

Generalized conformal realizations of Kac-Moody algebras

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To the memory of Issai Kantor

Abstract

We generalize the Kantor-Koecher-Tits construction, which associates a Lie algebra to any Jordan algebra. This gives a generalization of the conformal transformations in a $(p + q)$ -dimensional spacetime to a nonlinear realization of $\mathfrak{so}(p + n, q + n)$, for arbitrary n , with a linearly realized subalgebra $\mathfrak{so}(p, q)$. For Minkowski spacetimes of 3, 4, 6, 10 dimensions, the corresponding triple systems can be constructed from the Jordan algebras of hermitian 2×2 matrices over the division algebras \mathbb{R} , \mathbb{C} , \mathbb{H} , \mathbb{O} , respectively. We show that this construction can also be applied to 3×3 matrices and then gives rise to the exceptional Lie algebras \mathfrak{f}_4 , \mathfrak{e}_6 , \mathfrak{e}_7 , \mathfrak{e}_8 , as well as to their affine, hyperbolic and further extensions. In particular, this leads to a new realization of the indefinite Kac-Moody algebras \mathfrak{e}_{10} and \mathfrak{e}_{11} .

1 Introduction and motivation

Jordan algebras were originally studied in order to understand the foundations of quantum mechanics [1, 2]. Even though the hope of applications to "relativistic and nuclear phenomena" has not been fulfilled, Jordan algebras have turned out to play an important role in fundamental physics through their connection to Lie algebras. The origin of this connection lies in the observation that the triple product

$$(x, y, z) \mapsto [[x, \tau(y)], z] \tag{1.1}$$

in the subspace \mathfrak{g}_{-1} of a 3-graded Lie algebra $\mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$, where τ is an involution $\mathfrak{g}_{-1} \rightarrow \mathfrak{g}_1$, has the same general properties as the triple product

$$(x, y, z) \mapsto (xy)z + x(yz) - y(xz) \tag{1.2}$$

formed from the multiplication in a Jordan algebra. In the Kantor-Koecher-Tits construction [3–5], based on this observation, any Jordan algebra gives rise to a 3-graded Lie algebra, which is its *conformal* algebra, and two subalgebras that arise naturally in this construction are its reduced *structure* algebra and *derivation* algebra. Jordan algebras can be used to define generalized spacetimes in a way such that these Lie algebras generate conformal transformations, Lorentz transformations and rotations, respectively [6–9]. The conformal groups will then admit positive energy unitary representations only for *formally real* (or *Euclidean*) Jordan algebras, which were the original Jordan algebras introduced in [1], and classified (the simple ones) in [2]. The classification involves the four (normed) division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ [10, 11].

The formally real Jordan algebras $H_3(\mathbb{K})$ consist of 3×3 hermitian matrices over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ and their derivation, reduced structure and conformal algebras constitute the first three rows in a *magic square* of Lie algebras [12–14]. They also give rise to *Freudenthal triple systems* [15, 16], to which the exceptional Lie algebras $\mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$ in the fourth row are associated. This construction of Freudenthal triple systems and the associated *quasiconformal* algebras is possible also for another family of formally real Jordan algebras, which is infinite and parametrized by a positive integer d . Their quasiconformal algebras are respectively $\mathfrak{so}(4, d + 2)$ for all positive integers d . For $d = 3, 4, 6, 10$, they constitute the fourth row of another 'magic square' [17, 18], consisting of pseudo-orthogonal algebras, with $\mathfrak{so}(d - 1), \mathfrak{so}(1, d - 1), \mathfrak{so}(2, d)$ in the first three rows. These are the derivation, reduced structure and conformal algebras of the formally real Jordan algebras $H_2(\mathbb{K})$, consisting of 2×2 hermitian matrices over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

The formally real Jordan algebras of *degree three* [19], which are those that admit quasiconformal algebras, are also those that arise in Maxwell-Einstein supergravity theories [20–23]. In five dimensions, the scalar fields parametrize a manifold, which, if it is a symmetric space, has the form G/H , where G is the global symmetry group

of the Lagrangian and H its maximal compact subgroup. Further conditions on the manifold imply that G must be the reduced structure group of a formally real Jordan algebra of degree three. By reduction to four and three dimensions, it extends to the conformal and quasiconformal group, respectively, and these groups also act as spectrum generating symmetry groups in five and four dimensions, respectively [24,25].

The conformal algebra comes with a non-linear realization, where the three subspaces \mathfrak{g}_{-1} , \mathfrak{g}_0 and \mathfrak{g}_1 consist of operators which are respectively constant, linear or quadratic. For a generalized spacetime, these are the generators of translations (constant), Lorentz transformations together with dilatations (linear) and special conformal transformations (quadratic). The operators act on the subspace \mathfrak{g}_{-1} , which can be identified with the Jordan algebra, or the generalized spacetime. This *conformal realization* can be applied to any 3-graded Lie algebra, and the Jordan algebras are then generalized to *Jordan triple systems* [26]. Similarly, the quasiconformal algebra comes with a 5-grading, where the additional subspaces $\mathfrak{g}_{\pm 2}$ are one-dimensional, and, as was shown in [24], it has a *quasiconformal realization* on the subspace $\mathfrak{g}_{-1} + \mathfrak{g}_{-2}$ that can be applied to any Lie algebra with such a 5-grading. In [27], we gave a new nonlinear realization, based on *Kantor triple systems* [28, 29] instead of Freudenthal triple systems. This nonlinear realization is more general than the previous ones, in the sense that it can be applied to a Lie algebra with an arbitrary 5-grading, including the special cases of a 3-graded Lie algebra (where $\mathfrak{g}_{\pm 2} = 0$) and a 5-graded Lie algebra with one-dimensional subspaces $\mathfrak{g}_{\pm 2}$.

In the quasiconformal realization, $\mathfrak{so}(4, d+2)$ has $\mathfrak{so}(2, d) \oplus \mathfrak{gl}(2, \mathbb{R})$ as a linearly realized subalgebra, and can therefore naturally be generalized to $\mathfrak{so}(p+2, q+2)$ with a linearly realized subalgebra $\mathfrak{so}(p, q) \oplus \mathfrak{gl}(2, \mathbb{R})$ for arbitrary positive integers p, q . This quasiconformal realization was given in [30] (explicitly for $\mathfrak{so}(4, 12)$) and [31].

In the present paper, we generalize the quasiconformal algebras $\mathfrak{so}(4, d+2)$ further to $\mathfrak{so}(p+n, q+n)$ with a linearly realized subalgebra $\mathfrak{so}(p, q) \oplus \mathfrak{gl}(n, \mathbb{R})$, for arbitrary positive integers p, q, n . Instead of the quasiconformal realization, which only works for $n = 2$, we use the more general one described in [27]. For signatures $(p, q) = (1, 2), (1, 3), (1, 5), (1, 9)$, the corresponding Kantor triple systems can be constructed from the Jordan algebras $H_2(\mathbb{K})$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively, and this gives a generalization of the Kantor-Koecher-Tits construction, which, to our knowledge, has not appeared before. When we apply this construction to the Jordan algebras $H_3(\mathbb{K})$, we get instead the exceptional Lie algebras $\mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$ for $n = 2$, their affine and hyperbolic extensions for $n = 3$ and $n = 4$, respectively, and further extensions for $n \geq 5$. In particular, $\mathbb{K} = \mathbb{O}$ yields generically the Kac-Moody algebras \mathfrak{e}_{6+n} . Thus, for $n = 4$ and $n = 5$, the resulting algebras are, respectively, \mathfrak{e}_{10} and \mathfrak{e}_{11} , which both (but in different approaches) are conjectured to be symmetries underlying M-theory [32, 33].

1.1 Outline

The paper is organized as follows. Section 2 provides, briefly, the background about the classification of simple formally real Jordan algebras, the Kantor-Koecher-Tits construction and the magic square of Lie algebras. In section 3, we describe how a Lie algebra can be given a grading by any of its simple roots, and how graded Lie algebras give rise to generalized Jordan triple systems. Conversely, in section 4 we show how any generalized Jordan triple system gives rise to a graded Lie algebra. This is the usual generalization of the Kantor-Koecher-Tits construction from Jordan triple systems to generalized Jordan triple systems [29, 34], even though our description is new, but it generalizes further when we replace the elements in the generalized Jordan triple system by n -tuples of elements. In section 5, we confirm the general results in the case of pseudo-orthogonal algebras and connect them to the Jordan algebras $H_2(\mathbb{K})$. The conclusion that the same construction applied to $H_3(\mathbb{K})$ gives the exceptional algebras \mathfrak{f}_4 , \mathfrak{e}_6 , \mathfrak{e}_7 , \mathfrak{e}_8 and their extensions, is finally made in section 6.

2 Jordan algebras

A *Jordan algebra* is a commutative algebra that satisfies the *Jordan identity*

$$a^2 \circ (b \circ a) = (a^2 \circ b) \circ a. \quad (2.1)$$

Any associative algebra becomes a Jordan algebra if we replace the product by its symmetric part

$$a \circ b = \frac{1}{2}(ab + ba). \quad (2.2)$$

A real Jordan algebra is *formally real* if it is finite-dimensional and satisfies

$$a^2 + b^2 = 0 \Leftrightarrow a = b = 0. \quad (2.3)$$

Any simple formally real Jordan algebra is isomorphic to one of the following [2, 10, 11],

- (i) the algebra \mathbb{R} of real numbers,
- (ii) the direct sum $J(U)$ of \mathbb{R} and a Euclidean space U where the real number 1 is identity element and the product of two elements $a, b \in U$ is equal to their inner product, $a \circ b = (a, b)$.
- (iii) the algebras $H_n(\mathbb{K})$ of all hermitian $n \times n$ matrices over one of the division algebras $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ for $n \geq 3$, with the product (2.2),
- (iv) the algebra $H_3(\mathbb{O})$ of all hermitian 3×3 matrices over the division algebra \mathbb{O} of octonions, with the product (2.2).

