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Superconformal M2-branes and generalized Jordan triple systems

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ABSTRACT: Three-dimensional conformal theories with six supersymmetries and $SU(4)$ R-symmetry describing stacks of M2-branes are here proposed to be related to generalized Jordan triple systems. Writing the four-index structure constants in an appropriate form, the Chern-Simons part of the action immediately suggests a connection to such triple systems. In this note we show that the whole theory with six manifest supersymmetries can be naturally expressed in terms of structure constants of generalized Jordan triple systems. We comment on the associated graded Lie algebra, which corresponds to an extension of the gauge group.

KEYWORDS: String theory, M-theory, Branes, Chern-Simons theory.

1. Introduction

A three-dimensional maximally ($\mathcal{N} = 8$) superconformal theory was recently constructed by Bagger, Lambert and Gustavsson (BLG) in [1, 2, 3, 4]. The BLG theory was originally proposed to describe multiple M2-branes. An interesting aspect of this theory is that it contains a Chern-Simons term [5] making the BLG theory potentially interesting also for condensed matter applications. The multiple M2-brane interpretation has, however, met with a number of problems having to do with the algebraic structure on which the theory is based. The theory contains a kind of four-index structure constant for a three-algebra with a Euclidean metric. This three-algebra has, however, been proven [6, 7] to have basically only one realization, $\mathcal{A}_4$, related to the ordinary Lie algebra $so(4)$ through its totally antisymmetric epsilon tensor. This is limiting the role of this theory to stacks of two M2-branes [8, 9].

By relaxing the assumption that the metric on the algebra should be positive definite [10] any Lie algebra can be accommodated. The drawback of using a degenerate metric is that it produces a set of field equations which cannot be integrated to a Lagrangian if the zero norm mode is not assumed constant. This subsequently led to a number of attempts to use a non-degenerate but Lorentzian metric [11, 12, 13]. Again there are problems; these theories make sense only provided the negative norm modes can be rendered harmless. Even when this is the case they are of real interest only if they contain genuine M2-physics instead of just providing a reformulation of the D2-brane. For some recent results in this direction, see [14, 15, 16, 17].

From the work of [10] it was also clear that the structure constants need not be totally antisymmetric. This might be interesting since this property seems to be part of the reason why only one realization, related to $SO(4)$, of the fundamental identity can be constructed in the Euclidean case. In fact, as realized by Aharony, Bergman, Jafferis and Maldacena (ABJM) [18], by reducing the number of linearly realized supersymmetries from the maximal $\mathcal{N} = 8$ to $\mathcal{N} = 6$ this no-go theorem can be avoided. Following [19], the authors of [18] used a construction with the fields in the bi-fundamental representation of $U(N) \times U(N)$ and without any reference to the four-index structure constants. However, in a work following this Bagger and Lambert [20] pointed out that if reinstating the four-index structure constants there are interesting implications for their antisymmetry properties. In particular, six supersymmetries are compatible with structure constants which are not totally antisymmetric.

The purpose of this note is to write the structure constants in yet another form which suggests the possibility of relating them to certain algebraic structures, known as generalized Jordan triple systems.

The paper is organized as follows. In section two we review the ABJM theory and present the Lagrangian in terms of four-index structure constants as described in [20]. In section three we then provide a reformulation of this theory in terms of structure

constants adapted to triple systems. Some relevant aspects of generalized Jordan triple systems and the associated graded Lie algebra are summarized in section four. The last section contains conclusions and some further comments.

2. The ABJM M2 theory

The BLG theory contains three different fields; the two propagating ones $X^I{}_a$ and Ψ_a, which are three-dimensional scalars and spinors, respectively, and the auxiliary gauge field $\tilde{A}_\mu{}^a{}_b$. Here the indices a, b, ... are connected to the three-algebra and some n-dimensional basis T^a, while the I, J, K, ... indices are $SO(8)$ vector indices. The spinors transform under a spinor representation of $SO(8)$ but the corresponding index is not written out explicitly. Indices μ, ν, ... are vector indices on the flat M2-brane world volume.

