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Quantum simplicial geometry in the group field theory formalism: reconsidering the Barrett–Crane model

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Abstract. A dual formulation of group field theories, obtained by a Fourier transform mapping functions on a group to functions on its Lie algebra, has recently been proposed. In the case of the Ooguri model for SO(4) BF theory, the variables of the dual field variables are thus $\mathfrak{so}(4)$ bivectors, which have a direct interpretation as the discrete B variables. Here we study a modification of the model by means of a constraint operator implementing the simplicity of the bivectors in such a way that projected fields describe metric tetrahedra. This involves an extension of the usual group field theory (GFT) framework, where boundary operators are labeled by projected spin network states. By construction, the Feynman amplitudes are simplicial path integrals for constrained BF theory. We show that the spin foam formulation of these amplitudes corresponds to a variant of the Barrett–Crane model for quantum gravity. We then re-examine the arguments against the Barrett–Crane model(s) in light of our construction.

Contents

1. Introduction

Group field theories (GFTs) represent a second quantized framework for both spin networks and simplicial geometry [6, 7, 9], field theories on group manifolds (or Lie algebras) producing Feynman amplitudes which can be equivalently expressed as simplicial gravity path integrals [5] or spin foam models [10], in turn covariant formulations of spin networks dynamics [1][3]. Most spin foam (and GFT) models for 4d gravity are obtained starting from models describing topological BF theory and adding constraints on the 2-form B field; imposing it comes from a tetrad 1-form as $B = *e \wedge e$. This is based on the Plebanski formulation [11, 12] of general relativity as a constrained BF theory. A classically equivalent modification of the same action is obtained by adding a topological term to it, which vanishes on-shell, with a coupling constant named the Immirzi parameter [11]. This gives the so-called Holst action, which is the classical starting point of loop quantum gravity (LQG) [1]. The construction of spin foam amplitudes takes place in a simplicial context, by first assigning to the two-dimensional (2D) faces of a simplicial complex data representing the discrete analogue of the fields of BF theory, that is, Lie algebra elements representing the B field, and group elements defining a discrete connection, and then defining a quantum amplitude imposing the Plebanski constraints on them. Ideally, one would want to constrain the discrete data at the classical level and then quantize the resulting geometric structures, as one would do in a simplicial path integral. However, this has proven very difficult until now, and the usual route [4, 17, 22] starts with a quantization of topological BF theory and proceeds by imposing the constraints at the quantum level, that is, at the level of quantum states, hoping to obtain the correct result nevertheless, and despite the ambiguities

[3] They can also be seen, from a more statistical field theory perspective, as a higher dimensional generalization of matrix models [2] and a particular class of tensor models [3].

involved when proceeding in this way. More precisely, existing models [22] (which exist in both Euclidean and Minkowskian signatures) result from the imposition of constraints via the solution of operator equations (e.g. via a master constraint technique) or from constraining the quantum labels of BF coherent states, identified with the quantum analogue of the bivector fields of this theory, and are further distinguished by the presence and value of the Immirzi parameter. In the absence of the Immirzi parameter γ (or in the $\gamma = \infty$ sector of Holst theory), one gets the Barrett–Crane (BC) model [17] by using the operator method and the so-called FK model [22] via the coherent state method. With finite $\gamma > 1$, the operator method gives the so-called EPRL model [22] or an extension of the FK model. These two models coincide remarkably for $\gamma < 1$, and in particular for $\gamma = 0$ (corresponding to the 'topological sector' of Holst gravity). The new models thus succeed in incorporating the Immirzi parameter into the spin foam (and GFT [19, 20, 54]) formalism, attempts at which began very early [37, 38].

In parallel with the above results, the simplicial path integral route has also been explored, providing important insights [21, 30]. In particular, the analysis in [21] clarified to a great extent the (quantum) simplicial geometry of the BC model. The results we present in this paper can be seen as a new take on, and a generalization of, the results presented there. A decisive impulse to this line of research was given recently by the definition of a metric representation of GFTs [5][4], in terms of B variables, and the proof of the exact duality of simplicial path integrals and spin foam models, within a GFT context, based on non-commutative tools [13–16], then applied also to LQG states [31]. Despite being phrased in terms of simplicial geometry and in seemingly classical terms, as appropriate for a path integral quantization, this method of quantization necessarily relies on a certain choice of quantization map (which dictates, for example, the choices of operator orderings). The one that seems [15] to be at the root of the non-commutative Fourier transform and star product on which these results rely is the Duflo quantization map [34]. This map is the most mathematically natural one for systems whose classical phase space is (based on) the cotangent bundle of some group manifold [36], has been successfully applied to the path integral formulation of the Chern–Simons theory [35] and is argued to be relevant also for LQG [36]. Further support for the interpretation and use of these tools comes from their application to the much simpler case of the quantization of a particle on the sphere, where they allow us to derive a complete and correct canonical path integral formulation of the dynamics [33][5].

The EPRL and FK models [22] have now replaced the BC model in the interest of the spin foam community. On the one hand, this is due to the mentioned success in incorporating the Immirzi parameter into the formalism, and to the consequent close relation between the boundary states associated with these new models and those of canonical LQG. On the other, this is due to a variety of results and arguments that have been put forward as implying the failure of the BC model in correctly describing quantum geometry and thus in representing a compelling candidate for a quantum gravity model[6].

These (numerous) arguments (related to each other) concern: the boundary states of the model (used also to define the vertex amplitude) and the quantization procedure used to obtain such states [22], the role of degenerate geometries [26], the possible lack of (discrete analogues

[4] For earlier attempts aiming at the same result, see [24].

[5] This metric formulation also allows us to identify diffeomorphism transformations at the GFT level, and to link nicely their features across the canonical, spin foam and simplicial gravity formulation of the same theory [32].

[6] These arguments mostly refer to the BC *vertex* in the spin representation, and to the choice of data entering it, also because there is no definite consensus on the other ingredients entering the model, i.e. 'measure' factors.

of) needed secondary second class constraints [39, 40] and, most importantly, the encoding of simplicial geometry in the quantum amplitudes.

This last issue, often referred to as the 'ultralocality of the BC model' , can be articulated as

(a) 4-simplices speak only through face representations, i.e. triangle areas [25];
(b) bivectors associated with the same triangle in different simplices are not identified [21, 22];
(c) normal vectors to the same tetahedron seen in different 4-simplices are uncorrelated [5, 43];
(d) the simplicity constraints are imposed in a non-covariant fashion [5];
(e) because of this non-covariance, there are missing constraints over the connection variables [5].

The last two issues had been identified in [5], in a first application of the non-commutative metric formalism to GFT gravity models, which has the advantage of making the simplicial geometry behind spin foam models manifest. In this paper, we take further advantage of this representation to re-analyse the imposition of the simplicity constraints without the Immirzi parameter in a GFT context, and thus in the simplicial gravity path integral based on BF theory appearing as the Feynman amplitude of the theory. We actually *solve* the above two issues, and keep under control the rest of the encoding of simplicial geometry at the quantum level, by means of a simple generalization of the GFT formalism itself. This generalization has also the advantage of recasting the boundary states of the resulting model explicitly in the form of *projected spin networks* [23, 42]. Moreover, we show that the resulting model is *unique*, in a sense we will specify. The Feynman (spin foam) amplitudes turn out to be still a variant of the BC model. We are thus led to re-analyse the existing criticisms on this model.

While we leave the inclusion of the Immirzi parameter into the formalism to a forthcoming paper [48], we also give a model for the topological sector of Plebanski gravity and compare it with the so-called EPR spin foam model.

We work in the GFT context because it is the most complete setting to study the dynamics of spin networks and simplicial structures. But it should be clear that the entire construction could be carried through directly at the level of simplicial path integrals (as done in [21]) or spin foam amplitudes, if one so prefers.

2. Group field theories in the metric representation

In this section, we briefly recall the construction of the metric formulation of GFTs [5]. We start with the GFT model for 4D SO(4) BF theory and then discuss, in the metric framework, the implementation of the simplicity constraints [5], leading to the (usual versions of the) BC model.

2.1. GFT for BF theory

The GFT for SO(4) BF theory is defined in terms of a field $\varphi_{1234} := \varphi(g_1, \ldots, g_4)$ on four copies of the group, satisfying the gauge invariance condition

$$\forall h \in \mathrm{SO}(4), \quad \varphi(g_1, \ldots, g_4) = \int \mathrm{d}h\, \varphi(hg_1, hg_2, hg_3, hg_4). \tag{1}$$

The dynamics is governed by the action

$$S = \frac{1}{2}\int [\mathrm{d}g_i]^4 \varphi^2_{1234} - \frac{\lambda}{5!}\int [\mathrm{d}g_i]^{10} \varphi_{1234}\,\varphi_{4567}\,\varphi_{7389}\,\varphi_{962\,10}\,\varphi_{10\,851},$$

where $[\mathrm{d}g_i]^4$ and $[\mathrm{d}g_i]^{10}$ denote the Haar measures on four and ten copies of SO(4). The combinatorics of the interaction term is that of a 4-simplex (the ten indices $i = 1, \ldots, 10$ corresponding to its ten triangles), while the kinetic terms dictate the gluing rules for 4-simplices along tetrahedra. The quantum theory is defined by the perturbative expansion in λ of the partition function and, by construction, the Feynman diagrams are 2-complexes dual to 4D simplicial complexes: vertices, links and faces (2-cells) of the graph are dual to 4-simplices, tetrahedra and triangles of the simplicial complex, respectively. Using harmonic analysis on SO(4), the Feynman amplitudes take the form of the Ooguri state sum model [27].

The metric formulation [5] of the same model is based on the group Fourier transform of the GFT field. The SO(3) group Fourier transform [13–16], which maps functions on the group to functions on its Lie algebra $\mathfrak{su}(2)$, straightforwardly extends to functions of (several copies of) $\mathrm{SO}(4) \sim \mathrm{SU}^-(2) \times \mathrm{SU}^+(2)/\mathbb{Z}_2$; it is invertible on even functions[7] $f(g) = f(-g)$. The group Fourier transform of the field φ is thus the function on four copies of the Lie algebra $\mathfrak{so}(4)$ given by

$$\widehat{\varphi}(x_1, \ldots, x_4) \equiv \int [\mathrm{d}g]^4\, \varphi(g_1, \ldots, g_4)\, e_{g_1}(x_1) \cdots e_{g_4}(x_4), \qquad x_i \in \mathfrak{so}(4) \sim \mathbb{R}^6, \tag{2}$$

where $[\mathrm{d}g]^4$ is the normalized Haar measure on $\mathrm{SO}(4)^4$. The plane waves $\mathrm{e}_g : \mathfrak{so}(4) \to \mathrm{U}(1)$ are the functions defined as $\mathrm{e}_g(x) = \mathrm{e}^{\mathrm{i}\mathrm{Tr} x g}$, where Tr is trace and is the fundamental representation of SO(4). Explicitly, using the decomposition of the Lie algebra and group elements $x = (x_-, x_+)$, $g = (g_-, g_+)$ into the left and right components $x_\pm \in \mathfrak{su}(2)$, $g_\pm \in \mathrm{SU}(2)$, and the parametrization $g_\pm = \mathrm{e}^{\theta \vec{n}\cdot\vec{\tau}}$ and $x^\pm = \vec{x}\cdot\vec{\tau}$ of these components in terms of the (anti-Hermitian) $\mathfrak{su}(2)$ generators $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$, the plane waves read

$$e_g(x) = \mathrm{e}^{\mathrm{i}\mathrm{Tr} x_- g_-} \mathrm{e}^{\mathrm{i}\mathrm{Tr} x_+ g_+},$$

where the trace is defined as $\mathrm{Tr}\,\tau_i\tau_j = -\delta_{ij}$.

The space of dual fields inherits by duality a non-trivial (non-commutative) pointwise product from the convolution product on the group. It is defined on plane waves as

$$(e_g \star e_{g'})(x) := e_{gg'}(x), \tag{3}$$

and extends component-wise to the tensor product of four plane-waves and by linearity to the whole image of the Fourier transform.

The Lie algebra variables of the dual field $\widehat{\varphi}$ are interpreted, geometrically, as the bivectors associated with a triangle, in each tetrahedron, in the standard discretized version of BF theory. Upon Fourier transform, the gauge invariance condition (1) translates into a closure constraint $\widehat{C}$ for the bivectors associated with each tetrahedron (field), obtained as the group Fourier transform of the gauge invariance projector $\mathcal{P}$:

$$\mathcal{P}\varphi(g_1, \ldots, g_4) = \int \mathrm{d}h\, \varphi(hg_1, \ldots, hg_4), \quad \to \quad \widehat{\mathcal{P}\varphi} = \widehat{C} \star \varphi, \tag{4}$$

[7] In the following, we thus assume further invariance of the Ooguri field under $g_i \to -g_i$ in each of the variables.

where $\widehat{C}(x_1, \ldots, x_4) = \delta_0(\sum_{n=1}^4 x_n)$, and δ_0 is the element $x = 0$ of the family of functions δ_x defined as

$$\delta_x(y) := \int \mathrm{d}h\, e_{h^{-1}}(x) e_h(y). \tag{5}$$

These functions play the role of Dirac distributions in the non-commutative setting, as

$$\int \mathrm{d}^6 y\, (\delta_x \star f)(y) = f(x), \tag{6}$$

where $\mathrm{d}^6 y$ is the standard Lebesgue measure on $\mathfrak{so}(4) \sim \mathbb{R}^6$.

