

Extreme Bowen–York initial data

Sergio Dain^{1,2} and María E Gabach Clément¹

¹ Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba, Ciudad Universitaria, (5000) Córdoba, Argentina

² Max Planck Institute for Gravitational Physics, (Albert Einstein Institute), Am Mühlenberg 1, D-14476 Potsdam, Germany

Received 17 June 2008, in final form 14 October 2008

Published 19 January 2009

Online at stacks.iop.org/CQG/26/035020

Abstract

The Bowen–York family of spinning black hole initial data depends essentially on one, positive, free parameter. The extreme limit corresponds to making this parameter equal to zero. This choice represents a singular limit for the constraint equations. We prove that in this limit a new solution of the constraint equations is obtained. These initial data have similar properties to the extreme Kerr and Reissner–Nordström black hole initial data. In particular, in this limit one of the asymptotic ends changes from asymptotically flat to cylindrical. The existence proof is constructive, we actually show that a sequence of Bowen–York data converges to the extreme solution.

PACS numbers: 04.20.Dw, 04.20.Ex, 04.70.Bw

1. Introduction

The Kerr–Newman black hole depends on the three parameters, m , q and J , the mass, the electric charge and the angular momentum of the spacetime, respectively. They satisfy the following well-known inequality

$$m^2 \geq q^2 + \frac{J^2}{m^2}. \quad (1)$$

This inequality can be written in the following form in which the mass appears only on the left-hand side of the equation, and on the right-hand side we have all the ‘charges’

$$m^2 \geq \frac{q^2 + \sqrt{q^4 + 4J^2}}{2}. \quad (2)$$

The extreme Kerr–Newman black hole is defined by the equality in (2)

$$m^2 = \frac{q^2 + \sqrt{q^4 + 4J^2}}{2}. \quad (3)$$

For fixed values of q and J , we can interpret the extreme black hole as the black hole with the minimum mass. In other words, the extreme black hole has the maximum amount of charge

and angular momentum per mass unit allowed for given values of q and J . This variational interpretation of extreme black holes generalizes to non-stationary, axially symmetric black holes ([14–16]). It is convenient to define a parameter μ which measures how far a black hole is with respect to the extreme case. In the stationary case, assuming that m , q and J satisfy inequality (2), μ is given by

$$\mu = \sqrt{m^2 - \frac{q^2 + \sqrt{q^4 + 4J^2}}{2}}. \quad (4)$$

Note that μ has unit of mass. The extreme limit corresponds to $\mu = 0$. For the Schwarzschild solution we have $\mu = m$.

In the extreme limit the global structure of the spacetime changes (see [10]). Particularly relevant for the study of black holes as an initial value problem is the change in the structure of Cauchy surfaces, and hence initial data set, in this limit. The slices $t = \text{constant}$ in Boyer–Lindquist coordinates represent Cauchy surfaces for the Kerr–Newman black hole. For $\mu > 0$ these slices have two isometrical asymptotically flat ends. In the extreme limit $\mu = 0$ one of the ends changes from asymptotically flat to cylindrical. Also, for $\mu > 0$ the Cauchy surfaces contain an apparent horizon (in this case, due to the symmetry, it is also a minimal surface). In the extreme case they do not contain any apparent horizons or minimal surfaces.

We can characterize a black hole spacetime by an initial data set. Then, it is possible to define an analog to the extreme limit discussed above for more general (in particular, non-stationary) black holes families. In [19], the extreme limit for the Bowen–York family of spinning black holes initial data [5] was defined. Here we will restrict our considerations to a subset of this family, which describes non-stationary, axisymmetric, black holes with angular momentum and zero linear momentum. Having fixed the angular momentum, this subset of the Bowen–York family depends on one parameter, which is the analog of the μ parameter defined in (4). As for the Kerr–Newman black hole, the extreme limit in this case also corresponds to $\mu \rightarrow 0$. The problem is that these data are not given explicitly. They are prescribed as solutions of a nonlinear elliptic equation (essentially, the Hamiltonian constraint) with appropriate boundary conditions. For the case $\mu > 0$ it is well known that this equation has a unique solution. However, the value $\mu = 0$ represents a singular limit for this equation. The values of the mass and the area of the apparent horizon were computed numerically in this limit in [12]. It was shown that the value of these quantities is well defined in the limit. In [19] the behavior of the whole solution was explored numerically in this limit. The numerical calculations indicate that in the limit a new solution is obtained (see also [26]). The purpose of this paper is to prove this. Namely, we will prove that the sequence $\mu \rightarrow 0$ of Bowen–York spinning black hole data converges to a limit solution. We call this new solution of the constraint equations the extreme Bowen–York data. We also prove that the solution (as it was indicated numerically in [19]) has a similar behavior to the extreme Kerr–Newman initial data discussed above.