Using these fields one can write down $\mathcal{N} = 8$ supersymmetry transformation rules and covariant field equations. This is possible without introducing a metric on the three-algebra. In such a situation the position of the indices on the structure constants is fixed as $f^{abc}{}_d$. The corresponding fundamental identity needed for supersymmetry and gauge invariance then reads [1, 2, 3, 4],

$$f^{abc}{}_g f^{efg}{}_d = 3 f^{ef[a}{}_g f^{bc]g}{}_d \,, \tag{2.1}$$

which can be written in the following alternative but equivalent form [10],

$$f^{[abc}{}_g f^{e]fg}{}_d = 0 \,. \tag{2.2}$$

The construction of a Lagrangian requires the introduction of a metric on the three-algebra. As discussed above, if one wants to describe more general Lie algebras than $so(4)$, this metric must be degenerate [10] or non-degenerate but indefinite [11, 12, 13]. Finally, to construct an action one also needs to introduce the basic gauge field $A_{\mu ab}$ [1] which is related to the previously defined gauge field and structure constants as follows,

$$\tilde{A}_\mu{}^a{}_b = A_{\mu cd} f^{cda}{}_b \,. \tag{2.3}$$

The BLG Lagrangian is [3]

$$\mathcal{L} = -\frac{1}{2}(D_\mu X^{Ia})(D^\mu X^I{}_a) + \frac{i}{2}\bar{\Psi}^a \gamma^\mu D_\mu \Psi_a + \frac{i}{4}\bar{\Psi}_b \Gamma_{IJ} X^I{}_c X^J{}_d \Psi_a f^{abcd} \tag{2.4}$$
$$-V + \frac{1}{2}\varepsilon^{\mu\nu\lambda}\left(f^{abcd} A_{\mu ab}\partial_\nu A_{\lambda cd} + \frac{2}{3} f^{cda}{}_g f^{efgb} A_{\mu ab} A_{\nu cd} A_{\lambda ef} \right) \,,$$

where the potential is given by

$$V = \frac{1}{12} f^{abcd} f^{efg}{}_d X^I{}_a X^J{}_b X^K{}_c X^I{}_e X^J{}_f X^K{}_g \,. \tag{2.5}$$

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[1]However, already gauge invariance of the field equations requires this gauge field [10].

Note that in terms of $\tilde{A}$ the Chern-Simons term becomes

$$\mathcal{L}_{CS} = \frac{1}{2}\varepsilon^{\mu\nu\lambda}\left(A_{\mu ab}\partial_\nu\tilde{A}_\lambda{}^{ab} + \frac{2}{3}A_\mu{}^a{}_b\tilde{A}_\nu{}^b{}_c\tilde{A}_\lambda{}^c{}_a\right) \tag{2.6}$$

and that the fundamental identity implies that, in the last term, the structure constants can be associated with any two of the three vector fields.

Following ABJM [18] we now rewrite this in a form which has only six manifest supersymmetries and manifest $SU(4)$ R-symmetry. As emphasized by these authors, this is naturally done using matter fields in the bi-fundamental representation [19] of $U(N) \times U(N)$, and no reference to three-algebras and their structure constants is needed. However, for the purpose of this note we need to reinstate the four-index structure constants. Fortunately, this was discussed in detail in a recent work by Bagger and Lambert [20].

The ABJM action is expressed in terms of complex scalar fields Z_a^A and spinors Ψ_{Aa} with the capital indices transforming in fundamental and anti-fundamental representations of the $SU(4)$ R-symmetry, respectively. If rewritten in terms of four-index structure constants as done in [20], the ABJM action reads