The division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ are real algebras of dimension 1, 2, 4, 8, respectively. The quaternions \mathbb{H} and the octonions \mathbb{O} are generalizations of the complex numbers in the sense that they are spanned by the real numbers together with, respectively, three or seven *imaginary units* e_i that square to -1 . In this way hermitian matrices over \mathbb{C} can be generalized to \mathbb{H} and \mathbb{O} . Both \mathbb{H} and \mathbb{O} are non-commutative algebras, since the imaginary units anticommute, and \mathbb{O} is furthermore non-associative. For $\dim U = 2, 3, 5, 9$, the formally real Jordan algebra $J(U)$ is isomorphic to the algebra $H_2(\mathbb{K})$ of all hermitian 2×2 matrices over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively, with the product (2.2). An isomorphism is given by considering the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -e_i \\ e_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (2.4)$$

($i = 1, 2, \dots, \dim \mathbb{K} - 1$) as an orthonormal basis of U . (For $\mathbb{K} = \mathbb{C}$ the matrices (2.4) are thus the usual Pauli sigma matrices.) If we replace the inner product in U by one which is not positive-definite, we still get a Jordan algebra $J(U)$, but it will not be formally real. The same holds if we replace $\mathbb{K} = \mathbb{C}, \mathbb{H}, \mathbb{O}$ by the 'split' algebra \mathbb{K}^s which is obtained by changing the square of, respectively, 1, 2, 4 imaginary units from -1 to 1, but otherwise leaving the multiplication table unchanged. Due to the non-associativity of \mathbb{O} , we get no Jordan algebras $H_n(\mathbb{O})$ for $n \geq 3$, other than the *exceptional* Jordan algebra $H_3(\mathbb{O})$. As we will see next, it is related to the exceptional Lie algebras.

2.1 The Kantor-Koecher-Tits construction

Let J be a Jordan algebra with identity element e . The subalgebra $\mathfrak{str} J$ of $\text{End } J$, generated by all (left) multiplications

$$a_L : J \rightarrow J, \quad x \mapsto ax, \quad (2.5)$$

where $a \in J$, is the *structure algebra* of J . It can be shown [35] that the subalgebra $\mathfrak{der} J$ of $\text{End } J$, consisting of all *derivations* of J , is spanned by all commutators $[a_L, b_L]$ in $\mathfrak{str} J$, and that the structure algebra as a vector space is the direct sum of the derivation algebra $\mathfrak{der} J$ and a subspace spanned by a_L for all $a \in J$. The reduced structure algebra $\mathfrak{str}' J$ is the quotient $(\mathfrak{str} J)/\mathfrak{h}$ where \mathfrak{h} is the one-dimensional ideal spanned by e_L . In the Kantor-Koecher-Tits construction [3–5], the structure algebra gets extended to a 3-graded *conformal* algebra $\mathfrak{con} J$ consisting not only of linear operators, but also of quadratic and constant operators. (The conformal algebra $\mathfrak{con} J$ will be defined in section 5.1 as a special case of the Lie algebra associated to a generalized Jordan triple system, which is the subject of section 4.) In this way, the real exceptional Jordan algebra $H_3(\mathbb{O})$ gives rise to real forms of exceptional Lie

algebras. We have

$$\begin{aligned}
\mathfrak{der} H_3(\mathbb{O}) &= \mathfrak{f}_{4(-52)}, \\
\mathfrak{str}' H_3(\mathbb{O}) &= \mathfrak{e}_{6(-26)}, \\
\mathfrak{con} H_3(\mathbb{O}) &= \mathfrak{e}_{7(-25)}.
\end{aligned} \tag{2.6}$$

In this paper, we will present a generalization of the Kantor-Koecher-Tits construction, which leads to a continuation of the sequence above to \mathfrak{e}_8 , \mathfrak{e}_9 , \mathfrak{e}_{10} , and so on. There is already a way to include \mathfrak{e}_8 in a unified construction of exceptional Lie algebras from the four division algebras \mathbb{R} , \mathbb{C} , \mathbb{H} , \mathbb{O} , which we will review next, but to our knowledge, there has not yet been any successful attempt to involve also its infinite-dimensional extensions in this context.

2.2 Magic squares

For any pair $(\mathbb{K}, \mathbb{K}')$ of division algebras \mathbb{R} , \mathbb{C} , \mathbb{H} , \mathbb{O} , we consider the vector space

$$M(\mathbb{K}, \mathbb{K}') = \mathfrak{der} H_3(\mathbb{K}) \oplus (H_3'(\mathbb{K}) \otimes \text{Im } \mathbb{K}') \oplus \mathfrak{der} \mathbb{K}' \tag{2.7}$$

where $H_n'(\mathbb{K})$ is the subspace of $H_n(\mathbb{K})$ consisting of traceless matrices and $\text{Im } \mathbb{K}'$ is the subspace of \mathbb{K}' spanned by all imaginary units.

We can now define a certain Lie bracket [18] on $M(\mathbb{K}, \mathbb{K}')$ such that we get the following square of Lie algebras $M(\mathbb{K}, \mathbb{K}')$.

$\mathbb{K} \setminus \mathbb{K}'$	\mathbb{R}	\mathbb{C}	\mathbb{H}	\mathbb{O}
\mathbb{R}	\mathfrak{a}_1	\mathfrak{a}_2	\mathfrak{c}_3	\mathfrak{f}_4
\mathbb{C}	\mathfrak{a}_2	$\mathfrak{a}_2 \oplus \mathfrak{a}_2$	\mathfrak{a}_5	\mathfrak{e}_6
\mathbb{H}	\mathfrak{c}_3	\mathfrak{a}_5	\mathfrak{d}_6	\mathfrak{e}_7
\mathbb{O}	\mathfrak{f}_4	\mathfrak{e}_6	\mathfrak{e}_7	\mathfrak{e}_8

For simplicity, we do not write out the expression for the Lie bracket here, and we only specify the complex Lie algebras. In this magic square, the real Lie algebras would actually be the compact forms of the complex Lie algebras that we have specified, but we also get other magic squares of real Lie algebras if we replace \mathbb{K} or \mathbb{K}' by the corresponding 'split' algebra \mathbb{C}^s , \mathbb{H}^s , \mathbb{O}^s . When \mathbb{K} is split and \mathbb{K}' non-split, we get the derivation, reduced structure and conformal algebras of $H_3(\mathbb{K}')$ as the first three rows (and in particular (2.6) in the last column). When \mathbb{K} and \mathbb{K}' are both split, we get the split real forms of the complex Lie algebras above.

We focus on the 3×3 subsquare in the lower right corner, consisting of simply-laced algebras, with the following Dynkin diagrams.

$\mathbb{K} \setminus \mathbb{K}'$	\mathbb{C}	\mathbb{H}	\mathbb{O}
\mathbb{C}			
\mathbb{H}			
\mathbb{O}			

The Dynkin diagrams in the last row are drawn with one black node each. The outermost node next to it represents the simple root that generates (which we will explain in the next section) the unique 5-grading where the subspaces \mathfrak{g}_{-2} are one-dimensional. This 5-grading gives rise to the corresponding Freudenthal triple system. Deleting this node gives the algebra in the row above, which thus is an ideal of \mathfrak{g}_0 in this 5-grading, and deleting also the black node itself take us yet another row upwards in the square.

What we have described here is Tits' construction of the magic square [12]. There are also other constructions [14, 18, 36] where the symmetry $M(\mathbb{K}, \mathbb{K}') = M(\mathbb{K}', \mathbb{K})$ is manifest. (However, the construction by Kantor [36] is different in the sense that the real versions [37] do not contain any compact forms.)

In the same way as the exceptional Jordan algebra $H_3(\mathbb{O})$ gives rise to exceptional Lie algebras, the Jordan algebras $H_2(\mathbb{K})$ give rise to pseudo-orthogonal Lie algebras,

$$\begin{aligned}
\mathfrak{der} H_2(\mathbb{K}) &= \mathfrak{so}(d-1), \\
\mathfrak{str}' H_2(\mathbb{K}) &= \mathfrak{so}(1, d-1), \\
\mathfrak{con} H_2(\mathbb{K}) &= \mathfrak{so}(2, d),
\end{aligned} \tag{2.8}$$

where $d = \dim \mathbb{K} + 2 = 3, 4, 6, 10$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively. If we replace $H_3(\mathbb{K})$ by $H_2(\mathbb{K})$ and $\mathfrak{der} \mathbb{K}'$ by $\mathfrak{so}(\text{Im } \mathbb{K}')$ in (2.7), we get the vector space

$$L(\mathbb{K}, \mathbb{K}') = \mathfrak{der} H_2(\mathbb{K}) \oplus (H_2'(\mathbb{K}) \otimes \text{Im } \mathbb{K}') \oplus \mathfrak{so}(\text{Im } \mathbb{K}'). \tag{2.9}$$

If we then define a Lie bracket [18] on $L(\mathbb{K}, \mathbb{K}')$ similar to the one on $M(\mathbb{K}, \mathbb{K}')$, we get the subalgebra

$$L(\mathbb{K}, \mathbb{K}') = \mathfrak{so}(\mathbb{K} \oplus \mathbb{K}') \tag{2.10}$$

of $M(\mathbb{K}, \mathbb{K}')$, and we find the algebras (2.8) in the first three rows of the following square of Lie algebras $L(\mathbb{K}, \mathbb{K}')$.

