Math. Z.
DOI 10.1007/s00209-009-0476-0
Mathematische Zeitschrift

Remark on the anisotropic prescribed mean curvature equation on arbitrary domains

Thomas Marquardt

Received: 1 December 2008 / Accepted: 22 December 2008

Abstract In this article we consider the Dirichlet problem for hypersurfaces of anisotropic prescribed mean curvature $H = H(x, u, N)$ depending on $x \in \Omega \subset \mathbb{R}^n$, the height u of the hypersurface $M = \operatorname{graph} u$ over Ω and the unit normal N to M at (x, u). We give a condition relating H and the mean curvature of $\partial\Omega$ that guarantees the existence of smooth solutions even for not necessarily convex domains.

Keywords Dirichlet problem · Anisotropic prescribed mean curvature · Boundary gradient estimate

Mathematics Subject Classification (2000) 35J25 · 53A10

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and M a hypersurface over Ω given as the graph of $u : \Omega \to \mathbb{R}$. The Dirichlet problem of prescribed anisotropic mean curvature is given by

$$\frac{1}{n}\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = H\,(x, u, N(Du)) \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega.$$

The left-hand side of the first equation is the average of the principal curvatures of M, i.e. the mean curvature. $H = H(x, z, N)$ is a given function depending on the point $(x, u) \in M$ and the direction of the upper normal to M at that point, given by

$$N(Du) = \frac{1}{\sqrt{1+|Du|^2}}\begin{pmatrix} -Du \\ 1 \end{pmatrix}.$$

T. Marquardt (✉)
Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut,
Am Mühlenberg 1, 14476 Golm, Germany
e-mail: thomas.marquardt@aei.mpg.de

Published online: 22 January 2009

Early results of Serrin [4] for $H = H(x)$ are summarised in [3], where existence was proved using a fixed point argument and Hölder estimates for the gradient, reducing the proof to C^1-a priori estimates.

In the general case $H = H(x, z, N)$ either the condition

$$\int_\Omega \sup_{z\in\mathbb{R}, N\in S^n} |H(x, z, N)|^n \,\mathrm{d}x \;<\; \omega_n \tag{1}$$

or the assumption $D_z H \geq 0$ together with

$$\left|\int_\Omega H\,(x, 0, N(D\eta))\,\eta\,\mathrm{d}x\right| \;\leq\; \frac{1-\varepsilon}{n}\int_\Omega |D\eta|\,\mathrm{d}x \quad \forall\,\eta \in C_0^1(\Omega) \tag{2}$$

for some $\varepsilon > 0$ ensure a bound on u. The result is well known in the case that (1) holds and can be established by applying the method of Stampacchia [5] otherwise. A condition comparable to (2) was first imposed by Giaquinta [2].

To derive a gradient estimate it is favourable to impose for some $\beta > 0$ the structure condition

$$(sc)\begin{cases} H_1,\ H_2 \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^{n+1}) \cap C^{1,\beta}(\Omega \times \mathbb{R} \times \mathbb{R}^{n+1}), \\ H(x, z, N) \;=\; H_1(x, z, N) \;+\; H_2(x, z, N)N_{n+1}, \quad D_z H_1 \geq 0 \text{ on } \Omega \times \mathbb{R} \times S^n_+. \end{cases}$$

This condition was first used by Bergner [1]. He estimated Du in the interior in terms of the boundary gradient using a maximum principle. The boundary gradient estimate he obtained was only valid in convex domains.

The main result of this paper is Theorem 1, where we prove a boundary gradient estimate assuming an a priori bound for u, (sc) and the Serrin type condition

$$H_{\partial\Omega}(y) \;\geq\; \frac{n}{n-1}\,|H_1\,(y, \varphi(y), \gamma(y))| \quad \forall\, y \in \partial\Omega, \tag{3}$$

where γ is the inward pointing unit normal of $\partial\Omega$ considered as a vector in $\mathbb{R}^{n+1}$. As a consequence we obtain existence of smooth solutions in Corollary 1 even for not necessarily convex domains.