The combinatorial structure of the GFT action in the metric representation is the same as in group one, while group convolution is replaced by the $\star$-product. Using the short notation $\widehat{\varphi}_{1234} := \widehat{\varphi}(x_1, \ldots, x_4)$, we can write the action as

$$S[\widehat{\varphi}] = \int [\mathrm{d}^6 x_i]^4 \widehat{\varphi}_{1234} \star \widehat{\varphi}_{1234} + \lambda \int [\mathrm{d}^6 x_i]^{10} \widehat{\varphi}_{1234} \star \widehat{\varphi}_{4567} \star \widehat{\varphi}_{7389} \star \widehat{\varphi}_{96210}, \tag{7}$$

where it is understood that $\star$-products relate repeated lower indices as $\widehat{\phi}_i \star \widehat{\phi}_i := (\widehat{\phi} \star \widehat{\phi}_-)(x_i)$, with $\widehat{\phi}_-(x) = \widehat{\phi}(-x)$.

In this representation, the Feynman amplitudes of the GFT can be computed by sticking together with the $\star$-product the propagator and vertex functions given by

$$P(x, x') = \int [\mathrm{d}h] \prod_{i=1}^4 \delta_{-x_i}(x_i'), \qquad V(x, x') = \int \prod_{\ell=1}^5 [\mathrm{d}h_\ell] \prod_{i=1}^{10} (\delta_{-x_i^\ell} \star e_{h_\ell h_{\ell'}^{-1}})(x_i^{\ell'}). \tag{8}$$

The vertex function has 20 Lie algebra variables: two variables $x_i^\ell, x_i^{\ell'}$ for each of the ten triangles of a 4-simplex, corresponding to the two tetrahedra ℓ, ℓ' sharing that triangle. The vertex function encodes the identification of the bivectors associated with the same triangle in different tetrahedral frames [5], up to parallel transport between these frames given by the group elements $h_\ell h_{\ell'}^{-1}$; the sign difference reflects the fact, in an oriented 4-simplex, a triangle inherits opposite orientations from the two tetrahedra sharing it.

The Feynman amplitudes in this representation are simplicial path integrals for BF theory, involving an integration over one SO(4) variable h_l for each link of the Feynman graph, equivalently of each tetrahedron of the dual simplicial complex, which is interpreted as parallel transport between the two 4-simplices sharing that tetrahedron; and a single $\mathfrak{so}(4)$ variable x_t for each triangle t of the dual simplicial complex:

$$\mathcal{A}_\mathcal{G} = \int \prod_l \mathrm{d}h_l \prod_e [\mathrm{d}^6 x_t]\, \mathrm{e}^{\mathrm{i} \sum_t \mathrm{Tr}\, x_t H_t}, \tag{9}$$

$\mathrm{d}h_l$ is the Haar measure on SO(4) and $\mathrm{d}^6 x_t$ is the Lebesgue measure on $\mathfrak{so}(4) \sim \mathbb{R}^6$. The group element $H_t := \overrightarrow{\prod}_{l \in \partial f_t} h_l$ is the ordered product of group elements h_l on the links on the boundary of the face (2-cell) f_t of the graph dual to the triangle t, giving a measure of the discrete curvature associated with it. Such a discrete curvature is calculated for a given orientation of f_t, and a given 'reference' vertex in its boundary; the final amplitude is independent of this choice. The exponential corresponds to the product of plane waves $\prod_t \mathrm{e}_{H_t}(x_t)$; its argument $\sum_t \mathrm{Tr}\, x_t H_t$ is the unconstraint discrete BF action.

Importantly, the expression generalizes naturally [5] to the case of open Feynman diagrams (GFT transition amplitudes), i.e. simplicial complexes with boundary, to give BF path integrals with fixed B variables on the boundary, with appropriate boundary terms [50].

2.2. Imposing the simplicity constraints

This new representation of the Ooguri model for BF theory provides a convenient starting point for imposing in a geometrically transparent manner the discrete simplicity constraints that turn BF theory into 4D simplicial gravity. The simplicity conditions of Plebanski gravity [28] (or the gravitational sector of Holst gravity [29]), thus without any Immirzi parameter, ensure that the B-field can be expressed in terms of 1-form fields e_I as $B^{IJ} = \varepsilon^{IJKL} e_I \wedge e_J$. At the discrete level, part of these conditions (in their linearized form [22, 39, 47]) are implemented by requiring that, for each tetrahedron in the simplicial complex, the four bivectors B_j^{IJ} associated with its four triangles $j = 1, \ldots, 4$, expressed in the reference frame of the tetrahedron, all lie in the hyperplane orthogonal to some normal vector in $\mathbb{R}^4$. Concretely, using the canonical decomposition of bivectors into self-dual and anti-self-dual $\mathfrak{su}(2)$-components $B^{\pm i}\tau_i$ with $2B^{\pm i} = \frac{1}{2}\epsilon^{ijk}B_{jk} + B^{0i}$, the conditions are imposed by requiring the existence of $k \in \mathrm{SU}(2)$ ($\sim \mathcal{S}^3$) independent of j such that

$$\forall j, \quad B_j^+ = -kB_j^-k^{-1}. \tag{10}$$

Back to GFT, where the role of the discrete B is played by the field variables x_j in the metric representation, the condition (10) can be implemented on the GFT field by defining the projector $\widehat{\varphi} \mapsto \widehat{S}_k \star \widehat{\varphi}$, where $\widehat{S}_k$ is the function of four $\mathfrak{so}(4)$-variables defined in terms of the non-commutative δ functions (5) as

$$\widehat{S}_k(x_1, \ldots, x_4) = \prod_{j=1}^{4} \delta_{-kx_j^-k^{-1}}(x_j^+). \tag{11}$$

Such a projector imposes strongly the relations (10) on the field variables, since the non-commutative δ functions act as the Dirac distribution for the $\star$-product.

Upon group Fourier transform, one can show that

$$\widehat{S}_k \star \widehat{\varphi} = \int_{\mathrm{SO}(4)^4} [\mathrm{d}g_j]^4 \, (S_k\varphi)(g_j j)\, e_{g_1} \cdots e_{g_4}, \tag{12}$$

where S_k projects onto fields on the product of four copies of the homogeneous space $\mathcal{S}^3 \sim \mathrm{SO}(4)/\mathrm{SO}(3)_k$, $\mathrm{SO}(3)_k$ being the stabilizer group of k seen as a vector[8] of $\mathbb{R}^4$; namely, using the decomposition $g = (g^-, g^+)$ into self-dual and anti-self-dual components:

$$(S_k\varphi)(g) := \int_{\mathrm{SO}(3)_k^4} [\mathrm{d}u_j]^4 \, \varphi(k^{-1}u_j k g_j^-, u_j g_j^+).$$

The case $k = 1$ reproduces the standard BC projector [19, 20]. This constraint is then imposed on each GFT field, i.e. on each tetrahedron of the simplicial complex, in its associated frame. The task of correctly parallel transporting these conditions in the other frame associated with other tetrahedra and simplices of the simplicial complex is accomplished by propagator and vertex GFT functions.

[8] Thanks to the isomorphism $\mathrm{SO}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)/\mathbb{Z}_2$, each element of $\mathrm{SO}(4)$ can be represented as $g = (g^-, g^+)$, with $g^\pm \in \mathrm{SU}(2)$. The map $\mathrm{SO}(4) \to \mathrm{SU}(2)$, $g \mapsto g^+(g^-)^{-1}$, with kernel $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$, realizes the identification of the homogeneous space $\mathcal{S}^3 \sim \mathrm{SO}(4)/\mathrm{SO}(3)$ or the set of normal vectors in $\mathbb{R}^4$, and the group $\mathrm{SU}(2)$. Through this identification, the rotation by $g = (g^-, g^+)$ of a normal vector represented by $k \in \mathrm{SU}(2)$ is represented by $g^+k(g^-)^{-1}$. The stabilizer group $\mathrm{SO}(3)_k$ of k is thus the set of $g = (g^-, g^+) \in \mathrm{SO}(4)$ such that $g^- = k^{-1}g^+k$, i.e. of the form $g = (k^{-1}uk, u)$, with $u \in \mathrm{SU}(2)$.

The above amounts to, roughly speaking, half of the full simplicity conditions ensuring geometricity of the bivectors. The other half, often linked to the secondary constraints arising from the canonical analysis [39, 51], can be shown to be imposed automatically if the closure condition is imposed on *all* tetrahedra in the simplicial complex (see, e.g., [47]), and if bivectors are correctly parallel transported across frames. This condition is imposed on GFT fields already at the level of the action, via the 'closure' projector $\widehat{C}$ given by (4), dual to the gauge invariance projector. Thus, it is imposed on *every* tetrahedron of the simplicial complex. Therefore, we do not impose further conditions on the bivectors in what follows.

By combining the BC projector $\widehat{S}_1$ with closure, one can build up the field $\widehat{\Psi} := \widehat{S}_1 \star \widehat{C} \star \widehat{\varphi}$, defining a geometric tetrahedron. However, one immediately notices that, because the two functions $\widehat{S}_k$ and $\widehat{C}$ do not $\star$-commute, the product $\widehat{S}_k \star \widehat{C}$ does not act as a projector. This fact can be shown to be the manifestation of a geometric inconsistency in the way the resulting model imposes the simplicity constraints.

In fact, given $h \in \mathrm{SO}(4)$, one has

$$(e_h \star \widehat{S}_k)(x) = (\widehat{S}_{h \triangleright k} \star e_h)(x) \tag{13}$$

with $h \triangleright k := h^+ k (h^-)^{-1}$. This expresses the fact that, after rotation by h, simple bivectors with respect to the normal k become simple with respect to the *rotated* normal $h \triangleright k$. Imposing the simplicity condition everywhere with the same normal k simply misses the fact that, under rotation h, normals should be rotated as well (which strengthens the relation between the normal variables and the gauge connection). This means that the simplicity constraints are imposed *non-covariantly* by this procedure.

But it is exactly this procedure that defines the usual versions of the BC models. In fact, combining the interaction term

$$\frac{\lambda}{5!} \int \widehat{\Psi}_{1234} \star \widehat{\Psi}_{4567} \star \widehat{\Psi}_{7389} \star \widehat{\Psi}_{962\,10} \star \widehat{\Psi}_{10\,851}, \tag{14}$$

where $\widehat{\Psi} := \widehat{S}_1 \star \widehat{C} \star \widehat{\varphi}$, with the possible kinetic terms

$$\frac{1}{2} \int \widehat{\Psi}^{\star 2}_{1234}, \quad \frac{1}{2} \int (\widehat{C} \star \widehat{\varphi})^{\star 2}_{1234} \quad \text{or} \quad \frac{1}{2} \int \widehat{\varphi}^{\star 2}_{1234} \tag{15}$$

gives the versions of the BC model derived in [19], [20] or [21], respectively. The existence of different versions, all differing, in spin foam representations, by edge (tetrahedra) amplitudes only, is then understood to be due precisely to the non-covariant imposition of the simplicity constraints (see [50]).

In this metric representation, the GFT Feynman amplitudes of the constrained theories defined by (14) and (15) take the form of simplicial gravity path integrals, with a measure weighted by a constrained BF action [5]. A more general version of this calculation is detailed in the next section, so we do not repeat it here. The spin foam representation is then obtained by the successive application of the group Fourier transform and of the Peter–Weyl decomposition to the same amplitudes, and by the integration of all but the representation variables to encode the dynamics of the quantum simplicial geometry.

One can then analyse the encoding of simplicial geometric conditions in this model directly from the form of the GFT action, and test the criticisms mentioned in the introduction. Here we only note that the model couples the bivector variables x across simplices and that the interplay of simplicity constraints and parallel transport (following the covariance constraint) leads to further conditions involving both bivector and connection variables. It would be tempting to

interpret them as a discrete version of the secondary constraints of the canonical theory, but this cannot be done in a rigorous way at this stage. At the same time, normals to the same tetrahedron, as seen in different 4-simplices, are *not* correlated, as they should on geometric grounds, and this in turn implies a missing geometric condition on connection variables h_ℓ—the one resulting from the requirement that it correctly transports normal vectors across simplices.

We postpone a more detailed discussion of the simplicial geometric aspects to the last section. But first, we will introduce a modification of the above construction in order to solve the issue of non-covariant imposition of the simplicity constraints, and thus the consequent lack of constraints on the discrete connection and of correlations among normal variables. This is the goal of the next section.