$\mathbb{K} \setminus \mathbb{K}'$	\mathbb{R}	\mathbb{C}	\mathbb{H}	\mathbb{O}
\mathbb{R}	$\mathfrak{so}(2)$	$\mathfrak{so}(3)$	$\mathfrak{so}(5)$	$\mathfrak{so}(9)$
\mathbb{C}^s	$\mathfrak{so}(1, 2)$	$\mathfrak{so}(1, 3)$	$\mathfrak{so}(1, 5)$	$\mathfrak{so}(1, 9)$
\mathbb{H}^s	$\mathfrak{so}(2, 3)$	$\mathfrak{so}(2, 4)$	$\mathfrak{so}(2, 6)$	$\mathfrak{so}(2, 10)$
\mathbb{O}^s	$\mathfrak{so}(4, 5)$	$\mathfrak{so}(4, 6)$	$\mathfrak{so}(4, 8)$	$\mathfrak{so}(4, 12)$

We will come back to the Jordan algebras $H_2(\mathbb{K})$ in section 5.1 when we apply our generalization of the Kantor-Koecher-Tits construction. We will show that it works also for $H_3(\mathbb{K})$, but we will for simplicity only consider the $H_2(\mathbb{K})$ case explicitly.

3 Kac-Moody algebras

In this section we will briefly recall how a complex Kac-Moody algebra can be constructed from its (generalized) Cartan matrix, or equivalently, from its Dynkin diagram. For details, we refer to [38]. We will assume that the determinant of the Cartan matrix is non-zero, which in particular means that we leave the affine case for now, but we will come back to it in section 6. The Kac-Moody algebra will then be finite-dimensional (or simply *finite*) if the determinant is positive, and infinite-dimensional (or *indefinite*) if the determinant is negative.

The Cartan matrix is of type $r \times r$, where r is the *rank* of the Lie algebra. Its entries are integers satisfying $A^{ii} = 2$ (no summation) and

$$i \neq j \Rightarrow A^{ij} \leq 0, \quad A^{ij} = 0 \Leftrightarrow A^{ji} = 0 \quad (3.1)$$

for $i, j = 1, 2, \dots, r$. The Dynkin diagram consists of r nodes, and two nodes i, j are connected by a line if $A^{ij} = A^{ji} = -1$, but disconnected if $A^{ij} = A^{ji} = 0$ (these are the only two cases that we will consider).

In the construction of a Lie algebra from its Cartan matrix, one starts with $3r$ generators e_i, f_i, h_i satisfying the *Chevalley relations* (no summation)

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} h_j, & [h_i, h_j] &= 0, \\ [h_i, e_j] &= A_{ij} e_j, & [h_i, f_j] &= -A_{ij} f_j. \end{aligned} \quad (3.2)$$

The elements h_i span the abelian *Cartan subalgebra* \mathfrak{g}_0 . Further basis elements of \mathfrak{g} will then be multiple commutators of either e_i or f_i , generated by these elements modulo the *Serre relations* (no summation)

$$(\text{ad } e_i)^{1-A^{ji}} e_j = 0, \quad (\text{ad } f_i)^{1-A^{ji}} f_j = 0. \quad (3.3)$$

It follows from (3.2) that these multiple commutators (as well as the elements e_i and f_i themselves) are eigenvectors of $\text{ad } h$ for any $h \in \mathfrak{g}_0$, and thus each of them defines an element α in the dual space of \mathfrak{g}_0 , such that $\alpha(h)$ is the corresponding eigenvalue. These elements α are the *roots* of \mathfrak{g} and the eigenvectors are called *root vectors*. In particular, e_i are root vectors of the *simple roots* α_i , which form a basis of the dual space of \mathfrak{g}_0 . In this basis, an arbitrary root $\mu = \mu^i \alpha_i$ has integer components μ^i , either all non-negative (if α is a *positive root*) or all non-positive (if α is a *negative root*).

For finite Kac-Moody algebras, the space of root vectors corresponding to any root is one-dimensional. Furthermore, if α is a root, then $-\alpha$ is a root as well, but no other multiples of α . For any positive root α of a finite Kac-Moody algebra \mathfrak{g} , we let e_α and f_α be root vectors corresponding to α and $-\alpha$, respectively, such that they are multiple commutators of e_i or f_i . (This requirement fixes the normalization up to a sign.) Thus a basis of \mathfrak{g} is formed by these root vectors e_α, f_α for all positive roots α , and by the Cartan elements h_i for all $i = 1, 2, \dots, r$.

3.1 Graded Lie algebras

A Lie algebra \mathfrak{g} is *graded*, or has a *grading*, if it is the direct sum of subspaces $\mathfrak{g}_k \subset \mathfrak{g}$ for all integers k , such that $[\mathfrak{g}_m, \mathfrak{g}_n] \subset \mathfrak{g}_{m+n}$ for all integers m, n . If there is a positive integer m such that $\mathfrak{g}_{\pm m} \neq 0$ but $\mathfrak{g}_{\pm k} = 0$ for all $k > m$, then the Lie algebra \mathfrak{g} is $(2m + 1)$ -*graded*. We will occasionally use the notation $\mathfrak{g}_\pm = \mathfrak{g}_{\pm 1} + \mathfrak{g}_{\pm 2} + \dots$.

Any simple root α_i of a Kac-Moody algebra \mathfrak{g} *generates* a grading of \mathfrak{g} , such that \mathfrak{g}_k is spanned by all root vectors e_μ or f_μ with the component $\mu^i = -k$ (the minus sign is a convention) corresponding to α_i in the basis of simple roots, and, if $k = 0$, by the Cartan elements h_j .

A *graded involution* τ on the Lie algebra \mathfrak{g} is an automorphism such that $\tau^2(x) = x$ for any $x \in \mathfrak{g}$ and $\tau(\mathfrak{g}_k) = \mathfrak{g}_{-k}$ for any integer k . The simplest example of a graded involution in a graded Kac-Moody algebra is given by $e_\alpha \leftrightarrow \pm f_\alpha$ and $h_i \leftrightarrow -h_i$. (With the minus sign, this is the *Chevalley involution*.)

On the subspace \mathfrak{g}_{-1} of a graded Lie algebra \mathfrak{g} with a graded involution τ , we can define a triple product, that is, a trilinear map $(\mathfrak{g}_{-1})^3 \rightarrow \mathfrak{g}_{-1}$, given by

$$(x, y, z) \mapsto (xyz) = [[x, \tau(y)], z]. \quad (3.4)$$

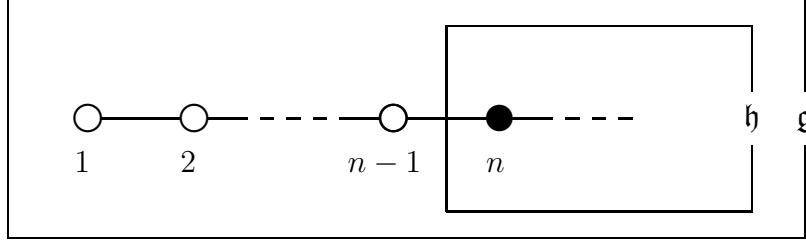
Then, due to the Jacobi identity and the fact that τ is an involution, this triple product will satisfy the identity

$$(uv(xyz)) - (xy(uvz)) = ((uvx)yz) - (x(vuy)z), \quad (3.5)$$

which means that \mathfrak{g}_{-1} is a *generalized Jordan triple system*. As this name suggests, and as we mentioned already in the introduction, this kind of triple systems is related to Jordan algebras. We will explain the relation in more detail in section 5.1.

3.2 Extensions of graded Lie algebras

We will now generalize the situation in the last row of the magic square to the case when an arbitrary Kac-Moody algebra \mathfrak{h} is extended by an \mathfrak{a}_{n-1} algebra, for an arbitrary integer $n \geq 2$, to a Kac-Moody algebra \mathfrak{g} with the following Dynkin diagram.



In the fourth row of the magic square on page 7, we thus have $n = 1$, and especially in the lower right corner, $\mathfrak{g} = \mathfrak{e}_8$ and $\mathfrak{h} = \mathfrak{e}_7$. The root corresponding to the black node n , which \mathfrak{g} and \mathfrak{h} have in common, generates a grading of \mathfrak{g} as well as of \mathfrak{h} . (It can be connected to more than one node in the Dynkin diagram of \mathfrak{h} .) We want to investigate how the generalized Jordan triple systems \mathfrak{g}_{-1} and \mathfrak{h}_{-1} , corresponding to these gradings, are related to each other.