The Bowen–York spinning black hole initial data have been extensively used in numerical relativity (see the review article [13]). The extreme Bowen–York data constructed here represent the data with the maximum amount of angular momentum per mass unit in this family and hence they are suitable for modeling highly spinning black holes. There exist astrophysical scenarios where it is expected that highly spinning black holes are relevant. In particular, the data presented here are useful in the study of the recent discovered kicks in the collision of two black holes (see [7, 8, 19, 21, 22, 24, 26]). In this process, the final recoil velocity depends on the angular momentum of the black holes and it is maximal for nearly extreme black holes (see [19]).

As a final comment, we mention that asymptotically flat Riemannian manifolds have been extensively studied in general relativity in connection with the constraint equations (see the review article [3]). On the other hand, very little is known about manifolds with cylindrical ends which appear naturally in extreme black holes. The solution presented here represents a non-stationary and non-trivial example of such manifolds.

This paper is organized as follows. In section 2 we present our main result given by theorem 2.1 and we discuss its implications. The proof of this theorem is split into section 3, 4 and 5. Possible generalizations and further studies are discussed in section 6. Finally, in the two sections of the appendix we prove the version of the maximum principle used in the paper and give the explicit expression of a lower bound for the solution that can be useful in numerical calculations.

2. Main result

Let us review the Bowen–York spinning black hole initial data [5] with ‘puncture’ boundary conditions [6]. The three-dimensional manifold is given by $\mathbb{R}^3 \setminus \{0\}$. On $\mathbb{R}^3 \setminus \{0\}$ the metric h_{ij} and the second fundamental form K_{ij} are given by

$$h_{ij} = \Phi^4 \delta_{ij}, \quad K_{ij} = \Phi^{-2} \sigma_{ij}, \quad (5)$$

where δ_{ij} is the flat metric and the tensor σ_{ij} is given by

$$\sigma_{ij} = \frac{6}{r^3} n_{(i} \epsilon_{j)kl} J^k n^l, \quad (6)$$

where r is the spherical radius, n^i the corresponding radial unit normal vector, ϵ_{ijk} the flat volume element and J_k an arbitrary constant vector. In this equation the indices are moved with the flat metric δ_{ij} .

The conformal factor Φ satisfies the following nonlinear elliptic equation in $\mathbb{R}^3 \setminus \{0\}$

$$\Delta \Phi = F(x, \Phi), \quad (7)$$

where

$$F(x, \Phi) = -\frac{9J^2 \sin^2 \theta}{4r^6 \Phi^7}, \quad (8)$$

and $J^2 = J_i J_j \delta^{ij}$, Δ is the flat Laplacian and x denotes spherical coordinates (r, θ) .

Boundary conditions for black holes are prescribed as follows. For a given parameter $\mu > 0$, define the function u_μ on \mathbb{R}^3 , by

$$\Phi_\mu := 1 + \frac{\mu}{2r} + u_\mu. \quad (9)$$

Inserting this definition into equation (7) we obtain the following equation for u_μ :

$$\Delta u_\mu = F(x, \Phi_\mu), \quad (10)$$

where

$$F(x, \Phi_\mu) = -\frac{9J^2 \sin^2 \theta}{4r^6 \left(1 + \frac{\mu}{2r} + u_\mu\right)^7}. \quad (11)$$

Then, equation (10) is solved in \mathbb{R}^3 for a function $u_\mu \in C^2$, subject to the asymptotic behavior

$$u_\mu \rightarrow 0 \quad \text{as } r \rightarrow \infty. \quad (12)$$

For every $\mu > 0$ there exists a unique solution $u_\mu \in C^2(\mathbb{R}^3)$ of (10) such that it satisfies (12). Note that u_μ is C^2 even at the origin. A proof of this result was given in [6] based on [9]. It is

also possible to prove this result using a suitable adapted version of the sub and supersolution theorem presented in [11] or using a compactification of \mathbb{R}^3 like the existence theorems in [4], [17].

Note that equation (10) depends, in principle, on two parameters, J and μ . There exists however a scale invariance for this equation (see [19]), and hence the solution depends non-trivially only on one parameter. We chose to fix J and vary μ .

In the rest of this paper we will denote by u_μ the unique solution of (10), with boundary condition (12) for any given $\mu > 0$. We have that $u_\mu \geq 0$ and $u_\mu \in C^{2,\alpha}(\mathbb{R}^3)$, where $C^{k,\alpha}(\mathbb{R}^3)$ denotes Hölder spaces (see, for example, [20] for definition and properties of these functional spaces).

The total angular momentum of the data is given by J and the total mass m is given by

$$m = \mu + \frac{1}{2\pi} \int_{\mathbb{R}^3} \frac{9J^2 \sin^2 \theta}{4r^6 (1 + \frac{\mu}{2r} + u_\mu)^7} dx. \tag{13}$$

Note that the mass cannot be *a priori* explicitly calculated as a function of μ and J since it involves the solution u_μ .