$$\begin{aligned}
\mathcal{L} = &-\frac{1}{2}(D_\mu Z_a^A)(D^\mu \bar{Z}_A^a) - \frac{i}{2}\bar{\Psi}^{Aa}\Gamma^\mu D_\mu \Psi_A^a \\
&-\frac{i}{2}f^{abcd}\bar{\Psi}_d^A\Psi_{Aa}Z_b^B\bar{Z}_{Bc} + if^{abcd}\bar{\Psi}_d^A\Psi_{Ba}Z_b^B\bar{Z}_{Ac} \\
&-\frac{i}{4}\epsilon_{ABCD}f^{abcd}\bar{\Psi}_c^A\Psi_d^B Z_a^C Z_b^D - \frac{i}{4}\epsilon^{ABCD}f^{cdab}\bar{\Psi}_{Ac}\Psi_{Bd}\bar{Z}_{Ca}\bar{Z}_{Bd} \\
&-V + \frac{1}{2}\epsilon^{\mu\nu\lambda}(f^{abcd}A_{\mu ab}\partial_\nu A_{\lambda cd} + \frac{2}{3}f^{cda}{}_g f^{efgb}A_{\mu ab}A_{\nu cd}A_{\lambda ef}),
\end{aligned} \tag{2.7}$$

where the potential can be written

$$V = \frac{1}{3}\Upsilon_{Bd}^{CD}\bar{\Upsilon}_{CD}^{Bd}, \tag{2.8}$$

$$\Upsilon_{Bd}^{CD} = f^{abc}{}_d Z_a^C Z_b^D \bar{Z}_{Bc} + f^{abc}{}_d \delta_B^{[C} Z_a^{D]} Z_b^E \bar{Z}_{Ec}. \tag{2.9}$$

In order to write this action one needs a metric on the three-algebra to raise and lower three-algebra indices. The structure constants appearing in this formulation of the $\mathcal{N} = 6$ ABJM theory [20] are antisymmetric in the first pair of indices as well as in the second pair while complex conjugation is defined to interchange the two pairs of indices.

As we will see below the need for a metric can be eliminated by writing the structure constants as $f^{ab}{}_{cd}$ or $f^a{}_b{}^c{}_d$ (which we will see later are in fact related to each other). This will also require the introduction of a graded Lie algebra in a way that will be explained in the next section.

3. Structure constants adapted to triple systems

Our next goal is to try to relate the M2-brane to generalized Jordan triple systems. The first step is to rewrite the $\mathcal{N}= 6$ M2-theory as formulated at the end of the previous section in terms of structure constants with two upper and two lower indices, which are antisymmetric in each pair separately,

$$f^{ab}{}_{cd} = f^{[ab]}{}_{cd} = f^{ab}{}_{[cd]}\,. \tag{3.1}$$

The crucial difference between our approach and the one used in [20] is that we do not consider the fields Z^A, Ψ_A as elements in the same three-algebra as their complex conjugates Z_A, Ψ^A. Rather, we are dealing with two vector spaces g_1 and g_{-1}, with bases T^a and T_a, respectively. These two vector spaces generate a graded Lie algebra g. We do not use any metric on g_1 and g_{-1} to raise and lower indices, but we use an antilinear involution τ on g to go between the subspaces, $\tau(T^a) = T_a$. We also use a bilinear form on g to contract upper and lower indices. We will describe this graded Lie algebra in more detail in the next section. Here we just define the components of the fields Z^A, Ψ_A in g_1 to have the index structure Z_a^A, Ψ_{Aa}. The components of $\tau(Z^A)$, $\tau(\Psi_A)$ in g_{-1} are then the complex conjugates $\bar{Z}_A^a$, Ψ^{Aa}. That it is natural to place the indices like this can be seen from rewriting the Bagger-Lambert version the ABJM action as follows,

$$\begin{aligned}
\mathcal{L} = &-\frac{1}{2}(D_\mu Z_a^A)(D^\mu \bar{Z}_A^a) - \frac{i}{2}\bar{\Psi}^{Aa}\gamma^\mu D_\mu \Psi_{Aa} \\
&-\frac{i}{2}f^{ab}{}_{cd}\bar{\Psi}^{Ad}\Psi_{Aa}Z_b^B \bar{Z}_B^c + if^{ab}{}_{cd}\bar{\Psi}^{Ad}\Psi_{Ba}Z_b^B \bar{Z}_A^c \\
&-\frac{i}{4}\epsilon_{ABCD}f^{ab}{}_{cd}\bar{\Psi}^{Ac}\Psi^{Bd}Z_a^C Z_b^D - \frac{i}{4}\epsilon^{ABCD}f^{cd}{}_{ab}\bar{\Psi}_{Ac}\Psi_{Bd}\bar{Z}_C^a \bar{Z}_B^b \\
&-V + \frac{1}{4}\epsilon^{\mu\nu\lambda}(f^{ab}{}_{cd}A_{\mu b}^d\partial_\nu A_{\lambda a}^c + \frac{2}{3}f^{bd}{}_{gc}f^{gf}{}_{ae}A_{\mu b}^a A_{\nu d}^c A_{\lambda f}^e)\,,
\end{aligned} \tag{3.2}$$

