frac{1}{2}\right)$.

2.3 Fields and correlators

We now want to become more precise about how the limit of the fields is taken. We focus here on the Neveu-Schwarz sector, the construction in the Ramond sector is analogous.

For the fields $\Phi_{q,n}$ with $0 < |q| < 1$ we proceed as follows. We first define averaged fields,

$$\Phi^{\epsilon,k}_{q,n} = \frac{1}{|N(q,\epsilon,k)|} \sum_{\substack{m \in N(q,\epsilon,k) \\ l = |m| + 2n}} \phi_{l,m} \, , \tag{2.20}$$

where the set $N(q,\epsilon,k)$ contains all labels m such that the corresponding charge q_m is close to q, more precisely

$$N(q,\epsilon,k) = \left\{ m \,\middle|\, q - \frac{\epsilon}{2} < -\frac{m}{k+2} < q + \frac{\epsilon}{2} \right\} \ . \tag{2.21}$$

The cardinality of the set is

$$|N(q,\epsilon,k)| = \epsilon(k+2) + \mathcal{O}(1) \ . \tag{2.22}$$

We assume that ϵ is small enough such that $|q| \pm \frac{\epsilon}{2}$ is still between 0 and 1.

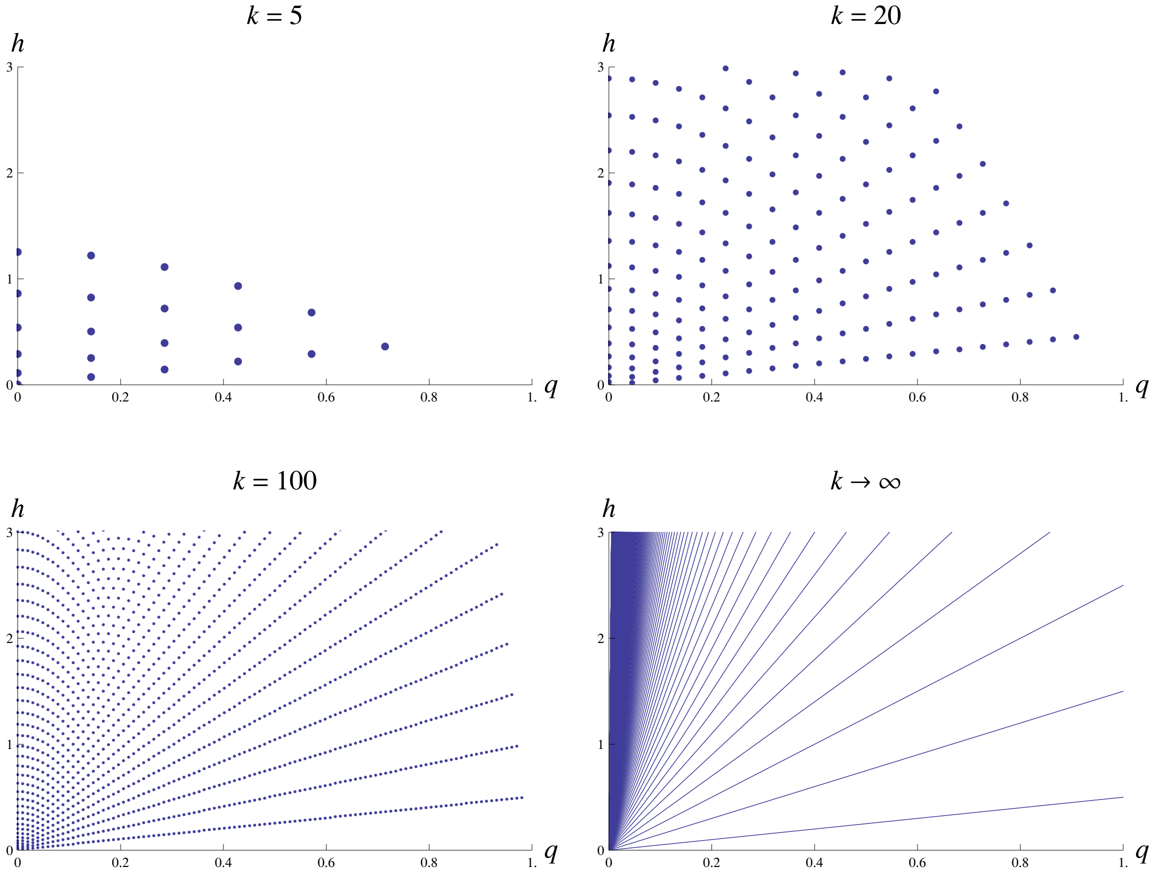

Figure 1. Behaviour of the spectrum of primary fields in the Neveu-Schwarz sector for large levels k: when one plots the values of the conformal weight h and of the $U(1)$ charge q as dots in the h-q-plane, one observes that the points assemble along straight lines starting from the origin. Notice that we only plotted the points corresponding to positive charge q (the negative charged part is just the mirror picture) and we truncated the conformal weights by $h \leq 3$.

The correlator of fields in the limit theory is then defined as

$$\langle \Phi_{q_1,n_1}(z_1,\bar{z}_1)\cdots\Phi_{q_r,n_r}(z_r,\bar{z}_r)\rangle = \lim_{\epsilon\to 0}\lim_{k\to\infty}\beta(k)^2\alpha(k)^r\langle \Phi^{\epsilon,k}_{q_1,n_1}(z_1,\bar{z}_1)\cdots\Phi^{\epsilon,k}_{q_r,n_r}(z_r,\bar{z}_r)\rangle\,, \tag{2.23}$$

where $\beta(k)^2$ is a factor that can be used to change the normalisation of the correlator in the limit (which corresponds to a rescaling of the vacuum by a factor $\beta(k)$), while $\alpha(k)$ is a factor that is used to change the normalisation of the fields while taking the limit.[3] The k-dependence of α and β are determined such that we obtain finite correlators in the limit. Obviously we need at least two correlators with a different number of fields to determine the k-dependence of both factors α and β.

[3]We could allow α to depend also on the field labels q, n, but it will turn out that this is not necessary.

Let us now analyse the two-point function. We normalise the fields in the minimal models such that

$$\langle \phi_{l_1,m_1}(z_1)\phi_{l_2,m_2}(z_2)\rangle = \delta_{l_1,l_2}\delta_{m_1,-m_2}\frac{1}{|z_1-z_2|^{4h_{l_1,m_1}}} \ . \tag{2.24}$$

The two-point function in the limit theory then becomes

$$\begin{aligned}\langle \Phi_{q_1,n_1}(z_1,\bar z_1)\Phi_{q_2,n_2}(z_2,\bar z_2)\rangle &= \lim_{\epsilon\to 0}\lim_{k\to\infty}\alpha(k)^2\beta(k)^2\langle \Phi^{\epsilon,k}_{q_1,n_1}(z_1,\bar z_1)\Phi^{\epsilon,k}_{q_2,n_2}(z_2,\bar z_2)\rangle & (2.25)\\ &= \lim_{\epsilon\to 0}\lim_{k\to\infty}\frac{\alpha(k)^2\beta(k)^2}{\epsilon^2(k+2)^2}\sum_{m\in N(q_1,\epsilon,k)\cap N(-q_2,\epsilon,k)}\frac{\delta_{n_1,n_2}}{|z_1-z_2|^{4h_{|m|+2n_1,m}}}. & (2.26)\end{aligned}$$

The conformal weight $h_{|m|+2n_1,m}$ approaches $h_{n_1}(q_1)=(2n_1+1)|q_1|/2$ in the limit, and the sum over m can be replaced by the cardinality of the overlap,

$$|N(q_1,\epsilon,k)\cap N(-q_2,\epsilon,k)| = (k+2)(\epsilon-|q_1+q_2|)\theta(\epsilon-|q_1+q_2|)+\mathcal{O}(1)\,, \tag{2.27}$$

where $\theta(x)$ is the Heaviside function being 1 for positive x, and 0 otherwise. In the limit $\epsilon\to 0$ we obtain a δ-distribution,

$$\frac{\epsilon-|x|}{\epsilon^2}\,\theta(\epsilon-|x|)\ \to\ \delta(x)\ . \tag{2.28}$$

With the choice

$$\alpha(k)\beta(k)=\sqrt{k+2} \tag{2.29}$$

to absorb the k-dependent pre-factor, we find the two-point function in a standard normalisation,

$$\langle \Phi_{q_1,n_1}(z_1,\bar z_1)\Phi_{q_2,n_2}(z_2,\bar z_2)\rangle = \delta_{n_1,n_2}\delta(q_1+q_2)\frac{1}{|z_1-z_2|^{4h_{n_1}(q_1)}}\ . \tag{2.30}$$

Let us conclude by briefly discussing the limit procedure. One might worry that the outcome depends on the precise ϵ-prescription of the limits of correlators. A conceptually clearer procedure would be to directly define correlators of smeared fields,

$$\Phi_n[f](z,\bar z) = \int dq\, f(q)\Phi_{q,n}(z,\bar z)\,, \tag{2.31}$$

by the prescription

$$\begin{aligned}\langle \Phi_{n_1}[f_1](z_1,\bar z_1)\cdots\Phi_{n_r}[f_r](z_r,\bar z_r)\rangle &= \lim_{k\to\infty}\beta(k)^2\left(\frac{\alpha(k)}{k+2}\right)^r\sum_{\{m_i\}}f_1\big(-\tfrac{m_1}{k+2}\big)\cdots f_r\big(-\tfrac{m_r}{k+2}\big)\\ &\quad\times\langle \phi_{|m_1|+2n_1,m_1}(z_1,\bar z_1)\cdots\phi_{|m_r|+2n_r,m_r}(z_r,\bar z_r)\rangle\ . \end{aligned}\tag{2.32}$$

In this framework one would recover the correlators of the fields $\Phi_{q,n}$ by letting the test functions f_i approach delta functions. Our prescription in (2.23) corresponds to a special choice for a family of test functions,

$$f_i(q)=\frac{1}{\epsilon}\theta(\epsilon/2-|q-q_i|)\,, \tag{2.33}$$

which approach $\delta(q-q_i)$ in the limit $\epsilon\to 0$, but the result does not depend on this choice.

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3 Three-point functions

In addition to the spectrum the three-point functions constitute the fundamental data of a conformal field theory. In an $N = 2$ superconformal theory, all three-point functions can be derived from the correlators of three (super-)primary fields together with the correlators involving two primaries and one superdescendant field [19, 20]. In this section we will analyse the limit of these correlators, which will also fix the normalisation factors $\alpha(k)$ and $\beta(k)$.

3.1 Correlators of primary fields

The correlators of three primary fields in minimal models have been determined in [19] (they are closely related to the three-point functions of the $SU(2)$ Wess-Zumino-Witten model derived in [21, 22]). Similar methods allow the computation of correlators involving superdescendants (see appendix C) that we will discuss later. The correlator of three primary fields in the Neveu-Schwarz sector in a model with diagonal spectrum reads [19]

$$\begin{aligned}&\langle\phi_{l_1,m_1}(z_1,\bar{z}_1)\phi_{l_2,m_2}(z_2,\bar{z}_2)\phi_{l_3,m_3}(z_3,\bar{z}_3)\rangle\\&\qquad = C(\{l_i,m_i\})\delta_{m_1+m_2+m_3,0}|z_{12}|^{2(h_3-h_1-h_2)}|z_{13}|^{2(h_2-h_1-h_3)}|z_{23}|^{2(h_1-h_2-h_3)}\end{aligned} \tag{3.1}$$

with

$$C(\{l_i,m_i\}) = \begin{pmatrix} \frac{l_1}{2} & \frac{l_2}{2} & \frac{l_3}{2} \\ \frac{m_1}{2} & \frac{m_2}{2} & \frac{m_3}{2} \end{pmatrix}^2 \sqrt{(l_1+1)(l_2+1)(l_3+1)}\, d_{l_1,l_2,l_3}\ . \tag{3.2}$$

Here, $\begin{pmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix}$ denotes the Wigner 3j-symbols, and d_{l_1,l_2,l_3} is a product of Gamma functions,

$$d^2_{l_1,l_2,l_3} = \frac{\Gamma(1+\rho)}{\Gamma(1-\rho)} P^2\left(\tfrac{l_1+l_2+l_3+2}{2}\right) \prod_{k=1}^{3} \frac{\Gamma(1-\rho(l_k+1))}{\Gamma(1+\rho(l_k+1))} \frac{P^2\left(\frac{l_1+l_2+l_3-2l_k}{2}\right)}{P^2(l_k)} \tag{3.3}$$

with

$$\rho = \frac{1}{k+2} \quad , \quad P(l) = \prod_{j=1}^{l} \frac{\Gamma(1+j\rho)}{\Gamma(1-j\rho)}\ . \tag{3.4}$$

We want to understand the limit[4] of this expression when $k \to \infty$ while the labels l_i and m_i grow such that the conformal weight h and the $U(1)$ charge q stay constant. In particular we have

$$l_i = |m_i| + 2n_i \quad \text{and} \quad m_i = -q_{(m_i)}(k+2)\,, \tag{3.5}$$

where n_i is a fixed integer, and $q_{(m_i)}$ lies in an ϵ-interval around q_i, hence it stays approximately constant in the limit.

The Wigner 3j-symbols enforce the condition $m_1+m_2+m_3 = 0$ as well as $l_{i_1} \leq l_{i_2}+l_{i_3}$ for any permutation $\{i_1,i_2,i_3\}$ of $\{1,2,3\}$. For definiteness we assume now that

$$m_1, m_2 \geq 0 \quad , \quad m_3 = -m_1-m_2 \leq 0\ . \tag{3.6}$$

[4] In [23] a related limit of WZW models $SU(2)_k$ has been considered.

For large $|m_i|$ the conditions on the l_i translate into a single condition on the n_i,

$$l_3 \leq l_1 + l_2 \Rightarrow n_3 \leq n_1 + n_2 \ . \tag{3.7}$$

When we consider the asymptotic behaviour of the three-point coefficient (3.2) for large k, there are two parts which have to be treated carefully. One is the Wigner 3j-symbol whose limit will be discussed in appendix A. The other is the limit of the products of Gamma functions, where $P(l)$ becomes an infinite product when l goes to infinity. However, the infinite products in the numerator and denominator cancel and leave a finite product in the limit as we will show in the following.