2 Results

For a domain $\Omega \subset \mathbb{R}^n$ with C^2-boundary we define the boundary strip

$$\Gamma_1 \;:=\; \left\{x \in \overline{\Omega} \mid \mathrm{d}(x) := \mathrm{dist}(x, \partial\Omega) \;<\; d_1\right\}$$

with $d_1 \leq 1$ such that $\mathrm{d} \in C^2(\overline{\Gamma}_1)$ (cf. [3, Lemma 14.16]). Using $a := \sup_{\Gamma_1} |\varphi|$, $m := \sup_\Omega |u|$ and

$$\begin{aligned} h_i &:= \sup_{\Gamma_1\times[-m,m]\times S^n} |H_i(x, z, N)| \quad \text{for } i = 1, 2, \\ h_{1,X} &:= \sup_{\Gamma_1\times[-a,a]\times S^n} |D_X H_1(x, z, N)| \quad \text{for } X = x, z, N, \end{aligned}$$

we obtain the following result:

Theorem 1 (Boundary gradient estimate) *Let $n \geq 2$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary. Suppose that $\varphi \in C^2(\overline{\Omega})$. Let $u \in C^{2,\alpha}(\overline{\Omega})$ be a solution of the Dirichlet problem of anisotropic prescribed mean curvature with an a priori bound m. If H satisfies (sc) and (3), then we have*

$$\sup_{\partial\Omega} |Du| \leq \sup_{\partial\Omega} |D\varphi| + \frac{1}{d_1}\left(1 + 2\|\varphi\|_{C^1(\overline{\Gamma}_1)}\right) e^{(1+C)(a+m)}$$

with $C = C\left(n, \Omega, \|\,\mathrm{d}\,\|_{C^2(\overline{\Gamma}_1)}, \|\varphi\|_{C^2(\overline{\Gamma}_1)}, h_1, h_2, h_{1,x}, h_{1,z}, h_{1,N}\right)$.

Remark 1 In the case $H = H(x, z)$, condition (3) reduces to the Serrin condition which is known to be sharp.

Corollary 1 (Existence and Uniqueness) *Let $n \geq 2$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$-boundary and $\varphi \in C^{2,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. Suppose that H satisfies (sc), (1) and (3). Then the Dirichlet problem for the equation of anisotropic prescribed mean curvature has a solution $u \in C^{2,\alpha}(\overline{\Omega})$. In the case $D_z H \geq 0$ condition (1) can be replaced by (2). Furthermore if $D_z H \geq 0$ the solution is unique.*

Remark 2 To generalise the result of Corollary 1 to boundary values $\varphi \in C^0(\partial\Omega)$ one can approximate φ uniformly by functions $\varphi_m \in C^{2,\alpha}(\overline{\Omega})$. Demanding $D_z H \geq 0$ and applying a comparison principle, the hereby obtained solutions $u_m \in C^{2,\alpha}(\overline{\Omega})$ converge uniformly to some function $u \in C^0(\overline{\Omega})$ with $u = \varphi$ on $\partial\Omega$. Interior gradient estimates which follow from [6], Theorem 1.1 (one can easily see that they even hold in the case $H_{1,z} \geq 0$), together with interior Hölder and Schauder estimates imply by Arzelà-Ascoli that $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ and $Qu = 0$.

Remark 3 Translating solutions $u(x, t) = u_0(x)+t$ of mean curvature flow have the property that u_0 satisfies our Dirichlet problem with $H_1 \equiv 0$ and $H_2 \equiv 1/n$. Hence Theorem 1 yields a boundary gradient estimate for such solutions in any mean convex domain. The necessary bound on u_0 can be shown explicitly, such that neither (1) nor (2) is needed.

Proof of Corollary 1 The conditions on Ω, φ and H allow us to apply [3], Theorem 13.8 which reduces the existence proof to the derivation of C^1-a priori bounds for solutions $u \in C^{2,\alpha}(\overline{\Omega})$. Condition (1) on H together with [3], Theorem 10.5 yields the a priori estimate of u. In the case $D_z H \geq 0$ this estimate can be obtained from (2) using the method of Stampacchia [5]. The estimate of Du in the interior can be established using the structure condition (sc) and a maximum principle for the gradient as in [1]. Finally the boundary gradient estimate follows from Theorem 1, (sc) and condition (3). Uniqueness is obtained from $D_z H \geq 0$ by applying a comparison principle. □