As described in section 3.1, a basis of \mathfrak{h}_{-1} consists of all root vectors e_μ such that the component of μ corresponding to α_{n+1} in the basis of simple roots is equal to one. A basis of \mathfrak{g}_{-1} consists of all such basis elements e_μ of \mathfrak{h}_{-1} together with all commutators $[e^i, e_\mu]$, where e^i , for $i = 1, 2, \dots, n-1$, is the root vector

$$e^i = [\dots [[e_i, e_{i+1}], e_{i+2}], \dots, e_{n-1}] \quad (3.6)$$

of the \mathfrak{a}_{n-1} subalgebra of \mathfrak{g} . We also define the root vector

$$f^i = [\dots [[f_i, f_{i+1}], f_{i+2}], \dots, f_{n-1}] \quad (3.7)$$

for the corresponding negative root, and the element

$$h^i = h_i + h_{i+1} + h_{i+2} + \dots + h_{n-1} \quad (3.8)$$

in the Cartan subalgebra of \mathfrak{g} , such that

$$[e^i, f^i] = h^i, \quad [h^i, e^i] = 2e^i, \quad [h^i, f^i] = -2f^i, \quad (3.9)$$

(no summation). If $i \neq j$, then

$$[h^i, e^j] = e^j, \quad [h^i, f^j] = -f^j, \quad (3.10)$$

while $[e^i, e^j]$ is either zero or a root vector of \mathfrak{g} that does not belong to \mathfrak{g}_{-1} . (We stress the difference between having the indices i, j, \dots on e, f, h upstairs and downstairs. The root vectors e_i correspond to the simple roots of the \mathfrak{a}_{n-1} subalgebra, while the root vectors e^i correspond to roots of the \mathfrak{a}_{n-1} subalgebra for which the component corresponding to the simple root α_{n-1} is equal to one, and these roots are not simple, except for α_{n-1} itself.) Using the relations (3.9)–(3.10) we get

$$\begin{aligned} [[e^i, f^j], e_\mu] &= -\delta^{ij}e_\mu, & [[e^i, f^j], f_\nu] &= \delta^{ij}f_\nu, \\ [[e_\mu, f_\nu], e^i] &= -\delta_{\mu\nu}e^i, & [[e_\mu, f_\nu], f^j] &= \delta_{\mu\nu}f^j, \end{aligned} \quad (3.11)$$

and then

$$\begin{aligned} [[e^i, e_\mu], [f^j, f_\nu]] &= -\delta^{ij}[e_\mu, f_\nu] - \delta_{\mu\nu}[e^i, f^j], \\ [[e^i, e_\mu], f_\nu] &= \delta_{\mu\nu}e^i, & [e_\mu, [f^j, f_\nu]] &= -\delta_{\mu\nu}f^j. \end{aligned} \quad (3.12)$$

Finally we have

$$[[e^i, f^j], e^k] = \delta^{jk}e^i + \delta^{ji}e^k. \quad (3.13)$$

For any graded involution τ , we define a bilinear form on \mathfrak{h}_{-1} associated to τ by $(e_\mu, \tau(f_\nu)) = \delta_{\mu\nu}$. Then, from (3.11)–(3.12) we get

$$\begin{aligned} [[e_\mu, \tau(e_\nu)], e^i] &= -(e_\mu, e_\nu)e^i, & [[e_\mu, \tau(e_\nu)], f^j] &= (e_\mu, e_\nu)f^j, \\ [[e^i, e_\mu], [f^j, \tau(e_\nu)]] &= -\delta^{ij}[e_\mu, \tau(e_\nu)] - (e_\mu, e_\nu)[e^i, f^j], \\ [[e^i, e_\mu], \tau(e_\nu)] &= (e_\mu, e_\nu)e^i, & [e_\mu, [f^j, \tau(e_\nu)]] &= -(e_\mu, e_\nu)f^j, \end{aligned} \quad (3.14)$$

We consider now the direct sum $(\mathfrak{h}_{-1})^n$ of n vector spaces, each isomorphic to \mathfrak{h}_{-1} , and write a general element in $(\mathfrak{h}_{-1})^n$ as $(x_1)^1 + (x_2)^2 + \dots + (x_{n+1})^n$, where x_1, x_2, \dots are elements in \mathfrak{h}_{-1} . Using the relations above it is straightforward to prove the following theorem.

Theorem 3.1. *The vector space $(\mathfrak{h}_{-1})^n$, together with the triple product given by*

$$(x^a y^b z^c) = \delta^{ab}[[x, \tau(y)], z]^c - \delta^{ab}(x, y)z^c + \delta^{bc}(x, y)z^a \quad (3.15)$$

for $a, b, \dots = 1, 2, \dots, n$ and $x, y, z \in \mathfrak{h}_{-1}$, is a triple system isomorphic to the triple system \mathfrak{g}_{-1} with the triple product

$$(uvw) = [[u, \tau(v)], w], \quad (3.16)$$

where the involution τ is extended from \mathfrak{h} to \mathfrak{g} by $\tau(e^i) = -f^i$. Thus $(\mathfrak{h}_{-1})^n$ is a generalized Jordan triple system, as well as \mathfrak{g}_{-1} .

Proof. The theorem says that there is a one-to-one linear map

$$\psi : (\mathfrak{h}_{-1})^n \rightarrow \mathfrak{g}_{-1} \quad (3.17)$$

such that

$$\psi((uvw)) = (\psi(u)\psi(v)\psi(w)) \quad (3.18)$$

for all $u, v, w \in (\mathfrak{h}_{-1})^n$. It is easy to see that the map ψ defined by $e_\mu^1 \mapsto e_\mu$ and $(e_\mu)^{i+1} \mapsto [e^i, e_\mu]$ for $i = 1, 2, \dots, n-1$ is one-to-one. To prove that it is an isomorphism, it suffices to show (3.18) when u, v, w are basis elements $e_\mu^i, e_\nu^j, e_\rho^k$, but this must be done case by case for each of i, j, k equal to one or not, which means eight different cases. Therefore (although each case is simple) a complete proof would be quite lengthy, and we leave it as an exercise for the reader. \square

4 The Lie algebra associated to a generalized Jordan triple system

In the end of section 3.1 we saw that any graded Lie algebra with a graded involution gives rise to a generalized Jordan triple system. In this section, we will show the converse, that any generalized Jordan triple system gives rise to a graded Lie algebra with a graded involution. The associated Lie algebra has been defined in different way by Kantor [29] and called the *Kantor algebra* [39].

We recall from section 3.1 that a generalized Jordan triple system is a triple system that satisfies the identity

$$(uv(xyz)) - (xy(uvz)) = ((uvx)yz) - (x(vuy)z). \quad (4.1)$$

For any pair of elements x, y in a generalized Jordan triple system T , we define the linear map

$$s_{xy} : T \rightarrow T, \quad s_{xy}(z) = (xyz). \quad (4.2)$$

Thus (4.1) (for all z) can be written

$$[s_{uv}, s_{xy}] = s_{(uvx)y} - s_{x(vuy)}. \quad (4.3)$$

For any $x \in T$, we also define the linear map

$$v_x : T \rightarrow \text{End } T, \quad v_x(y) = s_{xy}, \quad (4.4)$$

which we will use in the following subsection.

4.1 Construction

Let T be a vector space and set $\tilde{U}_0 = \text{End } T$. For $k < 0$, define \tilde{U}_k recursively as the vector space of all linear maps from T to \tilde{U}_{k+1} . Let \tilde{U}_- be the direct sum of all these vector spaces,

$$\tilde{U}_- = \tilde{U}_{-1} \oplus \tilde{U}_{-2} \oplus \cdots \quad (4.5)$$

and define a graded Lie algebra structure on \tilde{U}_- recursively by the relations

$$[u, v] = (\text{ad } u) \circ v - (\text{ad } v) \circ u. \quad (4.6)$$

Assume now that T is a generalized Jordan triple system. Let U_0 be the subspace of \tilde{U}_0 spanned by s_{uv} for all $u, v \in T$, and let U_- be the subspace of \tilde{U}_- generated by v_x for all $x \in T$. Furthermore, let U_+ be a Lie algebra isomorphic to U_- , with the isomorphism denoted by

$$* : U_- \rightarrow U_+, \quad u \mapsto u^*. \quad (4.7)$$

Thus U_+ is generated by v_x^* for all $x \in T$. Consider the vector space

$$L(T) = U_- \oplus U_0 \oplus U_+. \quad (4.8)$$

We can extend the Lie algebra structures on each of these subspaces to a Lie algebra structure on the whole of $L(T)$, by the relations

$$[s_{xy}, v_z] = v_{(xyz)}, \quad [v_x, v_y^*] = s_{xy}, \quad [s_{xy}, v_z^*] = -v_{(yxz)}. \quad (4.9)$$

Furthermore, we can extend the isomorphism $*$ between the subalgebras U_- and U_+ to a graded involution on the Lie algebra $L(T)$. On U_+ , it is given by the inverse of the original isomorphism, $(u^*)^* = u$, and on U_0 by $s_{xy}^* = -s_{yx}$.

Theorem 4.1. *Let \mathfrak{g} be a graded simple Lie algebra, generated by its subspaces $\mathfrak{g}_{\pm 1}$, with a graded involution τ . Let \mathfrak{g}_{-1} be the generalized Jordan triple system derived from \mathfrak{g} by*

$$(uvw) = [[u, \tau(v)], w]. \quad (4.10)$$

Then the Lie algebra $L(\mathfrak{g}_{-1})$ is isomorphic to \mathfrak{g} .

Proof. Define the linear map $\varphi : \mathfrak{g} \rightarrow L(\mathfrak{g}_{-1})$, with $\mathfrak{g}_k \rightarrow U_k$ for all integers k , recursively by

$$\begin{aligned} u \in \mathfrak{g}_- : \quad & \varphi(u)(x) = \varphi([u, \tau(x)]), \\ s \in \mathfrak{g}_0 : \quad & \varphi(s)(x) = [s, x], \\ \tau(u) \in \mathfrak{g}_+ : \quad & \varphi(\tau(u)) = \varphi(u)^*, \end{aligned} \quad (4.11)$$

where $x \in \mathfrak{g}_{-1}$. We will show that φ is an isomorphism.