where the potential now takes the form

$$V = \frac{1}{3}\Upsilon_{Bd}^{CD}\bar{\Upsilon}_{CD}^{Bd}\,, \tag{3.3}$$

$$\Upsilon_{Bd}^{CD} = f^{ab}{}_{cd}Z_a^C Z_b^D \bar{Z}_B^c + f^{ab}{}_{cd}\delta_B^{[C}Z_a^{D]}Z_b^E \bar{Z}_E^c\,. \tag{3.4}$$

This action can be shown to be $\mathcal{N} = 6$ supersymmetric provided that

$$f^{a[b}{}_{dc}f^{e]d}{}_{gh} = f^{be}{}_{d[g}f^{ad}{}_{h]c}\,. \tag{3.5}$$

One immediate way to see that this identity is relevant is to consider the Chern-Simons term

$$\mathcal{L}_{CS} = \frac{1}{2}\varepsilon^{\mu\nu\lambda}\left(A_{\mu a}^b\partial_\nu \tilde{A}_{\lambda b}^a + \frac{2}{3}A_{\mu b}^a\tilde{A}_{\nu c}^b\tilde{A}_{\lambda a}^c\right)\,, \tag{3.6}$$

where we use vector fields $A^a_{\mu b}$ and

$$\tilde{A}^a_{\mu b} = f^{ac}{}_{bd} A^d_{\mu c} \,. \tag{3.7}$$

The above identity for $f^{ab}{}_{cd}$ then follows from the observation that when deriving the field equation the variation of each vector field must provide an identical contribution to the answer. Note that also the Chern-Simons field without tilde has an upper and a lower index which is not the case in previous treatments of the M2-brane system. As we will see in the next section these structure constants and the identity they satisfy hint at a connection to generalized Jordan triple systems.

In terms of structure constants of generalized Jordan triple systems the transformation rules for the six supersymmetries, parametrized by the complex self-dual three-dimensional spinor ϵ_{AB}, read

$$\delta Z^A_a = i\bar{\epsilon}^{AB} \Psi_{Ba} \,, \tag{3.8}$$

$$\delta \Psi_{Bd} = \gamma^\mu D_\mu Z^A_d \epsilon_{AB} + f^{ab}{}_{cd} Z^C_a Z^D_b \bar{Z}^c_B \epsilon_{CD} - f^{ab}{}_{cd} Z^A_a Z^C_b \bar{Z}^c_C \epsilon_{AB} \,, \tag{3.9}$$

while the Chern-Simons one-form transforms as follows,

$$\delta A^a_{\mu b} = -i\bar{\epsilon}_{AB} \gamma_\mu \Psi^{Aa} Z^B_b + i\bar{\epsilon}^{AB} \gamma_\mu \Psi_{Ab} \bar{Z}_{Ba} \,. \tag{3.10}$$

To prove that the Lagrangian has six supersymmetries only requires the use of the above identity for the structure constants $f^{ab}{}_{cd}$. In order to see how this identity arises in generalized Jordan triple systems, we need to discuss some further aspects of the underlying graded Lie algebra.