Firstly we look at the following ratio of products of Gamma functions,

$$\begin{aligned} \frac{P(\frac{l_1+l_2+l_3+2}{2})}{P(l_3)} &= \frac{\prod_{j=1}^{m_1+m_2+n_1+n_2+n_3+1} \frac{\Gamma(1+j\rho)}{\Gamma(1-j\rho)}}{\prod_{j=1}^{m_1+m_2+2n_3} \frac{\Gamma(1+j\rho)}{\Gamma(1-j\rho)}} & (3.8)\\ &= \prod_{j=m_1+m_2+2n_3+1}^{m_1+m_2+n_1+n_2+n_3+1} \frac{\Gamma(1+j\rho)}{\Gamma(1-j\rho)} & (3.9)\\ &= \left(\frac{\Gamma(1+q_{(m_3)})}{\Gamma(1-q_{(m_3)})}\right)^{n_1+n_2-n_3+1} \left(1+\mathcal{O}(\tfrac{1}{k})\right) \ . & (3.10) \end{aligned}$$

Similarly we have

$$\begin{aligned} \frac{P(\frac{-l_1+l_2+l_3}{2})}{P(l_2)} &= \left(\frac{\Gamma(1+q_{(m_2)})}{\Gamma(1-q_{(m_2)})}\right)^{n_1+n_2-n_3} \left(1+\mathcal{O}(\tfrac{1}{k})\right) & (3.11)\\ \frac{P(\frac{l_1-l_2+l_3}{2})}{P(l_1)} &= \left(\frac{\Gamma(1+q_{(m_1)})}{\Gamma(1-q_{(m_1)})}\right)^{n_1+n_2-n_3} \left(1+\mathcal{O}(\tfrac{1}{k})\right) & (3.12) \end{aligned}$$

and

$$P(\tfrac{l_1+l_2-l_3}{2}) = 1 + \mathcal{O}(\tfrac{1}{k}) \ . \tag{3.13}$$

In total, the coefficient d_{l_1,l_2,l_3} behaves in the limit[5] as

$$d_{l_1,l_2,l_3} = \left(\prod_{j=1}^{3} \frac{\Gamma(1+q_{(m_j)})}{\Gamma(1-q_{(m_j)})}\right)^{-\frac{1}{2}\sum_{i=1}^{3}\sigma_i(2n_i+1)} \left(1+\mathcal{O}(\tfrac{1}{k})\right) \ . \tag{3.14}$$

Here, $\sigma_i = \mathrm{sgn}(q_i)$ denotes the sign of the corresponding charge. In this form the expression is valid without any assumptions on which of the charges are positive or negative.

The asymptotic behaviour of the 3j-symbols is derived in appendix A. For m_i linearly growing with k and $m_1, m_2 \geq 0$, $m_3 \leq 0$, it is given by (see (A.17))

$$\begin{pmatrix} \frac{|m_1|}{2}+n_1 & \frac{|m_2|}{2}+n_2 & \frac{|m_3|}{2}+n_3 \\ \frac{m_1}{2} & \frac{m_2}{2} & \frac{m_3}{2} \end{pmatrix} = (-1)^{m_1+n_3+n_2} \left(|m_3|\right)^{-1/2} d^J_{M',M}(\beta) \cdot \left(1+\mathcal{O}(\tfrac{1}{k})\right), \tag{3.15}$$

[5]Note that this result can also be obtained by using the asymptotic formula for P given in (B.6).

where $d^J_{M',M}(\beta)$ is the Wigner d-matrix and

$$\cos\beta = \frac{|m_1|-|m_2|}{|m_1|+|m_2|}\,,\ J=\frac{n_1+n_2}{2}\,,\ M'=-\frac{n_1+n_2}{2}+n_3\,,\ M=\frac{n_1+n_2}{2}-n_2\,. \quad (3.16)$$

Putting everything together, the three-point coefficient $C(\{l_i,m_i\})$ given in (3.2) has the limiting behaviour

$$C(\{l_i,m_i\}) \sim (k+2)^{1/2}\mathcal{C}(\{q_{(m_i)},n_i\})\,, \quad (3.17)$$

where $\mathcal{C}$ is a smooth function of the charges q_i. For $q_1,q_2<0$ and $q_3>0$ it is given by

$$\mathcal{C}(\{q_i,n_i\}) = \left(\frac{|q_1q_2|}{|q_3|}\right)^{1/2}(d^J_{M',M}(\beta))^2\left(\prod_{j=1}^{3}\frac{\Gamma(1+q_j)}{\Gamma(1-q_j)}\right)^{n_1+n_2-n_3+\frac{1}{2}}\,, \quad (3.18)$$

with $\cos\beta=\frac{|q_1|-|q_2|}{|q_1|+|q_2|}$ and J,M,M' given in (3.16). Notice that $\mathcal{C}$ in this case is non-zero only for $n_1+n_2\geq n_3$.

Now we are ready to work out the limit of the 3-point function. By definition it is given by

$$\begin{aligned}
&\langle\Phi_{q_1,n_1}(z_1,\bar z_1)\Phi_{q_2,n_2}(z_2,\bar z_2)\Phi_{q_3,n_3}(z_3,\bar z_3)\rangle\\
&\quad=\lim_{\epsilon\to0}\lim_{k\to\infty}\beta(k)^2\alpha(k)^3\langle\Phi^{\epsilon,k}_{q_1,n_1}(z_1,\bar z_1)\Phi^{\epsilon,k}_{q_2,n_2}(z_2,\bar z_2)\Phi^{\epsilon,k}_{q_3,n_3}(z_3,\bar z_3)\rangle \quad (3.19)\\
&\quad=\lim_{\epsilon\to0}\lim_{k\to\infty}\frac{\beta(k)^2\alpha(k)^3}{\epsilon^3(k+2)^3}\sum_{\{m_i\in N(q_i,\epsilon,k)\}}C(\{|m_i|+n_i,m_i\})\delta_{m_1+m_2+m_3,0}\\
&\qquad\times|z_{12}|^{2(h_3-h_1-h_2)}|z_{13}|^{2(h_2-h_1-h_3)}|z_{23}|^{2(h_1-h_2-h_3)}\,. \quad (3.20)
\end{aligned}$$

We already determined the limit of the three-point coefficient, so it only remains to determine the factor that originates from the summation over the labels m_i, i.e. the cardinality of the set

$$N_{123}=\{(m_1,m_2,m_3)\in N(q_1,\epsilon,k)\times N(q_2,\epsilon,k)\times N(q_3,\epsilon,k):m_1+m_2+m_3=0\}\,. \quad (3.21)$$

It is given by

$$|N_{123}|=(k+2)^2\epsilon^2 f(\tfrac{1}{\epsilon}\textstyle\sum_i q_i)+\mathcal{O}(k+2)\,, \quad (3.22)$$

where the function f is defined as

$$f(x)=\begin{cases}0 & \text{for}\quad x<-\frac{3}{2}\\ \frac{1}{2}(x+\frac{3}{2})^2 & \text{for}\ -\frac{3}{2}<x<-\frac{1}{2}\\ \frac{3}{4}-x^2 & \text{for}\ -\frac{1}{2}<x<\frac{1}{2}\\ \frac{1}{2}(x-\frac{3}{2})^2 & \text{for}\quad \frac{1}{2}<x<\frac{3}{2}\\ 0 & \text{for}\quad \frac{3}{2}<x\ .\end{cases} \quad (3.23)$$

The function f is displayed in figure 2, it has the property

$$\int dx\, f(x)=1\,. \quad (3.24)$$

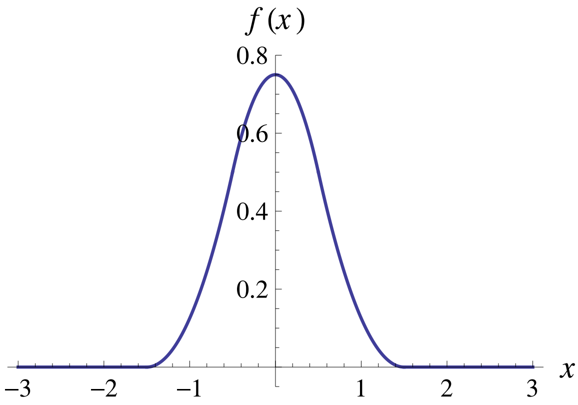

Figure 2. An illustration of the function f defined in (3.23).

When we finally take the limit, we observe that the function f leads to a delta distribution for the sum of the charges,

$$\frac{1}{\epsilon} f\left(\tfrac{1}{\epsilon}\textstyle\sum_i q_i\right) \to \delta\left(\textstyle\sum_i q_i\right) \ . \tag{3.25}$$

Using the condition (2.29) we can absorb the remaining k-dependence by setting

$$\alpha(k) = (k+2)^{-1/2} \quad , \quad \beta(k) = (k+2) \ . \tag{3.26}$$

The total result is then

$$\begin{aligned} &\langle \Phi_{q_1,n_1}(z_1,\bar{z}_1)\Phi_{q_2,n_2}(z_2,\bar{z}_2)\Phi_{q_3,n_3}(z_3,\bar{z}_3)\rangle \\ &\qquad = \mathcal{C}(\{q_i,n_i\})\delta\left(\textstyle\sum_i q_i\right)|z_{12}|^{2(h_3-h_1-h_2)}|z_{13}|^{2(h_2-h_1-h_3)}|z_{23}|^{2(h_1-h_2-h_3)} \end{aligned} \tag{3.27}$$

with $\mathcal{C}$ given in (3.18).

3.2 Correlators involving superdescendants

Now we want to show that also the three-point function of two primaries and one superdescendant (which corresponds to the odd fusion channel [19]) has a well-defined limit. We will limit ourselves to the case of a superdescendant obtained by acting with G^+, the discussion for G^--descendants is analogous. As derived in appendix C such a correlator is given by (see (C.23))

$$
\begin{aligned}
&\langle (\bar{G}^{+}_{-\frac{1}{2}} G^{+}_{-\frac{1}{2}}\phi_{l_1,m_1})(z_1,\bar{z}_1)\phi_{l_2,m_2}(z_2,\bar{z}_2)\phi_{l_3,m_3}(z_3,\bar{z}_3)\rangle \\
&\quad = \frac{k+2}{2(n_1+1)(l_1-n_1)} \begin{pmatrix} \frac{l_2+m_2}{2} \\ \frac{l_1-m_1}{2}+1 \end{pmatrix} \begin{pmatrix} \frac{l_1+l_2-m_1-m_2}{2}+1 \\ \frac{l_1-m_1}{2}+1 \end{pmatrix} \begin{pmatrix} l_1 \\ \frac{l_1-m_1}{2}+1 \end{pmatrix}^{-1} \\
&\quad\quad \times \begin{pmatrix} \frac{k-l_1}{2} & \frac{l_2}{2} & \frac{l_3}{2} \\ -\frac{k-l_1}{2} & \frac{m_1+m_2-l_1}{2}-1 & \frac{m_3}{2} \end{pmatrix}^2 \sqrt{(k-l_1+1)(l_2+1)(l_3+1)}\, d_{k-l_1,l_2,l_3} \\
&\quad\quad \times |z_{12}|^{2(h_{l_3,m_3}-(h_{l_1,m_1}+1/2)-h_{l_2,m_2})}|z_{23}|^{2((h_{l_1,m_1}+1/2)-h_{l_2,m_2}-h_{l_3,m_3})} \\
&\quad\quad \times |z_{13}|^{2(h_{l_2,m_2}-(h_{l_1,m_1}+1/2)-h_{l_3,m_3})}\,, \qquad (3.28)
\end{aligned}
$$

where $l_i \geq |m_i|$ and we assume that $m_1, m_2 > 0$ and $m_3 < 0$.

To determine the limit we first simplify the prefactor (that we call A) in (3.28) by expressing the 3j-symbol with the help of (A.5),

$$
\begin{aligned}
A &= \frac{k+2}{2(n_1+1)(l_1-n_1)} \begin{pmatrix} \frac{l_2+m_2}{2} \\ \frac{l_1-m_1}{2}+1 \end{pmatrix} \begin{pmatrix} \frac{l_1+l_2-m_1-m_2}{2}+1 \\ \frac{l_1-m_1}{2}+1 \end{pmatrix} \begin{pmatrix} l_1 \\ \frac{l_1-m_1}{2}+1 \end{pmatrix}^{-1} \\
&\quad \times \begin{pmatrix} \frac{k-l_1}{2} & \frac{l_2}{2} & \frac{l_3}{2} \\ -\frac{k-l_1}{2} & \frac{m_1+m_2-l_1}{2}-1 & \frac{m_3}{2} \end{pmatrix}^2 \\
&= \frac{k+2}{2(n_1+1)(l_1-n_1)} \frac{(\frac{l_1+m_1}{2}-1)!}{l_1!(\frac{l_1-m_1}{2}+1)!} \\
&\quad \times \frac{(\frac{-k+l_1+l_2+l_3}{2})!(\frac{l_3+m_3}{2})!(\frac{l_2+m_2}{2})!(k-l_1)!}{(\frac{k-l_1+l_2+l_3}{2}+1)!(\frac{k-l_1-l_2+l_3}{2})!(\frac{k-l_1+l_2-l_3}{2})!(\frac{l_3-m_3}{2})!(\frac{l_2-m_2}{2})!}\,. \qquad (3.29)
\end{aligned}
$$