- φ is injective

Suppose that r and s are elements in \mathfrak{g}_0 such that $\varphi(r) = \varphi(s)$. Then $[r - s, x] = 0$ for all $x \in \mathfrak{g}_{-1}$, which means that $[r - s, \mathfrak{g}_{-}] = 0$ since \mathfrak{g}_{-} is generated by \mathfrak{g}_{-1} . But then the proper subspace

$$\sum_{k \in \mathbb{N}} (\text{ad } (\mathfrak{g}_0 + \mathfrak{g}_+))^k (r - s) \subset \mathfrak{g}_+ + \mathfrak{g}_0 \quad (4.12)$$

of \mathfrak{g} is an ideal. Since \mathfrak{g} is simple, it must be zero, but $r - s$ is an element of this subspace, so $r = s$. Suppose now that u and v are elements in \mathfrak{g}_{-} with $\varphi(u) = \varphi(v)$. Then $\varphi([u, \tau(x)]) = \varphi([v, \tau(x)])$ for all $x \in \mathfrak{g}_{-1}$, and by induction we can now show that this implies $[u - v, \tau(x)] = 0$ for all $x \in \mathfrak{g}_{-1}$. Now we can use the same argument as before (but with \mathfrak{g}_+ replaced by \mathfrak{g}_{-}) to show that u and v must be equal. The case $u, v \in \mathfrak{g}_+$ then easily follows by

$$\varphi(\tau(u)) - \varphi(\tau(v)) = \varphi(u)^* - \varphi(v)^* = (\varphi(u) - \varphi(v))^*. \quad (4.13)$$

- φ is a homomorphism

It is sufficient to show this when $u \in \mathfrak{g}_i$ and $v \in \mathfrak{g}_j$ for all integers i, j and we will do it by induction over $|i| + |j|$. One easily checks that $\varphi([u, v]) = [\varphi(u), \varphi(v)]$ when $|i| + |j| \leq 1$. Thus suppose that this is true if $|i| + |j| = p$ for some integer $p \geq 1$. For $i, j < 0$ we now have

$$\begin{aligned} [\varphi(u), \varphi(v)](x) &= [\varphi(u), \varphi(v)(x)] - [\varphi(v), \varphi(u)(x)] \\ &= [\varphi(u), \varphi([v, \tau(x)])] - [\varphi(v), \varphi([u, \tau(x)])] \\ &= \varphi([u, [v, \tau(x)]] - [v, [u, \tau(x)]]) = \varphi([u, v])(x) \end{aligned} \quad (4.14)$$

by the assumption of induction in the third step, and by the Jacobi identity in the last one. We use this for the case $i, j > 0$, where we have

$$\begin{aligned} \varphi([\tau(u), \tau(v)]) &= \varphi(\tau([u, v])) = \varphi([u, v])^* = [\varphi(u), \varphi(v)]^* \\ &= [\varphi(u)^*, \varphi(v)^*] = [\varphi(\tau(u)), \varphi(\tau(v))]. \end{aligned} \quad (4.15)$$

Finally, we consider the case where $i \geq 0$ and $j \leq 0$. Again, we show it by induction over $|i| + |j|$, which means $i - j$ in this case. One easily checks that it is true when $i = 1$ and $j = -1$, so we can assume that $j \leq -2$ (or, analogously, $i \geq 2$). Then v can be written as a sum of elements $[x, y]$ where $x \in \mathfrak{g}_m$ and $y \in \mathfrak{g}_n$ for $j < m, n < 0$. We consider one such term and, using what we have already proven, we get

$$\begin{aligned} [\varphi(u), \varphi([x, y])] &= [\varphi(u), [\varphi(x), \varphi(y)]] \\ &= [[\varphi(u), \varphi(x)], \varphi(y)] - [[\varphi(u), \varphi(y)], \varphi(x)] \\ &= [\varphi([u, x]), \varphi(y)] - [\varphi([u, y]), \varphi(x)] \\ &= \varphi([[u, x], y] - [[u, y], x]) = \varphi([u, [x, y]]) \end{aligned} \quad (4.16)$$

by the assumption of induction in the third and fourth steps.

- φ is surjective

Since φ is a homomorphism, this follows from the fact that \mathfrak{g} and $L(\mathfrak{g}_{-1})$ are generated by $\mathfrak{g}_{\pm 1}$ and $U_{\pm 1}$, respectively. The proof is complete. \square

As we will see, the theorem is useful when a generalized Jordan triple system T happens to be isomorphic to \mathfrak{g}_{-1} , because it then tells us how to construct \mathfrak{g} from T .

4.2 Realization

The construction of the Lie algebra in the previous subsection may seem rather abstract, where the elements are linear operators acting on vector spaces of other linear operators, which in turn act on other vector spaces, and so on. However, once the Lie algebra is constructed, it can also be realized in a way such that the elements act on the same vector space, but in general non-linearly, and there is a very simple formula for this, as we will see in this subsection.

Let V be the direct sum of (infinitely many) vector spaces V_1, V_2, \dots . We write an element $v \in V$ as $v = v_1 + v_2 + \dots$, where $v_k \in V_k$, for $k = 1, 2, \dots$. With an operator on V of order p we mean a map $f : V \rightarrow V$ such that for any $i = 1, 2, \dots$, there is a symmetric $(p_1 + p_2 + \dots)$ -linear map

$$F_i : V^{p_1} \times V^{p_2} \times \dots \rightarrow V_i, \quad (4.17)$$

where $p_1 + 2p_2 + 3p_3 + \dots = i + p$, that satisfies

$$f(v)_i = F_i(v_1, v_1, \dots, v_1; v_2, v_2, \dots, v_2; \dots). \quad (4.18)$$

We define the composition $f \circ g$ of such an operator f and another operator g , of order q , as the operator of order $p + q$ given by

$$\begin{aligned} (f \circ g)_i(v) &= p_1 F_i(g(v)_1, v_1, \dots, v_1; v_2, v_2, \dots, v_2; \dots) \\ &\quad + p_2 F_i(v_1, v_1, \dots, v_1; g(v)_2, v_2, \dots, v_2; \dots) + \dots \end{aligned} \quad (4.19)$$

for all $i = 1, 2, \dots$, and a Lie bracket as usual by $[f, g] = f \circ g - g \circ f$. Let M_p be the vector space of all operators on V of order p , and let $M(V)$ be the direct sum of all M_p for all integers p (note that they can also be negative). It follows that $M(V)$ is a graded Lie algebra. It is isomorphic to the Lie algebra of all vector fields $f^i \partial_i$ on V , where $f \in M(V)$, with an isomorphism given by

$$f \mapsto -f^i \partial_i. \quad (4.20)$$

Any graded Lie algebra $\mathfrak{g} = \mathfrak{g}_- + \mathfrak{g}_0 + \mathfrak{g}_+$ is isomorphic to a subalgebra of $M(\mathfrak{g}_-)$. It can be shown [40, 41] that an injective homomorphism $\chi : \mathfrak{g} \rightarrow M(\mathfrak{g}_-)$ is given by

$$\chi(u) : x \mapsto \left(\frac{\text{ad } x}{1 - e^{-\text{ad } x}} P e^{-\text{ad } x} \right) (u), \quad (4.21)$$

where P is the projection onto U_- along $U_0 + U_+$, and the ratio should be considered as the power series

$$\frac{\text{ad } x}{1 - e^{-\text{ad } x}} = 1 + \frac{\text{ad } x}{2} + \frac{(\text{ad } x)^2}{12} - \frac{(\text{ad } x)^4}{720} + \dots . \quad (4.22)$$

4.3 Examples

We will now illustrate the ideas in two cases, where the generalized Jordan triple system satisfies further conditions.

First of all, we assume that the triple systems are such that if $(xyz) = 0$ for all y, z , then $x = 0$. This allows us to identify x with v_x for all x in the triple system, that is, we can identify U_{-1} with T and we can consider any element $[v_x, v_y] \in U_{-2}$ as a linear map on T , which we denote by $\langle x, y \rangle$. Since

$$\begin{aligned} [v_x, v_y](z) &= [v_x, v_y(z)] - [v_y, v_x(z)] \\ &= [v_x, s_{yz}] - [v_y, s_{xz}] = v_{(xzy)} - v_{(yzx)}, \end{aligned} \quad (4.23)$$

this linear map is given by

$$\langle x, y \rangle(z) = (xzy) - (yzx). \quad (4.24)$$

A generalized Jordan triple system is generalized in the sense that this linear map does not have to be zero – in a *Jordan triple system* [26], the triple product (xyz) is by definition symmetric in x and z . Accordingly, the Lie algebra associated to a Jordan triple system is 3-graded, $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$, and it can be realized on its subspace \mathfrak{g}_{-1} by applying the formula (4.21). Everything that is left from the power series expansion (4.22) is then the identity map,

$$\frac{\text{ad } x}{1 - e^{-\text{ad } x}} = 1, \quad (4.25)$$

and we get

$$\begin{aligned} u \in \mathfrak{g}_{-1} : \quad & x \mapsto Pu = u \\ [u, \tau(v)] \in \mathfrak{g}_0 : \quad & x \mapsto P([u, \tau(v)] - [x, [u, \tau(v)]]) = -[x, [\tau(u), v]] \\ & = [s_{uv}, x] = (uvx), \\ \tau(u) \in \mathfrak{g}_1 : \quad & x \mapsto P(\tau(u) - [x, \tau(u)] + \frac{1}{2}[x, [x, \tau(u)]]) = \frac{1}{2}[x, [x, \tau(u)]] \\ & = -\frac{1}{2}[s_{xu}, x] = \frac{1}{2}(xux). \end{aligned} \quad (4.26)$$

This is the *conformal realization* of \mathfrak{g} on \mathfrak{g}_{-1} .