4. Triple systems and graded Lie algebras

Consider first $sl(n)$, with generators $K^a{}_b$, and two vector spaces g_1 and g_{-1}, of dimension n. We let $sl(n)$ act on g_1 and g_{-1} in the fundamental and antifundamental representation, respectively:

$$[K^a{}_b, T^c] = \delta_b{}^c T^a, \qquad\qquad [K^a{}_b, T_c] = -\delta^a{}_c T_b \,. \tag{4.1}$$

We write the action as a commutator since we want to include g_1 and g_{-1} in the Lie algebra structure, in a way such that $[g_{-1}, g_1] \subset sl(n)$. For this we introduce the structure constants $f^a{}_b{}^c{}_d$,

$$[T^a, T_b] = f^a{}_b{}^c{}_d K_c{}^d \equiv S^a{}_b \tag{4.2}$$

and from (4.1) we get

$$[S^a{}_b, T^c] = f^a{}_b{}^c{}_d T^d, \qquad\qquad [S^a{}_b, T_c] = -f^a{}_b{}^d{}_c T_d \,. \tag{4.3}$$

We thus have $[[T^a, T_b], T^c] = f^a{}_b{}^c{}_d T^d$, and analogously we define the structure constants $f_a{}^b{}_c{}^d$ by $[[T_a, T^b], T_c] = f_a{}^b{}_c{}^d T_d$. It follows from (4.3) that

$$f_a{}^b{}_c{}^d = f^b{}_a{}^d{}_c.$$

(4.4)

Let g_0 be the subspace of $sl(n)$ spanned by all generators $S^a{}_b$. For the Jacobi identity

$$[[T^a, T_b], T^c] - [[T^c, T_b], T^a] = [[T^a, T^c], T_b]$$

(4.5)

to hold, the structure constants must satisfy the identity

$$f^a{}_b{}^f{}_g f^c{}_d{}^e{}_f = f^c{}_d{}^f{}_g f^a{}_b{}^e{}_f + f^a{}_b{}^c{}_f f^f{}_d{}^e{}_g - f_b{}^a{}_d{}^f f^c{}_f{}^e{}_g.$$

(4.6)

Then g_0 form a subalgebra of $sl(n)$ with the commutation relations

$$[S^a{}_b, S^c{}_d] = f^a{}_b{}^c{}_e S^e{}_d - f_b{}^a{}_d{}^e S^c{}_e.$$

(4.7)

We call any Lie algebra g generated by a subspace $g_{-1} + g_0 + g_1$ of this form a *graded Lie algebra*. It can in general be written as a direct sum of subspaces g_k for all integers k, such that

$$[g_i, g_j] \subseteq g_{i+j}$$

(4.8)

for all integers i, j (with the possibility that $g_k = 0$ for all sufficiently large $|k|$). We call k the *level* of the elements in g_k. Let τ be the restriction of the Chevalley involution on $sl(n)$ to g_0. Then τ can then be extended by $\tau(T^a) = T_a$ to a graded involution on the whole of g, such that $\tau(g_k) \subseteq g_{-k}$ for all integers k. It follows from this property, together with (4.8), that g_1 closes under the triple product

$$(abc) = [[a, \tau(b)], c],$$

(4.9)

(and likewise for g_{-1}). Thus $f^a{}_b{}^c{}_d$ are the 'structure constants' of the triple system, and the identity (4.6) can now be expressed as

$$(ab(xyz)) - (xy(abz)) = ((abx)yz) - (x(bay)z).$$

(4.10)

This is the definition of a generalized Jordan triple system. The special case of Jordan triple systems arises when $[g_1, g_1] = [g_{-1}, g_{-1}] = 0$. This means that the Lie algebra g is 3-graded and it follows by the Jacobi identity that the triple product (abc) is symmetric in a and c. For any generalized Jordan triple system T, there is an associated graded Lie algebra g, which is an extension of the vector space $g_{-1} + g_0 + g_1$ that we described above [21]. We stress that the Lie algebra associated to a generalized Jordan triple system is the whole graded Lie algebra g, and not only the subalgebra g_0, which (in the case of three-algebras) is what Bagger and Lambert called 'the associated Lie algebra' in [3]. In the case when g is finite-dimensional,