In the limit we set $l_i = |m_i| + 2n_i$ where the n_i are kept constant, and the m_i are sent to infinity growing linearly in k. By using (A.11) we get

$$
\begin{aligned}
A &= \frac{k+2}{2(n_1+1)(m_1+n_1)} \frac{n_3!}{(n_1+1)!(n_2)!(-n_1-n_2+n_3-1)!} \\
&\quad \times \frac{(m_1+n_1-1)!(k-m_1-2n_1)!(m_2+n_2)!(-m_3+n_1+n_2+n_3+1)!}{(m_1+2n_1)!(k-m_1-n_1+n_2-n_3+1)!(m_2-n_1+n_2+n_3)!(-m_3+n_3)!} \\
&= \frac{1}{2(k+2)(n_1+1)} \frac{n_3!}{(n_1+1)!n_2!(-n_1-n_2+n_3-1)!} \\
&\quad \times |q_1|^{-n_1-2}|1+q_1|^{-n_1-n_2+n_3-1}|q_2|^{n_1-n_3}|q_3|^{n_1+n_2+1}\left(1+\mathcal{O}(\tfrac{1}{k})\right)\,, \qquad (3.30)
\end{aligned}
$$

where $q_i = -\frac{m_i}{k+2}$ is kept fixed in the limit. By similar arguments as before we can evaluate the asymptotic form of d_{k-l_1,l_2,l_3} to be

$$
d_{k-l_1,l_2,l_3} = \left(\frac{\Gamma(1+|1+q_1|)\Gamma(1-|q_2|)\Gamma(1+|q_3|)}{\Gamma(1-|1+q_1|)\Gamma(1+|q_2|)\Gamma(1-|q_3|)}\right)^{n_1+n_2-n_3+\frac{1}{2}} \cdot \left(1+\mathcal{O}(\tfrac{1}{k+2})\right)\,. \qquad (3.31)
$$

The final result for the three-point correlator of two primaries and one superdescendant in the limit theory is then given by

$$
\begin{aligned}
&\langle (G^+_{-\frac{1}{2}}\bar{G}^+_{-\frac{1}{2}}\Phi_{q_1,n_1})(z_1,\bar{z}_1)\Phi_{q_2,n_2}(z_2,\bar{z}_2)\Phi_{q_3,n_3}(z_3,\bar{z}_3)\rangle = \\
&\quad = \frac{1}{2(n_1+1)}\,\frac{n_3!}{(n_1+1)!n_2!(n_3-n_1-n_2-1)!}|1+q_1|^{-n_1+n_2+n_3-\frac{1}{2}}|q_2|^{n_1-n_3+\frac{1}{2}}|q_3|^{n_1+n_2+\frac{3}{2}} \\
&\qquad \times |q_1|^{-n_1-2}\left(\frac{\Gamma(1+|1+q_1|)\Gamma(1-|q_2|)\Gamma(1+|q_3|)}{\Gamma(1-|1+q_1|)\Gamma(1+|q_2|)\Gamma(1-|q_3|)}\right)^{n_1+n_2-n_3+\frac{1}{2}} \\
&\qquad \times \delta(1+q_1+q_2+q_3)|z_{12}|^{2(h_3-h_1-\frac{1}{2}-h_2)}|z_{13}|^{2(h_2-h_1-\frac{1}{2}-h_3)}|z_{23}|^{2(h_1+\frac{1}{2}-h_2-h_3)}\,, \qquad (3.32)
\end{aligned}
$$

where we assumed that $q_1, q_2 < 0$ and $q_3 > 0$. The generalisation to other cases is straightforward. As in a superconformal theory all three-point functions are determined if the three-point correlators of three primaries and the correlators of two primaries and one superdescendant are given, this result shows that all three-point functions of the limit theory are well defined.

4 Boundary conditions and one-point functions

In this section we investigate the limit of bulk one-point functions on the upper half plane. In the minimal models there are two types of maximally symmetric boundary conditions, called A-type and B-type [24]. In a diagonal model, only chargeless fields can couple to B-type boundary conditions, so that we do not find any B-type boundary conditions in the limit theory. On the other hand we will discover two families of A-type boundary conditions in the limit.

The A-type boundary conditions are labelled by the same labels as the representations of the bosonic subalgebra of the superconformal algebra, (L,M,S), where L is an integer satisfying $0 \leq L \leq k$, M is a $(2k+4)$-periodic integer, and S is a 4-periodic integer such that $L+M+S$ is even. Labels are identified according to (2.3).

Boundary states $|L,M,S\rangle$ are given by linear combinations of Ishibashi states [24],

$$
|L,M,S\rangle = \sum_{(l,m,s)} \frac{S_{(L,M,S)(l,m,s)}}{\sqrt{S_{(0,0,0)(l,m,s)}}}|l,m,s\rangle\!\rangle\,, \qquad (4.1)
$$

where S is the modular S-matrix of the $N=2$ superconformal algebra,

$$
S_{(L,M,S)(l,m,s)} = \frac{1}{k+2}\sin\frac{\pi(l+1)(L+1)}{k+2}e^{-\pi i(\frac{sS}{2}-\frac{mM}{k+2})}\,. \qquad (4.2)
$$

The coefficients of the boundary states determine the bulk one-point functions on the upper half plane with boundary condition $\alpha = (L,M,S)$ on the real axis. For a primary field $\phi_{l,m}$ it is given by

$$
\langle \phi_{l,m}(z,\bar{z})\rangle_{(L,M,S)} = \frac{S_{(L,M,S)(l,m,0)}}{\sqrt{S_{(0,0,0)(l,m,0)}}}|z-\bar{z}|^{-2h_{l,m}}\,. \qquad (4.3)
$$

Writing out the one-point function we get

$$\langle\phi_{l,m}(z,\bar{z})\rangle_{(L,M,S)} = (k+2)^{-1/2}\frac{\sin\frac{\pi(l+1)(L+1)}{k+2}}{\sqrt{\sin\frac{\pi(l+1)}{k+2}}}e^{\pi i\frac{mM}{k+2}}|z-\bar{z}|^{-2h_{l,m}} \ . \tag{4.4}$$

When we take the limit $k\to\infty$ we have some freedom of what to do with the boundary labels. There are two natural choices: either we keep the boundary labels constant in the limit, or we scale them in the same way as we scale the field labels. Both lead to sensible expressions as we will see shortly.

4.1 Discrete boundary conditions

First we will take the limit such that the boundary labels are kept fixed. The one-point function in the limit is then

$$\begin{aligned}
&\langle\Phi_{q,n}(z,\bar{z})\rangle_{(L,M,S)} = \\
&\quad = \lim_{\epsilon\to 0}\lim_{k\to\infty}\alpha(k)\beta(k)\langle\Phi^{\epsilon,k}_{q,n}(z,\bar{z})\rangle_{(L,M,S)} \\
&\quad = \lim_{\epsilon\to 0}\lim_{k\to\infty}\frac{\alpha(k)\beta(k)}{\epsilon(k+2)^{\frac{3}{2}}}\sum_{m\in N(q,\epsilon,k)}\frac{\sin\frac{\pi(|m|+2n+1)(L+1)}{k+2}}{\sqrt{\sin\frac{\pi(|m|+2n+1)}{k+2}}}e^{\pi i\frac{mM}{k+2}}|z-\bar{z}|^{-2h_{|m|+2n,m}} \\
&\quad = \frac{\sin(\pi|q|(L+1))}{\sqrt{\sin(\pi|q|)}}e^{-\pi iqM}|z-\bar{z}|^{-2h_n(q)} \ .
\end{aligned} \tag{4.5}$$

For the Ramond fields one finds

$$\langle\Psi^0_q(z,\bar{z})\rangle_{(L,M,S)} = \frac{\sin(\pi|\frac{1}{2}-q|(L+1))}{\sqrt{\sin(\pi|\frac{1}{2}-q|)}}e^{\pi i(\frac{1}{2}-q)M}e^{-\pi i\frac{S}{2}}|z-\bar{z}|^{-1/4} \tag{4.6}$$

$$\langle\Psi^\pm_{q,n}(z,\bar{z})\rangle_{(L,M,S)} = \frac{\sin(\pi|\frac{1}{2}\mp q|(L+1))}{\sqrt{\sin(\pi|\frac{1}{2}\mp q|)}}e^{\pi i(\pm\frac{1}{2}-q)M}e^{\mp\pi i\frac{S}{2}}|z-\bar{z}|^{-2h^\pm_n(q)} \ . \tag{4.7}$$

These boundary conditions are not independent. Using the trigonometric identity

$$\sin\left(\pi|q|(L+1)\right) = \sin\left(\pi|q|\right)\sum_{j=0}^{L}e^{i\pi q(L-2j)} \ , \tag{4.8}$$

we see that

$$\langle\ \cdot\ \rangle_{(L,M,S)} = \sum_{j=0}^{L}\langle\ \cdot\ \rangle_{(0,M+L-2j,S)} \ . \tag{4.9}$$

All boundary conditions are therefore superpositions of boundary conditions with $L=0$, and the elementary boundary conditions are $(0,M,S)$. This can be compared to the situation in minimal models before taking the limit, where all boundary conditions can be obtained by boundary renormalisation group flows from superpositions of those with $L=0$ [24, 25]. These flows become shorter when the level k grows, and in the limit the boundary conditions can be identified.

4.2 Continuous boundary conditions

Now we will scale the boundary labels in the same way as we did for the field labels. We introduce a continuous parameter Q, $0 < |Q| < 1$, and a discrete parameter $N \in \mathbb{N}_0$, and instead of considering fixed boundary labels in the limit, we consider a sequence of boundary conditions $B_k(Q,N)$ of the form

$$B_k(Q,N) = (|\lfloor -Q(k+2)\rfloor| + 2N, \lfloor -Q(k+2)\rfloor, 0)\,, \tag{4.10}$$

where $\lfloor x\rfloor$ denotes the greatest integer smaller or equal to x. The one-point function in the limit is then

$$\begin{aligned}\langle\Phi_{q,n}(z,\bar z)\rangle_{(Q,N)} &= \lim_{\epsilon\to 0}\lim_{k\to\infty}\alpha(k)\beta(k)\langle\Phi^{\epsilon,k}_{q,n}(z,\bar z)\rangle_{B_k(Q,N)} & (4.11)\\ &= \lim_{\epsilon\to 0}\lim_{k\to\infty}\frac{\alpha(k)\beta(k)}{\epsilon(k+2)^{\frac{3}{2}}}\sum_{m\in N(q,\epsilon,k)}\frac{\sin\frac{\pi(|m|+2n+1)(|\lfloor -Q(k+2)\rfloor|+2N+1)}{k+2}}{\sqrt{\sin\frac{\pi(|m|+2n+1)}{k+2}}}\\ &\qquad\times e^{\pi i\frac{m\lfloor -Q(k+2)\rfloor}{k+2}}|z-\bar z|^{-2h_{|m|+2n,m}}\,. & (4.12)\end{aligned}$$

We observe that the arguments of the sine function in the numerator and of the exponential diverge when k is sent to infinity, so that we get strongly oscillating expressions. Their combination behaves as

$$\begin{aligned}&2i\sin\frac{\pi(|m|+2n+1)(|\lfloor -Q(k+2)\rfloor|+2N+1)}{k+2}e^{\pi i\frac{m\lfloor -Q(k+2)\rfloor}{k+2}}\\ &\sim\Big(e^{i\frac{\pi|m||\lfloor -Q(k+2)\rfloor|}{k+2}}e^{i\frac{\pi[|m|(2N+1)+(2n+1)|\lfloor -Q(k+2)\rfloor|]}{k+2}}-e^{-i\frac{\pi|m||\lfloor -Q(k+2)\rfloor|}{k+2}}e^{-i\frac{\pi[|m|(2N+1)+(2n+1)|\lfloor -Q(k+2)\rfloor|]}{k+2}}\Big)\\ &\quad\times e^{i\frac{\pi m\lfloor -Q(k+2)\rfloor}{k+2}} \qquad (4.13)\\ &\sim\begin{cases}\Big(e^{2i\frac{\pi|m||\lfloor -Q(k+2)\rfloor|}{k+2}}e^{i\pi[|q|(2N+1)+(2n+1)|Q|]}-e^{-i\pi[|q|(2N+1)+(2n+1)|Q|]}\Big) & \text{for } qQ>0\\ \Big(e^{i\pi[|q|(2N+1)+(2n+1)|Q|]}-e^{-2i\frac{\pi|m||\lfloor -Q(k+2)\rfloor|}{k+2}}e^{-i\pi[|q|(2N+1)+(2n+1)|Q|]}\Big) & \text{for } qQ<0\,.\end{cases}\end{aligned} \tag{4.14}$$

Upon taking the average over m the strongly oscillating term is suppressed, and in the limit only the other term survives. The final result is therefore

$$\langle\Phi_{q,n}(z,\bar z)\rangle_{(Q,N)} = \frac{1}{2i\sqrt{\sin(\pi|q|)}}|z-\bar z|^{-2h_n(q)}\times\begin{cases}-e^{-i\pi[|q|(2N+1)+(2n+1)|Q|]} & \text{for } qQ>0\\ e^{i\pi[|q|(2N+1)+(2n+1)|Q|]} & \text{for } qQ<0\,.\end{cases} \tag{4.15}$$

Similarly, in the Ramond sector we find

$$\langle\Psi^0_q(z,\bar z)\rangle_{(Q,N)} = \frac{e^{-\pi i\frac{S}{2}}}{2i\sqrt{\sin(\pi|\frac12-q|)}}|z-\bar z|^{-1/4}\times\begin{cases}-e^{-i\pi|\frac12-q|(2N+1)} & \text{for } Q>0\\ e^{i\pi|\frac12-q|(2N+1)} & \text{for } Q<0\end{cases} \tag{4.16}$$

$$\langle\Psi^\pm_{q,n}(z,\bar z)\rangle_{(Q,N)} = \frac{e^{\mp\pi i\frac{S}{2}}|z-\bar z|^{-2h_n^\pm(q)}}{2i\sqrt{\sin(\pi|q\mp\frac12|)}}\times\begin{cases}-e^{-i\pi[|q\mp\frac12|(2N+1)+2n|Q|]} & \text{for } (q\mp\frac12)Q>0\\ e^{i\pi[|q\mp\frac12|(2N+1)+2n|Q|]} & \text{for } (q\mp\frac12)Q<0\,.\end{cases} \tag{4.17}$$

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5 Fields of charge zero

When we consider the limit of the spectrum (see figure 1), we observe that there are also fields of charge zero in the minimal model, which do not contribute to the averaged fields $\Phi_{q,n}$. This raises the question whether we can define another class of fields in the limit theory that arises from chargeless fields in the minimal model.