As we said in the introduction, we are interested in studying the limit

$$\lim_{\mu \rightarrow 0} u_\mu. \tag{14}$$

The corresponding equation becomes

$$\Delta u_0 = - \frac{9J^2 \sin^2 \theta}{4r^6 (1 + u_0)^7}. \tag{15}$$

We remark that when $\mu > 0$, the right-hand side of (10) is bounded in \mathbb{R}^3 (this is of course related with the fact the solution u_μ is regular at the origin for $\mu > 0$). Whereas in the extreme case, $\mu = 0$, it becomes singular at the origin, and hence we cannot expect the solution u_0 to be regular at the origin.

The following theorem constitutes the main result of the present paper. To formulate the theorem we will use weighted Sobolev spaces, denoted by $H'^{2,\delta}$, defined in [2] (see equation (69) in section 5).

Theorem 2.1.

(i) *There exists a solution u_0 of equation (15) in $\mathbb{R}^3 \setminus \{0\}$ such that $u_0 \in C^\infty(\mathbb{R}^3 \setminus \{0\})$ and u_0 satisfies the following bounds:*

$$u_0^- \leq u_0 \leq u_0^+, \tag{16}$$

where the functions u_0^+ and u_0^- are explicitly given by

$$u_0^+ = \sqrt{1 + \frac{|q|}{r}} - 1, \quad |q| = \sqrt{3|J|}, \tag{17}$$

and

$$u_0^- = Y_{00}(\theta)\chi_1(r) - \frac{Y_{20}(\theta)}{5^{3/2}}\chi_2(r). \tag{18}$$

Here Y_{00} and Y_{20} are spherical harmonics (see equation (B.5)) and $\chi_1(r)$ and $\chi_2(r)$ are elementary functions given explicitly in the appendix (equations (B.13) and (B.14)).

(ii) *In addition, we have that $u_0 \in H'^{2,\delta}$ for $-1 < \delta < -1/2$ and u_0 is the limit of the sequence*

$$\lim_{\mu \rightarrow 0} u_\mu = u_0, \tag{19}$$

in the norm $H'^{2,\delta}$.

The bounds (16) obtained in part (i) of theorem 2.1 imply

$$u_0 = O(r^{-1}) \quad \text{as } r \rightarrow \infty, \quad u_0 = O(r^{-1/2}) \quad \text{as } r \rightarrow 0. \quad (20)$$

These bounds show that the limit solution u_0 behaves differently near the origin from the sequence's members u_μ . This behavior confirms the numerical calculations presented in [19] and [26]. This is also related to the change of one of the ends from asymptotically flat to cylindrical in the extreme limit. To see this, we calculate the area of the 2-surfaces $r = \text{constant}$ with respect to the physical metric h_{ij} defined in (5). The area A is given by

$$A_\mu(r) = 2\pi r^2 \int_0^\pi \Phi_\mu^4 \sin \theta \, d\theta. \quad (21)$$

It is well known that for $\mu > 0$ the surface $r = \mu/2$ is a minimal surface. Also, for $\mu > 0$ we have

$$\lim_{r \rightarrow \infty} A_\mu(r) = \lim_{r \rightarrow 0} A_\mu(r) = \infty, \quad (22)$$

which reflects the fact that the data has two asymptotically flat ends. Moreover, these asymptotic regions are isometrical and are connected by the minimal surface at $r = \mu/2$. The situation changes in the extreme limit. Using the bounds (16) we can obtain the following bounds for the area in this limit:

$$0 < 2.37\pi|J| \leq A_0(0) \leq 12\pi|J|. \quad (23)$$

We see that the point $r = 0$ has finite, non-zero area. This shows that $r = 0$ is not an asymptotically flat end. It is a cylindrical end similar to that present in extreme Kerr and extreme Reissner–Nordström. On the other hand, the behavior as $r \rightarrow \infty$ is identical in both the non-extreme and the extreme cases. That is, this end is always asymptotically flat.

Having described some similarities between extreme Bowen–York and extreme Kerr data, it is also worth mentioning some differences. It is not *a priori* obvious that in the limit we do not obtain extreme Kerr data. However, this follows from the theorem proved in [28], because our data are conformally flat and there are no conformally flat slices in Kerr (including the extreme limit). Moreover (using the theorem proved in [16]) we also conclude that for the Bowen–York data the strict inequality $\sqrt{|J|} < m$ holds (cf equation (2)), where m is the total mass of the data (for extreme Kerr we have $\sqrt{|J|} = m$). This is also explicitly observed in numerical computations (see [12–18]), where bounds for the ratio $\sqrt{|J|}/m$ have been found.

Note that in part (i) of theorem 2.1 nothing is said about the behavior of the derivatives of u_0 near the origin and the fall-off near infinity. The behavior of the derivatives of u_0 in these regions is analyzed in part (ii) with the weighted Sobolev spaces. In particular, these spaces provide a norm for the convergence of the sequence and its derivatives in \mathbb{R}^3 .