3. Extended GFT formalism

We have seen that some of the problematic issues in the standard formulation of the BC model have to do with the way normal vectors to each tetrahedron are treated. A simple generalization of the GFT field on which both the Ooguri and BC models are based leads to an easy solution of them.

3.1. Tetrahedra normals and gauge covariance

The required step is to promote the would-be normal vector to the tetrahedron to a dynamical variable $k \in S^3 \simeq \mathrm{SU}(2)$, by adding it as an independent argument to the Ooguri field, representing the tetrahedron[9]. Thus, in this extended formalism, GFT are defined in terms of fields on $\mathfrak{so}(4)^4 \times \mathrm{SU}(2)$:

$$\widehat{\varphi}_k(x_1, \ldots, x_4) := \widehat{\varphi}(x_1, \ldots, x_4; k) = \int [\mathrm{d}g]^4 \, \varphi(g_1, \ldots, g_4; k) \, e_{g_1}(x_1) e_{g_2}(x_2) e_{g_3}(x_3) e_{g_4}(x_4). \tag{16}$$

In adopting the new field as our basic dynamical variable in a new formulation of quantum 4D BF theory, we still want to impose the local gauge invariance of the simplicial theory, as the invariance under change of local frame in each tetrahedron. This time, however, the local rotation should simultaneously rotate the bivectors and the normal vectors. In other words, the gauge invariance condition for the Ooguri field is now replaced by a gauge covariance of the SO(4) arguments with respect to the normal k, with the overall 5-argument field being invariant:

$$\varphi(hg_1, \ldots, hg_4; h \triangleright k) = \varphi(g_1, \ldots, g_4; k) \quad \forall h \in \mathrm{SO}(4), \tag{17}$$

where $h \triangleright k := h^+ k (h^-)^{-1}$ denotes the action of the SO(4)-rotation $h := (h^-, h^+)$ on the normal vector[10] $k \in \mathrm{SU}(2) \sim \mathcal{S}^3$.

To express this invariance in the metric representation, let us introduce the family of functions labeled by $h \in \mathrm{SO}(4)$, given by a product of four SO(4) plane waves $e_h(x) := e_{h^-}(x^-) e^+_h(x^+)$[11]:

$$\widehat{C}_h(x_1, \ldots, x_4) = e_h(x_1) \cdots e_h(x_4). \tag{18}$$

[9] Obviously, before any geometricity condition is imposed, constraining the bivector variables to be triangle area bivectors and relating them to the normal vector, any geometric interpretation is to be understood as purely suggestive.

[10] See footnote 8 in section 2.2.

[11] Note that these are the same functions used to impose gauge invariance in the Ooguri BF model.

Upon group Fourier transform, the gauge covariance condition (17) reads

$$(\widehat{C}_h \star \widehat{\varphi}_{h^{-1}\triangleright k})(x_1, \ldots, x_4) = \widehat{\varphi}_k(x_1, \ldots, x_4) \qquad \forall h \in \mathrm{SO}(4). \tag{19}$$

This condition is implemented by means of the constraint projector:

$$\widehat{C}\widehat{\varphi}_k := \int \mathrm{d}h\, \widehat{C}_h \star \widehat{\varphi}_{h^{-1}\triangleright k}. \tag{20}$$

Two important points must be noted here. First, the SO(4)-rotation invariance (17) of the extended field induces an invariance under the stabilizer group $\mathrm{SO}(3)_k$ of the normal k, affecting only the first four arguments of the GFT field. Upon Peter–Weyl decomposition of the φ_k, this means that the field modes are labeled by one normal vector k, four irreducible SO(4) representations (given by pairs of SU(2) spins $J_i = (j_i^+, j_i^-)$, $i = 1, \ldots, 4$), each of which can be further decomposed into $\mathrm{SO}(3)_k$ representations k_i; and a four-valent $\mathrm{SO}(3)_k$ intertwiner, labeled by a SU(2) spin j, contracting these representations. In terms of the dual spin network vertex, the labeling thus corresponds to an SO(4) representation in each link, and a normal vector k and an $\mathrm{SO}(3)_k$ intertwiner at the node. We thus obtain, as basis for the Peter–Weyl decomposition of the GFT field, the functions given in terms of the SU(2) Wigner matrices $D^j(g)$ as

$$\Psi^{(J_i,k_i,j)}_{m_i,m_i'}(g_i; k) = \left(\prod_{i=1}^{4} D^{j_i^+}_{n_i m_i}(g_i^+) D^{j_i^-}_{n_i' m_i'}(g_i^-) C^{j_i^+ j_i^- k_i}_{n_i n_i' p_i}\right) (\iota_j)^{k_i}_{p_i}(k), \tag{21}$$

where repeated lower indices are summed over. The indices (m_i, m_i') label tensors in the SO(4) representation J_i; $C^{j_i^+ j_i^- k_i}_{n_i n_i' p_i}$ are SU(2) Clebsch–Gordan coefficients and $(\iota_j)^{k_i}_{p_i}(k)$ are the matrix elements of an invariant $\mathrm{SO}(3)_k$ intertwiner between the four representations k_i, labeled by the intermediate representation j. Using the fact that $\varphi_k(g_1, \ldots, g_4) = \varphi(\bar{k}^{-1} g_1, \ldots, \bar{k}^{-1} g_4)$, where $\bar{k}$ is the SO(4) rotation with self-dual/anti-self-dual decomposition $(1, k)$, one can equivalently use the basis functions

$$\Psi^{(J_i,k_i,j)}_{m_i,m_i'}(\bar{k}^{-1} g_i; 1) = \left(\prod_{i=1}^{4} D^{j_i^+}_{n_i m_i}(g_i^+) D^{j_i^-}_{n_i' m_i'}(k^{-1} g_i^-) C^{j_i^+ j_i^- k_i}_{n_i n_i' p_i}\right) (\iota_j)^{k_i}_{p_i}, \tag{22}$$

written in terms of a four-valent SU(2) intertwiner $(\iota_j)^{k_i}$ independent of the normals.

Such basis functions correspond to (a single vertex of) projected spin networks [23, 42, 49]. The convolution of GFT fields to define generic observables, and thus generic boundary spin network states at the level of transition amplitudes, will still be defined with respect to the SO(4) connection. The need for a generalization of the standard gauge invariance condition on quantum states (Gauss law) in the quantization of Plebanski gravity, and in the presence of linearized simplicity constraints, which then leads to projected spin network states, has been argued from a canonical continuum point of view in [55] (see also [39, 40, 47]).

The second point is that, despite this generalization, it is easy to check that the following action of the invariant (under generalized gauge invariance) field φ_k:

$$S[\varphi_k] = \frac{1}{2}\int \varphi^2_{1234,k} - \frac{\lambda}{5!}\int \varphi_{1234,k_1}\, \varphi_{4567,k_2}\, \varphi_{7389,k_3}\, \varphi_{962\,10,k_4}\, \varphi_{10\,851 k_5}, \tag{23}$$

where we used the shorthand notation $\varphi_{1234,k} := \varphi_k(g_1, \ldots, g_4)$, defines the same amplitudes as the original Ooguri model and thus still corresponds at the perturbative level to quantum simplicial BF theory.

3.2. Simplicity constraints and geometricity projector

To obtain a geometric theory in the extended framework, we would like to impose the Plebanski simplicity constraints on the model (23). Just like in the previous section, one can implement these constraints by means of the functions $\widehat{S}_k$ labeled by $k \in \mathrm{SU}(2)$, defined in (11), where this time the label k is coupled to the additional field variable. We thus define a simplicity projector $\widehat{\varphi} \mapsto \widehat{S}\widehat{\varphi}$ acting on the extended fields $\widehat{\varphi}_k(x_1, \ldots, x_4) := \widehat{\varphi}(x_1, \ldots, x_4; k)$ as

$$(\widehat{S}\,\widehat{\varphi})_k := \widehat{S}_k \star \widehat{\varphi}_k. \tag{24}$$

Thus the only modification with respect to the GFT formulations of the BC model is that now the simplicity projections have to be defined imposing orthogonality of bivectors with respect to the normal vector that comes as a new argument of the field. This leads, however, to a very important difference with respect to the previous case.

In fact, remarkably, this simplicity projector commutes with the action of a rotation, in the sense that, given $h \in \mathrm{SO}(4)$,

$$\widehat{C}_h \star (\widehat{S}\,\widehat{\varphi})_{h^{-1}\triangleright k} = \widehat{C}_h \star \widehat{S}_{h^{-1}\triangleright k} \star \widehat{\varphi}_{h^{-1}\triangleright k} = \widehat{S}_k \star \widehat{C}_h \star \widehat{\varphi}_{h^{-1}\triangleright k} = \widehat{S}(\widehat{C}_h \star \widehat{\varphi}_{h^{-1}\triangleright k}),$$

where $h^{-1} \triangleright k := h^- k (h^+)^{-1}$ and $\widehat{C}_h$ are the functions of $\mathfrak{so}(4)^4$ defined in (18). The first term corresponds to the action of a rotation h on the projected field $\widehat{S}\widehat{\varphi}$; the last term corresponds to the projection by $\widehat{S}$ of the rotated extended field $\widehat{C}_h \star \widehat{\varphi}_{h^{-1}\triangleright k}$. The simplicity projector acts on the rotated field by imposing simplicity with respect to the rotated normal vector. Thus, the equality just states that rotating by h bivectors that are simple with respect to a normal k gives bivectors that are simple with respect to the rotated normal $h^{-1} \triangleright k$.

This commutation property is an important property, because it means that the model constructed in these terms will not suffer from the 'lack of covariance' noted in [5] in the various versions of the BC model, and that the simplicity constraints are now going to be correctly imposed in the different frames across the simplicial complex, obtained as a Feynman diagram of the GFT model in the following section.

This has a further crucial consequence. Simplicity and gauge covariance are encoded into two projectors $\widehat{S}$ and $\widehat{C}$, defined in (11) and (20), which are now commuting: their product therefore defines a *projector* $\widehat{G} = \widehat{S}\widehat{C} = \widehat{C}\widehat{S}$, with $G^2 = 1$. Explicitly, in the metric representation, this projector acts as

$$(\widehat{G}\widehat{\varphi})(x_j; k) = \int_{\mathrm{SO}(4)} \mathrm{d}h \left[\prod_{j=1}^{4} \delta_{-k x_j^- k^{-1}}(x_j^+)\right] \star [e_h(x_1) \cdots e_h(x_4)] \star \varphi(x_1, \ldots, x_4; h^- k (h^+)^{-1}). \tag{25}$$

Upon group Fourier transform, it acts dually on extended group fields as

$$(G\varphi)(g_j; k) = \int_{\mathrm{SO}(4)} \mathrm{d}h \left(\prod_j \int_{\mathrm{SU}(2)} \mathrm{d}u_j\right) \varphi((h^- k^{-1} u_j k g_j^-, h^+ u_j g_j^+); h^+ k (h^-)^{-1}). \tag{26}$$

The geometric nature of the tetrahedron corresponding to the GFT field $\widehat{\varphi}$ is then enforced by the single projector G that we can call *geometricity projector*. It is then imposing this projector on the generalized fields $\widehat{\varphi}$ that one can construct a GFT (and spin foam) model of 4D quantum gravity based on the (linear) Plebanski formulation of classical gravity, free of the mentioned problematic features (from the point of view of simplicial geometry) of the usual formulations of the BC model. This is what we do in the following section.

4. A revised model for four-dimensional (4D) quantum gravity with no Immirzi parameter

4.1. The GFT model and its Feynman amplitudes

We start from the extended Ooguri model introduced in section 3 and use the geometricity projector G combining both (extended) closure and simplicity constraints in a single operator. We define a projected field $\widehat{\Psi} := \widehat{G}\widehat{\varphi}$, representing a geometric (non-commutative) tetrahedron, whose geometry is characterized by four area bivectors x_j and one normal vector k.

The GFT model is defined by the action

$$S(\widehat{\Psi}) = \frac{1}{2}\int [\mathrm{d}^6 x_i]^4 \int \mathrm{d}k\, \widehat{\Psi}^{\star 2}_{1234,k} + \frac{\lambda}{5!}\int [\mathrm{d}^6 x_i]^{10}[\mathrm{d}k_a]^5 \times \widehat{\Psi}_{1234,k_a} \star \widehat{\Psi}_{4567,k_b} \star \widehat{\Psi}_{7389,k_c} \star \widehat{\Psi}_{962\,10,k_d} \star \widehat{\Psi}_{10\,851,k_e}, \tag{27}$$

where $\mathrm{d}^6 x_i$ is the Lebesgue measure on $\mathfrak{so}(4) \sim \mathbb{R}^6$ and $\mathrm{d}k$ is the Haar measure on SU(2). It is understood, just like in the previous sections, that $\star$-products relate repeated lower indices as $\hat{\Psi}_i \star \hat{\Psi}_i := (\hat{\Psi} \star \hat{\Psi}_-)(x_i)$, with $\hat{\Psi}_-(x) = \hat{\Psi}(-x)$.