Following our general strategy of defining fields in the limit theory leads to an ansatz for chargeless fields which does not seem to give a sensible result. Therefore we will follow a different ansatz in section 5.2. The latter one appears to give a sensible class of fields which however is decoupled from the fields $\Phi_{q,n}$.

5.1 First ansatz: average of approximately chargeless fields

In the spirit of our general construction, we should try to define possible chargeless fields by averaging over fields that approximate a given conformal weight h and a $U(1)$ charge $q = 0$ in the limit. In contrast to our analysis in section 2 the label m now has to stay small compared to k, so that we do not need a strong fine-tuning in the growth of l and m to get a finite weight h (recall that the difference $l - |m| = 2n$ stays finite in that case). Instead both terms in the formula (2.8) for the weight $h_{l,m,0}$, the one depending on l and the one depending on m contribute to the weight on an equal footing. Therefore we are led to introduce labels y, p, μ such that

$$h = \frac{y^2}{4} \quad , \quad l = p\sqrt{k+2} \quad , \quad m = v\sqrt{k+2}\,. \tag{5.1}$$

For large quantum numbers l we then get the relation

$$y^2 = p^2 - v^2\,. \tag{5.2}$$

The condition that the charge $q = -\frac{m}{k+2}$ is close to zero, $|q| < \epsilon/2$, translates into a condition on v,

$$|v| < \frac{1}{2}\epsilon\sqrt{k+2}\,. \tag{5.3}$$

We define the set of labels $\hat{N}(y,\delta,\epsilon,k)$ that correspond to fields with charge approximately zero, and conformal weight close to $y^2/4$,

$$\hat{N}(y,\delta,\epsilon,k) = \left\{(l,m) \middle| |m| \leq l,\ |m| < \tfrac{\epsilon}{2}(k+2),\ |y - 2\sqrt{h_{l,m}}| < \tfrac{\delta}{2}\right\} \tag{5.4}$$

$$= \left\{(l,m) \middle| |m| \leq l,\ |v| < \tfrac{\epsilon}{2}\sqrt{k+2},\ |y - \sqrt{p^2 - v^2}| < \tfrac{\delta}{2}\right\}. \tag{5.5}$$

The cardinality of $\hat{N}$ for large level is given by

$$|\hat{N}(y,\delta,\epsilon,k)| = (k+2)y\delta \log\frac{\epsilon^2(k+2)}{y^2}\left(1 + \mathcal{O}(\delta)\right) + \mathcal{O}(\log(k+2))\,, \tag{5.6}$$

where in the leading term in $(k+2)$ we only stated the linear term in δ.

This suggests to define the averaged fields

$$\hat{\Phi}_y^{\delta,\epsilon,k} := \frac{1}{|\hat{N}(y,\delta,\epsilon,k)|}\sum_{(l,m)\in\hat{N}(y,\delta,\epsilon,k)} \phi_{l,m}\,. \tag{5.7}$$

Introducing a normalisation factor $\hat{\alpha}(y,k)$, we obtain the two-point function for the limit fields $\hat{\Phi}_y$,

$$\begin{aligned} &\langle \hat{\Phi}_{y_1}(z_1,\bar{z}_1)\hat{\Phi}_{y_2}(z_2,\bar{z}_2)\rangle \\ &\quad = \lim_{\epsilon,\delta\to 0}\lim_{k\to\infty}\frac{\hat{\alpha}(y_1,k)\hat{\alpha}(y_2,k)\beta(k)^2}{|\hat{N}(y_1,\delta,\epsilon,k)||\hat{N}(y_2,\delta,\epsilon,k)|}\sum_{(l_i,m_i)\in\hat{N}(y_i,\delta,\epsilon,k)}\frac{\delta_{l_1,l_2}\delta_{m_1,-m_2}}{|z_1-z_2|^{4h_{l_1,m_1}}} &(5.8)\\ &\quad = \frac{\delta(y_1-y_2)}{|z_1-z_2|^{y_1^2}}\,, &(5.9)\end{aligned}$$

for $\hat{\alpha}(y,k)=\sqrt{y\frac{\log(k+2)}{k+2}}$. We observe that in contrast to the analysis in section 2 the normalisation factor $\hat{\alpha}$ now depends on the field label.

This is only the first oddity of this construction. The main problem of this ansatz is that the fields $\hat{\Phi}_y$ come out as an average over fields with different values of p and v, whereas the correlators heavily depend on p and v, so that the fields over which we average do not tend to have a similar behaviour in the limit. This is against the spirit of the limiting procedure because we only want to combine fields that have similar behaviour. One can now explicitly check that correlators involving these fields do not have a well-defined limit. For example one can easily check that the one-point function of $\hat{\Phi}_y$ diverges for the $(0,0,0)$ boundary condition.

All in all, these results suggest that our ansatz for $\hat{\Phi}$ does not lead to a sensible field in the limit theory. On the other hand, the failure mainly resulted from the attempt to average over fields that do not behave similarly in the limit. Instead we will now try to define fields where we keep the quantum number $m=0$ fixed in the limit. These fields tend to behave much better in the limit, although they decouple from the charged fields $\Phi_{q,n}$. Their behaviour seems to point towards the existence of another consistent limit theory that is decoupled from the one that we discussed before. This will be further explored in [26].

5.2 Second ansatz starting from exactly chargeless fields

Again we introduce a label $p>0$ such that $h=\frac{p^2}{4}$. For fields $\phi_{l,0}$ that approach a conformal weight h the label l behaves as

$$l = p\sqrt{k+2}-1+\mathcal{O}(1/k^{1/2})\,, \tag{5.10}$$

it grows with the square root of k. In addition to the spectrum concentrated on lines of slope $(2n+1)$ that we found in section 2, we thus can try to define another continuous class of fields $\tilde{\Phi}_p$ that have U(1)-charge 0 and conformal weight $h(p)=p^2/4$.

Similarly we set up fields $\tilde{\Psi}^{\pm}_p$ in the Ramond sector with $q=\pm\frac{1}{2}$ and $h=\frac{1}{8}+\frac{p^2}{4}$. They arise from fields $\psi^{\pm}_{l,0}$ with $l\approx p\sqrt{k+2}$.

We now want to analyse correlators of the fields $\tilde{\Phi}_p$ of zero charge in the limit theory. As we have seen in (5.10), those fields have to be defined in terms of minimal model fields $\phi_{l,0}$, where the label l grows with the square root of $k+2$. We introduce the averaged fields

$$\tilde{\Phi}^{\epsilon,k}_p = \frac{1}{\left|\tilde{N}(p,\epsilon,k)\right|}\sum_{l\in\tilde{N}(p,\epsilon,k)}\phi_{l,0}\ . \tag{5.11}$$

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The set $\tilde{N}(p,\epsilon,k)$ contains those labels l such that the corresponding conformal weight is close to $h(p)=p^2/4$,

$$\tilde{N}(p,\epsilon,k)=\left\{l : p-\frac{\epsilon}{2}<\frac{l}{\sqrt{k+2}}<p+\frac{\epsilon}{2}\right\} . \tag{5.12}$$

For large k its cardinality is

$$\left|\tilde{N}(p,\epsilon,k)\right|=\epsilon\sqrt{k+2}+\mathcal{O}(1) . \tag{5.13}$$

Here we assumed $p>0$ and ϵ small enough such that $p-\frac{\epsilon}{2}>0$.

The two-point function of such fields in the limit theory is then

$$\langle\tilde{\Phi}_{p_1}(z_1,\bar{z}_1)\tilde{\Phi}_{p_2}(z_2,\bar{z}_2)\rangle=\lim_{\epsilon\to 0}\lim_{k\to\infty}\tilde{\alpha}(k)^2\beta(k)^2\langle\tilde{\Phi}^{\epsilon,k}_{p_1}(z_1,\bar{z}_1)\tilde{\Phi}^{\epsilon,k}_{p_2}(z_2,\bar{z}_2)\rangle \tag{5.14}$$

$$=\lim_{\epsilon\to 0}\lim_{k\to\infty}\frac{\tilde{\alpha}(k)^2\beta(k)^2}{\epsilon^2(k+2)}\sum_{l\in\tilde{N}(p_1,\epsilon,k)\cap\tilde{N}(p_2,\epsilon,k)}\frac{1}{|z_1-z_2|^{4h_{l,0}}} . \tag{5.15}$$

We introduced a new normalisation factor $\tilde{\alpha}(k)$ for the charge zero fields. The conformal weight $h_{l,0}$ approaches $h(p_1)=p_1^2/4$, and the sum can be replaced by the cardinality of the overlap,

$$\left|\tilde{N}(p_1,\epsilon,k)\cap\tilde{N}(p_2,\epsilon,k)\right|=\sqrt{k+2}(\epsilon-|p_1-p_2|)\theta(\epsilon-|p_1-p_2|)+\mathcal{O}(1) . \tag{5.16}$$

By using (2.28) and choosing

$$\tilde{\alpha}(k)\beta(k)=(k+2)^{1/4} , \tag{5.17}$$

we finally obtain

$$\left\langle\tilde{\Phi}_{p_1}(z_1,\bar{z}_1)\tilde{\Phi}_{p_2}(z_2,\bar{z}_2)\right\rangle=\delta(p_1-p_2)\frac{1}{|z_1-z_2|^{4h(p_1)}} . \tag{5.18}$$

A similar analysis can be done for the Ramond fields.

5.3 Boundary conditions

The one-point functions of the fields $\tilde{\Phi}_p$ are zero for the discrete class of boundary conditions that we analysed before. For the continuous class the one-point functions oscillate strongly with the label l, and after averaging they tend to zero as well. The construction of the chargeless fields however also suggests another way of defining boundary conditions in the limit theory. Namely we can scale the boundary labels in analogy to the charge zero fields $\tilde{\Phi}$,

$$\tilde{B}_k(P)=(\lfloor\sqrt{k+2}P\rfloor,0,0) . \tag{5.19}$$

For such boundary conditions we find trivial one-point functions for the fields $\Phi_{q,n}$,

$$\langle\Phi_{q,n}(z,\bar{z})\rangle_P=0 , \tag{5.20}$$

but a non-trivial result for the fields $\tilde{\Phi}_p$,

$$\langle \tilde{\Phi}_p(z,\bar{z})\rangle_P = \frac{\sin(\pi p P)}{\sqrt{\pi p}} |z-\bar{z}|^{-2h} \ . \tag{5.21}$$

Similarly we find in the Ramond sector

$$\langle \Psi^0_{q,n}\rangle_P = 0 = \langle \Psi^{\pm}_{q,n}\rangle \ , \tag{5.22}$$

and

$$\left\langle \tilde{\Psi}^{\pm}_p \right\rangle_P = \frac{\sin(\pi p P)}{\sqrt{\pi p}} \ . \tag{5.23}$$

These positive results are encouraging to continue the analysis of the fields $\tilde{\Phi}_p$.

We can also check whether we can define sensible B-type boundary conditions. They only couple to fields with opposite left- and right-moving $U(1)$ charges. As we are considering a diagonal theory with equal left- and right-moving quantum numbers, only fields of charge zero can couple to a B-type boundary condition in our case. In the minimal models the B-type boundary conditions [24] are labelled only by two labels L, S, where $0 \leq L \leq k$ and S is identified modulo 2. The boundary states are built from the B-type Ishibashi states by

$$|L,S\rangle = \sum_l (2k+4)^{1/2} \frac{S_{(L,0,0)(l,0,0)}}{\sqrt{S_{(L,0,0)(l,0,0)}}} (-1)^{l/2} \left(|l,0,0\rangle\rangle_B + e^{-i\pi S} |l,0,2\rangle\rangle_B \right) \ . \tag{5.24}$$

In particular this means that we have the one-point functions

$$\langle \phi_{l,m}(z,\bar{z})\rangle_{(L,S)} = \sqrt{2}\, \frac{\sin \frac{\pi(l+1)(L+1)}{k+2}}{\sqrt{\sin \frac{\pi(l+1)}{k+2}}} (-1)^{l/2}\, \delta_{m,0} |z-\bar{z}|^{-2h_{l,0}} \ . \tag{5.25}$$

Keeping the boundary label fixed while taking the limit, the one-point function vanishes because of the oscillating sign $(-1)^{l/2}$. On the other hand, if we redefined the fields $\phi_{l,0}$ by the factor $(-1)^{l/2}$, we would obtain modified fields $\tilde{\Phi}^{(\mathrm{mod})}_p$ in the limit theory with finite one-point functions

$$\langle \tilde{\Phi}^{(\mathrm{mod})}_p(z,\bar{z})\rangle_{(L,S)} = \sqrt{2\pi p}\, (L+1) |z-\bar{z}|^{-2h} \ . \tag{5.26}$$

Obviously, we have the relation

$$\langle \ \cdot \ \rangle_{(L,S)} = (L+1) \langle \ \cdot \ \rangle_{(0,S)} \ , \tag{5.27}$$

which means that all boundary conditions are just superpositions of the one with $L=0$. Again this reflects the fact that in the minimal models all B-type boundary conditions can be obtained by a boundary renormalisation group flow from superpositions of boundary conditions with $L=0$ [27].

At this point it is hard to decide which of the fields, $\tilde{\Phi}_p$ or $\tilde{\Phi}^{(\mathrm{mod})}_p$, is the better definition. $\tilde{\Phi}^{(\mathrm{mod})}_p$ has a non-trivial one-point function for B-type boundary conditions, on the other hand its one-point function in the presence of the A-type boundary conditions (5.19) labelled by P vanishes due to the oscillating sign.

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5.4 Three-point functions

We now want to determine the three-point correlation functions involving the fields $\tilde{\Phi}_p$, which have charge zero. Due to charge conservation the correlator $\left\langle \Phi\tilde{\Phi}\tilde{\Phi} \right\rangle$ is manifestly zero, so we only have to consider the combinations $\left\langle \Phi\Phi\tilde{\Phi} \right\rangle$ and $\left\langle \tilde{\Phi}\tilde{\Phi}\tilde{\Phi} \right\rangle$.