Finally, let us mention three important points which we were unable to analyze at the moment. The first one is uniqueness of the solution u_0 . We have not proven that this solution is unique in $H'^{2,\delta}$ or in other suitable functional space. The second point is related with the behavior of the total mass in the sequence u_μ . The numerical calculations show that the mass decreases as $\mu \rightarrow 0$ (see [18–19]). This is, of course, the main reason why we call this solution the extreme Bowen–York data. However, we did not prove this analytically. Finally the third point is concerned with the existence of minimal surfaces and horizons. For $\mu > 0$ the spinning Bowen–York data contain a minimal surface located at $r = \mu/2$ which is also an apparent horizon. This follows because the data are symmetric under an inversion of the form $r \rightarrow \mu^2/(4r)$ (see [5]). However, in the singular limit $\mu \rightarrow 0$ this inversion symmetry is lost. The heuristic picture is that the minimal surface moves toward the end $r = 0$ as μ decreases and it disappears in the limit $\mu \rightarrow 0$. We conjecture that the extreme solution does not have

any minimal surface or apparent horizon (in analogy with the extreme Kerr–Newman black hole). This is also indicated in numerical calculations. But we were unable to show this.

The proof of theorem 2.1 falls naturally into three parts presented in section 3, 4 and 5. The plan of this proof is presented below.

Proof. We first prove that the sequence u_μ is pointwise monotonically increasing as μ decreases. This is proved in lemma 3.1. Then, we show that there exists a function u_0^+ , independent of μ , which is an upper bound to this sequence for all μ . See theorem 4.1. This theorem constitutes the most important part of the proof. From this upper bound we construct a lower bound u_0^- in lemma 4.2. Combining these lemmas and using standard elliptic estimates for the Laplacian on open balls which do not contain the origin we prove that the limit (19) exists and u_0 is smooth outside the origin. See lemma 5.1. This proves the part (i) of the theorem. Finally, part (ii) is proved in lemma 5.2. \square

3. Monotonicity

The function $F(x, \Phi)$ defined by (8) is non-decreasing in Φ . This fact, together with the maximum principle for the Laplace operator, will allow us to prove the monotonicity of the sequence u_μ with respect to the parameter μ .

The non-decreasing property of F is conveniently written in the following way. Let Φ_1, Φ_2 be positive functions such that $\Phi_1 \geq \Phi_2$, then we have

$$F(x, \Phi_1) - F(x, \Phi_2) = (\Phi_1 - \Phi_2)H(\Phi_2, \Phi_1) \geq 0, \quad (24)$$

where we have defined the function $H(\Phi_2, \Phi_1) = H(\Phi_1, \Phi_2)$ as

$$H(\Phi_2, \Phi_1) = \frac{9J^2 \sin^2 \theta}{4r^6} \sum_{i=0}^6 \Phi_1^{i-7} \Phi_2^{-1-i} \geq 0, \quad (25)$$

and we have used the following elementary identity for real numbers a and b :

$$\frac{1}{a^p} - \frac{1}{b^p} = (b - a) \sum_{i=0}^{p-1} a^{i-p} b^{-1-i}. \quad (26)$$

In our case the functions Φ are given by (9) with $\mu > 0$, and since $u_\mu \geq 0$ for $\mu > 0$, from (9) we obtain an upper bound for H

$$|H(\Phi_{\mu_2}, \Phi_{\mu_1})| \leq \frac{9J^2 r^2 \sin^2 \theta}{4} \sum_{i=0}^6 \left(r + \frac{\mu_1}{2}\right)^{i-7} \left(r + \frac{\mu_2}{2}\right)^{-1-i}, \quad (27)$$

which shows that H is bounded in \mathbb{R}^3 if $\mu_1, \mu_2 > 0$. Taking an upper bound, independent of μ , on the right-hand side of (27) we obtain the following bound for H :

$$|H(\Phi_{\mu_2}, \Phi_{\mu_1})| \leq \frac{63J^2 \sin^2 \theta}{4r^6}. \quad (28)$$

Note that this bound diverges at the origin.

The main result of this section is summarized in the following lemma.

Lemma 3.1. *Assume $\mu_1 \geq \mu_2 > 0$ then we have $u_{\mu_1}(x) \leq u_{\mu_2}(x)$ in \mathbb{R}^3 .*

Proof. Define w by

$$w(x) = u_{\mu_2}(x) - u_{\mu_1}(x). \quad (29)$$

Using equation (10), we obtain that w satisfies the equation

$$\Delta w = F(x, \Phi_{\mu_2}) - F(x, \Phi_{\mu_1}). \quad (30)$$

We use (24) to write this equation in the following form:

$$\Delta w - wH(\Phi_{\mu_2}, \Phi_{\mu_1}) = \frac{\mu_2 - \mu_1}{2r} H(\Phi_{\mu_2}, \Phi_{\mu_1}), \quad (31)$$

where H is given by (25). Since $H \geq 0$ and by hypothesis we have $\mu_2 - \mu_1 \leq 0$, then the right-hand side of (31) is negative. We also have that $w \rightarrow 0$ as $r \rightarrow \infty$ (because of (12)). Hence, we can apply the maximum principle for the Laplace operator (theorem A.1 in the appendix), to conclude that $w \geq 0$ in \mathbb{R}^3 . We emphasize that this theorem can be applied because H is bounded in \mathbb{R}^3 when $\mu_1, \mu_2 > 0$. \square

Remarkably, the sequence Φ_μ has the opposite behavior as the sequence u_μ , namely Φ_μ is increasing with respect to μ . This is proved in the following lemma.