We now turn to *Kantor triple systems* [28] (or generalized Jordan triple systems of *second order* [29]). These are generalized Jordan triple systems that in addition to the condition (4.1) satisfy the identity

$$\langle \langle u, v \rangle(x), y \rangle = \langle (yxu), v \rangle - \langle (y xv), u \rangle. \quad (4.27)$$

It follows that the Lie algebra associated to a Kantor triple system is 5-graded, and the only part of (4.22) that we have to keep is

$$\frac{\text{ad } x}{1 - e^{-\text{ad } x}} = 1 + \frac{\text{ad } x}{2}. \quad (4.28)$$

Then we get

$$\begin{aligned} [u, v] \in \mathfrak{g}_{-2} : & \quad z + Z \mapsto \langle u, v \rangle, \\ u \in \mathfrak{g}_{-1} : & \quad z + Z \mapsto u + \frac{1}{2} \langle u, z \rangle, \\ [u, \tau(v)] \in \mathfrak{g}_0 : & \quad z + Z \mapsto (uvz) - \langle u, Z(v) \rangle, \\ \tau(u) \in \mathfrak{g}_1 : & \quad z + Z \mapsto -\frac{1}{2}(zuz) - Z(u) \\ & \quad + \frac{1}{12} \langle (zuz), z \rangle - \frac{1}{2} \langle Z(u), z \rangle, \\ \tau(u), \tau(v) \in \mathfrak{g}_2 : & \quad z + Z \mapsto -\frac{1}{6}(z \langle u, v \rangle(z)z) - Z(\langle u, v \rangle(z)) \\ & \quad + \frac{1}{24} \langle (z \langle u, v \rangle(z)z), z \rangle + \langle Z(u), Z(v) \rangle, \end{aligned} \quad (4.29)$$

where $z \in \mathfrak{g}_{-1}$ and $Z \in \mathfrak{g}_{-2}$, which is the same realization as in [27], apart from a rescaling of the elements in \mathfrak{g}_{-2} by a factor of two.

5 Applications to pseudo-orthogonal algebras

In this section, we will apply the general considerations in section 3, and the generalized conformal realization in section 4.3, to pseudo-orthogonal algebras. First we recall some basic facts. Let V be a real vector space with an inner product. The real Lie group $SO(V)$ consists of all endomorphisms F of V which preserve the inner product,

$$(F(u), F(v)) = (u, v) \quad (5.1)$$

for all $u, v \in V$. The corresponding real Lie algebra $\mathfrak{so}(V)$ consists of all endomorphisms f of V which are antisymmetric with respect to the inner product,

$$(f(u), v) + (u, f(v)) = 0 \quad (5.2)$$

for all $u, v \in V$. If V is non-degenerate and finite-dimensional with signature (p, q) , then we can identify $SO(V)$ with the real Lie group $SO(p, q)$ consisting of all real $(p+q) \times (p+q)$ matrices X such that

$$X^t \eta X = \eta, \quad \det X = 1, \quad (5.3)$$

where η is the diagonal matrix associated to the inner product. Correspondingly, we can identify $\mathfrak{so}(V)$ with the real Lie algebra $\mathfrak{so}(p, q)$ consisting of all $(p+q) \times (p+q)$ matrices x such that

$$x^t \eta + \eta x = 0. \quad (5.4)$$

In other words, $\mathfrak{so}(p, q)$ consists of all real matrices of the form

$$x = \begin{pmatrix} a & c^t \\ c & b \end{pmatrix}, \quad (5.5)$$

where a and b are orthogonal $p \times p$ and $q \times q$ matrices respectively. These groups and algebras are said to be *pseudo-orthogonal* or, if $p = 0$, *orthogonal*, written simply $SO(q)$ and $\mathfrak{so}(q)$.

We consider now the pseudo-orthogonal algebra $\mathfrak{so}(p+n, q+n)$, with the inner product given by

$$\eta = \text{diag}(\underbrace{-1, \dots, -1}_p, \underbrace{+1, \dots, +1}_q, \underbrace{-1, \dots, -1}_n, \underbrace{+1, \dots, +1}_n), \quad (5.6)$$

for some arbitrary positive integers n, p, q . It is spanned by all matrices G^I_J , where the entry in row L , column K is given by

$$(G^I_J)^K_L = \delta^I_L \delta^K_J - \eta^{IK} \eta_{JL}, \quad (5.7)$$

and $I, J, \dots = 0, 1, \dots, p+q+2n-1$. It follows that $(G^I_J)^t = G^J_I$. If $I \neq J$, then the entry of G^I_J in row I , column J is 1 while the entry in row J , column I is ± 1 and all the others are zero. If $I = J$, then $G^I_J = 0$. These matrices satisfy the commutation relations

$$[G^I_J, G^K_L] = \delta^I_L G^K_J - \delta^K_J G^I_L + \eta^{IK} \eta_{JM} G^M_L - \eta_{JL} \eta^{IM} G^K_M, \quad (5.8)$$

and all those G^I_J with $I < J$ form a basis of $\mathfrak{so}(p+n, q+n)$.

For $\mu, \nu, \dots = 0, 1, \dots, p+q-1$ and $a, b, \dots = 1, 2, \dots, n$, with $\mu < \nu$ and $a < b$, we take the linear combinations

$$\begin{aligned} K_{ab} &= \frac{1}{2}(-G_{b+m+n}^{a+m+n} + G_{b+m+n}^{a+m} - G_{b+m}^{a+m+n} + G_{b+m}^{a+m}), \\ K^\mu_a &= -G_{a+m+n}^\mu - G_{a+m}^\mu, \\ D^a_b &= \frac{1}{2}(G_{b+m+n}^{a+m+n} + G_{b+m+n}^{a+m} + G_{b+m}^{a+m+n} + G_{b+m}^{a+m}), \\ P_\mu^a &= -G_\mu^{a+m+n} - G_\mu^{a+m}, \\ P^{ab} &= \frac{1}{2}(-G_{b+m+n}^{a+m+n} - G_{b+m+n}^{a+m} + G_{b+m}^{a+m+n} + G_{b+m}^{a+m}), \end{aligned} \quad (5.9)$$

as a new basis, where we have set $m = p+q-1$ for convenience. We note that K_{ab} and P^{ab} vanish when $n = 1$, since they are antisymmetric in the indices a, b . The basis elements (5.9) satisfy the commutation relations

$$[G^\mu_\nu, D^a_b] = [G^\mu_\nu, P^{ab}] = [G^\mu_\nu, K_{ab}] = 0,$$

$$[G_\nu^\mu, G_\sigma^\rho] = \delta_\sigma^\mu G_\nu^\rho - \delta_\nu^\rho G_\sigma^\mu + \eta^{\mu\rho}\eta_{\nu\lambda}G_\sigma^\lambda - \eta_{\nu\sigma}\eta^{\mu\lambda}G_\lambda^\rho,$$

$$[D_b^a, D_d^c] = \delta_d^a D_b^c - \delta_b^c D_d^a,$$

$$[P_\mu^a, K_\nu^b] = 2(\delta_b^a G_\mu^\nu - \delta_\nu^\mu D_b^a),$$

$$\begin{aligned} [G_\nu^\mu, P_\rho^a] &= \delta_\rho^\mu P_\nu^a - \eta_{\nu\rho}\eta^{\mu\lambda}P_\lambda^a, & [G_\nu^\mu, K_\rho^a] &= -\delta_\rho^\nu K_\mu^a + \eta^{\mu\rho}\eta_{\nu\lambda}K_\lambda^a, \\ [D_b^a, P_\mu^c] &= -\delta_b^c P_\mu^a, & [D_b^a, K_\mu^c] &= \delta_c^a K_\mu^b, \\ [P_\mu^a, P_\nu^b] &= -2\eta_{\mu\nu}P^{ab}, & [K_\mu^a, K_\nu^b] &= -2\eta^{\mu\nu}K_{ab}, \\ [P^{ab}, P_\mu^c] &= 0, & [K_{ab}, K_\mu^c] &= 0, \end{aligned}$$

$$\begin{aligned} [D_b^a, P^{cd}] &= \delta_b^d P^{ac} - \delta_b^c P^{ad}, & [D_b^a, K_{cd}] &= \delta_c^a K_{bd} - \delta_d^a K_{bc}, \\ [K_\mu^a, P^{bc}] &= \delta_c^a \eta^{\mu\lambda}P_\lambda^b - \delta_b^a \eta^{\mu\lambda}P_\lambda^c, & [P_\mu^a, K_{bc}] &= \delta_c^a \eta_{\mu\lambda}K_\lambda^b - \delta_b^a \eta_{\mu\lambda}K_\lambda^c, \\ [P^{ab}, P^{cd}] &= 0, & [K_{ab}, K_{cd}] &= 0, \end{aligned}$$

$$[P^{ab}, K_{cd}] = \delta_c^a D_b^d - \delta_b^c D_d^a - \delta_d^a D_b^c + \delta_b^d D_c^a. \quad (5.10)$$

We see that $\mathfrak{so}(p+n, q+n)$ has the following 5-grading, which reduces to a 3-grading when $n = 1$.

subspace	\mathfrak{g}_{-2}	\mathfrak{g}_{-1}	\mathfrak{g}_0	\mathfrak{g}_1	\mathfrak{g}_2
basis	P^{ab}	P_μ^a	G_ν^μ, D_b^a	K_μ^a	K_{ab}

Furthermore, we see that D_b^a satisfy the commutation relations for $\mathfrak{gl}(n, \mathbb{R})$. Since they also commute with G_ν^μ , we have

$$\mathfrak{g}_0 = \mathfrak{so}(p, q) \oplus \mathfrak{gl}(n, \mathbb{R}) \quad (5.11)$$

as a direct sum of subalgebras. Finally, a graded involution τ is given by

$$\tau(P_\mu^a) = \eta_{\mu\nu}K_\nu^a = -G_\mu^{a+m+n} + G_\mu^{a+m}, \quad (5.12)$$

and it follows that

$$\tau(K_\mu^a) = \eta^{\mu\nu}P_\nu^a = -G_{a+m+n}^\mu + G_{a+m}^\mu, \quad (5.13)$$

$$\begin{aligned}
\tau(D^a{}_b) &= -D^b{}_a, & \tau(K_{ab}) &= P^{ab}, \\
\tau(G^\mu{}_\nu) &= G^\mu{}_\nu, & \tau(P^{ab}) &= K_{ab}.
\end{aligned} \tag{5.14}$$