Let us start with the mixed correlator. We have

$$\begin{aligned}
&\langle \Phi_{q_1,n_1}(z_1,\bar{z}_1)\Phi_{q_2,n_2}(z_2,\bar{z}_2)\tilde{\Phi}_p(z_3,\bar{z}_3)\rangle \\
&\quad = \lim_{\epsilon\to 0}\lim_{k\to\infty} \beta(k)^2\alpha(k)^2\tilde{\alpha}(k)\langle \Phi^{\epsilon,k}_{q_1,n_1}(z_1,\bar{z}_1)\Phi^{\epsilon,k}_{q_2,n_2}(z_2,\bar{z}_2)\tilde{\Phi}^{\epsilon,k}_p(z_3,\bar{z}_3)\rangle \qquad (5.28)\\
&\quad = \lim_{\epsilon\to 0}\lim_{k\to\infty} \frac{\beta(k)^2\alpha(k)^2\tilde{\alpha}(k)}{\epsilon^3(k+2)^{5/2}} \sum_{\{m_i\in N(q_i,\epsilon,k)\}}\ \sum_{l\in\tilde{N}(p,\epsilon,k)} \delta_{m_1+m_2,0}\\
&\qquad \times C(|m_1|+2n_1,m_1;|m_1|+2n_2,-m_1;l,0)\\
&\qquad \times |z_{12}|^{2(h_3-h_1-h_2)}|z_{13}|^{2(h_2-h_1-h_3)}|z_{23}|^{2(h_1-h_2-h_3)}\,. \qquad (5.29)
\end{aligned}$$

Let us assume that $q_1<0$ such that $m_1>0$ and $m_2=-m_1<0$. To continue we have to determine the asymptotic behaviour of the three-point coefficient

$$\begin{aligned}
C(|m_1|+2n_1,m_1;|m_1|+2n_2,-m_1;l,0) &= \begin{pmatrix} \frac{m_1}{2}+n_1 & \frac{m_1}{2}+n_2 & \frac{l}{2}\\ \frac{m_1}{2} & -\frac{m_1}{2} & 0\end{pmatrix}^2\\
&\quad \times \sqrt{(|m_1|+2n_1+1)(|m_2|+2n_2+1)(l+1)}\,d_{m_1+2n_1,m_1+2n_2,l}\,. \qquad (5.30)
\end{aligned}$$

The coefficient $d_{m_1+2n_1,m_1+2n_2,l}$ behaves as

$$\begin{aligned}
d_{m_1+2n_1,m_1+2n_2,l} &= \frac{\Gamma(1+\rho)}{\Gamma(1-\rho)}P^2(m_1+n_1+n_2+\tfrac{l}{2}+1)\\
&\quad\times\frac{\Gamma(1-\rho(m_1+2n_1+1))\Gamma(1-\rho(m_1+2n_2+1))\Gamma(1-\rho(l+1))}{\Gamma(1+\rho(m_1+2n_1+1))\Gamma(1+\rho(m_1+2n_2+1))\Gamma(1+\rho(l+1))}\\
&\quad\times\frac{P^2(-n_1+n_2+\frac{l}{2})P^2(n_1-n_2+\frac{l}{2})P^2(m_1+n_1+n_2-\frac{l}{2})}{P^2(m_1+2n_1)P^2(m_1+2n_2)P^2(l)}\,. \qquad (5.31)
\end{aligned}$$

For the asymptotics of the functions P we use the result of appendix B. From (B.6) we get

$$\frac{P(m_1+n_1+n_2+\frac{l}{2}+1)P(m_1+n_1+n_2-\frac{l}{2})}{P(m_1+2n_1)P(m_1+2n_2)} \to \frac{\Gamma(1+|q_1|)}{\Gamma(1-|q_1|)}e^{\frac{1}{4}p^2(\psi(1+|q_1|)+\psi(1-|q_1|))}\,, \qquad (5.32)$$

where $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$ is the Digamma function. Similarly

$$\frac{P(-n_1+n_2+\frac{l}{2})P(n_1-n_2+\frac{l}{2})}{P(l)} \to \exp\left(\frac{p^2\gamma}{2}\right)\,, \qquad (5.33)$$

where $\gamma=\psi(1)$ denotes the Euler-Mascheroni constant. The coefficient $d_{m_1+2n_1,m_1+2n_2,l}$ thus has the limit

$$d_{m_1+2n_1,m_1+2n_2,l} \to e^{\frac{1}{2}p^2(2\gamma+\psi(1+|q_1|)+\psi(1-|q_1|))}\,. \qquad (5.34)$$

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The 3j-symbol behaves as (see (A.23) in appendix A)

$$\begin{pmatrix} \frac{m_1}{2}+n_1 & \frac{m_1}{2}+n_2 & \frac{l}{2} \\ \frac{m_1}{2} & -\frac{m_1}{2} & 0 \end{pmatrix}^2 = \frac{(n_1!n_2!)^{-1}}{|m_1|}\left(\frac{4|q_1|}{p^2}\right)^{-(n_1+n_2)}\left[{}_2F_0\left(-n_1,-n_2;-\frac{4|q_1|}{p^2}\right)\right]^2 \times\left(1+\mathcal{O}(k^{-1/2})\right) \ . \tag{5.35}$$

In total the three-point coefficient C has the behaviour

$$C(|m_1|+2n_1,m_1;|m_1|+2n_2,-m_1;l,0)\sim(k+2)^{1/4}\mathcal{C}_1(q_1,n_1,n_2,p) \tag{5.36}$$

with the regular function $\mathcal{C}_1$ given by

$$\mathcal{C}_1(q_1,n_1,n_2,p)=e^{\frac{1}{2}p^2(2\gamma+\psi(1+|q_1|)+\psi(1-|q_1|))}\,(n_1!n_2!)^{-1}\,p^{\frac{1}{2}}\left(\frac{4|q_1|}{p^2}\right)^{-(n_1+n_2)} \times\left[{}_2F_0\left(-n_1,-n_2;-\frac{4|q_1|}{p^2}\right)\right]^2 \ . \tag{5.37}$$

We finally arrive at the following expression for the three point function:

$$\langle\Phi_{q_1,n_1}(z_1,\bar z_1)\Phi_{q_2,n_2}(z_2,\bar z_2)\tilde\Phi_p(z_3,\bar z_3)\rangle=\lim_{k\to\infty}\frac{\beta(k)^2\alpha(k)^2\tilde\alpha(k)}{(k+2)^{3/4}} \times\delta(q_1+q_2)\mathcal{C}_1(q_1,n_1,n_2,p)|z_{12}|^{2(h_3-h_1-h_2)}|z_{13}|^{2(h_2-h_1-h_3)}|z_{23}|^{2(h_1-h_2-h_3)} \ . \tag{5.38}$$

This three-point function encodes the coupling between two fields Φ and one field $\tilde\Phi$. It still contains the normalisation factors. The factor in front can be evaluated as

$$\frac{\beta(k)^2\alpha(k)^2\tilde\alpha(k)}{(k+2)^{3/4}}=(k+2)^{-1/2} \ . \tag{5.39}$$

The correlator is therefore suppressed, and fields $\tilde\Phi_p$ cannot appear in the operator product expansion of two fields Φ_{q_i,n_i}. Hence, the fields $\tilde\Phi_p$ decouple from the fields $\Phi_{q,n}$. Had we instead used the modified fields $\tilde\Phi_p^{(\mathrm{mod})}$ that we introduced before eq. (5.26), we would have encountered an additional suppression from the oscillating factor $(-1)^{l/2}$.

Next we look at the correlator involving only $\tilde\Phi$. We find

$$\begin{aligned}&\left\langle\tilde\Phi_{p_1}(z_1,\bar z_1)\tilde\Phi_{p_2}(z_2,\bar z_2)\tilde\Phi_{p_3}(z_3,\bar z_3)\right\rangle\\&=\lim_{\epsilon\to0}\lim_{k\to\infty}\beta(k)^2\tilde\alpha(k)^3\left\langle\tilde\Phi^{\epsilon,k}_{p_1}(z_1,\bar z_1)\tilde\Phi^{\epsilon,k}_{p_2}(z_2,\bar z_2)\tilde\Phi^{\epsilon,k}_{p_3}(z_3,\bar z_3)\right\rangle &(5.40)\\&=\lim_{\epsilon\to0}\lim_{k\to\infty}\frac{\beta(k)^2\tilde\alpha(k)^3}{\epsilon^3(k+2)^{5/2}}\sum_{\{l_i\in\tilde N(p_i,\epsilon,k)\}}C(\{l_i,0\})\\&\qquad\times|z_{12}|^{2(h_3-h_1-h_2)}|z_{13}|^{2(h_2-h_1-h_3)}|z_{23}|^{2(h_1-h_2-h_3)} \ . &(5.41)\end{aligned}$$

The three-point coefficient C is given by

$$C(\{l_i,0\})=\begin{pmatrix}\frac{l_1}{2}&\frac{l_2}{2}&\frac{l_3}{2}\\0&0&0\end{pmatrix}^2\sqrt{(l_1+1)(l_2+1)(l_3+1)}\,d_{l_1,l_2,l_3} \ . \tag{5.42}$$

Using the asymptotic formula (B.6) for the function P that occurs in the coefficients d, we find

$$d_{l_1,l_2,l_3} \to 1 \;. \tag{5.43}$$

The average[6] of the square of the 3j-symbol behaves as (see (A.27))

$$\begin{pmatrix} l_1/2 & l_2/2 & l_3/2 \\ 0 & 0 & 0 \end{pmatrix}^2_{\mathrm{av}} \sim (k+2)^{-1}\frac{4}{\pi} \\ \times \left((p_1+p_2+p_3)(-p_1+p_2+p_3)(p_1-p_2+p_3)(p_1+p_2-p_3)\right)^{-1/2} \;. \tag{5.44}$$

Notice that the 3j-symbol vanishes if the argument of the square root becomes negative. In total the three-point coefficient C becomes

$$C(\{l_i,0\}) \sim (k+2)^{-1/4}\mathcal{C}_2(\{p_i\})\,, \tag{5.45}$$

with $\mathcal{C}_2$ given by

$$\mathcal{C}_2(\{p_i\}) = \frac{4}{\pi}\left(\frac{p_1p_2p_3}{(p_1+p_2+p_3)(-p_1+p_2+p_3)(p_1-p_2+p_3)(p_1+p_2-p_3)}\right)^{\frac{1}{2}} \;. \tag{5.46}$$

The limit of the three-point function is

$$\left\langle \tilde{\Phi}_{p_1}(z_1,\bar{z}_1)\tilde{\Phi}_{p_2}(z_2,\bar{z}_2)\tilde{\Phi}_{p_3}(z_3,\bar{z}_3)\right\rangle = \lim_{k\to\infty}\frac{\beta(k)^2\tilde{\alpha}(k)^3}{(k+2)^{1/4}} \\ \times \mathcal{C}_2(\{p_i\})|z_{12}|^{2(h_3-h_1-h_2)}|z_{13}|^{2(h_2-h_1-h_3)}|z_{23}|^{2(h_1-h_2-h_3)} \;. \tag{5.47}$$

Again we encounter a problem with the global factor that is given by

$$\frac{\beta(k)^2\tilde{\alpha}(k)^3}{(k+2)^{1/4}} = (k+2)^{-1/2}\,, \tag{5.48}$$

so that also this three-point function is suppressed. Note that we would have obtained the same result for the modified fields $\tilde{\Phi}_p^{(\mathrm{mod})}$ that we introduced before eq. (5.26), because the 3j-symbol involved is non-zero only for $l_1+l_2+l_3$ even, so that the sign $(-1)^{l_1+l_2+l_3}$ that appears in the computation of the correlator of the three fields $\tilde{\Phi}_p^{(\mathrm{mod})}$ is trivial.

How should we interpret these results? The vanishing of the three-point function $\langle\tilde{\Phi}\Phi\Phi\rangle$ tells us that the fields $\tilde{\Phi}$ and Φ decouple — in the operator product expansion (OPE) of two fields Φ there will never appear a field $\tilde{\Phi}$, and on the other hand in the OPE of two fields $\tilde{\Phi}$ there will never be a field Φ because of charge conservation. This means that there might be two different limiting theories, one involving the fields Φ and one that includes the fields $\tilde{\Phi}$. If this is true, then we can use a different normalisation factor β for the vacuum in the theory of the fields $\tilde{\Phi}$, thus rendering the three-point function $\langle\tilde{\Phi}\tilde{\Phi}\tilde{\Phi}\rangle$ finite. This will be further explored in [26].

[6] Note that the 3j-symbol in question oscillates rapidly if one varies the l_i. Due to the summation over the l_i we can insert the average value of the 3j-symbol squared.

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6 Conclusions

In this article we have analysed the limit of $N = (2,2)$ minimal models at central charge $c = 3$. In the Neveu-Schwarz sector we have identified fields $\Phi_{q,n}$ that are labelled by their non-zero $U(1)$ charge q $(0 < |q| < 1)$ and by a discrete label $n \geq 0$. We have computed the three-point functions of such fields by taking an appropriate limit of the correlators in the minimal models. We have also identified boundary conditions in the limit theory that lead to well-defined disc one-point functions for the fields $\Phi_{q,n}$. Although we have not checked crossing symmetry of the three-point functions, our results strongly suggest that the limit theory exists as a consistent conformal field theory.

In section 5 we discussed the question whether there are additional fields of zero charge. Our results indicate that there could be such fields $\tilde{\Phi}_p$, but they completely decouple from the charged fields $\Phi_{q,n}$. This points towards the existence of a second limit theory containing only chargeless fields. This second theory would arise by a different limit procedure where in addition to the weight h the label m is kept fixed. The simplicity of the three-point function (5.47) suggests that this second limit theory might well be the theory of two free bosons and fermions.