Lemma 3.2. *Assume $\mu_2 \geq \mu_1 > 0$ then we have $\Phi_{\mu_2}(x) \geq \Phi_{\mu_1}(x)$ in $\mathbb{R}^3 \setminus \{0\}$.*

Proof. The proof is similar to that in the previous lemma, however since Φ_μ is singular at the origin we need to exclude this point from the domains. First, we note that for $\mu_2 = \mu_1$ we have $\Phi_{\mu_2} = \Phi_{\mu_1}$ because the solution u_μ is unique. Hence, in the following we assume $\mu_2 > \mu_1 > 0$.

In order to handle the singularity of Φ_μ at the origin we choose a domain Ω defined as $\mathbb{R}^3 \setminus B_\epsilon$ where B_ϵ is a small ball of radius ϵ centered at the origin. As before, we define w as the difference

$$w = \Phi_{\mu_2} - \Phi_{\mu_1} = \frac{\mu_2 - \mu_1}{2r} + u_{\mu_2} - u_{\mu_1}. \quad (32)$$

Then, we have

$$\Delta w - wH(\Phi_{\mu_1}, \Phi_{\mu_2}) = 0, \quad (33)$$

where H is given by (25). Since u_μ is bounded in \mathbb{R}^3 for $\mu > 0$ then the first term on the right-hand side of (32) will dominate for sufficiently small r . Hence, for $\mu_2 > \mu_1 > 0$ there exists ϵ sufficiently small such that w is positive on ∂B_ϵ . Consider equation (32) on Ω . Since w goes to zero at infinity, we can apply the maximum principle (theorem A.1 in the appendix) in Ω to obtain $w \geq 0$. \square

4. Bounds

In this section we give bounds for the sequence u_μ . The main result of the section is given by theorem 4.1 where we construct an upper bound u_0^+ , based on the Reissner–Nordström black hole initial data, which does not depend on μ . The lower bound is then directly constructed using this upper bound in lemma 4.2.

The Reissner–Nordström black hole will play an important role in what follows. Let us review it. The Reissner–Nordström metric is characterized by two parameters: the mass m and the electric charge q . This metric describes a black hole if $|q| \leq m$. When $|q| = m$ the solution is called the extreme Reissner–Nordström black hole. Take a slice $t = \text{constant}$ in the canonical coordinates and let r be the isotropical radius on this slice. The intrinsic metric on the slice is conformally flat, i.e. it has the form (5) where the conformal factor is denoted by Φ_μ^+ (the reason for the + in the notation will become clear later on) and it is explicitly given by

$$\Phi_\mu^+ = \sqrt{1 + \frac{m}{r} + \frac{\mu^2}{4r^2}}, \quad (34)$$

where the parameter μ is defined in terms of m and q by (4) (with $J = 0$), that is

$$m = \sqrt{\mu^2 + q^2}. \quad (35)$$

Note that, when q is fixed, m decreases as μ goes to zero. We also define the function $u_\mu^+(x)$ by

$$\Phi_\mu^+ = 1 + \frac{\mu}{2r} + u_\mu^+(x), \quad (36)$$

that is, we have

$$u_\mu^+(x) = \sqrt{1 + \frac{m}{r} + \frac{\mu^2}{4r^2}} - 1 - \frac{\mu}{2r}. \quad (37)$$

The extreme limit corresponds to $\mu = 0$, in this limit the solution is denoted by u_0^+ , we have

$$u_0^+(x) = \sqrt{1 + \frac{|q|}{r}} - 1. \quad (38)$$

As a consequence of the constraint equations the function u_μ^+ satisfies

$$\Delta u_\mu^+ = -\frac{q^2}{4r^4(\Phi_\mu^+)^3}. \quad (39)$$

We have $u_\mu^+ \geq 0$. From the explicit expression (37) we deduce that the sequence u_μ^+ is increasing as $\mu \rightarrow 0$ and it is bounded by the extreme solution u_0^+ , that is

$$u_\mu^+(x) < u_0^+(x), \quad (40)$$

for all $\mu > 0$. Also, $u_\mu^+(x)$ is smooth on $\mathbb{R}^3 \setminus \{0\}$ and, for $\mu > 0$, we have $u_\mu^+ \in C^1(\mathbb{R}^3)$ (but it is not C^2 at the origin). The values of the function and its derivative at the origin are given by

$$u_\mu^+(0) = \frac{m}{\mu} - 1 \quad \frac{du_\mu^+}{dr}(0) = -\frac{q^2}{\mu^3}. \quad (41)$$

Note that both values diverge as $\mu \rightarrow 0$. In fact the limit function u_0^+ diverges as $r^{-1/2}$ near the origin. We want to prove that a similar behavior occurs for the Bowen–York case.