We can immediately note four main features of the model so defined:

Just as in the Ooguri model and in the standard BC model(s), bivector variables associated with the same triangle in different tetrahedra (fields) within the same 4-simplex, as well as across different 4-simplices, are identified in the frame of the same 4-simplex, to which the above writing refers. When expanding the gauge covariance constraint in integral form and writing explicitly the delta-functions relating the arguments of the fields, one sees that the same bivector variables are now expressed in the frames associated with the various tetrahedra, and are related by the discrete connection introduced by covariance constraint (see also [5, 24, 47, 51]).

Just as in the extended Ooguri model, normal vectors to tetrahedra are coupled only indirectly in each 4-simplex, through their relation with bivector variables imposed by the generalized covariance condition and by the simplicity constraints $\widehat{S}_k$. On the other hand, the normal vectors associated with the same tetrahedron in the two different 4-simplices sharing it are strictly identified by the kinetic term *in the reference frame associated with the tetrahedron* (this becomes apparent when making explicit the form of the interaction and kinetic functions and writing the generalized closure constraint in integral form).

The use of the generalized geometricity projector $\widehat{G}$ imposes all the necessary geometric conditions on the bivector variables in each tetrahedron. Note that the closure of the four field variables, and hence of the bivectors associated with the four triangles of a tetrahedron, is recovered by integration over the normals. In fact, such integration makes the insertion of a closure constraint in front of each field $\hat{\Psi}_{1234,k}$ in the interaction term redundant, since

$$\int \mathrm{d}k \delta(x_1 + \cdots + x_4) \star \hat{\Psi}(x_1, \ldots, x_4; k) = \int \mathrm{d}k \hat{\Psi}(x_1, \ldots, x_4; k). \tag{28}$$

The nature of the operator $\widehat{G}$ as the product of two commuting projectors leads to a unique definition of the constrained model. By contrast, the standard BC projector does not commute with gauge invariance, giving rise to some ambiguities in the form of edge amplitudes of the spin foam model. Here, for example, the use of the interaction term

of (27) with any of the kinetic terms:

$$\frac{1}{2}\int \widehat{\Psi}^{\star 2}_{1234,k}, \quad \frac{1}{2}\int (C \triangleright \widehat{\varphi})^{\star 2}_{1234,k}, \quad \frac{1}{2}\int (\widehat{S}_k \triangleright \widehat{\varphi})^{\star 2}_{1234,k} \quad \text{or} \quad \frac{1}{2}\int \widehat{\varphi}^{\star 2}_{1234,k}$$

leads to the same amplitudes upon perturbative expansion. At the same time, it should be stressed that any of the above choices would lead, strictly speaking, to the definition of a different field theory, for what concerns the related space of fields, the classical equations of motion, etc.

The Feynman amplitudes of model (27), in the metric representation, can be computed with the propagator and vertex functions given by[12]

$$P(x_i, x_i'; k, k') = \delta_{-x_i}(x_i')\,\delta(k'k^{-1}) \qquad V(x_i^\ell, x_i^{\ell'}; k_\ell, k_{\ell'})$$
$$= \int \prod_{\ell=1}^{4} \mathrm{d}h_\ell \prod_{i=1}^{6} (\delta_{-x_i^\ell} \star S_{k_\ell} \star e_{h_\ell h_{\ell'}^{-1}} \star S_{k_{\ell'}})(x_i^{\ell'}), \tag{29}$$

where $S_k(x) := \delta_{-kx^{-}k^{-1}}(x^{+})$ and $\mathrm{d}h_\ell$ is the Haar measure on SO(4). Note that the vertex does not encode any correlation among normal vectors for different tetrahedra. The propagator, instead, presents a fifth strand corresponding to the identification of normal vectors in the two 4-simplices sharing the same tetrahedron.

The calculation of the Feynman amplitudes is similar to the topological case [5]. In building up the diagram, propagator and vertex strands are joined to one another using the $\star$-product. The amplitudes are integrals over the bivector and holonomy variables, and the normals. In terms of the simplicial complex dual to the Feynman graph, bivectors x_t^τ are labeled by a couple {triangle t, tetrahedron τ} with $t \subset \tau$; holonomies $h_{\sigma\tau}$ are labeled by a couple {4-simplex σ, tetrahedron τ} with $\tau \subset \sigma$; the normals are labeled by tetrahedra as k_τ.

Under the integration over the normals and holonomy variables, the amplitude of a closed graph factorizes into a product of contributions $\mathcal{A}_t[h_{\sigma\tau}, k_\tau]$ for each *face* of the graph, and hence for each *triangle* t of the dual simplicial complex, which take the form of a cyclic $\star$-product:

$$\mathcal{A}_t[h_{\sigma\tau}, k_\tau] = \int \prod_{j=0}^{N_t} [\mathrm{d}^6 x_t^j]\, \vec{\bigstar}_{j=0}^{N_t} (\delta_{x_t^j} \star S_{k_j} \star e_{h_{jj+1}})(x_t^{j+1}). \tag{30}$$

The integer $j \in \{0, \ldots, N_t\}$ labels the tetrahedra sharing the triangle t, or the links of the boundary of the face f_t dual to t in the Feynman graph. The expression is defined for a given ordering of these tetrahedra, induced by a choice of orientation of f_t and a reference point in its boundary. $h_{jj+1} := h_{\sigma j}^{-1} h_{\sigma j+1}$ is the holonomy from the tetrahedron j to the tetrahedron $j+1$ through the 4-simplex σ sharing them. By convention we set $x_{N+1} := x_0$. The variables $x_t^j \in \mathfrak{so}(4)$ correspond to the same bivector associated with the triangle t, expressed in the various reference frames of the tetrahedra j. The measure features both the simplicity constraints for the bivector at the level of each tetrahedron and the parallel transports of (constrained) bivectors and normal vectors across simplices, via the discrete connection.

The Feynman amplitude can be written as a simplicial path integral, by first integrating, in each face contribution $\mathcal{A}_t[h_{\sigma\tau}, k_\tau]$, over all bivectors x_t^j save one $x_t := x_0$ corresponding to a 'reference' tetrahedron frame $j = 0$; and by a repeat use of the commutation relation

[12] This corresponds to the choice: $S(\widehat{\varphi}) = \frac{1}{2}\int \widehat{\varphi}^{\star 2}_{1234,k} + \frac{\lambda}{5!}\int \widehat{\Psi}_{1234,k_a} \star \widehat{\Psi}_{4567,k_b} \star \widehat{\Psi}_{7389,k_c} \star \widehat{\Psi}_{962\,10,k_d} \star \widehat{\Psi}_{10\,851,k_e}$.

$e_h \star S_k = S_{h\triangleright k} \star e_h$, with $h \triangleright k = h^+ k h^{-1}$, to place all the plane waves to the right of the integrand. We obtain the following expression for the graph amplitude:

$$\mathcal{A}_\mathcal{G} = \int \prod_{\langle\sigma\tau\rangle} \mathrm{d}h_{\sigma\tau} \prod_t \mathrm{d}^6 x_t \left[\int \prod_\tau \mathrm{d}k_\tau \prod_t \vec{\bigstar}_{j=0}^{N_t} S_{h_{0j}\triangleright k_j}(x_t) \right] \star \mathrm{e}^{\mathrm{i}\sum_t \mathrm{Tr}\, x_t H_t}, \tag{31}$$

where the holonomy $h_{0j} = h_{01}, \ldots, h_{j-1\,j}$ parallel-transports the reference tetrahedron 0 to the tetrahedron j (with $h_{00} = 1$) and $H_t := h_{01}, \ldots, h_{N0}$ is the holonomy along the boundary of the face f_t of $\mathcal{G}$ dual to the triangle t, calculated from the reference tetrahedron frame. The measures $\mathrm{d}h_{\tau\sigma}$, $\mathrm{d}^6 x_i$ and $\mathrm{d}k_\tau$ are, respectively, the Haar measure on SO(4), the Lebesgue measure on $\mathfrak{so}(4) \sim \mathbb{R}^6$ and the Haar measure on SU(2). The exponential corresponds to the product of plane waves $\prod_t \mathrm{e}_{H_t}(x_t)$; its argument $\sum_t \mathrm{Tr} x_t H_t$ is the unconstraint discrete BF action.

This amplitude takes the form of a (non-commutative) simplicial path integral for a constrained BF theory. The measure features the simplicity of the bivector x_t of a triangle in the frame of all tetrahedra sharing the triangle. Thus, $S_{h_{0j}\triangleright k_j}(x_t)$ imposes the simplicity of x_t with respect to the $h_{0j} \triangleright k_j$, which is the normal to the tetrahedron j parallel-transported to the reference frame 0. This is the same as requiring the simplicity with respect to k_j of the bivector parallel-transported to the frame of j.

Note that the above amplitude automatically encodes the closure constraint for the bivectors within each tetrahedron. This is a direct consequence of (extended) gauge invariance and the integration over the normals. Let us nevertheless check it explicitly by picking a tetrahedron τ and considering the variables $x_{t_i}, i = 1, \ldots, 4$, associated with its four boundary triangles, namely bivectors that expressed the reference tetrahedron frame of each triangle. The closure constraint says that the sum of the four bivectors, once parallel-transported to the frame of τ, is zero; in the non-commutative formalism, it is imposed using the non-commutative delta-function (5). Using a suitable gauge transformation, we can always suppose that τ is the reference tetrahedron for triangles t_i. By appropriately flipping signs $x_{t_i} \mapsto -x_{t_i}$, we can also assume that the orientations of the faces f_{t_i} chosen to calculate the holonomies induce the same orientation of the common link dual to τ. Let us insert the closure constraint $\delta(\sum_i x_{t_i}) := \int \mathrm{d}g \prod_i e_g(x_i)$ on the left of the simplicity functions into the integrand of (31), using the star product. The term depending on x_{t_i} in this integrand then becomes

$$\int \mathrm{d}g \prod_{i=1}^4 e_g(x_i) \star \left[\vec{\bigstar}_{j=0}^{N_{t_i}} S_{h_{0j}\triangleright k_j}(x_{t_i}) \right] \star \mathrm{e}^{\mathrm{i}\sum_i \mathrm{Tr}\, x_{t_i} H_{t_i}} = \int \mathrm{d}g \prod_{i=1}^4 \vec{\bigstar}_{j=0}^{N_{t_i}} S_{g h_{0j}\triangleright k_j}(x_{t_i}) \star \mathrm{e}^{\mathrm{i}\sum_i \mathrm{Tr}\, x_{t_i} g H_{t_i}},$$

where the equality follows from the commutation relation (13) between plane waves and simplicity functions and the definition of star product of plane waves. Now, the dependence upon g in the integrand can be removed by the change of variables $k_\tau \mapsto g k_\tau$, $h_{\sigma_\tau \tau} \mapsto h_{\sigma_\tau \tau} g^{-1}$, where σ_τ is the 4-simplex[13] to which both τ and its successor in the ordered lists $j = 0, \ldots, N_{t_i}$ of tetrahedra sharing t_i, belong. This shows that the insertion of the closure constraint is redundant and hence that such constraints are already implemented in the simplicial path integral.

The integrand of (31), as a function of the bivectors, holonomies and normals, is invariant under the following gauge transformations, generated by an SO(4) element g_τ for each tetrahedron:

$$h_{\sigma\tau} \mapsto h_{\sigma\tau} g_\tau^{-1}, \quad k_\tau \mapsto g_\tau^+ k_\tau (g_\tau^-)^{-1}, \quad x_t \mapsto g_{\tau(t)}^{-1} x_t g_{\tau(t)}, \tag{32}$$

[13] σ_τ is also the 4-simplex dual to the 'target' vertex of the oriented link dual to τ.

where $\tau(t)$ is the reference tetrahedron for the triangle t and $g_\tau^\pm$ arises from the self-dual/anti-self-dual decomposition of g_τ. This corresponds to SO(4) rotation of the tetrahedral frames. The integrand is also invariant under rotation of the simplex frames acting only on holonomies $h_{\sigma\tau} \mapsto g_\sigma h_{\sigma\tau}$ (all the other variables being expressed in tetrahedral frames). Choosing $g_\tau = (g_\tau^+, g_\tau^-) := (1, k_\tau^{-1})$ as gauge parameters corresponds to the 'time gauge', which fixes the normals to the north pole of S^3. This shows that the integrals over the normals k_τ drop out of the amplitude (31). Making explicit the simplicity functions and denoting $\bar{h}_{0j} := h_{0j}^+ (h_{0j}^-)^{-1}$, we obtain

$$\mathcal{A}_\mathcal{G} = \int \prod_{\langle\sigma\tau\rangle} \mathrm{d}h_{\sigma\tau} \prod_t \mathrm{d}^6 x_t \left[\prod_t \vec{\bigstar}_{j=0}^{N_t} \delta_{-\bar{h}_{0j} x_t^- \bar{h}_{0j}^{-1}}(x_t^+) \right] \star \, \mathrm{e}^{\mathrm{i} \sum_t \mathrm{Tr}\, x_t H_t}. \tag{33}$$

Note that gauge fixing the normals is only possible for amplitudes of closed graphs; the amplitude of open graphs, dual to simplicial complexes with boundary, has an explicit dependence on the normals to the boundary tetrahedra. We illustrate this in the case of the simplex boundary state in section 4.3.