It is interesting to compare the results to the less supersymmetric situations. In a recent article [28], Gaberdiel and Suchanek argued that the limit of Virasoro minimal models at central charge $c = 1$ (the Runkel-Watts theory [1]) can be understood as a continuous orbifold of a free compact boson. A similar construction is proposed for other limit theories that are based on families of diagonal cosets. These results suggest that such limit theories could in general be related to free theories, and that the kind of non-rationality that one encounters in such limits is similar to the non-rationality that arises from the existence of a continuum of twisted sectors. Although the construction of Gaberdiel and Suchanek cannot be applied directly to the $N = 2$ case, because the coset structure is different, one might still suspect that the limit theory is related to a free orbifold. We plan to investigate this point in a subsequent publication [26].

In less supersymmetric situations, it has turned out that the limit theories are related to Liouville or more general conformal Toda theories. In [29, 30] it was shown that the limit of Virasoro minimal models coincides with the $c = 1$ limit of Liouville theory; similarly, the limit of $N = 1$ minimal models is related to $N = 1$ Liouville theory [3], and the limit of W_n minimal models to $SU(n)$ conformal Toda theories [2]. One might therefore wonder whether the $N = 2$ limit theories are related to $N = 2$ Liouville theory (see e.g. [31]) — or equivalently to its mirror [32, 33], the supersymmetric "cigar". When one compares these theories, one can observe that the so-called discrete representations in the Liouville spectrum precisely reproduce the spectrum of our limit theory. It would be interesting to work out this relation further.

Further clarification of the $N = 2$ limit theory will also come from a geometric point of view. In [24] a sigma model interpretation of the minimal models is given, which makes it possible to understand the limit of large levels also geometrically. This will be analysed in a forthcoming publication [26]. Finally one might consider the limit also in the framework of Landau-Ginzburg models, for which a similar limit has been mentioned in [34]. It would be interesting to work this out in more detail.

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Limits of $N = 2$ models recently have been discussed [15, 16] in the context of a duality of supersymmetric higher spin theories on AdS_3 backgrounds and two dimensional superconformal theories. There one does not only take the level k to infinity, but also the label n of the coset $\mathrm{SU}(n+1)_k/U(n)$ (the minimal models correspond to $n = 1$). Taking first $k \to \infty$ and then n corresponds to the case of zero 't Hooft coupling. It would be interesting to extend our analysis also to the case of $n > 1$.

Acknowledgments

We thank Bianca Dittrich, Matthias Gaberdiel, Ilarion Melnikov, Daniele Oriti and Volker Schomerus for useful discussions and valuable comments.

A Asymptotics of 3j-symbols

We want to approximate the Wigner 3j-symbols in the limit of large quantum numbers, in a specific range of parameters defined by the limiting procedure which is described in the core of this paper.

A.1 Notations and preliminaries

To set up our notations, let us briefly state the definition of the Clebsch-Gordan coefficients. A spin j representation V_j of $su(2)$ with standard generators J_i satisfying $[J_i, J_j] = i\epsilon_{ijk}J_k$ has a natural basis consisting of the eigenvectors $|j,\mu\rangle$ of the generator J_3 with eigenvalue μ. The tensor product of two irreducible representations can be decomposed into irreducible representations of the diagonal subalgebra,

$$V_{j_1} \otimes V_{j_2} = \bigoplus_j V_j\,, \tag{A.1}$$

where $|j_1 - j_2| \leq j \leq j_1 + j_2$ and $j + j_1 + j_2$ is an integer. The Clebsch-Gordan coefficients

$$\langle j_1, \mu_1; j_2, \mu_2 | j_1, j_2, j, \mu\rangle \tag{A.2}$$

are then given by the overlap of the two natural sets of basis vectors.

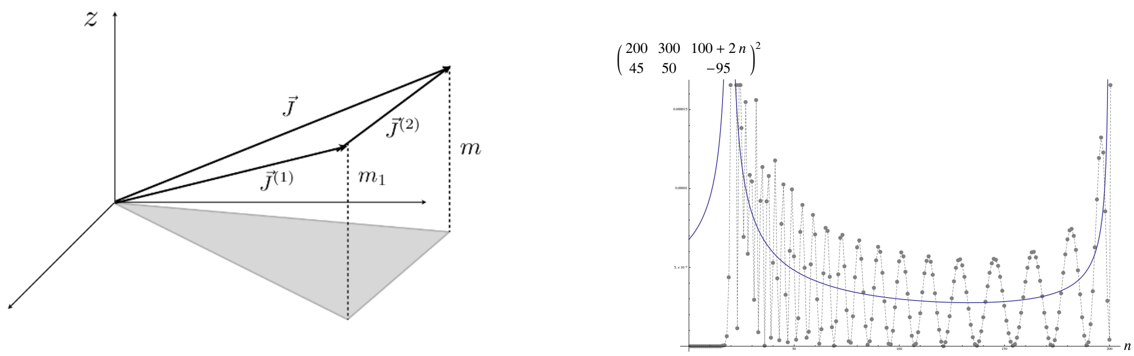
Closely related are the Wigner 3j-symbols that are defined as

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix} := \frac{(-1)^{j_1-j_2-\mu_3}}{\sqrt{2j_3+1}} \langle j_1, \mu_1; j_2, \mu_2 | j_1, j_2, j_3, -\mu_3\rangle\,, \tag{A.3}$$

with the choice of conventions: $\mu_3 = -\mu = -\mu_1 - \mu_2$.

An explicit expression was obtained by Racah in [35] (see e.g. [36, section 8.2, eq.3]),

$$\begin{aligned}\begin{pmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix} &= (-1)^{j_1-j_2-\mu_3} \left(\frac{(j_1+j_2-j_3)!(j_1-j_2+j_3)!(-j_1+j_2+j_3)!}{(j_1+j_2+j_3+1)!}\right)^{1/2} \\ &\quad\times \left[(j_1+\mu_1)!(j_1-\mu_1)!(j_2+\mu_2)!(j_2-\mu_2)!(j_3+\mu_3)!(j_3-\mu_3)!\right]^{1/2} \\ &\quad\times \sum_z \frac{(-1)^z}{z!(j_1+j_2-j_3-z)!(j_1-\mu_1-z)!(j_2+\mu_2-z)!(j_3-j_2+\mu_1+z)!(j_3-j_1-\mu_2+z)!}\,,\end{aligned} \tag{A.4}$$



(a) The shaded region is the projection of the triangle formed by the classical vectors on the x-y plane.

(b) The blue line is a plot coming from the Wigner estimate (A.7). The points connected by dashed lines are the exact values of the 3j-symbols. n ranges from 0 to 200.

Figure 3. Wigner approximation.

where the sum over z runs over all the values for which the arguments of the factorials in the denominator are non-negative. In particular, this formula provides a simple expression if one of the labels μ_i is extremal, e.g.

$$\begin{aligned}\begin{pmatrix} j_1 & j_2 & j_3 \\ -j_1 & \mu_2 & \mu_3 \end{pmatrix} &= \begin{pmatrix} j_3 & j_1 & j_2 \\ \mu_3 & -j_1 & \mu_2 \end{pmatrix} \\ &= (-1)^{j_3-j_1-\mu_2} \left(\frac{(-j_1+j_2+j_3)!(j_3+\mu_3)!(j_2+\mu_2)!(2j_1)!}{(j_1-j_2+j_3)!(j_1+j_2-j_3)!(j_3-\mu_3)!(j_2-\mu_2)!(j_1+j_2+j_3+1)!} \right)^{\frac{1}{2}} .\end{aligned} \tag{A.5}$$

A.2 Wigner's estimate

For large quantum numbers one expects the Clebsch-Gordan coefficients to be related to the classical problem of adding angular momenta. This issue has first been discussed by Wigner in [37]. To each quantum angular momentum specified by j_i, μ_i we therefore associate a vector $\vec{J}^{(i)}$ of length squared $|\vec{J}^{(i)}|^2 = j(j+1)$ and with specified z-component $J_z^{(i)} = \mu_i$. The x- and y- component are not specified. Classically such angular momenta can be coupled to zero if they satisfy the condition $\vec{J}^{(1)} + \vec{J}^{(2)} + \vec{J}^{(3)} = 0$. If this is the case, the triangle their projections form in the x-y-plane (see figure 3 (a)) has an area

$$A = \frac{1}{4}\sqrt{(\lambda_1+\lambda_2+\lambda_3)(-\lambda_1+\lambda_2+\lambda_3)(\lambda_1-\lambda_2+\lambda_3)(\lambda_1+\lambda_2-\lambda_3)}\,, \tag{A.6}$$

where $\lambda_i = \sqrt{|\vec{J}^{(i)}|^2 - |J_z^{(i)}|^2} = \sqrt{j_i(j_i+1) - \mu_i^2}$ are the lengths of the projections of $\vec{J}^{(i)}$ in the x-y-plane. The quantum numbers are then said to lie in a *classically allowed* region. If there are no associated vectors that can be added to zero, they belong to a *classically forbidden* region; in that case the "area" A in (A.6) is imaginary. If the projected triangle degenerates ($A = 0$), they are said to be in the *transition region*.

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Wigner gave an estimate of the averaged semiclassical behaviour of the Clebsch-Gordan coefficient in the allowed region [37],

$$|\langle j_1,\mu_1;j_2,\mu_2|j_1,j_2,j,\mu\rangle|^2_{\text{averaged}} \approx \frac{2j+1}{4\pi|A|}\,. \tag{A.7}$$

One naturally expects (and it is shown numerically e.g. in [38]) that the accuracy of the approximation goes down when the area A is small compared to the typical length squared of the vectors $\vec{J}^{(i)}$. For more discussions of the semi-classical asymptotics of the Wigner 3j-symbols see e.g. [38, 39].

For our main application, namely to determine the limit of the three-point function for the fields $\Phi_{q,n}$, we will see that we are precisely in this transition region, and we have to follow a different route to deal with the limit. For the correlator of the fields $\tilde{\Phi}_p$, however, we are in the classically allowed region, and the Wigner estimate applies.

A.3 Asymptotics for the correlators of charged fields

When we discuss the limit of the three-point functions for the charged fields Φ_{q_i,n_i} we are led to consider the asymptotics of the 3j-symbols for quantum numbers[7] $j_i = |\mu_i| + n_i$, where n_i is kept fixed, and the $|\mu_i|$ grow linearly in a parameter k.

The 3j-symbol vanishes unless the usual conditions on the addition of angular momenta are satisfied, namely

$$\mu_1+\mu_2+\mu_3 = 0 \quad \text{and} \quad j_{i_1}+j_{i_2} \geq j_{i_3} \tag{A.8}$$

for any permutation i_1, i_2, i_3 of $1, 2, 3$. If we assume $\mu_1, \mu_2 > 0$ and $\mu_3 < 0$, then for large $|\mu_i|$ the conditions on the j_i reduce to one condition $n_1 + n_2 \geq n_3$.

Because the z-components of the angular momenta $\vec{J}^{(i)}$ are close to maximal in our case, their projections to the x-y-plane are short and have lengths

$$\lambda_i = \sqrt{|\mu_i|(2n_i+1)+n(n+1)}\,, \tag{A.9}$$

which only grow with the square root of k. This means that the quantity A given in (A.6), which describes the area of the triangle in the x-y-plane provided it exists, is relatively small. Thus we are in the transition region between the classically allowed and the classically forbidden region, and cannot use the classical Wigner estimate.

Instead we can get the asymptotic behaviour directly from the Racah formula (A.4). Firstly, we have to understand the range of z in the sum in (A.4). In the limit of large μ_i we see that the arguments $j_2+\mu_2-z$ and $j-j_2+\mu_1+z$ do not constrain the sum since they are both surely positive. Bounds to the summation range are given by the other factorials in the denominator of equation (A.4), and the summation range is

$$\mathcal{I} := \{z \in \mathbb{Z}\,|\, z \geq 0,\ z \geq n_1-n_3,\ z \leq n_1+n_2-n_3,\ z \leq n_1\}\,. \tag{A.10}$$

Even in the limit of large $|\mu_i|$ the summation range stays finite, and its lower bound is either zero or $n_1 - n_3$ depending on its sign.

[7] In the main text we use $l_i = 2j_i$ and $m_i = 2\mu_i$.

The 3j-symbols can be rewritten as

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix} = (-1)^{j_1-j_2-\mu_3}$$
$$\times \underbrace{\left(\frac{[n_1+n_2-n_3]!}{[2(|\mu_1|+|\mu_2|)+n_1+n_2+n_3+1]!}\right)^{1/2} \times \left((n_1)!(n_2)!(n_3)![2(|\mu_1|+|\mu_2|)+n_3]!\right)^{1/2}}_{\mathbf{N}}$$
$$\times \sum_{z\in\mathcal{I}} \underbrace{\frac{\left([2|\mu_1|+n_1-n_2+n_3]![2|\mu_1|+n_1]!\right)^{1/2}}{\left([2|\mu_1|+n_3-n_2+z]!\right)^{1/2}\left([2|\mu_1|+n_3-n_2+z]!\right)^{1/2}}}_{\mathbf{A}}$$
$$\times \underbrace{\frac{\left([2|\mu_2|+n_2-n_1+n_3]![2|\mu_2|+n_2]!\right)^{1/2}}{\left([2|\mu_2|+n_2-z]!\right)^{1/2}\left([2|\mu_2|+n_2-z]!\right)^{1/2}}}_{\mathbf{B}}$$
$$\times \underbrace{\frac{(-1)^z}{z!}\frac{1}{[n_1+n_2-n_3-z]![n_1-z]![n_3-n_1+z]!}}_{\mathbf{C}} \ .$$

Using the fact that k is large we are able to recast parts A and B using that

$$\frac{(K+a)!}{K!} = \frac{K!}{K!}\times(K+1)\dots(K+a) = K^a\left(1+\mathcal{O}\left(\frac{1}{K}\right)\right) \qquad \text{for large } K \tag{A.11}$$

so that the leading contributions read

$$A \approx (2|\mu_1|)^{n_1+\frac{n_2-n_3}{2}-z}, \qquad B \approx (2|\mu_2|)^{z-\frac{n_1-n_3}{2}} \ . \tag{A.12}$$

Similarly, the factor N can be approximated by

$$N \approx (2|\mu_1|+2|\mu_2|)^{-\frac{n_1+n_2+1}{2}} \times \sqrt{n_1!n_2!n_3![n_1+n_2-n_3]!} \ . \tag{A.13}$$