The following constitutes the main result of this section.

Theorem 4.1. *Assume that*

$$|q| \geq \sqrt{3|J|}. \quad (42)$$

Then, for all $\mu > 0$, we have

$$u_\mu(x) \leq u_\mu^+(x) < u_0^+(x), \quad (43)$$

where u_μ^+ and u_0^+ are given by (37) and (38) respectively.

Proof. From (39) and assuming that condition (42) holds, we obtain

$$\Delta u_\mu^+ = -\frac{q^2}{4r^4(\Phi_\mu^+)^3} \leq -\frac{9J^2 \sin^2 \theta}{q^2 4r^4(\Phi_\mu^+)^3}. \quad (44)$$

Then, since $m \geq |q|$, we have

$$(\Phi_\mu^+)^4 \geq \left(1 + \frac{|q|}{r}\right)^2 \geq \frac{q^2}{r^2}, \quad (45)$$

which gives us

$$\Delta u_\mu^+ \leq -\frac{9J^2 \sin^2 \theta}{4r^6(\Phi_\mu^+)^7} = F(x, \Phi_\mu^+). \quad (46)$$

Now, we define the difference

$$w = u_\mu^+ - u_\mu. \quad (47)$$

Using equation (15) and (46) we obtain

$$\Delta w \leq F(x, \Phi_\mu^+) - F(x, \Phi_\mu). \quad (48)$$

We use formula (24) to conclude that

$$\Delta w - wH(\Phi_\mu^+, \Phi_\mu) \leq 0. \quad (49)$$

Note that the function w is not C^2 at the origin because u_μ^+ is not C^2 , and hence it does not satisfy the inequality (49) in the classical sense at the origin. However, we have $w \in H_{\text{loc}}^1$ (in fact w is C^1 because u_μ^+ is C^1) and then it satisfies (49) in the weak sense also at the origin. We also have that w goes to zero as $r \rightarrow \infty$. Hence, we can apply the maximum principle (theorem A.1 in the appendix) to conclude that $w \geq 0$. \square

Since F is non-decreasing, once an upper bound is found for the sequence, the construction of a lower bound is straightforward. Namely, we define $u_\mu^-(x)$ as the solution of the following linear Poisson equation:

$$\Delta u_\mu^- = F(x, \Phi_\mu^+) = -\frac{9J^2 \sin^2 \theta}{4r^6 \left(1 + \frac{m}{r} + \frac{\mu^2}{4r^2}\right)^{7/2}}, \quad (50)$$

with the fall-off condition

$$\lim_{r \rightarrow \infty} u_\mu^- = 0. \quad (51)$$

Lemma 4.2. *Let u_μ^- be the solution of (50) with the asymptotic condition (51). We have that for all $\mu > 0$*

$$u_\mu^-(x) \leq u_\mu(x), \quad (52)$$

and

$$\frac{\mu}{2r} + u_\mu^-(x) \geq u_0^-(x). \quad (53)$$

The function u_0^- has the following behavior:

$$u_0^-(x) = \frac{C_1}{r} + O(r^{-2}), \quad \text{as } r \rightarrow \infty, \quad (54)$$

$$u_0^-(x) = \frac{C_2}{\sqrt{r}} + O(1), \quad \text{as } r \rightarrow 0, \quad (55)$$

where $C_1, C_2 > 0$.

Proof. The solution can be explicitly constructed using the fundamental solution (or Green's function) of the Laplacian (see the appendix). From the standard elliptic estimates (or directly from the explicit expression) we deduce that $u_\mu^- \in C^{2,\alpha}(\mathbb{R}^3)$ for $\mu > 0$.

Let us prove inequality (52). As usual we take the difference $w = u_\mu - u_\mu^-$, then, using equation (50), we have

$$\Delta w = F(x, \Phi_\mu) - F(x, \Phi_\mu^+) = (u_\mu - u_\mu^+)H(\Phi_\mu, \Phi_\mu^+). \quad (56)$$

Since $u_\mu - u_\mu^+ \leq 0$ by lemma 4.1 we obtain $\Delta w \leq 0$ and then by the maximum principle we get $w \geq 0$.

To prove inequality (53) we use a similar argument as in the proof of lemma 3.2. Note that we can in principle deduce (53) from the explicit expression for u_μ^- , however the formula is so complicated that this is not straightforward.