Proof of Theorem 1 We will use the elliptic operator $Qv := a^{ij}(Dv)D_{ij}v + b(x, v, Dv)$ with

$$\begin{aligned}
a^{ij}(p) &:= (1 + |p|^2)\delta_{ij} - p_i p_j, \\
b(x, z, p) &:= -n(1 + |p|^2)\left(|p - D\varphi| H_1(x, z, N(p)) + b_0(x, p)\right), \\
b_0(x, p) &:= \left((1 + |p|^2)^{1/2} - |p - D\varphi|\right) H_1(x, u(x), N(p)) + H_2(x, u(x), N(p)).
\end{aligned}$$

Note that solutions u of the prescribed mean curvature equation satisfy $Qu = 0$ and that by $D_z H_1 \geq 0$, $b(x, z, p)$ is decreasing in z. We will prove that for some function ψ, which will be determined at the end, $w^{\pm} := \varphi \pm \psi \circ \mathrm{d}$ are global upper and lower barriers for u and Q in Γ_1. Therefore we need to show that the following holds:

$$w^{\pm} = u \quad \text{on } \partial\Omega, \quad w^- \le u \le w^+ \quad \text{on } \partial\Gamma_1 \backslash \partial\Omega \quad \pm Q w^{\pm} < 0 \quad \text{in } \Gamma_1 \backslash \partial\Omega. \qquad (*)$$

Assuming $\psi'(\mathrm{d}) \ge 1$, $\psi''(\mathrm{d}) \le 0$ and using [3], Lemma 14.17 we first get

$$\begin{aligned}
&\pm a^{ij}(Dw^{\pm}) D_{ij} w^{\pm} \\
&\quad \le -\psi'(\mathrm{d})(1 + |Dw^{\pm}|^2)(n-1) H_{\partial\Omega}(y) + n^2 \sup_{\Gamma_1} |D\varphi|^2 \sup_{\Gamma_1} |D^2 \mathrm{d}\,| \psi'(\mathrm{d})^2 \\
&\qquad + \psi''(\mathrm{d}) + 3n^2 \Big(1 + \sup_{\Gamma_1} |D\varphi|\Big)^2 \sup_{\Gamma_1} |D^2 \varphi| \psi'(\mathrm{d})^2 \\
&\quad =: -\psi'(\mathrm{d})(1 + |Dw^{\pm}|^2)(n-1) H_{\partial\Omega}(y) + c_1 \psi'(\mathrm{d})^2 + \psi''(\mathrm{d}),
\end{aligned}$$

where $y \in \partial\Omega$, $\mathrm{d}(x) = |x - y|$ and $H_{\partial\Omega}$ denotes the mean curvature of $\partial\Omega$. Using this estimate in combination with

$$n|b_0(x, Dw^{\pm})|(1 + |Dw^{\pm}|^2) \le 2n(h_1 + h_2)\Big(1 + \sup_{\Gamma_1} |D\varphi|\Big)^3 \psi'(\mathrm{d})^2 =: c_2 \psi'(\mathrm{d})^2,$$

$$n\psi'(\mathrm{d})(1 + |Dw^{\pm}|^2) \le 2n\Big(1 + \sup_{\Gamma_1} |D\varphi|\Big)^2 \psi'(\mathrm{d})^3 =: c_3 \psi'(\mathrm{d})^3$$

and the monotonicity of $H_1(x, z, N)$ in z we obtain for $\psi(\mathrm{d}) \ge 0$:

$$\begin{aligned}
\pm Q(w^{\pm}) &\le -\psi'(\mathrm{d})(1 + |Dw^{\pm}|^2)(n-1) H_{\partial\Omega}(y) + c_1 \psi'(\mathrm{d})^2 + \psi''(\mathrm{d}) \\
&\quad \mp n(1 + |Dw^{\pm}|^2)|Dw^{\pm} - D\varphi| H_1(x, w^{\pm}, N(Dw^{\pm})) \\
&\quad \mp n(1 + |Dw^{\pm}|^2) b_0(x, Dw^{\pm}) \\
&\le -\psi'(\mathrm{d})(1 + |Dw^{\pm}|^2) \left[(n-1) H_{\partial\Omega}(y) \pm n H_1(y, \varphi(y), \mp\gamma(y))\right] \\
&\quad + c_3 \psi'(\mathrm{d})^3 \left| H_1(y, \varphi(y), \mp\gamma(y)) - H_1\left(x, \varphi(x), N(Dw^{\pm}(x))\right)\right| \\
&\quad + (c_1 + c_2)\psi'(\mathrm{d})^2 + \psi''(\mathrm{d}).
\end{aligned}$$