The 3j-symbol then reads

$$\begin{aligned}\begin{pmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix} &= (-1)^{2|\mu_1|+n_1-n_2}\sqrt{n_1!n_2!n_3![n_1+n_2-n_3]!}(2|\mu_1|+2|\mu_2|)^{-\frac{n_1+n_2+1}{2}} \\ &\times \sum_{z\in\mathcal{I}}(-1)^z\frac{1}{z![n_1+n_2-n_3-z]![n_1-z]![n_3-n_1+z]!}(2|\mu_1|)^{n_1+\frac{n_2-n_3}{2}-z}(2|\mu_2|)^{z-\frac{n_1-n_3}{2}} \ .\end{aligned} \tag{A.14}$$

Introducing the notation

$$J = \frac{n_1+n_2}{2} \quad , \quad M = \frac{n_1-n_2}{2} \quad , \quad M' = -\frac{n_1+n_2}{2}+n_3\,, \tag{A.15}$$

and

$$\cos\beta = \frac{|\mu_1|-|\mu_2|}{|\mu_1|+|\mu_2|}\,, \tag{A.16}$$

we can express the asymptotic form of the 3j-symbol as

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix} \approx (-1)^{2|\mu_1|+n_3-n_2}(2|\mu_1|+2|\mu_2|)^{-\frac{1}{2}} d^J_{M',M}(\beta) \ . \tag{A.17}$$

Here, $d^J_{M',M}(\beta)$ denotes the Wigner d-matrix [36, 37],

$$\begin{aligned} d^J_{M',M}(\beta) &= \sqrt{(J+M')!(J-M')!(J+M)!(J-M)!} \\ &\times \sum_z \frac{(-1)^{M'-M+z}}{(J+M-z)!z!(M'-M+z)!(J-M'-z)!} \left(\cos\frac{\beta}{2}\right)^{2J+M-M'-2z} \left(\sin\frac{\beta}{2}\right)^{M'-M+2z} \ . \end{aligned} \tag{A.18}$$

The Wigner d-matrix is expressible in terms of standard ${}_2F_1$ hypergeometric functions. More precisely, for $n_1 \leq n_3$, we find

$$\begin{aligned} \begin{pmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix} &= (-1)^{2\mu_1+n_1-n_2} \frac{1}{(n_3-n_1)!} \sqrt{\frac{n_2!n_3!}{n_1!(n_1+n_2-n_3)!}} (2|\mu_1|+2|\mu_2|)^{-\frac{1}{2}} \\ &\times \frac{|\mu_1|^{n_1+\frac{n_2-n_3}{2}} |\mu_2|^{\frac{n_3-n_1}{2}}}{(|\mu_1|+|\mu_2|)^{\frac{n_1+n_2}{2}}} \, {}_2F_1\left(n_3-n_2-n_1, -n_1; n_3-n_1+1; -\frac{|\mu_2|}{|\mu_1|}\right)(1+\mathcal{O}(1/k)) \ , \end{aligned} \tag{A.19}$$

whereas for $n_1 \geq n_3$ we have

$$\begin{aligned} \begin{pmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix} &= (-1)^{2\mu_1+n_1-n_2} \frac{1}{(n_1-n_3)!} \sqrt{\frac{n_1!}{n_2!n_3!}} (2|\mu_1|+2|\mu_2|)^{-\frac{1}{2}} \\ &\times \frac{|\mu_1|^{\frac{n_2+n_3}{2}} |\mu_2|^{\frac{n_1-n_3}{2}}}{(|\mu_1|+|\mu_2|)^{\frac{n_1+n_2}{2}}} \, {}_2F_1\left(-n_3, -n_2; n_1-n_3+1; -\frac{|\mu_2|}{|\mu_1|}\right)(1+\mathcal{O}(1/k)) \ . \end{aligned} \tag{A.20}$$

A.4 Asymptotics for the mixed correlators

Now we want to study the asymptotics of the 3j-symbol when two of the (j,μ) pairs behave as before, i.e. $j_i = |\mu_i| + n_i$ $(i = 1,2)$ with fixed non-negative integers n_i, and the quantum numbers μ_i grow linearly with the parameter k. For the third coloumn we choose $\mu_3 = 0$ and j_3 grows with the square root of k. As the labels μ_i have to add up to zero, we have $\mu_2 = -\mu_1$ and we choose μ_1 to be positive.

From the Racah formula (A.4) we find

$$\begin{aligned} \begin{pmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & 0 \end{pmatrix} &= (-1)^{n_1-n_2} (n_1!n_2!)^{1/2} \\ &\times \sum_z (-1)^z \left(\frac{(2\mu_1+n_1+n_2-j_3)!(2\mu_1+n_1)!(2\mu_1+n_2)!}{(2\mu_1+n_1+n_2+j_3+1)![(2\mu_1+n_1+n_2-j_3-z)!]^2}\right)^{1/2} \\ &\times \left(\frac{(j_3+n_1-n_2)!(j_3-n_1+n_2)![j_3!]^2}{[(j_3-n_1+z)!(j_3-n_2+z)!]^2}\right)^{1/2} \frac{1}{z!(n_1-z)!(n_2-z)!} \ , \end{aligned} \tag{A.21}$$

where the sum runs from $z = 0$ to $z = \min(n_1, n_2)$. As in the previous discussion, the ratios of the factorials growing with $2\mu_1$ and also the ratios of the factorials growing with j_3 can be approximated by (A.11), and we obtain

$$\begin{aligned}
&\begin{pmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & 0 \end{pmatrix} \\
&= (-1)^{n_1-n_2} \frac{(n_1!n_2!)^{1/2}}{(2\mu_1)^{1/2}} \sum_z \frac{(-1)^z}{z!(n_1-z)!(n_2-z)!} \left(\frac{2\mu_1}{j_3^2}\right)^{z-\frac{n_1+n_2}{2}} \left(1+\mathcal{O}(k^{-1/2})\right) &\text{(A.22)} \\
&= (-1)^{n_1-n_2} \frac{(n_1!n_2!)^{-1/2}}{(2\mu_1)^{1/2}} \left(\frac{2\mu_1}{j_3^2}\right)^{-\frac{n_1+n_2}{2}} {}_2F_0\left(-n_1, -n_2; -\frac{2\mu_1}{j_3^2}\right) \left(1+\mathcal{O}(k^{-1/2})\right), &\text{(A.23)}
\end{aligned}$$

where ${}_2F_0$ denotes the corresponding hypergeometric function.

A.5 Asymptotics for the correlators of uncharged fields

Our third region of interest has all $\mu_i = 0$, and the j_i are growing at the same rate, proportional to the square root of k. The corresponding 3j-symbols are given by [36, section 8.5, eq.32]

$$\begin{aligned}
\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} &= (-1)^{\frac{j_1+j_2+j_3}{2}} \left(\frac{(-j_1+j_2+j_3)!(j_1-j_2+j_3)!(j_1+j_2-j_3)!}{(j_1+j_2+j_3+1)!}\right)^{1/2} \\
&\times \frac{\left(\frac{j_1+j_2+j_3}{2}\right)!}{\left(\frac{-j_1+j_2+j_3}{2}\right)! \left(\frac{j_1-j_2+j_3}{2}\right)! \left(\frac{j_1+j_2-j_3}{2}\right)!} \qquad \text{(A.24)}
\end{aligned}$$

if $|j_1 - j_2| \le j_3 \le j_1 + j_2$ and if $j_1 + j_2 + j_3$ is an even integer, otherwise it vanishes.

Let us for a moment assume that $j_1 + j_2 + j_3$ is even. To analyse the behaviour of the 3j-symbol we use Stirling's formula for the factorial,

$$n! = \sqrt{2\pi n}\, n^n e^{-n}(1 + \mathcal{O}(1/n)) \,. \qquad \text{(A.25)}$$

We find

$$\begin{aligned}
&\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{\frac{j_1+j_2+j_3}{2}} \sqrt{\frac{2}{\pi}} \\
&\quad \times \left((j_1+j_2+j_3)(-j_1+j_2+j_3)(j_1-j_2+j_3)(j_1+j_2-j_3)\right)^{-1/4} (1+\mathcal{O}(k^{-1/2})) \,. \\
&\hfill \text{(A.26)}
\end{aligned}$$

For the computations in the main text we are interested in the averaged value of the square of the 3j-symbol. In the allowed region, i.e. where $|j_1 - j_2| \le j_3 \le j_1 + j_2$, every second 3j-symbol vanishes due to the constraint that $j_1 + j_2 + j_3$ should be even. Therefore we obtain

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix}^2_{\mathrm{av}} \approx \frac{1}{\pi} \left((j_1+j_2+j_3)(-j_1+j_2+j_3)(j_1-j_2+j_3)(j_1+j_2-j_3)\right)^{-1/2} \,. \qquad \text{(A.27)}$$

This precisely equals the Wigner estimate (A.7), with the area A given in (A.6).

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B Asymptotics of products of Gamma functions

The three-point coefficient contains products of Gamma functions of the form (see (3.4))

$$P(l) = \prod_{j=1}^{l} \frac{\Gamma(1+j\rho)}{\Gamma(1-j\rho)}\,, \tag{B.1}$$

where $\rho = 1/(k+2)$. When we take the limit $k \to \infty$, also the quantum numbers become large, so that we have to determine the asymptotics of $P(l)$ for large l and k.

We write $l = f/\rho$, where f tends towards a constant f_0 in the limit,

$$\lim_{k\to\infty} f = f_0 \quad , \quad 0 \leq f_0 < 1\ . \tag{B.2}$$

We then have

$$\begin{aligned} P(f/\rho) &= \exp\left(\sum_{j=1}^{f\rho^{-1}} \log \frac{\Gamma(1+j\rho)}{\Gamma(1-j\rho)} \right) && \text{(B.3)} \\ &= \exp\left(\rho^{-1} \int_0^f \log \frac{\Gamma(1+x)}{\Gamma(1-x)} dx + \frac{1}{2} \log \frac{\Gamma(1+f)}{\Gamma(1-f)} + \mathcal{O}(\rho) \right)\,, && \text{(B.4)} \end{aligned}$$

where we employed the Euler-MacLaurin sum formula (see e.g. [40]). The integral is given by (see e.g. [41])

$$\int_0^f \log \frac{\Gamma(1+x)}{\Gamma(1-x)} dx = -f^2 + f \log \frac{\Gamma(1+f)}{\Gamma(1-f)} - \log\left[G(1+f)G(1-f)\right]\,, \tag{B.5}$$

where G is the Barnes G-function.[8]

When we write $f = f_0 + f_1$, where f_1 goes to zero in the limit, we obtain the asymptotic formula

$$\begin{aligned} P(f/\rho) = \exp\Bigg(&\rho^{-1}\left(-f_0^2 + f_0 \log \frac{\Gamma(1+f_0)}{\Gamma(1-f_0)} - \log\left[G(1+f_0)G(1-f_0)\right] \right) \\ &+ (\rho^{-1} f_1 + \tfrac{1}{2}) \log \frac{\Gamma(1+f_0)}{\Gamma(1-f_0)} + \frac{\rho^{-1} f_1^2}{2} \left(\psi(1+f_0) + \psi(1-f_0)\right) + \mathcal{O}(f_1, \rho, \rho^{-1} f_1^3) \Bigg)\,. \end{aligned} \tag{B.6}$$

Here, $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ denotes the Digamma function.

C Odd channel three-point functions

In this section we consider three-point functions of two primaries and one superdescendant field in minimal models. In the coset model description they can be derived from the three-point function of the $SU(2)$ WZW models as it has been done for the three-point function of three primaries in [19]. To this end one has to realise the superdescendants explicitly as

[8] G is related to the Barnes double gamma function $\Gamma_2(z; b_1, b_2)$ by $G(z) = \sqrt{2\pi}(\Gamma_2(z; 1, 1))^{-1}$.

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descendants in the $SU(2)$ model and determine the corresponding correlators. Although the computation is straightforward, to our knowledge these results have not appeared in the literature before.

For explicitness let us consider the Neveu-Schwarz correlator

$$\langle(\bar{G}^+_{-\frac{1}{2}}G^+_{-\frac{1}{2}}\phi_{l_1,m_1})(z_1,\bar{z}_1)\phi_{l_2,m_2}(z_2,\bar{z}_2)\phi_{l_3,m_3}(z_3,\bar{z}_3)\rangle\,, \tag{C.1}$$

where we assume that $|m_i| \leq l_i$ and $m_1 > 0$. Due to charge conservation a non-zero correlator has to satisfy

$$1 - \frac{m_1}{k+2} - \frac{m_2}{k+2} - \frac{m_3}{k+2} = 0\,. \tag{C.2}$$

In the coset description we have

$$\bar{G}^+_{-\frac{1}{2}}G^+_{-\frac{1}{2}}|l,m,0\rangle = \left(\frac{l(l+2)-m(m-2)}{2(k+2)}\right)^{-1}|l,m,2\rangle \tag{C.3}$$

for $-l+2 \leq m \leq l$. Notice that as in the main text we have chosen the diagonal minimal models with equal holomorphic and anti-holomorphic quantum numbers ($\bar{m} = m$). To relate the above three-point function to a correlator in the $SU(2)$ model we have to use the field identification

$$|l_1,m_1,2\rangle = |k-l_1,m_1-k-2,0\rangle = |\tilde{l}_1,-\tilde{l}_1-2n_1-2,0\rangle\,, \tag{C.4}$$

where we set $\tilde{l}_1 = k - l_1$ and $n_1 = \frac{l_1-|m_1|}{2}$. Then the $U(1)$ part of the coset trivially factorises (all three labels s_i are 0, and the new labels m_i add up to zero, $(m_1-k-2)+m_2+m_3 = 0$, which corresponds to the charge conservation condition (C.2)). The coset state $|\tilde{l}_1,-\tilde{l}_1-2(n_1+1),0\rangle$ comes from the state

$$\zeta_{l_1,n_1} = \gamma^{-1}_{l_1,n_1}(J^-_{-1})^{n_1+1}(\bar{J}^-_{-1})^{n_1+1}|\tilde{l}_1,-\tilde{l}_1,-\tilde{l}_1\rangle_{SU(2)} \tag{C.5}$$

in the $SU(2)$ model, where γ_{l_1,n_1} is a normalisation factor to ensure that $|\zeta_{l_1,n_1}|^2 = 1$. The conventions for the $SU(2)$ current algebra that we use here are given by

$$[J^+_m,J^-_n] = 2J^0_{m+n} + km\delta_{m+n,0} \quad , \quad [J^0_m,J^\pm_n] = \pm J^\pm_{m+n}\,. \tag{C.6}$$