Finally, the fall-off behavior (54)–(55) is obtained from the explicit expression of u_0^- given in the appendix (see equation (B.15) and (B.16)). \square

Note that the sequence u_μ^- is monotonic in μ , as the Bowen–York sequence u_μ . Namely, for $\mu_1 \geq \mu_2 \geq 0$, we obtain

$$u_{\mu_1}^-(x) \leq u_{\mu_2}^-(x) \leq u_0^-(x), \tag{57}$$

and also for $\mu_1 > \mu_2 > 0$ we have

$$\Phi_{\mu_1}^-(x) \geq \Phi_{\mu_2}^-(x) \geq \Phi_0^-(x) \tag{58}$$

where Φ^- is defined as

$$\Phi_\mu^- = 1 + \frac{\mu}{2r} + u_\mu^-. \tag{59}$$

5. Convergence

In this section we prove that the sequence u_μ converges in the limit $\mu \rightarrow 0$. We begin with the interior convergence. We will make use of Lebesgue spaces L^2 and Sobolev spaces H^2 (for definition and properties of these functional spaces see, for example, [20]).

Lemma 5.1. *Let U be an arbitrary open ball contained in $\mathbb{R}^3 \setminus \{0\}$. Then the sequence u_μ converges in the $H^2(U)$ norm. Moreover, the limit function*

$$u_0 = \lim_{\mu \rightarrow 0} u_\mu \tag{60}$$

is a solution of equation (15) in U and $u_0 \in C^\infty(U)$.

Proof. Given the open ball U , there always exists an open ball U' contained in $\mathbb{R}^3 \setminus \{0\}$ such that $U \subset\subset U'$. The set U' is important in what follows, in order to use the interior elliptic estimate given by (62).

Choose $x \in U'$. Consider the sequence of real numbers $u_\mu(x)$ for $\mu \rightarrow 0$. By lemma 3.1 the sequence u_μ is non-decreasing as μ goes to zero, and by lemma 4.1 it is bounded from above (and the bound does not depend on μ) by $u_\mu(x) \leq u_0^+(x)$. Note that it is important that the closure of U' does not contain the origin $\{0\}$, since u_0^+ is not bounded there. It follows that the sequence converges pointwise to a limit $u_0(x)$. And then, the convergence in the L^2 norm follows from the dominated convergence theorem (see, e.g., [25]). Hence, the sequence u_μ is a Cauchy sequence in $L^2(U')$, i.e.

$$\lim_{\mu_1, \mu_2 \rightarrow 0} \|w\|_{L^2(U')} = 0, \tag{61}$$

where $w = u_{\mu_2} - u_{\mu_1}$.

To prove that the sequence u_μ is a Cauchy sequence in $H^2(U)$ we use the standard elliptic estimate for the Laplacian (see e.g. [20])

$$\|w\|_{H^2(U)} \leq C(\|\Delta w\|_{L^2(U')} + \|w\|_{L^2(U')}) \tag{62}$$

where the constant C depends only on U' and U .

The difference w satisfies equation (31), then we obtain

$$\|\Delta w\|_{L^2(U')} = \left\| Hw + H \frac{\mu_2 - \mu_1}{r} \right\|_{L^2(U')} \tag{63}$$

$$\leq \|Hw\|_{L^2(U')} + (\mu_1 - \mu_2) \left\| \frac{H}{r} \right\|_{L^2(U')}. \tag{64}$$

The functions H and H/r are bounded in U' (see equation (28)) by a constant independent of μ . Then, from the inequality (63) we obtain

$$\|\Delta w\|_{L^2(U')} \leq C(\|w\|_{L^2(U')} + (\mu_1 - \mu_2)) \tag{65}$$

where C does not depend on μ . Using the estimate (62) we finally get

$$\|w\|_{H^2(U)} \leq C(\|w\|_{L^2(U')} + (\mu_1 - \mu_2)). \tag{66}$$

From this inequality and the convergence in L^2 given by (61) we conclude that

$$\lim_{\mu_1, \mu_2 \rightarrow 0} \|w\|_{H^2(U)} = 0 \tag{67}$$

and hence $u_0 = \Phi_0 - 1 \in H^2(U)$. By the same argument we also have that u_0 is a strong solution (see [20] for the definition of strong solutions for elliptic equations) of equation (15) in U .

Using the standard elliptic estimates once again and iterating using equation (15) we get that $u_0 \in C^\infty(U)$. This iteration can be done as follows. By the Sobolev imbedding theorem we have that $u_0 \in C^\alpha(U)$. Then, it follows that $F(x, \Phi_0) \in C^\alpha(U)$. But then, by Hölder estimates for the Laplace operator (see [20]) it follows that $u_0 \in C^{2,\alpha}(U)$. We can iterate this argument to obtain that u_0 is smooth in U . \square