Thus all conditions are satisfied for \mathfrak{g}_{-1} to be a Kantor triple system K with the triple product

$$\begin{aligned}
(P_\mu{}^a P_\nu{}^b P_\rho{}^c) &= [[P_\mu{}^a, \tau(P_\nu{}^b)], P_\rho{}^c] = [[P_\mu{}^a, \eta_{\nu\lambda} K^\lambda{}_b], P_\rho{}^c] \\
&= -2\delta^{ab} \eta_{\mu\lambda} (\delta^\lambda{}_\rho P_\nu{}^c - \eta_{\nu\rho} \eta^{\lambda\kappa} P_\kappa{}^c) + 2\delta^{bc} \eta_{\mu\nu} P_\rho{}^a \\
&= 2\delta^{ab} (\eta_{\nu\rho} P_\mu{}^c - \eta_{\mu\rho} P_\nu{}^c) + 2\delta^{bc} \eta_{\mu\nu} P_\rho{}^a.
\end{aligned} \tag{5.15}$$

If we now insert (5.15) in (4.29) (but rescale the elements in \mathfrak{g}_{-2} according to [27]), and use the isomorphism (4.20), so that we identify any operator f with the vector field $-f^\mu{}_a \partial_\mu{}^a - f_{ab} \partial^{ab}$, then we get the realization

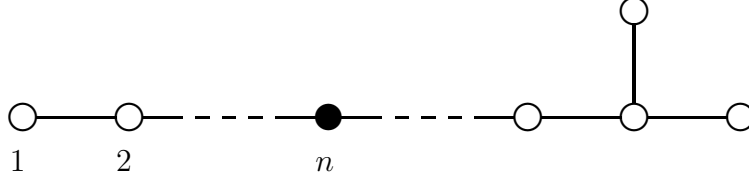
$$\begin{aligned}
P^{ab} &= -2\partial^{ab}, \\
P_\mu{}^a &= \partial_\mu{}^a - 2x_{\mu b} \partial^{ab}, \\
G^\mu{}_\nu &= x_{\nu a} \partial^{\mu a} - x^\mu{}_a \partial_\nu{}^a, \\
D^a{}_b &= x^\mu{}_b \partial_\mu{}^a + 2x_{bc} \partial^{ac}, \\
K^\mu{}_a &= -2x^\nu{}_a x^\mu{}_b \partial_\nu{}^b + x^\nu{}_a x_{\nu b} \partial^{\mu b} - x_{ab} \partial^{\mu b} \\
&\quad - 2x^\nu{}_a x^\mu{}_b x_{\nu c} \partial^{bc} + 2x_{ab} x^\mu{}_c \partial^{bc}, \\
K_{ab} &= x^\mu{}_a x^\nu{}_b x_{\mu c} \partial_\nu{}^c - x^\mu{}_b x^\nu{}_a x_{\mu c} \partial_\nu{}^c - x_{ac} x^\mu{}_b \partial_\mu{}^c + x_{bc} x^\mu{}_a \partial_\mu{}^c \\
&\quad + 2x^\mu{}_a x^\nu{}_b x_{\mu c} x_{\nu d} \partial^{cd} - 2x_{ac} x_{bd} \partial^{cd}.
\end{aligned} \tag{5.16}$$

Straightforward calculations show that these generators indeed satisfy the commutation relations (5.10). When $n = 1$, the $\mathfrak{gl}(n, \mathbb{R})$ indices a, b, \dots take only one value, so we can suppress them, and everything antisymmetric in these indices vanishes. We are then left with

$$\begin{aligned}
P_\mu &= \partial_\mu, \\
G^\mu{}_\nu &= x_\nu \partial^\mu - x^\mu \partial_\nu, \\
D &= x^\mu \partial_\mu, \\
K^\mu &= -2x^\nu x^\mu \partial_\nu + x^\nu x_\nu \partial_\mu,
\end{aligned} \tag{5.17}$$

which is the usual conformal realization.

We will now show that the 5-grading of $\mathfrak{so}(p+n, q+n)$ in this section is generated (as described in section 3) by the simple root corresponding to node n in the Dynkin diagram below of \mathfrak{d}_r for $p+q = 2r$. We will then show that the cases $n = 1$ and $n > 1$ are related in the way that we described in section 3.



For this we must relate the P_μ basis of \mathfrak{g}_{-1} used in this section to the basis consisting of root vectors. This relation will of course be different for different Lie algebras $\mathfrak{so}(p+n, q+n)$. We consider first the case $n = 1$. Below we give explicitly the relations (with a suitable choice of Cartan-Weyl generators) for two examples, $(p, q) = (5, 5)$ in the left table and $(p, q) = (1, 9)$ in the right table. We have indicated the roots by their coefficients in the basis of simple roots, corresponding to the nodes in the Dynkin diagram above. It is evident from these tables how to generalize them to arbitrary values of p, q (with $p + q$ even).

μ	e_μ	f_μ	μ	e_μ	f_μ
10000^0	$\frac{1}{2}(P_5 + P_0)$	$\frac{1}{2}(K^5 + K^0)$	10000^0	$\frac{1}{2}(P_5 + P_0)$	$\frac{1}{2}(K^5 + K^0)$
11000^0	$\frac{1}{2}(P_6 + P_1)$	$\frac{1}{2}(K^6 + K^1)$	11000^0	$\frac{1}{2}(P_6 - iP_1)$	$\frac{1}{2}(K^6 + iK^1)$
11100^0	$\frac{1}{2}(P_7 + P_2)$	$\frac{1}{2}(K^7 + K^2)$	11100^0	$\frac{1}{2}(P_7 - iP_2)$	$\frac{1}{2}(K^7 + iK^2)$
11110^0	$\frac{1}{2}(P_8 + P_3)$	$\frac{1}{2}(K^8 + K^3)$	11110^0	$\frac{1}{2}(P_8 - iP_3)$	$\frac{1}{2}(K^8 + iK^3)$
11111^0	$\frac{1}{2}(P_9 + P_4)$	$\frac{1}{2}(K^9 + K^4)$	11111^0	$\frac{1}{2}(P_9 - iP_4)$	$\frac{1}{2}(K^9 + iK^4)$
11110^1	$\frac{1}{2}(P_9 - P_4)$	$\frac{1}{2}(K^9 - K^4)$	11110^1	$\frac{1}{2}(P_9 + iP_4)$	$\frac{1}{2}(K^9 - iK^4)$
11111^1	$\frac{1}{2}(P_8 - P_3)$	$\frac{1}{2}(K^8 - K^3)$	11111^1	$\frac{1}{2}(P_8 + iP_3)$	$\frac{1}{2}(K^8 - iK^3)$
11121^1	$\frac{1}{2}(P_7 - P_2)$	$\frac{1}{2}(K^7 - K^2)$	11121^1	$\frac{1}{2}(P_7 + iP_2)$	$\frac{1}{2}(K^7 - iK^2)$
11221^1	$\frac{1}{2}(P_6 - P_1)$	$\frac{1}{2}(K^6 - K^1)$	11221^1	$\frac{1}{2}(P_6 + iP_1)$	$\frac{1}{2}(K^6 - iK^1)$
12221^1	$\frac{1}{2}(P_5 - P_0)$	$\frac{1}{2}(K^5 - K^0)$	12221^1	$\frac{1}{2}(P_5 - P_0)$	$\frac{1}{2}(K^5 - K^0)$

It follows that the bilinear form associated to the involution

$$\tau(P_\mu) = \eta_{\mu\nu}K^\nu, \quad (5.18)$$

is given in the P_μ basis by $(P_\mu, P_\nu) = 2\eta_{\mu\nu}$.

When we extend $\mathfrak{so}(p+1, q+1)$ to $\mathfrak{g} = \mathfrak{so}(p+n, q+n)$ for $n > 1$, we put a superscript 1 on P_μ in the expressions for the root vectors e_μ of the subalgebra $\mathfrak{h} = \mathfrak{so}(p+1, q+1)$ in \mathfrak{h}_{-1} (and a subscript 1 on K^μ). In the notation that we used in section 3, we have furthermore $e^a = D^a_1$ and $f^a = D^1_a$. Then the graded involution (5.12) on \mathfrak{g} is indeed the extension of the original involution (5.18) on \mathfrak{h} that we described in Theorem 3.1 and we get

$$\begin{aligned} (P_\mu^a P_\nu^b P_\rho^c) &= 2\delta^{ab}\eta_{\nu\rho}P_\mu^c - 2\delta^{ab}\eta_{\mu\rho}P_\nu^c + 2\delta^{bc}\eta_{\mu\nu}P_\rho^a \\ &= 2\delta^{ab}\eta_{\nu\rho}P_\mu^c - 2\delta^{ab}\eta_{\mu\rho}P_\nu^c + 2\delta^{ab}\eta_{\mu\nu}P_\rho^c - 2\delta^{ab}\eta_{\mu\nu}P_\rho^c + 2\delta^{bc}\eta_{\mu\nu}P_\rho^a \\ &= \delta^{ab}(P_\mu P_\nu P_\rho)^c - 2\delta^{ab}\eta_{\mu\nu}P_\rho^c + 2\delta^{bc}\eta_{\mu\nu}P_\rho^a \\ &= \delta^{ab}[[P_\mu, \tau(P_\nu)], P_\rho]^c - \delta^{ab}(P_\mu, P_\nu)P_\rho^c + \delta^{bc}(P_\mu, P_\nu)P_\rho^a \end{aligned} \quad (5.19)$$

as we should, according to Theorem 3.1.