To summarize, we have written the GFT Feynman amplitudes as simplicial path integrals with the explicit form of a constrained BF theory, with a clear form of (linear) simplicity constraints imposed in each tetrahedral frame, with respect to a given normal variable and correctly parallel-transported across frames by means of the gauge connection. The integration over the same connection imposes the closure constraint in every tetrahedron; this in turn implies the remaining (volume) simplicity constraints [22, 47]. The amplitude also encodes constraints on the connection $\{h_{\sigma\tau}\}$, following from simplicity, giving a specific measure on the space of connections.

4.2. Spin foam representation of the amplitudes

The same amplitudes can be written in spin foam representation, exploiting the exact duality between the metric and spin representation of the GFT [5]. This re-writing is obtained by inverse group Fourier transform—giving the pure lattice gauge theory formulation of the model involving only group elements [4])—and the Peter–Weyl decomposition of the amplitudes, leaving only a sum over representation of pure spin foam amplitudes.

For the present model the calculation goes as follows. We first note that, under the integration of the group elements $h_{\sigma\tau}$ in (33), each integral over x_t takes the form of an inverse SO(4) Fourier transform formula. Indeed, for a given set of holonomy variables $\{h_{\sigma\tau}\}$, the functions $\vec{\bigstar}_{j=0}^{N_t} \delta_{-\bar{h}_{0j} x_t^- \bar{h}_{0j}^{-1}}(x_t^+)$ inside the brackets, labeled by t, define functions of $\mathfrak{su}(2)$, which are the image by non-commutative Fourier transform of some functions $\mathcal{O}_t$ on SO(4):

$$\vec{\bigstar}_{j=0}^{N_t} \delta_{-\bar{h}_{0j} x_t^- \bar{h}_{0j}^{-1}}(x_t^+) = \int_{\mathrm{SO}(4)} \mathrm{d}g \, \mathcal{O}_t(g) e_g(x).$$

The inverse Fourier transform formula (see, e.g., [5]) gives

$$\mathcal{O}_t(g) = \int \mathrm{d}^3 x \, \vec{\bigstar}_{j=0}^{N_t} \delta_{-\bar{h}_{0j} x_t^- \bar{h}_{0j}^{-1}}(x_t^+) \star e_{g^{-1}}(x). \tag{34}$$

A quick comparison between this formula and the integrand of (33) leads to the following simple expression of the graph amplitude in terms of the group functions $\mathcal{O}_t$:

$$\mathcal{A}_\mathcal{G} = \int \prod_{\tau\sigma} \mathrm{d}h_{\tau\sigma} \prod_t \mathcal{O}_t(H_t^{-1}), \tag{35}$$

where we recall that $H_t := h_{01}, \ldots, h_{N0}$ is the holonomy around the triangle t.

Let us determine the functions $\mathcal{O}_t$. The plane-wave expansion of the non-commutative delta-functions gives

$$\vec{\bigstar}_{j=0}^{N}\,\delta_{-\bar{h}_{0j}x_t^-\bar{h}_{0j}^{-1}}(x_t^+) = \int \mathrm{d}g \int \prod_{j=1}^{N} \mathrm{d}g_j\; e_{g\prod_{j=1}^{N} h_{0j}^- g_j h_{0j}^{--1}}(x_-)\; e_{g\prod_{j=1}^{N} h_{0j}^+ g_j h_{0j}^{+-1}}(x_+)$$

$$= \int \prod_{\pm} \mathrm{d}g_\pm \int \prod_{j=1}^{N} \mathrm{d}g_j\; \delta(g_+ g_-^{-1}) e_{g_-\prod_{j=1}^{N} h_{0j}^- g_j h_{0j}^{--1}}(x_-)\; e_{g_+\prod_{j=1}^{N} h_{0j}^+ g_j h_{0j}^{+-1}}(x_+), \quad (36)$$

where the integrations are over SU(2). We omitted the t of N_t to simplify the notations. After the following change of variables:

$$g_- \;\mapsto\; g_- \prod_{j=1}^{N} h_{0j}^- g_j h_{0j}^{--1}, \qquad g_+ \;\mapsto\; g_+ \prod_{j=1}^{N} h_{0j}^+ g_j h_{0j}^{+-1}, \quad (37)$$

the expression (36) takes the explicit form of an SO(4) Fourier transform $\widehat{\mathcal{O}_t}(x) = \int \mathrm{d}g\, \mathcal{O}_t(g) e_g(x)$, where $\mathcal{O}_t(g)$ is given by

$$\mathcal{O}_t(g) = \int \prod_{j=1}^{N} \mathrm{d}g_j\; \delta\left(g_+ \left[\prod_{j=1}^{N} h_{0j}^+ g_j h_{0j}^{+-1} \right]^{-1} \prod_{j=1}^{N} h_{0j}^- g_j h_{0j}^{--1}\, g_-^{-1} \right). \quad (38)$$

The evaluation of this function for $g = H_t^{-1}$ gives

$$\mathcal{O}_t(H_t^{-1}) = \int \prod_{j=1}^{N} \mathrm{d}g_j\; \delta\left(\left[h_{01}^+ g_1 h_{12}^+, \ldots, g_N h_{N0}^+ \right]^{-1} h_{01}^- g_1 h_{12}^-, \ldots, g_N h_{N0}^- \right). \quad (39)$$

Next, we use the Plancherel decomposition of the SU(2) delta-function in terms of the characters $\chi^J(g) = \mathrm{Tr}\, D^J(g)$ in the SU(2)-representations labeled by $J \in \frac{1}{2}\mathbb{N}$, which also labels the simple SO(4) representation (J, J) [17, 44], to write

$$\mathcal{O}_t(H_t^{-1}) = \sum_J d_J \int \prod_{j=1}^{N} \mathrm{d}g_j\; \chi^J\left(\left[h_{01}^+ g_1 h_{12}^+, \ldots, g_N h_{N0}^+ \right]^{-1} h_{01}^- g_1 h_{12}^-, \ldots, g_N h_{N0}^- \right), \quad (40)$$

where $d_J = 2J + 1$. The integration over each group variables g_j in (39), which appears exactly two times in the argument of each character χ^J, is performed using N times the orthogonality relation of the representation matrices

$$\int \mathrm{d}g\; D^J_{mn}(g^{-1}) D^J_{pq}(g) = \frac{1}{d_J} \delta_{m,q} \delta_{n,p}.$$

In fact, if we denote $\chi_0^J := \chi^J(h_{01}^+ (h_{01}^-)^{-1}$ and, for $K = 1, \ldots, N$:

$$\chi_K^J := \chi^J\left(\left[h_{01}^+ g_1 h_{12}^+, \ldots, g_K h_{KK+1}^+ \right]^{-1} h_{01}^- g_1 h_{12}^-, \ldots, g_K h_{KK+1}^- \right)$$

(where by convention $N + 1 = 0$), one can show, using orthogonality, the following recursion relations:

$$\int \mathrm{d}g_K\, \chi_K^J = \frac{1}{d_J} \chi^J(h_{KK+1}^+ (h_{KK+1}^-)^{-1}) \chi_{K-1}^J. \quad (41)$$

We conclude by iteration:

$$\mathcal{O}_t(H_t^{-1}) = \sum_J \frac{d_J^2}{d_J^{N+1}} \prod_{j=0}^{N} \chi^J(h^+_{jj+1}(h^-_{jj+1})^{-1}), \tag{42}$$

where $N+1$ is the number of tetrahedra sharing the triangle t.

Using the form (35) of the Feynman amplitude (33) and the above expression for the functions $\mathcal{O}_t$, we finally obtain the spin foam amplitude:

$$\mathcal{A}_{\mathcal{G}} = \sum_{\{J_t\}} \prod_t d_{J_t}^2 \prod_\tau \frac{1}{\prod_{t\in\partial\tau} d_{J_t}} \prod_\sigma \{10J\}_\sigma. \tag{43}$$

The sum is over the SU(2) spins J_t labeling the triangles of the simplicial complex dual to $\mathcal{G}$ and the products are over the triangles t, the tetrahedra τ and the 4-simplices σ. The 4-simplex weight $\{10J\}_\sigma$ is the Barrett–Crane $10j$-symbol [44]:

$$\{10J\}_\sigma = \int \prod_v \mathrm{d}h_v \prod_{l=(vv')} \chi^{J_{t(l)}}(h^+_v(h^-_{v'})^{-1}). \tag{44}$$

In this expression, v label the five vertices and l the ten links of the graph dual to the boundary of the 4-simplex: a vertex is dual to a boundary tetrahedron, a link $l = (vv')$ is dual to a triangle $t(l)$ sharing two tetrahedra. The measure is over SO(4) elements $h_v := (h^+_v, h^-_v)$.

The model is thus a variant of the BC model with a specific edge amplitude, different from that of other versions of the BC model [50] (note, in particular, the absence of the norm of the BC intertwiner from the amplitudes), although identical to that obtained in [21], whose construction is indeed close to ours regarding both its motivations and techniques employed.

From both the construction and the detail of the calculation, it should be clear that, while the appearance of the $10j$-symbol is the direct result of the form of the simplicity constraints chosen (corresponding to the pure gravity sector of the Holst action or to the Plebanski formulation of gravity, with linear constraints), most of the information concerning the details of the simplicial geometry behind the model, in particular the extended closure condition and associated Lorentz covariance of quantum states, as well as the correlations of normal vectors across simplices, is now hidden in the edge amplitudes. The reason for this non-transparent form is obviously the fact that all the geometric variables in which this information is manifest, i.e. bivectors, gauge connection and normal vectors, have been integrated out.

4.3. Simplex boundary state

Taking the GFT Feynman amplitudes as the definition of a lattice model, we would like to discuss here the simplex boundary state. The vertex function (29) gives the amplitude for a single 4-simplex (represented by the vertex graph) with fixed boundary data $\{x_t^\tau, x_t^{\tau'}\}$ and $\{k_\tau\}$: the pair $x_t^\tau, x_t^{\tau'}$ both corresponds to bivectors associated with the triangle t, expressed in the frame of the two tetrahedra sharing it. Given ten such pairs, the vertex function encodes the relations between the bivectors of the same triangle expressed in different tetrahedral frames:

$$\int \prod_\tau \mathrm{d}h_\tau \prod_{t=\langle\tau\tau'\rangle} (\delta_{-x_t^\tau} \star S_{k_\tau} \star S_{h_{\tau\tau'}\triangleright k_{\tau'}} \star e_{h_{\tau\tau'}})(x_t^{\tau'}). \tag{45}$$

The product is over the triangles of the 4-simplex, each of which is shared by an (ordered) pair of tetrahedra. $h_{\tau\tau'} = h_\tau^{-1} h_{\tau'}$ is the parallel transport between τ and τ' and $S_k(x) := \delta_{-kx^{-}k^{-1}}(x^{+})$ are the simplicity functions written in terms of the non-commutative delta-functions (5).

An important remark is in order, here. Without the simplicity functions, the bivectors x_t^τ, x_t^τ would be identified up to (a sign and) parallel transport $h_{\tau\tau'}$ from τ to τ', as they should be classically. However, due to the specific properties of the star product, the presence of the simplicity functions relaxes this identification. Using the plane wave expansion of S_k, we indeed have

$$(\delta_{-x_t^\tau} \star S_{k_\tau} \star S_{h_{\tau\tau'} \triangleright k_{\tau'}} \star e_{h_{\tau\tau'}})(x_t^{\tau'}) = \int \mathrm{d}u_t^\tau \, \mathrm{d}u_t^\tau (\delta_{-x_t^\tau} \star e_{u_t^\tau h_{\tau\tau'} u_t^{\tau'}})(x_t^{\tau'}), \tag{46}$$

where the integration of the right-hand side is over the product $\mathrm{SO}(3)_\tau \times \mathrm{SO}(3)_{\tau'}$ of stabilizer groups of the normals k_τ and $k_{\tau'}$. The fact that the amplitude does not force the complete identification up to parallel transport of the bivectors across neighboring tetrahedra, also noted in [21] and argued to be the main shortcoming of the BC model, appears, in our formalism, to be due to the star product structure and thus the quantization map chosen for the constraints. As we argue also in the discussion in section 6, such a relaxation of parallel transport could be purely quantum effects, devoted to disappear in a suitable semi-classical limit. In fact, such a requirement serves as a guide for the very definition of such a limit, which should reach a regime in which the star product is approximated by a commutative product. A scale $\lambda > 0$ can be introduced in the amplitude via a modification of the group Fourier transform, where $\mathrm{SU}(2)$ group elements are parametrized as $g = \mathrm{e}^{\lambda \vec{p}\cdot\vec{\tau}}$ with $\vec{p} \in \mathbb{R}^3$ and $|\vec{p}| < 1/\lambda$, plane waves are replaced by $\mathrm{e}_g^\lambda(x) := \mathrm{e}^{\frac{i}{\lambda^2}\mathrm{Tr}\, xg}$, and the star product preserves the parameter [14]

$$\mathrm{e}^{\frac{i}{\lambda^2}\mathrm{Tr}\, xg} \star_\lambda \mathrm{e}^{\frac{i}{\lambda^2}\mathrm{Tr}\, xg'} = \mathrm{e}^{\frac{i}{\lambda^2}\mathrm{Tr}\, xgg'}.$$

In the limit $\lambda \to 0$, corresponding to a regime of low curvature, the star product can be approximated by the standard pointwise product of functions on $\mathfrak{so}(4) \sim \mathbb{R}^6$. In this parametrized framework, the parameter λ appears not only in front of the action in the path integral, making it suitable for defining the semi-classical regime, but also as a deformation parameter of the product used for the definition and the imposition of the constraints; this contributes to further quantum corrections, of which the above relaxation of parallel transport may be a manifestation. This possibility will be studied in more detail elsewhere.