Kantor showed that simplicity of g is equivalent to the notion of *K-simplicity* for T [21]. A generalized Jordan triple system T is K-simple if there are no proper nontrivial subspace U such that $(UUT) \subseteq U$ and $(TUU) \subseteq U$. We will assume that T is K-simple, and also that g_0 is semisimple. We can extend the Killing form κ on g_0 to the vector space $g_{-1} + g_0 + g_1$ by $\kappa(T^a, T_b) = \delta^a{}_b$ and to higher levels by invariance, provided that the structure constants satisfy

$$f^a{}_b{}^c{}_d = f^c{}_d{}^a{}_b. \tag{4.11}$$

In the construction of the graded Lie algebra g associated to a generalized Jordan triple system T, one defines elements $T^{ab} = [T^a, T^b]$ at level two, $T^{abc} = [[T^a, T^b], T^c]$ at level three, and so on, (and likewise at the negative levels) but impose also further symmetry conditions on the generators than just

$$0 = T^{(ab)cd\cdots} = T^{[abc]d\cdots}, \tag{4.12}$$

coming from antisymmetry of the Lie bracket and the Jacobi identity. With our assumptions here, these further conditions amounts to factoring out the remaining ideals that would be left in the free Lie algebra generated by T^a and T_a, which in turn come from the fact the bilinear invariant form is degenerate on the free Lie algebra. This can be done recursively. Suppose that the restriction of the bilinear form to the vector space $g_{-k+1} + \cdots + g_{k-1}$, for some k, is non-degenerate. Then $\kappa(x, y)$, where $x \in g_k$ and $y \in g_{-k}$, is a linear combination of terms

$$f^{a_1 \cdots a_k}{}_{b_1 \cdots b_{k-1} b_k} \equiv \kappa(T^{a_1 \cdots a_k}, T_{b_1 \cdots b_{k-1} b_k}) = -\kappa([T^{a_1 \cdots a_k}, T_{b_k}], T_{b_1 \cdots b_{k-1}}). \tag{4.13}$$

Using the structure constants for the triple product, this can be evaluated as

$$f^{a_1 \cdots a_k}{}_{b_1 \cdots b_k} = f^{a_1}{}_{b_1}{}^{a_2}{}_c f^{ca_3 \cdots a_k}{}_{b_2 \cdots b_k} - \sum f^{a_j}{}_{b_1}{}^{a_i}{}_c f^{c_{ij}}{}_{b_2 b_3 \cdots b_k}, \tag{4.14}$$

where the sum goes over all i, j such that $1 \leq i < j \leq k$ and c_{ij} denotes the sequence of indices obtained from $a_1 \cdots a_n$ by omitting a_j and replacing a_i by c, that is,

$$c_{ij} = a_1 \cdots a_{i-1} \, c \, a_{i+1} \cdots a_{j-1} \, a_{j+1} \cdots a_k. \tag{4.15}$$

From now on, we assume that the structure constants are antisymmetric in the first and third index. It then follows from (4.11) that they are antisymmetric also in the second and the fourth index:

$$f^a{}_b{}^c{}_d = -f^c{}_b{}^a{}_d = -f^a{}_d{}^c{}_b \tag{4.16}$$

and we have

$$f^{ab}{}_{cd} = 2f^a{}_c{}^b{}_d \tag{4.17}$$

from the Jacobi identity. The identity (4.6) becomes

$$f^{e[a}{}_{dc}f^{b]d}{}_{gh} = f^{ab}{}_{d[g}f^{ed}{}_{h]c}, \qquad (4.18)$$

which was the identity that we saw was needed for supersymmetry in the previous section. Furthermore, the first term on the right hand side of (4.14) coincide with the first term in the summation and in the case $k = 3$ the equation simplifies to

$$\begin{aligned} f^{abc}{}_{def} &= 2f^a{}_d{}^b{}_g f^{gc}{}_{ef} - f^c{}_d{}^a{}_g f^{gb}{}_{ef} - f^c{}_d{}^b{}_g f^{ag}{}_{ef} \\ &= f^{ab}{}_{dg}f^{gc}{}_{ef} - \tfrac{1}{2}f^{ca}{}_{dg}f^{gb}{}_{ef} - \tfrac{1}{2}f^{cb}{}_{dg}f^{ag}{}_{ef} \\ &= f^{ab}{}_{dg}f^{gc}{}_{ef} - f^{c[a}{}_{gd}f^{b]g}{}_{ef}. \end{aligned} \qquad (4.19)$$

We see that $f^{abc}{}_{def}$ is antisymmetric in the first two indices and vanishes upon antisymmetrization in the three upper indices (or the three lower ones), as it should according to the Jacobi identity. Continuing in this way, one can determine which symmetries the tensors at each level must have, and their commutation relations follow from the Jacobi identity.