The first term can be estimated using (1). For the second term we get from the condition $\psi'(\mathrm{d}) \ge 2\|\varphi\|_{C^1(\overline{\Gamma}_1)}$ and with the help of

$$\left| N(Dw^{\pm}(x)) \pm \gamma(y)\right| \le 4\Big(1 + \sup_{\Gamma_1} |D\varphi|\Big) \psi'(\mathrm{d}(x))^{-1}$$

the estimate

$$\begin{aligned}
&\left| H_1(y, \varphi(y), \mp\gamma(y)) - H_1\left(x, \varphi(x), N(Dw^{\pm}(x))\right)\right| \\
&\quad \le c(\Omega)\left(h_{1,x} + h_{1,z}\|\varphi\|_{C^1(\overline{\Gamma}_1)}\right) \mathrm{d}(x) + 4c(\Omega) h_{1,N} \left(1 + \|\varphi\|_{C^1(\overline{\Gamma}_1)}\right) \psi'(\mathrm{d}(x))^{-1} \\
&\quad =: r\,\mathrm{d} + s\psi'(\mathrm{d})^{-1}.
\end{aligned}$$

Therefore if $\psi'(\mathrm{d})\,\mathrm{d} \le 1$ we have finally

$$\pm Q(w^{\pm}) \le (c_1 + c_2 + c_3(r + s))\, \psi'(\mathrm{d})^2 + \psi''(\mathrm{d}) =: C\psi'(\mathrm{d})^2 + \psi''(\mathrm{d}).$$

Choosing

$$\psi : [0, \infty) \to \mathbb{R} : x \mapsto \psi(x) := \frac{\ln(1 + \lambda x)}{1 + C}, \qquad \lambda := \frac{e^{(1+C)(a+m)} - 1}{d_0}$$

we see that ψ satisfies $\psi(\mathrm{d}) \geq 0$, $\psi'(\mathrm{d})\,\mathrm{d} \leq 1$, $\psi''(\mathrm{d}) < 0$ and also $\psi'(\mathrm{d}) \geq 1 + 2\|\varphi\|_{C^1(\overline{\Gamma}_1)}$ for

$$d_0 := \frac{e^{(1+C)(a+m)} - 1}{(1+C)\left(1 + 2\|\varphi\|_{C^1(\overline{\Gamma}_0)}\right) e^{(1+C)(a+m)}} d_1.$$

From the definition of ψ and the calculations above we see that $w^{\pm}$ satisfy (*). From (*) we can derive that the following estimate holds:

$$\sup_{\partial\Omega} |Du| \leq \sup_{\partial\Omega} |D\varphi| \; + \; \psi'(0).$$

Thus the desired result follows. □

Acknowledgments The author wants to thank Prof. Dr. Gerhard Huisken for the fruitful conversations which lead to this result.

References

1. Bergner, M.: The Dirichlet problem for graphs of prescribed anisotropic mean curvature in $\mathbb{R}^{n+1}$. Int. Math. J. Anal. Appl. **28**, 149–166 (2008)
2. Giaquinta, M.: On the Dirichlet problem for surfaces of prescribed mean curvature. Manus. Math. **12**, 73–86 (1974)
3. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Springer, Berlin (2001)
4. Serrin, J.: Gradient estimates for solutions of nonlinear elliptic and parabolic equations. Contributions to Nonlinear Functional Analysis, pp. 565–601. Academic Press, New York (1971)
5. Stampacchia, G.: Equations elliptiques du second ordre à coefficients discontinus. Ann. Inst. Fourier **15**, 189–258 (1965)
6. Wang, X.-J.: Interior gradient estimates for mean curvature equations. Math. Z. **228**, 73–81 (1998)