Let us also point out that, if one defines, in the parametrized framework, the $\mathfrak{su}(2)$ non-commutative delta-function as $\delta = \frac{1}{\lambda^6}\int \mathrm{d}g\, e_g^\lambda$, then they formally become a true delta-function on $\mathfrak{su}(2) \sim \mathbb{R}^3$ in the limit $\lambda \to 0$. With simplicity functions defined as before $S_k(x) = \delta_{-kx^{-}k^{-1}}(x^{+})$, this means that, in this limit, the constraints in (45) impose, for each triangle t, the following condition on the bivector $x_t := x_t^{\tau'}$ and the connection:

$$-x_t^{+} = k_\tau x_t^{-} k_{\tau'} = [h_{\tau\tau'} \triangleright k_{\tau'}] x_t^{-} [h_{\tau\tau'} \triangleright k_{\tau'}]^{-1}.$$

In other words, the normals of the tetrahedra sharing t, expressed in the frame of τ, differ from an element in the stabilizer group of the x_t^{-}, namely $[h_{\tau\tau'} \triangleright k_{\tau'}] k_\tau^{-1} \in \mathrm{U}(1)_{x_t^{+}}$. This expresses the geometrical fact that the two normals belong the plane co-orthogonal to the bivector x_t (namely orthogonal to his Hodge dual, which defines the plane spanned by the triangle). In time gauge $k = 1$, this becomes a condition on the boost part of the parallel transports $h_{\tau\tau'}$ between adjacent tetrahedra.

The integration of the amplitude (45) over half of the bivector variables, namely, for each triangle $t = \langle \tau, \tau' \rangle$, the variable $x_t^{\tau'}$ gives a function over the normals and the ten remaining

variables $x_t := x_t^{\tau_t}$, which are bivectors of the triangle t in its reference tetrahedron τ_t. The resulting amplitude takes the form of a discretized (and non-commutative) path integral for constrained BF theory, for a single 4-simplex, with fixed boundary bivectors:

$$A(x_t; k_\tau) = \int \prod_t \left[\mathrm{d}g_t \, \mathrm{e}^{\mathrm{i}\mathrm{Tr}\, x_t g_t} \star S_{k_{\tau_t}}(x_t)\right] \star \int \prod_\tau \mathrm{d}h_\tau \prod_t S_{h_{\tau\tau'} \triangleright k_{\tau'}}(x_t)\, \delta(h_\tau g_t h_{\tau'}^{-1}). \tag{47}$$

In this expression, we made explicit the part of the constraint implementing the linear simplicity of the bivectors in their reference tetrahedron, and the part (also dependent on the bivectors) entering the definition of the measure over the connections. Note that the closure constraint for the bivectors is only implemented after integration over the normals. That a proper spin foam model should have this form, with the corresponding constraints modifying the measure over the connections, has been advocated in [39].

5. A model for the topological sector of Plebanski gravity

In this section, we sketch the construction of a GFT model for the so-called topological sector of Plebanski gravity as a constrained BF theory, where the constraints on the B-field impose that it can be expressed in terms of 1-forms $B^{IJ} = e^I \wedge e^J$ rather than $\epsilon^{IJKL} e_I \wedge e_J$. At the discrete level, they are implemented by requiring the same condition as in (10), but for the Hodge dual to B rather than B itself. In terms of B, the condition differs from (10) by a sign: there exists a $k \in \mathrm{SU}(2)$, such that $\forall j$, $B_j^+ = k B_j^- k^{-1}$.

This condition is obtained from the BF GFT in its extended formulation, by using an operator, acting on the extended field (in metric variables) as $\widehat{\varphi}_k \mapsto (\widehat{S}^{\mathrm{top}}\widehat{\varphi})_k := \widehat{S}_k^{\mathrm{top}} \star \widehat{\varphi}_k$, where $\widehat{S}_k^{\mathrm{top}}$ are the functions of four $\mathfrak{so}(4)$ variables, labeled by an SU(2) element k, defined in terms of the non-commutative δ functions (5) as

$$\widehat{S}_k^{\mathrm{top}}(x_1, \ldots, x_4) = \prod_{j=1}^{4} \delta_{k x_j^- k^{-1}}(x_j^+). \tag{48}$$

Upon group Fourier transform,

$$\widehat{S}_k^{\mathrm{top}} \star \widehat{\varphi}_k = \int_{\mathrm{SO}(4)^4} [\mathrm{d}g_j]^4 \, (S_k^{\mathrm{top}}\varphi_k)(g_j)\, e_{g_1} \ldots e_{g_4},$$

this simplicity operator acts as

$$(S_k^{\mathrm{top}}\varphi_k)(g) := \prod_j \int_{\mathrm{SO}(3)_k^4} [\mathrm{d}u_j]^4 \, \varphi_k(k^{-1}u_j^{-1} k g^-, u_j g^+), \tag{49}$$

where we used the decomposition $g = (g^-, g^+)$ into self-dual and anti-self-dual components.

Note that, unlike its analogue in the gravitational sector, the simplicity operator does not define a projector[14]. One can, however, check that it commutes with the extended gauge invariance projector $\widehat{C}$ in (17). This is because, just like their analogues $\widehat{S}_k$, the simplicity functions $\widehat{S}_k^{\mathrm{top}}$ satisfy the property that, given $h \in \mathrm{SO}(4)$,

$$e_h \star \widehat{S}_k^{\mathrm{top}} = \widehat{S}_{h \triangleright k}^{\mathrm{top}} \star e_h. \tag{50}$$

This property allows us to impose the simplicity constraint covariantly.

[14] There are ways of imposing such constraints via a projector: for example by acting on the field with $\widehat{S}_k^{\mathrm{top}}$ by right star multiplication on the components x^- and left on the components x^+; we will not investigate them in this paper.

A model can be defined by the action having the same form as (27), but now defined in terms of fields $\hat{\Psi} = \widehat{G}^{\rm top}\widehat{\varphi}$, with $\widehat{G}^{\rm top} = \widehat{S}^{\rm top}\widehat{C} = \widehat{C}\widehat{S}^{\rm top}$. Just as in the gravitational sector, the Feynman amplitudes take the form of simplicial path integrals for a constrained BF theory:

$$\mathcal{A}_{\mathcal{G}} = \int \prod_{\langle\sigma\tau\rangle} \mathrm{d}h_{\sigma\tau} \prod_t \mathrm{d}^6 x_t \left[\prod_t \vec{\bigstar}_{j=0}^{N_t} \delta_{\bar{h}_{0j} x_t^- \bar{h}_{0j}^{-1}}(x_t^+)\right] \star \mathrm{e}^{\mathrm{i}\sum_t \mathrm{Tr}\, x_t H_t}, \tag{51}$$

where the notations are the same as those in the previous section: for each triangle t, the integers $j = 0 \cdots N_t$ labels the tetrahedra sharing t, h_{0j} is the holonomy between the tetrahedra 0 and j and $\bar{h} := h^+(h^-)^{-1}$. $\mathcal{A}_{\mathcal{G}}$ is the generic amplitude of the closed graph expressed in time gauge, as (extended) gauge invariance allows us to remove the dependence upon the normals.

The spin foam expression of the model can be calculated directly from its simplicial path integral form, following the same route as in section 4.2. Here we will not describe this spin foam expression in detail. Let us, however, have a look at the action of the simplicity operator on the Peter–Weyl components on the fields. Gauge invariant fields φ_k expand into projected spin network vertex functions (21): for $k = 1$,

$$\Psi^{(J_i,k_i,j)}_{m_i,m_i'}(g_i) = \left(\prod_{i=1}^4 D^{j_i^+}_{n_i m_i}(g_i^+) D^{j_i^-}_{n_i' m_i'}(g_i^-) C^{j_i^+ j_i^- k_i}_{n_i n_i' p_i}\right) (\iota_j)^{k_i}_{p_i}. \tag{52}$$

We deduce from the orthogonality relations of the Wigner matrices

$$\int \mathrm{d}u\, D^{j^+}_{mn}(u^{-1}) D^{j^-}_{pq}(u) = \frac{1}{d_j^+} \delta_{j^+,j^-} \delta_{m,q} \delta_{n,p}$$

that simplicity (49) projects onto simple representations $j_i^+ = j_i^+$, just like in the gravitational sector, and exchanges the indices n_i and n_i' in $C^{j_i^+ j_i^- k_i}_{n_i n_i' p_i}$ in the formula (52). Given that the Clebsh–Gordan coefficients satisfy

$$C^{j,j,k}_{m,n,p} = (-1)^{2j+k} C^{j,j,k}_{n,m,p},$$

we conclude that $S^{\rm top}$ projects onto the simple SO(4) representation $J = (j, j)$, does not impose any restriction on the expansion of (j, j) into SU(2) irreducible $k = 0, \ldots, 2j$ and acts on each component (J, k) by multiplication by the phase $(-1)^{2j+k}$. The expansion into the SU(2) representations k is what defines the fusion coefficients characterizing the new models [22] and in particular the EPR model. In the EPR model, the expansion is restricted to spin $2k = j$. The model obtained here is manifestly different; the action of the simplicity function rather suggests a form analogous to the Ooguri model, as a sum over SO(4) representations of the products of $(15j)$ symbols, where the representations are restricted to be simple.

Thus, by following the same strategy as that in the previous sections, a model for the topological sector of Plebanski gravity can be proposed in the non-commutative metric formulation of GFT, where the classical constraints are imposed via non-commutative delta-functions. Just like in the gravity case, the amplitudes are simplicial path integrals with a clear geometric content. The resulting spin foam model is manifestly distinct, at least in the full quantum regime, from the one we would obtain by imposing the constraints on coherent state parameters [22].

6. Discussion

In this section, we summarize our results and in light of them, re-examine the arguments against the BC model, by which, from now on, we mean the amplitudes associated with a

given triangulation as obtained within the generalized GFT formalism presented above, whether written in the simplicial path integral or in the spin foam representation.

First of all, let us recapitulate the procedure we adopted, and the ingredients we put in.

We started from a GFT formulation of quantum BF theory in four dimensions. The GFT field represents the quantization of the data associated with a single tetrahedron in a simplicial decomposition of spacetime (to be generated by the GFT perturbative expansion) in BF theory. Its arguments represent the corresponding simplicial BF variables. We extended the standard formalism to the one including, for each tetrahedron, an additional variable valued in S^3, in preparation for the imposition of the geometric constraints that should turn this model into the one for 4D gravity. We then imposed onto such an extended GFT field an *extended covariance condition* affecting both the usual field arguments and the additional S^3 variable. This condition represents the gauge invariance constraint of (simplicial) BF theory, and introduces, when written in the form of an integral operator acting on a generic GFT field, the bulk gauge connection of the theory, encoding parallel transport among tetrahedral and simplex frames. In a geometric theory, this should also encode the *closure constraint* on the faces of the same tetrahedron.

After appropriate constraints, in fact, the GFT field should turn into a quantum description of a (quantum) geometric tetrahedron. The needed (simplicity) constraints can be split (both in the discrete and in the continuum setting) into two sets: diagonal and cross-diagonal simplicity, acting at the level of individual tetrahedra, and here imposed in their combined linear form [22, 47], and 'volume' simplicity constraints. The variables on which these constraints have to be imposed are exactly the Lie algebra elements appearing as arguments of the GFT fields in the metric representation, as confirmed by the GFT amplitudes written in the simplicial path integral representation. We then impose the first set of simplicity constraints, as well as the extended covariance (thus closure), as restrictions on the GFT field. Such restrictions then show up in the form of (non-commutative) delta-function insertions in the simplicial BF path integral, at the level of GFT Feynman amplitudes, for all tetrahedra in the associated triangulation, and in the frames of the tetrahedra, thus result in a covariant imposition of the constraints. Moreover, we know [22, 47] that the volume simplicity constraints follow automatically when parallel transport conditions among tetrahedral frames, simplicity and closure conditions hold. In turn, when all of these conditions hold, assuming non-degeneracy, the set of bivectors define a unique geometry for the simplicial complex, and thus can be inverted for the set of edge lengths. No classical geometric condition is thus missing in the construction.