If the triple product is antisymmetric in its first and third argument, and in addition K-simple, then this procedure will never terminate. This follows from the theorem that the Lie algebra g associated to a K-simple generalized Jordan triple system is simple. Indeed, if g is a simple finite-dimensional, then we can find a Cartan-Weyl basis with elements e, f, h, corresponding to a simple root, at level zero, one, and minus one, respectively. But then we would have

$$(e\tau(f)e) = [[e, f], e] = [h, e] = 2e. \qquad (4.20)$$

which must vanish since we assume the triple product to be antisymmetric in the first and the third argument. Thus, g seems not be a Kac-Moody algebra, although its construction is suggestive of this. We would rather suggest a Borcherds algebra, since these algebras allow for zero eigenvalues of the step operators under the adjoint action of the Cartan elements.

5. Conclusions and comments

This note is based on the observation that the $\mathcal{N} = 6$ ABJM theory can be written in terms of four-index structure constants $f^{ac}{}_{bd}$ which are antisymmetric only in the upper pair and the lower pair separately. The fundamental identity then takes the same form as the basic identity in a generalized Jordan triple system suggesting a connection to graded Lie algebras appearing in such systems. The position of the indices are related to the involution used in the triple system and the Lagrangian can then be constructed without introducing a metric.

We have been very general in the description of the Lie algebra associated to a generalized Jordan triple system. The example that it first of all should be applied to

is the three-algebra given by Bagger and Lambert in [20], for which the Lie algebra, The relation between their work and ours should be studied in detail.

Even if much of what we have presented in this note are reformulations of previous results, we think that our approach opens up new perspectives. We have interpreted the fields Z^A, Ψ_A as elements in g_1, their conjugates as elements in g_{-1}, and the gauge field A_μ as an element in g_0. Although we do not have any interpretation of the elements at higher (positive and negative) levels, we cannot set them to zero, because we need the triple product to be antisymmetric in the first and third argument. Therefore we believe that also the full algebra might play an important role in the theory of M2-branes. For example, it points out a new direction in which one could possibly search for the behavior $n^{3/2}$ that the degrees of freedom of n M2-branes are conjectured to exhibit. In any case, it would be interesting to see how fast the dimension of the Lie algebra grows as we go to higher levels. The algorithm that we have described for finding the corresponding g_0-representations would probably be easy to implement in a computer program.

There are many implications following from a relation between M2-brane systems and generalized Jordan triple systems. In particular, very little is known about the structure of such systems when the grading is infinite. Finite-dimensional cases are better known and many of their properties have been studied (for an overview of Jordan, Kantor and Freudenthal triple systems, we refer to [22]). For instance, in analogy with Freudenthal triple systems (see e.g. [23]), we may suspect that other graded systems might also be of interest in connection with minimal representations, spherical vectors and the associated automorphic forms. For previous attempts to use the theory of automorphic forms in the context of the M2-brane, see [24, 25].

Let us end by mentioning two other issues. The triple system construction leads to minimal representations via non-linear realizations of the full algebra. In [18] the authors argue that the M2-theory discussed here really has eight supersymmetries but that the last two are somehow realized non-linearly. The triple system may in fact also suggest how to derive non-linear realizations of the remaining two supersymmetries needed to obtain the maximal number of $\mathcal{N} = 8$ supersymmetries.

The second issue is the one of unitarity. Standard triple system constructions naturally lead to Lie algebras that appear in their split form although other forms are also possible. To achieve unitarity one may try to quantize the theory whereby an infinite-dimensional unitary minimal representation is realized on a Hilbert space. For an explicit example, see [26].

Acknowledgments

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