5.1 Connection to Jordan algebras

When $n = 1$, the triple product (5.15) becomes

$$(P_\mu P_\nu P_\rho) = 2\eta_{\nu\rho}P_\mu - 2\eta_{\mu\rho}P_\nu + 2\eta_{\mu\nu}P_\rho. \quad (5.20)$$

If we introduce an inner product in the vector space \mathfrak{g}_{-1} by $P_\mu \cdot P_\nu = \eta_{\mu\nu}$, then this can be written

$$(xyz) = 2(z \cdot y)x - 2(z \cdot x)y + 2(x \cdot y)z. \quad (5.21)$$

Let U be the subspace of \mathfrak{g}_{-1} spanned by P_i for $i = 1, 2, \dots, p+q-1$. Then we can consider \mathfrak{g}_{-1} as the Jordan algebra $J(U)$ (defined in section 2 for Euclidean spaces U), with the product

$$P_i \circ P_j = (P_i \cdot P_j)P_0 \quad (5.22)$$

for $i, j = 1, 2, \dots, p+q-1$, and P_0 as identity element. If we introduce a linear map

$$J(U) \rightarrow J(U), \quad z \mapsto \tilde{z}, \quad (5.23)$$

which changes sign on P_0 , but otherwise leaves the basis elements P_μ unchanged, then we can write the inner product as

$$2(u \cdot v) = u \circ \tilde{v} + v \circ \tilde{u}. \quad (5.24)$$

Inserting (5.24) in (5.21), we get

$$\begin{aligned}
(xyz) &= (\tilde{z} \circ y) \circ x - (\tilde{z} \circ x) \circ y + (\tilde{x} \circ y) \circ z \\
&\quad + (z \circ \tilde{y}) \circ x - (z \circ \tilde{x}) \circ y + (x \circ \tilde{y}) \circ z \\
&= [y, \tilde{z}, x] + [y, \tilde{x}, z] + (z \circ \tilde{y}) \circ x + (x \circ \tilde{y}) \circ z.
\end{aligned} \tag{5.25}$$

It is easy to see that the *associators* in the last line remain unchanged if we move the tilde from one element to another. Thus we get

$$\begin{aligned}
(xyz) &= [y, \tilde{z}, x] + [y, \tilde{x}, z] + (z \circ \tilde{y}) \circ x + (x \circ \tilde{y}) \circ z \\
&= [\tilde{y}, z, x] + [\tilde{y}, x, z] + (z \circ \tilde{y}) \circ x + (x \circ \tilde{y}) \circ z \\
&= 2(z \circ \tilde{y}) \circ x - 2(z \circ x) \circ \tilde{y} + 2(x \circ \tilde{y}) \circ z.
\end{aligned} \tag{5.26}$$

If we instead use the involution given by

$$\tau(\tilde{P}_\mu) = \eta_{\mu\nu} K^\nu, \tag{5.27}$$

then we can remove the tilde,

$$(xyz) = 2(z \circ y) \circ x - 2(z \circ x) \circ y + 2(x \circ y) \circ z. \tag{5.28}$$

Any Jordan algebra J is also a Jordan triple system with this triple product. The associated Lie algebra, defined by the construction in section 4, or in this case equivalently by the realization (4.26), is its conformal algebra $\mathbf{con} J$.

Consider now the case $p = 1$ (and still $n = 1$). Then U is a Euclidean space, and $J(U)$ is a formally real Jordan algebra. For $q = 2, 3, 5, 9$, we recall from section 2 that there is an isomorphism from $J(U)$ to $H_2(\mathbb{K})$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively, given by

$$P_0 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P_{i+1} \mapsto \begin{pmatrix} 0 & -e_i \\ e_i & 0 \end{pmatrix}, \quad P_{p+q-1} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.29}$$

for $i = 1, 2, \dots, p + q - 3$. (As before, e_i are the 'imaginary units' that anticommute and square to -1 .) The involution (5.27) becomes

$$\tau(P_\mu) = K^\mu \tag{5.30}$$

and we see from the tables on page 21 that the associated bilinear form on \mathfrak{g}_{-1} has the simple form $(x, y) = \text{tr}(x \circ y)$. If we instead consider the split form ($p = q$) for $p = 3, 5, 9$, then (5.29) is still an isomorphism, from $J(U)$ to $H_2(\mathbb{K}^s)$, where $\mathbb{K} = \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively. Furthermore, the bilinear form on \mathfrak{g}_{-1} , associated to the graded involution (5.27) still has the form $(x, y) = \text{tr}(x \circ y)$.

6 Conclusions

It is a well known fact that the conformal algebra $\mathbf{con} H_2(\mathbb{K})$ of the Jordan algebra $H_2(\mathbb{K})$ is $\mathfrak{so}(p+1, q+1)$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ and $(p, q) = (1, 2), (1, 3), (1, 5), (1, 9)$, respectively. This means that there is a 3-grading of $\mathfrak{so}(p+1, q+1)$ and a graded involution such that the associated Jordan triple system is isomorphic to $H_2(\mathbb{K})$ with the triple product

$$(xyz) = 2(z \circ y) \circ x - 2(z \circ x) \circ y + 2(x \circ y) \circ z. \quad (6.1)$$

From this 3-grading we get the conformal realization of $\mathfrak{so}(p+1, q+1)$.

In section 5 we have explicitly given the 3-grading of $\mathfrak{so}(p+1, q+1)$, the graded involution and the isomorphism to $H_2(\mathbb{K})$. We have also shown that the bilinear form associated to the graded involution in $H_2(\mathbb{K})$ is given by $(x, y) = \text{tr}(x \circ y)$ and that the 3-grading is generated by the simple root corresponding to the leftmost node in the Dynkin diagram (drawn as on page 21). If we add $n-1 \geq 1$ nodes to the left, then this simple root will instead generate a 5-grading of the resulting algebra $\mathfrak{so}(p+n, q+n)$. Theorem 3.1 tells us how the two triple systems, associated to the 5-graded Lie algebra $\mathfrak{so}(p+n, q+n)$ and its 3-graded subalgebra $\mathfrak{so}(p+1, q+1)$, respectively, are related to each other. The conclusion is that the generalized Jordan triple system associated to $\mathfrak{so}(p+n, q+n)$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ and $(p, q) = (1, 2), (1, 3), (1, 5), (1, 9)$, respectively, is isomorphic to $H_2(\mathbb{K})^n$ with the triple product

$$\begin{aligned} (x^a y^b z^c) &= 2\delta^{ab}((z \circ y) \circ x)^c - 2\delta^{ab}((z \circ x) \circ y)^c + 2\delta^{ab}((x \circ y) \circ z)^c \\ &\quad - \delta^{ab}(x, y)z^c + \delta^{bc}(x, y)z^a, \end{aligned} \quad (6.2)$$

where $a, b, c = 1, 2, \dots, n$ and $(x, y) = \text{tr}(x \circ y)$. The same holds if we replace these pseudo-orthogonal algebras by the split forms of the corresponding complex Lie algebras, and $\mathbb{C}, \mathbb{H}, \mathbb{O}$ by $\mathbb{C}^s, \mathbb{H}^s, \mathbb{O}^s$.

The idea that we have presented here can also be applied to the Jordan algebras $H_3(\mathbb{K})$. The corresponding conformal algebras were given in the third row of the magic square on page 6 (and for simplicity we here only consider the complex Lie algebras), for example $\mathbf{con} H_3(\mathbb{O}) = \mathfrak{e}_7$. The node next to the black one in each Dynkin diagram represented the simple root that generates the corresponding 3-grading. If we add one more node next to the black one, we get the exceptional Lie algebras in the fourth row. The node that we add happens to be the 'affine' one, which means that if we add even more nodes, then we get first the affine extension of the exceptional Lie algebra, and then the hyperbolic extension. The conclusion in this case is thus that the exceptional algebras $\mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$ are the Lie algebras associated to $H_3(\mathbb{K})^2$, with the triple product (6.2) for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively. Their affine and hyperbolic extensions are those associated to $H_3(\mathbb{K})^3$ and $H_3(\mathbb{K})^4$, respectively, while further extensions correspond generically to $H_3(\mathbb{K})^n$ for $n = 5, 6, \dots$

It remains to show that the bilinear form associated to the graded involution really is given by $(x, y) = \text{tr}(x \circ y)$, also in the case of 3×3 matrices. However, one can show that the triple product (6.2) indeed satisfies the definition of a generalized Jordan triple system when x, y, z are elements in $H_3(\mathbb{K})$ and $(x, y) = \text{tr}(x \circ y)$. In order to verify the guess, one would need the relation between the two bases that we use for \mathfrak{g}_{-1} in the 3-grading of $\mathfrak{g} = \mathbf{con} H_3(\mathbb{K})$, corresponding to the tables on page 21. However, $H_3(\mathbb{K})$ is more complicated than $H_2(\mathbb{K})$, and, to our knowledge, such a change of basis has never been given explicitly. This is also the reason why we have studied only the pseudo-orthogonal algebras associated to $H_2(\mathbb{K})$ in detail, leaving the $H_3(\mathbb{K})$ case for future work.

An important and interesting difference between the $H_2(\mathbb{K})^n$ and $H_3(\mathbb{K})^n$ cases is that the Lie algebra associated to $H_2(\mathbb{K})^n$ is 3-graded for $n = 1$ and then 5-graded for all $n \geq 2$, while the Lie algebra associated to $H_3(\mathbb{K})^n$ is 3-graded for $n = 1$ but 7-graded for $n = 2$, and for $n = 3, 4, 5, \dots$, we get infinitely many subspaces in the grading, since these Lie algebras are infinite-dimensional. In the affine case, we only get the corresponding current algebra directly in this construction, which means that the central element and the derivation must be added by hand. It would be interesting to find an interpretation of these elements in the Jordan algebra approach. Finally, concerning the hyperbolic case and further extensions, we hope that our new construction can give more information about these indefinite Kac-Moody algebras, which, in spite of a great interest from both mathematicians and physicists, are not yet fully understood.

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