The next step, however, is to quantize these conditions. As mentioned, this process is necessarily ambiguous and a choice of the quantization map must be made. And as mentioned, our procedure relies on the non-commutative Fourier transform and on the resulting star product used to compose the building blocks of the simplicial path integral, including the simplicity constraints and the parallel transport conditions. Thus we have implicitly used, in the path integral setting, the Duflo quantization map. Once more, support for this choice, besides the more abstractly mathematical one, comes from the results of its application to the pure BF theory [5, 32] as well as to simpler systems [33] (and, more recently, to canonical 3d gravity with cosmological constant [45]).

In this quantization, the closure condition of bivectors associated with the faces of each given tetrahedron is not manifest, in contrast to the standard formulation of BF theory, due to the extended form of gauge covariance. One may then want to include it explicitly, inserting non-commutative delta-functions at the level of the Feynman amplitudes of the model. However,

as noted also in [30] and remarked above, it is immediate to see that such an insertion would be redundant and could be re-absorbed in the integration over the discrete connection. The closure constraint has indeed already been implemented by this integration.

As we have shown in the previous sections, this quantization procedure also implied a characteristic interrelation between simplicity constraints and parallel transport, which leads to the covariant imposition of the first, resulting restrictions on the connection, and a consequent relaxation of parallel transport conditions. This fact is not affected by the presence or value of the Immirzi parameter, but once more is a necessary consequence of the quantization chosen. Actually, it seems to go beyond it and be a feature of the other spin foam models as well. The result is that the configurations summed over in the simplicial path integral, and thus in the spin foam model, are not *strictu sensu* classical geometries, due to unavoidable quantum corrections (it can also be understood to be a consequence of the specific path integral measure that our GFT procedure defines).

The end result, in this case, is a simplicial path integral with non-commutative variables, and in spin variables, a specific version of the BC spin foam amplitudes.

Let us then reconsider the issues concerning the quantum simplicial geometry behind the BC model, including the one mentioned above, as made manifest in the metric/simplicial gravity path integral representation. These are:

(a) 4-simplices speak only through face representations, i.e. triangle areas [25];
(b) bivectors associated with the same triangle in different simplices are not identified (after parallel transport) [21, 22];
(c) normal vectors to the same tetahedron seen in different 4-simplices are uncorrelated [5, 43];
(d) the simplicity constraints are imposed in a non-covariant fashion [5];
(e) because of this non-covariance, there are missing constraints over the connection variables [5].

The extended GFT formalism (with linear constraints and no Immirzi parameter) that we presented were meant exactly to solve the last three problems above, and they do.

The other points refer to the coupling of simplices and to the coupling of tetrahedra within simplices. The non-commutative metric formulation shows that the bivector variables associated with the ten triangles in each 4-simplex are correctly identified across different 4-simplices sharing the same tetrahedron (and thus the same triangles), thus addressing the first issue. The additional coupling of normal vectors, in turn covariantly related to the bivectors by the simplicity constraints and by the generalized closure relation, ensures that the correlations among simplices resulting from bivector identifications are not undone by the lack of correlations of normals or by missing conditions on the discrete connection. Obviously, if one expands the transition amplitudes by Peter–Weyl into (simple) SO(4) representations and then integrates out *all variables except the representations themselves* (interpreted as quantum numbers for the areas of the triangles), the amplitudes associated with individual 4-simplices will share only such remaining variables with one another. This simply means that the whole simplicial geometry behind the model and the exact correlations among geometric variables (bivectors and normals) in the same tetrahedron as seen by different (neighboring) 4-simplices have been encoded in a not-at-all-transparent way into the lower-dimensional amplitudes.

As we illustrated in section 4.3, point (b), argued in [21] to be a serious shortcoming of the BC model, has to do exactly with the relaxation of parallel transport resulting from the

interplay between simplicity constraints and gauge covariance, due to the properties of the non-commutative star product. In fact, using the decomposition of the simplicity constraints into plane waves, the star product allows us to recombine them with the plane wave corresponding to the unconstraint BF action, into a single effective action for the model. This is the action used also in [21]. Now, as observed in this work, the equations of motion for action do not force the complete identification up to parallel transport of the bivectors across neighboring tetrahedra. Is this a problem or a feature of the model, i.e. of the quantum amplitudes? We argue that it is a necessary feature (which will be present also in any extension of the same construction involving the Immirzi parameter), an indication of what quantum geometry may be and not necessarily the sign of a problem with it. Our view is that the effective action obtained using the star product should be interpreted as an effective quantum-corrected action indicating properties of the quantum configurations summed over in the path integral, which indeed should not be expected to be classical simplicial geometries, in general. As a consequence, the variations of the same action should not necessarily be interpreted as encoding the classical geometry behind the model, and their failure to reproduce classical simplicial geometry should not necessarily be interpreted as indicating that geometric conditions are missing among the ingredients leading to it. It just indicates how these necessary ingredients are affected by the quantization procedure. While we leave a detailed analysis of the semi-classical expansion of the amplitudes to future work, we can already argue that such expansion will involve approximating the star product with a commutative product, and thus a commutative limit of the equations of motion resulting from the effective quantum action appearing in the path integral. This is exactly what happens in the simpler case considered in [33]. In this approximation, the issue with parallel transport of bivectors disappears and classical simplicial geometry would then be recovered.

In light of the above, the criticisms of (the first reference of) [22], saying that bivectors are not identified across simplices and that they are in the new models, therefore, can also be reconsidered. On closer examination, one notices that what is not identified across different simplices in the BC amplitudes are the *coherent state parameters* associated with each triangle in the different 4-simplices, in quantum BF theory, on which a quantum version of the simplicity constraints is then imposed strongly (i.e. by means of delta-functions) in the FK model (and in its extension to finite γ). The point is, however, that the coherent state parameters can be identified with the continuous bivector variables of BF theory, *only* in a semi-classical sense [24], i.e. in the sense of mean values [22] or in the asymptotic regime of the spin labels [22, 53]. This is confirmed by explicit calculations of the group Fourier transform of Wigner representation matrices [16, 31, 33], in the coherent state basis, relating the coherent state parameters with the classical Lie algebra variables (for each SU(2) component). This gives, for large spins j: $D^{j}_{\vec{n}\vec{n}}(\vec{x}) \approx \delta(\vec{x} - j\vec{n})$. It is directly the continuous bivector (Lie algebra) variables, on the other hand, that appear in our simplicial path integral representation of the amplitudes, and on which the simplicity constraints are imposed, by means of (non-commutative) delta-functions, as appropriate in a path integral quantization [5, 24, 30]. As for the identification of bivectors across tetrahedra within a given simplex, the same remark applies to the FK model (the other available one in the absence of the Immirzi parameter): no strict identification up to parallel transport is imposed on the Lie algebra B variables and, as shown in [21], even the identification of coherent state parameters is imposed only in a semiclassical limit.

We now move on to discuss the other arguments that have been put forward against the BC model. We do so for the sake of completeness only. We have not dealt directly with them in

this paper, nor worked in the context in which these concerns were raised. We limit ourselves, therefore, to reconsider the evidence suggesting these concerns, in light of what we learned from our results.

As a preliminary step, we note a very general criticism raised [24, 30] against the BC model, which concerns other spin foam models as well. While the aim is to obtain a quantization of a classically constrained theory, i.e. the Plebanski formulation of gravity as a constrained BF theory, it results from a procedure that amounts to first quantizing the classical configurations of BF theory to get its quantum states (SO(4) spin networks) and then constraining such quantum states (usually in representation space). However, our metric representation of GFTs and spin foam models, providing their dual simplicial path integral representation, makes it possible to follow a more standard path integral procedure and avoid some ambiguities, and to keep under control which of the constraints are imposed and how, even though it still (necessarily) depends on a choice of quantization map, as we discussed[15].

A popular criticism of the BC model is that it results from imposing the simplicity constraints 'too strongly' at the quantum level [22]. Indeed, when the constraints to be imposed are second class, as is the case in the presence of Immirzi parameter or for quadratic simplicity constraints in general, they have to be imposed weakly in the quantum theory. Regarding this, we note two points. First of all, the model presented above imposes the simplicity constraints at the level of simplicial path integrals, by means of delta-functions, as is appropriate to do for both first and second class constraints. The nature of the constraints is manifested in their relation with other constraints and in the resulting further modification of the quantum measure. Secondly, while quadratic simplicity constraints are indeed second class, in the pure gravitational sector of the Holst gravity or in the Plebanski theory *with linear constraints*, the constraints are actually first class [41], so that they *have to* be imposed strongly. The only kind of 'weakening' of constraints that is consistent with the classical theory in our case is the one following from the non-commutative nature of the bivector variables themselves (whence the dependence on a quantization map[16]), which is taken into account by the $\star$-product and the associated group Fourier transform.

A related criticism [39, 40] is that the BC model does not seem to impose the secondary second class constraints that arise in the canonical analysis of the continuum theory [55, 56] (again when using quadratic simplicity constraints). This criticism applies also to all new spin foam models [22]. It is difficult to test any spin foam model in this respect, because none of them is derived following a canonical quantization procedure. Moreover, being based on a simplicial discretization, the standard canonical reasoning is difficult to apply and discrete counterparts of continuum secondary constraints are very difficult to define. A variant of this criticism adapted to the simplicial setting comes from the beautiful analysis of [51], and from the standard discretization of the simplicity constraints of the classical theory, whether linear or quadratic [17, 22, 47]. These analyses show that in the canonical setting it is necessary to impose *all* simplicity constraints, including the so-called volume or edge simplicity constraints, on top of the ones we discussed explicitly, and also the so-called 'gluing constraints' which are analogues of secondary continuum constraints, in order to ensure geometricity of the classical

[15] The same motivation is shared by the work [30], where however the non-commutativity of bivector variables, following from their conjugate nature to the non-Abelian discrete connection and consistent with the canonical phase space of both BF theory and LQG, is neglected.

[16] One may also wonder why this quantization prescription is already present in the *classical* GFT action; this, however, is simply a confirmation of its interpretation as a 'third quantization' formalism [52].

configurations appearing in the model. This is indeed a reason for concern, and a detailed analysis is certainly needed in the context of the model we presented and of the other available spin foam models. However, we emphasize that the non-commutative path integral formulation of spin foam models, to be developed also for the newer models, allows us to study in a very direct way which constraints one is imposing in the definition of the spin foam model. Secondly, it is known [22, 47] that, when imposed *at the covariant classical level in all tetrahedra* of the triangulation, the combination of diagonal, off-diagonal and closure constraints is enough to ensure full geometricity of the degrees of freedom included in the model, modulo degenerate configurations, if gauge covariance (parallel transport) is also imposed. Thirdly, we have noted how our covariant imposition of simplicity constraints involving the extra data encoded in the normals and the generalized form of the closure conditions imply further conditions on the normal vectors and on the discrete connection variables. The extra conditions affect the propagation of simplicity constraints from frame to frame across the triangulation and thus their correct imposition in all tetrahedra, including those lying on different hypersurfaces. Thus it could be that such extra conditions are equivalent to the canonical secondary constraints. However, further work is needed to confirm or refute this possibility.

Several criticisms concerned the boundary states of the BC model. Once more, more work is needed to properly define the boundary Hilbert space of the theory we presented. Still, our results offer some insight. It has been argued [22, 46] that states of the BC model do not have enough parameters to correspond to a 3-geometry. In other words, there would be too-restricted a set of commuting observables and in particular no label for the 3-volume (no intertwiner degree of freedom). In this respect, we can note that the boundary states of the BC model, as we have defined it, are given by graphs labeled by SO(4) representations (or Lie algebra elements (fluxes) [31] or group elements) on their links, subject to simplicity constraints, *and* one normal vector k for each node, subject to our generalized gauge invariance condition. They are indeed projected spin networks as we have noted [23, 39, 40, 42]. A heuristic interpretation of such a state is as the discrete analogue of canonical wave functions on the extended gravity phase space, including lapse and shift vector: $\Psi(h, N, N^i)$. The standard states of the BC model, labeled only by simple representations and by the BC intertwiner, arise *after* gauge invariance is imposed and normals are integrated out. However, this integration over normals cannot be done freely (i.e. on each vertex separately, in the boundary spin network graph) because of the requirement of covariance, and because normals are correlated across tetrahedra by means of the discrete connection (also subject to constraints, as we have seen). It would rather seem that the states of the model, now including the normals, possess all the geometric information to be expected, while it can of course still be true that particular linear combinations of states (as resulting, for example, from arbitrary integrations of normal degrees of freedom) do not correspond to the well-defined boundary data.