The primary states in the $SU(2)$ model are labelled by $|l,m,\bar{m}\rangle_{SU(2)}$, where in our conventions

$$J^0_0|l,m,\bar{m}\rangle_{SU(2)} = \frac{m}{2}|l,m,\bar{m}\rangle_{SU(2)} \tag{C.7}$$

$$\left((J^0_0)^2 + \frac{1}{2}J^+_0J^-_0 + \frac{1}{2}J^-_0J^+_0\right)|l,m,\bar{m}\rangle_{SU(2)} = \frac{l(l+2)}{4}|l,m,\bar{m}\rangle_{SU(2)} \tag{C.8}$$

$$J^\pm_0|l,m,\bar{m}\rangle_{SU(2)} = c_\pm(l,m)|l,m\pm 2,\bar{m}\rangle_{SU(2)}\,, \tag{C.9}$$

where

$$c_\pm(l,m) = \frac{1}{2}\sqrt{(l\mp m)(l\pm m+2)}\,. \tag{C.10}$$

Anologous relations hold for the operators $\bar{J}$. Now it is easy to check inductively that

$$\gamma_{l_1,n_1} = \frac{(n_1+1)!\, l_1!}{(l_1-n_1-1)!}\ . \tag{C.11}$$

The coefficient of the three-point function (C.1) therefore can be read off from the $SU(2)$ correlator

$$\begin{aligned} F =& \left(\frac{2(n_1+1)(l_1-n_1)\gamma_{l_1,n_1}}{(k+2)}\right)^{-1} \\ & \times \langle \left((J^-_{-1})^{n_1+1}(\bar{J}^-_{-1})^{n_1+1}\chi_{\tilde{l}_1,-\tilde{l}_1,-\tilde{l}_1}\right)(z_1,\bar{z}_1)\chi_{l_2,m_2,m_2}(z_2,\bar{z}_2)\chi_{l_3,m_3,m_3}(z_3,\bar{z}_3)\rangle\,, \end{aligned} \tag{C.12}$$

where we denoted the field corresponding to the state $|l,m,\bar{m}\rangle$ by $\chi_{l,m,\bar{m}}$. This correlator can be computed starting from the known three-point function for primary fields [21, 22],

$$\begin{aligned} &\langle \chi_{l_1,m_1,\bar{m}_1}(z_1,\bar{z}_1)\chi_{l_2,m_2,\bar{m}_2}(z_2,\bar{z}_2)\chi_{l_3,m_3,\bar{m}_3}(z_3,\bar{z}_3)\rangle = \begin{pmatrix} \frac{l_1}{2} & \frac{l_2}{2} & \frac{l_3}{2} \\ \frac{m_1}{2} & \frac{m_2}{2} & \frac{m_3}{2}\end{pmatrix}\begin{pmatrix} \frac{l_1}{2} & \frac{l_2}{2} & \frac{l_3}{2} \\ \frac{\bar{m}_1}{2} & \frac{\bar{m}_2}{2} & \frac{\bar{m}_3}{2}\end{pmatrix} \\ &\times \sqrt{(l_1+1)(l_2+1)(l_3+1)}\, d_{l_1,l_2,l_3}\, |z_{12}|^{2(h_{l_3}-h_{l_1}-h_{l_2})}|z_{13}|^{2(h_{l_2}-h_{l_1}-h_{l_3})}|z_{23}|^{2(h_{l_1}-h_{l_2}-h_{l_3})}\,, \end{aligned} \tag{C.13}$$

where d_{l_1,l_2,l_3} is given in (3.3), and the conformal weights are

$$h_l = \frac{l(l+2)}{4(k+2)}\ . \tag{C.14}$$

Correlators of descendant fields are then computed by the usual contour integral techniques. Let us start with the simple case that there is only one operator J^-_{-1} acting on $\chi_{\tilde{l}_1,-\tilde{l}_1,-\tilde{l}_1}$. We find

$$\begin{aligned} &\langle \left(J^-_{-1}\chi_{\tilde{l}_1,-\tilde{l}_1,-\tilde{l}_1}\right)(z_1,\bar{z}_1)\chi_{l_2,m_2,\bar{m}_2}(z_2,\bar{z}_2)\chi_{l_3,m_3,\bar{m}_3}(z_3,\bar{z}_3)\rangle \\ &\quad = \frac{1}{2\pi i}\oint_{z_1}\frac{dw}{w-z_1}\langle J^-(w)\chi_{\tilde{l}_1,-\tilde{l}_1,-\tilde{l}_1}(z_1,\bar{z}_1)\chi_{l_2,m_2,\bar{m}_2}(z_2,\bar{z}_2)\chi_{l_3,m_3,\bar{m}_3}(z_3,\bar{z}_3)\rangle \\ &\quad = -\frac{1}{2\pi i}\left(\oint_{z_2}+\oint_{z_3}\right)\frac{dw}{w-z_1}\langle J^-(w)\chi_{\tilde{l}_1,-\tilde{l}_1,-\tilde{l}_1}(z_1,\bar{z}_1)\chi_{l_2,m_2,\bar{m}_2}(z_2,\bar{z}_2)\chi_{l_3,m_3,\bar{m}_3}(z_3,\bar{z}_3)\rangle \\ &\quad = \frac{1}{z_{12}}\langle \chi_{\tilde{l}_1,-\tilde{l}_1,-\tilde{l}_1}(z_1,\bar{z}_1)(J^-_0\chi_{l_2,m_2,\bar{m}_2})(z_2,\bar{z}_2)\chi_{l_3,m_3,\bar{m}_3}(z_3,\bar{z}_3)\rangle \\ &\qquad + \frac{1}{z_{13}}\langle \chi_{\tilde{l}_1,-\tilde{l}_1,-\tilde{l}_1}(z_1,\bar{z}_1)\chi_{l_2,m_2,\bar{m}_2}(z_2,\bar{z}_2)(J^-_0\chi_{l_3,m_3,\bar{m}_3})(z_3,\bar{z}_3)\rangle \\ &\quad = \frac{1}{z_{12}}c_-(l_2,m_2)\langle \chi_{\tilde{l}_1,-\tilde{l}_1,-\tilde{l}_1}(z_1,\bar{z}_1)\chi_{l_2,m_2-2,\bar{m}_2}(z_2,\bar{z}_2)\chi_{l_3,m_3,\bar{m}_3}(z_3,\bar{z}_3)\rangle \\ &\qquad + \frac{1}{z_{13}}c_-(l_3,m_3)\langle \chi_{\tilde{l}_1,-\tilde{l}_1,-\tilde{l}_1}(z_1,\bar{z}_1)\chi_{l_2,m_2,\bar{m}_2}(z_2,\bar{z}_2)\chi_{l_3,m_3-2,\bar{m}_3}(z_3,\bar{z}_3)\rangle\ . \end{aligned} \tag{C.15}$$

Due to the shift relations of 3j-symbols [36, section 8.4, eq.5] we have

$$c_-(l_3,m_3)\begin{pmatrix} \frac{\tilde{l}_1}{2} & \frac{l_2}{2} & \frac{l_3}{2} \\ -\frac{\tilde{l}_1}{2} & \frac{m_2}{2} & \frac{m_3}{2}-1\end{pmatrix} = -c_-(l_2,m_2)\begin{pmatrix} \frac{\tilde{l}_1}{2} & \frac{l_2}{2} & \frac{l_3}{2} \\ -\frac{\tilde{l}_1}{2} & \frac{m_2}{2}-1 & \frac{m_3}{2}\end{pmatrix}\ . \tag{C.16}$$

Therefore

$$
\begin{aligned}
&\langle \left(J^-_{-1}\chi_{\tilde{l}_1,-\tilde{l}_1,-\tilde{l}_1}\right)(z_1,\bar{z}_1)\chi_{l_2,m_2,\bar{m}_2}(z_2,\bar{z}_2)\chi_{l_3,m_3,\bar{m}_3}(z_3,\bar{z}_3)\rangle \\
&\quad = \left(\frac{1}{z_{12}}-\frac{1}{z_{13}}\right) c_-(l_2,m_2)\langle\chi_{\tilde{l}_1,-\tilde{l}_1,-\tilde{l}_1}(z_1,\bar{z}_1)\chi_{l_2,m_2-2,\bar{m}_2}(z_2,\bar{z}_2)\chi_{l_3,m_3,\bar{m}_3}(z_3,\bar{z}_3)\rangle \quad \text{(C.17)}\\
&\quad = c_-(l_2,m_2)\begin{pmatrix} \frac{\tilde{l}_1}{2} & \frac{l_2}{2} & \frac{l_3}{2} \\ -\frac{\tilde{l}_1}{2} & \frac{m_2}{2}-1 & \frac{m_3}{2}\end{pmatrix}\begin{pmatrix} \frac{\tilde{l}_1}{2} & \frac{l_2}{2} & \frac{l_3}{2} \\ -\frac{\tilde{l}_1}{2} & \frac{\bar{m}_2}{2} & \frac{\bar{m}_3}{2}\end{pmatrix}\sqrt{(\tilde{l}_1+1)(l_2+1)(l_3+1)}\, d_{\tilde{l}_1,l_2,l_3} \\
&\qquad \times z_{12}^{h_{l_3}-(h_{\tilde{l}_1}+1)-h_{l_2}}\bar{z}_{12}^{h_{l_3}-h_{\tilde{l}_1}-h_{l_2}} z_{23}^{(h_{\tilde{l}_1}+1)-h_{l_2}-h_{l_3}}\bar{z}_{23}^{h_{\tilde{l}_1}-h_{l_2}-h_{l_3}} z_{13}^{h_{l_2}-(h_{\tilde{l}_1}+1)-h_{l_3}}\bar{z}_{13}^{h_{l_2}-h_{\tilde{l}_1}-h_{l_3}} \,.
\end{aligned} \tag{C.18}
$$

Following this procedure iteratively, one obtains an expression for the correlator given in (C.12),

$$
\begin{aligned}
F = &\left(\frac{2(n_1+1)(l_1-n_1)\gamma_{l_1,n_1}}{(k+2)}\right)^{-1}\left(\prod_{i=0}^{n_1} c_-(l_2,m_2-2i)\right)^2 \\
&\times \begin{pmatrix} \frac{\tilde{l}_1}{2} & \frac{l_2}{2} & \frac{l_3}{2} \\ -\frac{\tilde{l}_1}{2} & \frac{m_2}{2}-n_1-1 & \frac{m_3}{2}\end{pmatrix}^2 \sqrt{(\tilde{l}_1+1)(l_2+1)(l_3+1)}\, d_{\tilde{l}_1,l_2,l_3} \\
&\times |z_{12}|^{2(h_{l_3}-(h_{\tilde{l}_1}+n_1+1)-h_{l_2})}|z_{23}|^{2((h_{\tilde{l}_1}+n_1+1)-h_{l_2}-h_{l_3})}|z_{13}|^{2(h_{l_2}-(h_{\tilde{l}_1}+n_1+1)-h_{l_3})} \,.
\end{aligned} \tag{C.19}
$$

To extract the corresponding minimal model correlator (C.1) we only have to shift the conformal weights by the contribution $-\frac{m^2}{4(k+2)}$ of the $U(1)_{2(k+2)}$ part,

$$
h_{l_3} \to h_{l_3} - \frac{m_3^2}{4(k+2)} = h_{l_3,m_3} \tag{C.20}
$$

$$
h_{l_2} \to h_{l_2} - \frac{m_2^2}{4(k+2)} = h_{l_2,m_2} \tag{C.21}
$$

$$
h_{\tilde{l}_1} + n_1 + 1 \to h_{\tilde{l}_1} + n_1 + 1 - \frac{(m_1-k-2)^2}{4(k+2)} = h_{l_1,m_1} + \frac{1}{2} \,. \tag{C.22}
$$

After simplifying the prefactor in (C.19) we obtain our final result for the minimal model correlator (C.1),

$$
\begin{aligned}
&\langle (\bar{G}^+_{-\frac{1}{2}}G^+_{-\frac{1}{2}}\phi_{l_1,m_1})(z_1,\bar{z}_1)\phi_{l_2,m_2}(z_2,\bar{z}_2)\phi_{l_3,m_3}(z_3,\bar{z}_3)\rangle \\
&\quad = \frac{k+2}{2(n_1+1)(l_1-n_1)}\begin{pmatrix}\frac{l_2+m_2}{2} \\ n_1+1\end{pmatrix}\begin{pmatrix}\frac{l_2-m_2}{2}+n_1+1 \\ n_1+1\end{pmatrix}\begin{pmatrix}l_1 \\ n_1+1\end{pmatrix}^{-1} \\
&\qquad \times \begin{pmatrix} \frac{\tilde{l}_1}{2} & \frac{l_2}{2} & \frac{l_3}{2} \\ -\frac{\tilde{l}_1}{2} & \frac{m_2}{2}-n_1-1 & \frac{m_3}{2}\end{pmatrix}^2 \sqrt{(\tilde{l}_1+1)(l_2+1)(l_3+1)}\, d_{\tilde{l}_1,l_2,l_3} \\
&\qquad \times |z_{12}|^{2\left(h_{l_3,m_3}-(h_{l_1,m_1}+1/2)-h_{l_2,m_2}\right)}|z_{23}|^{2\left((h_{l_1,m_1}+1/2)-h_{l_2,m_2}-h_{l_3,m_3}\right)} \\
&\qquad \times |z_{13}|^{2\left(h_{l_2,m_2}-(h_{l_1,m_1}+1/2)-h_{l_3,m_3}\right)} \,.
\end{aligned} \tag{C.23}
$$

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