Clearly, this state space *does not* match that of LQG. This is based on SU(2) spin networks, which are labeled by an arbitrary SU(2) intertwiner and which, while embeddable in a covariant way in SO(4) using indeed the projected spin network formalism [39, 42, 49], are based on the Ashtekar connection that requires for its definition the introduction of the Immirzi parameter [1, 22]. Indeed, the stated failure of the BC model to reproduce the tensorial structure of the (lattice) graviton propagator [46], on closer examination, could be interpreted simply as a mismatch between the data encoded in LQG spin networks and used to define both the background boundary states and the same graviton observables, and the boundary states (and consequent dynamical data) of the BC model, which, as we have shown, should be taken to be

the above projected spin networks. However, no such matching can be expected for a theory that is independent of the Immirzi parameter from the start.

Finally, the inclusion of the Immirzi parameter is the most interesting feature of the new spin foam models [22]. A generalization of the non-commutative metric formalism presented here to include the Immirzi parameter is possible and will be presented elsewhere [48], together with the resulting new model. Still, two general points must be noted. Firstly, one could expect any new model constructed in this way to reproduce (one of) the new spin foam models only in a semi-classical approximation, having in mind both our results on the topological sector of the theory and the construction of the new models based on group coherent states. Secondly, at present our main reason for wanting the Immirzi parameter in a quantum gravity theory is because of its necessity in the LQG context. It would be good to have further confirmations, either at the classical or quantum level, of its importance for capturing the correct physics of a quantum spacetime.

Lastly, we note that an important open issue is the role that degenerate geometries play in all spin foam models, including the one we have presented. Not only do they drastically affect the imposition of simplicity constraints and thus the geometric nature of the configurations included in them [12, 51], but also they may end up dominating the quantum dynamics on entropic grounds [26]. It is also crucial to stress, however, as done in [46], that this issue has to be studied in the context of the computation of physical quantities and that the insertion of appropriate observables or boundary states can dramatically change the respective weight of proper and degenerate quantum geometries.

7. Conclusions

We have used the non-commutative metric formulation of group field theories and spin foam models to define a model of 4D quantum gravity as a constrained BF theory, without the Immirzi parameter. This involved a generalization of the usual GFT framework to include both Lie algebra or group elements, associated with triangles in the triangulation, and normal vectors associated with tetrahedra of the same. The generalization led naturally to projected spin network states and the associated covariance under Lorentz transformations. The model is *uniquely* defined thanks to the projector nature of the generalized *geometricity* operator we introduced at the GFT level, encoding both simplicity constraints and (generalized) gauge covariance. The resulting model, for which we exhibited both a complete simplicial path integral expression and a spin foam representation, turns out to be a variant of the BC model, characterized by specific lower-dimensional amplitudes. We also presented a similar construction for the topological sector of the Holst action, and discussed its relation with the EPR spin foam model. In light of the above results, we have then re-examined the arguments against the BC model(s), concluding that it can still be considered a plausible quantization of 4D gravity as far as the encoding of simplicial geometry is concerned and that further work is needed to either confirm or refute its validity.

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References

[1] Rovelli C 2007 *Quantum Gravity* (Cambridge: Cambridge University Press)
Thiemann T 2007 *Modern Canonical Quantum General Relativity* (Cambridge: Cambridge University Press)
[2] David F 1985 *Nucl. Phys.* B **257** 45
Ginsparg P 1991 Matrix models of 2-d gravity arXiv:hep-th/9112013
Di Francesco P, Ginsparg P H and Zinn-Justin J 1995 *Phys. Rep.* **254** 1 (arXiv:hep-th/9306153)
[3] Gross M 1992 *Nucl. Phys. Proc. Suppl.* A **25** 144–9
Ambjorn J, Durhuus B and Jonsson T 1991 *Mod. Phys. Lett.* A **6** 1133–46
Sasakura N 1991 *Mod. Phys. Lett.* A **6** 2613–24
Bonzom V, Gurau R, Riello A and Rivasseau V 2011 arXiv:1105.3122 [hep-th]
Gurau R and Rivasseau V 2011 arXiv:1101.4182 [gr-qc]
[4] Oriti D 2001 *Rep. Prog. Phys.* **64** 1489 (arXiv:gr-qc/0106091)
Perez A 2003 *Class. Quantum Grav.* **20** R43 (arXiv:gr-qc/0301113)
Oriti D 2003 *PhD Thesis* University of Cambridge (arXiv:gr-qc/0311066)
Perez A 2004 arXiv:gr-qc/0409061
[5] Baratin A and Oriti D 2010 *Phys. Rev. Lett.* **105** 221302 (arXiv:1002.4723 [hep-th])
[6] Oriti D 2009 *Approaches to Quantum Grav.* ed D Oriti (Cambridge: Cambridge University Press) (arXiv: gr-qc/0607032)
[7] Oriti D 2007 *Quantum Gravity* ed B Fauser, J Tolksdorf and E Zeidler (Basel: Birkhauser) (arXiv: gr-qc/0512103)
[8] Freidel L 2005 *Int. J. Phys.* **44** 1769–83 (arXiv:hep-th/0505016)
[9] Oriti D 2011 *Foundations of Space and Time* ed G Ellis and J Murugan (Cambridge: Cambridge University Press)
[10] Reisenberger M and Rovelli C 2001 *Class. Quantum Grav.* **18** 121–40 (arXiv:gr-qc/0002095)
[11] Peldan P 1994 *Class. Quantum Grav.* **11** 1087 (arXic:gr-qc/9305011)
[12] De Pietri R and Freidel L 1999 *Class. Quantum Grav.* **16** 2187 (arXiv:gr-qc/9804071)
Reisenberger M 1998 arXiv:gr-qc/9804061
[13] Joung E, Mourad J and Noui K 2008 arXiv:0806.4121 [hep-th]
[14] Freidel L and Livine E 2006 *Class. Quantum Grav.* **23** 2021 (arXiv:hep-th/0502106)
[15] Freidel L and Majid S 2008 *Class. Quantum Grav.* **25** 045006 (arXiv:hep-th/0601004)
[16] Livine E R 2009 *Class. Quantum Grav.* **26** 195014 (arXiv:0811.1462 [gr-qc])
[17] Barrett J W and Crane L 1998 *J. Math. Phys.* **39** 3296 (arXiv:gr-qc/9709028)
[18] Barrett J W and Crane L 2000 *Class. Quantum Grav.* **17** 3101 (arXiv:gr-qc/9904025)
[19] De Pietri R, Freidel L, Krasnov K and Rovelli C 2000 *Nucl. Phys.* B **574** 785 (arXiv:hep-th/9907154)
[20] Perez A and Rovelli C 2001 *Nucl. Phys.* B **599** 255 (arXiv:gr-qc/0006107)
[21] Bonzom V and Livine E 2008 arXiv:0812.3456
Bonzom V 2009 *Class. Quantum Grav.* **26** 155020 (arXiv:0903.0267)
Bonzom V 2009 arXiv:0905.1501
[22] Freidel L and Krasnov K 2008 *Class. Quantum Grav.* **25** 125018 (arXiv:0708.1595)
Engle J, Pereira R and Rovelli C 2007 *Phys. Rev. Lett.* **99** 161301 (arXiv:0705.2388)
Engle J, Pereira R and Rovelli C 2008 *Nucl. Phys.* B **798** 251 (arXiv:0708.1236)
Engle J, Livine E, Pereira R and Rovelli C 2008 *Nucl. Phys.* B **799** 136 (arXiv:0711.0146)
Livine E and Speziale S 2008 *Europhys. Lett.* **81** 50004 (arXiv:0708.1915 [gr-qc])
[23] Livine E R 2002 *Class. Quantum Grav.* **19** 5525–42 (arXiv:gr-qc/0207084)

[24] Oriti D 2006 *Phys. Rev.* D **73** 061502 (arXiv:gr-qc/0512069)
Oriti D and Tlas T 2008 *Class. Quantum Grav.* **25** 085011 (arXiv:0710.2679 [gr-qc])
Oriti D 2010 *Class. Quantum Grav.* **27** 145017 (arXiv:0902.3903 [gr-qc])
Oriti D and Tlas T 2010 *Class. Quantum Grav.* **27** 135018 (arXiv:0912.1546 [gr-qc])
[25] Barrett J W and Williams R M 1999 *Adv. Theor. Math. Phys.* **3** 209 (arXiv:gr-qc/9809032)
[26] Barrett J W and Steele C M 2003 *Class. Quantum Grav.* **20** 1341 (arXiv:gr-qc/0209023)
Baez J C, Christensen J D, Halford T R and Tsang D C 2002 *Class. Quantum Grav.* **19** 4627 (arXiv: gr-qc/0202017)
[27] Ooguri H 1992 *Mod. Phys. Lett.* A **7** 2799–810 (arXiv:hep-th/9205090)
[28] De Pietri R and Freidel L 1999 *Class. Quantum Grav.* **16** 2187–96 (arXiv:gr-qc/9804071)
Reisenberger M 1998 (arXiv:gr-qc/9804061)
[29] Holst S 1996 *Phys. Rev.* D **53** 5966–9 (arXiv:gr-qc/9511026)
[30] Han M and Thiemann T 2010 arXiv:1010.5444 [gr-qc]
[31] Baratin A, Dittrich B, Oriti D and Tambornino J 2010 arXiv:1004.3450 [hep-th]
[32] Baratin A, Girelli F and Oriti D 2011 *Phys. Rev.* D **83** 104051 (arXiv:1101.0590 [hep-th])
[33] Oriti D and Raasakka M 2011 *Phys. Rev.* D **84** 025003 (arXiv:1103.2098 [hep-th])
[34] Duflo M 1977 *Ann. Sci. Ecole Norm. Sup.* **10** 265–88
[35] Sahlmann H and Thiemann T 2011 arXiv:1101.1690 [gr-qc]
[36] Alekseev A, Polychronakos A P and Smedback M 2003 *Phys. Lett.* B **574** 296–300 (arXiv:hep-th/0004036)
[37] Livine E 2001 arXiv:gr-qc/0103081
[38] Livine E and Oriti D 2002 *Phys. Rev.* D **65** 044025 (arXiv:gr-qc/0104043)
[39] Alexandrov S 2010 *Phys. Rev.* D **82** 024024 (arXiv:1004.2260 [gr-qc])
[40] Alexandrov S 2008 *Phys. Rev.* D **78** 044033 (arXiv:0802.3389 [gr-qc])
[41] Engle J and Pereira R 2008 *Class. Quantum Grav.* **25** 105010 (arXiv:0710.5017 [gr-qc])
[42] Alexandrov S and Livine E 2003 *Phys. Rev.* D **67** 044009 (arXiv:gr-qc/0209105)
[43] Livine E and Oriti D 2007 *J. High Energy Phys.* JHEP02(2007)092 (arXiv:gr-qc/0512002)
[44] Barrett J W 1998 *Adv. Theor. Math. Phys.* **2** 593–600 (arXiv:math/9803063)
[45] Noui K, Perez A and Pranzetti D 2011 arXiv:1105.0439 [gr-qc]
[46] Bianchi E, Modesto L, Rovelli C and Speziale S 2006 *Class. Quantum Grav.* **23** 6989–7028 (arXiv: gr-qc/0604044)
Alesci E and Rovelli C 2007 *Phys. Rev.* D **76** 104012 (arXiv:0708.0883 [gr-qc])
[47] Gielen S and Oriti D 2010 *Class. Quantum Grav.* **27** 185017 (arXiv:1004.5371 [gr-qc])
[48] Baratin A and Oriti D to appear
[49] Dupuis M and Livine E 2010 *Phys. Rev.* D **82** 064044 (arXiv:1008.4093 [gr-qc])
[50] Oriti D 2002 *Phys. Lett.* B **532** 363–72 (arXiv:gr-qc/0201077)
[51] Dittrich B and Ryan J 2008 arXiv:0807.2806 [gr-qc]
Dittrich B and Ryan J 2010 *Phys. Rev.* D **82** 064026 (arXiv:1006.4295 [gr-qc])
[52] Gielen S and Oriti D 2011 arXiv:1102.2226 [gr-qc]
[53] Conrady F and Freidel L 2008 *Phys. Rev.* D **78** 104023 (arXiv:0809.2280 [gr-qc])
[54] Ben-Geloun J, Gurau R and Rivasseau V 2010 arXiv:1008.0354 [hep-th]
[55] Buffenoir E, Henneaux M, Noui K and Roche Ph 2004 *Class. Quantum Grav.* **21** 5203–20 (arXiv: gr-qc/0404041)
[56] Barros e Sa N 2001 *Int. J. Mod. Phys.* D **10** 261–72 (arXiv:gr-qc/0006013)