In the previous theorem we have not analyzed the fall-off of the solution u_0 at infinity and its behavior at the origin. In order to do so, more precise estimates are required. In particular, we need to make use of weighted Sobolev norms. We will use the weighted Sobolev spaces defined in [2] and denoted here by $H'^{k,\delta}$. The definitions of the corresponding norms are the following (we restrict ourselves to the case $p = 2$ and dimension 3):

$$\|f\|'_{L'^{2,\delta}} = \left(\int_{\mathbb{R}^3 \setminus \{0\}} |f|^2 r^{-2\delta-3} dx \right)^{1/2}, \tag{68}$$

and

$$\|f\|'_{H'^{k,\delta}} := \sum_0^k \|D^j f\|'_{L'^{2,\delta-j}}. \tag{69}$$

These functional spaces are relevant for our purpose because we have that

$$u_\mu^+(x) \in H'^{2,\delta} \quad \text{for } -1 < \delta < -1/2, \tag{70}$$

for all $\mu \geq 0$. We can understand the given range of δ by noting that the extreme Reissner-Nordström solution goes as $r^{-1/2}$ as $r \rightarrow 0$, and as r^{-1} as $r \rightarrow \infty$. It can also be seen that, if we consider only solutions with $\mu > 0$, then the allowed interval for δ expands to $(-1, 0)$ reflecting the fact that in this case, the functions are bounded at the origin.

Lemma 5.2. *The sequence u_μ is Cauchy in the norm $H'^{2,\delta}$ for $-1 < \delta < -1/2$.*

Proof. The proof is similar as in the previous lemma, the main difference is that we have to take into account the singular behavior of the functions at the origin.

We first note that the same argument presented above allows us to prove convergence in the weighted Lebesgue spaces $L'^{2,\delta}$. In effect, consider the sequence $u_\mu r^{-\delta-3/2}$ for $-1 < \delta < -1/2$. This sequence is pointwise bounded by $u_0^+ r^{-\delta-3/2}$ and monotonically increasing as the parameter μ goes to zero, which means that it is a.e. pointwise converging to

a function $u_0 r^{-\delta-3/2}$. Then, we can use the dominated convergence theorem (since $u_0^+ r^{-\delta-3/2}$ is summable in \mathbb{R}^3 for the given values of the weight δ) to find that the new sequence converges in $L^2(\mathbb{R}^3)$. But this implies that the original sequence u_μ converges in $L'^{2,\delta}$, with $\delta \in (-1, -1/2)$. That is

$$\lim_{\mu_1, \mu_2 \rightarrow 0} \|w\|_{L'^{2,\delta}} = 0, \tag{71}$$

where w is the difference introduced above in equation (29).

In order to prove that the sequence u_μ is a Cauchy sequence also in the weighted Sobolev space $H'^{2,\delta}$ with $\delta \in (-1, -1/2)$, we will apply the following estimate (see, e.g. [2]):

$$\|w\|_{H'^{2,\delta}} \leq C \|\Delta w\|_{L'^{2,\delta-2}}, \tag{72}$$

where the constant C depends only on δ .

As before, we obtain

$$\|\Delta w\|_{L'^{2,\delta-2}} = \left\| Hw + H \frac{\mu_2 - \mu_1}{r} \right\|_{L'^{2,\delta-2}} \tag{73}$$

$$\leq \|Hw\|_{L'^{2,\delta-2}} + (\mu_1 - \mu_2) \left\| \frac{H}{r} \right\|_{L'^{2,\delta-2}}. \tag{74}$$

From the definition of the norm $L'^{2,\delta}$ given in (68) we obtain

$$\|Hw\|_{L'^{2,\delta-2}} \leq \sup_{\mathbb{R}^3} |Hr^2| \|w\|_{L'^{2,\delta}}, \tag{75}$$

and hence, using (73) we have

$$\|\Delta w\|_{L'^{2,\delta-2}} \leq C \left(\sup_{\mathbb{R}^3} |Hr^2| \|w\|_{L'^{2,\delta}} + (\mu_1 - \mu_2) \left\| \frac{H}{r} \right\|_{L'^{2,\delta-2}} \right). \tag{76}$$

The crucial step in the proof is to bound, in (76), the corresponding norms of H and H/r . This point is where the weighted Sobolev spaces play a role, because these norms are not bounded in the standard Sobolev norms.

To bound Hr^2 we use

$$H \leq \frac{63J^2 \sin^2 \theta}{4r^6} 7(1 + u_0^-)^{-8}. \tag{77}$$

By theorem 4.2 we know that u_0^- goes to zero at infinity, hence H decays as r^{-6} . At the origin, by lemma 4.2, we know that $u_0^- = \mathcal{O}(r^{-1/2})$, therefore, H grows as r^{-2} . Hence the $r^2 H$ is finite for every value of the parameter μ .

For the other term we have

$$\left\| \frac{H}{r} \right\|_{L'^{2,\delta-2}} = \left(\int_{\mathbb{R}^3 \setminus \{0\}} \left| \frac{H}{r} \right|^2 r^{-2\delta+1} dx \right)^{1/2} \tag{78}$$

and, using again the lower bound as in (77) we find that this norm is also finite for $\delta \in (-1, -1/2)$. Then, we can write

$$\|w\|_{H'^{2,\delta}} \leq C (\|w\|_{L'^{2,\delta}} + (\mu_1 - \mu_2)), \tag{79}$$

where the constant C does not depend on μ . This and equation (71) give us, in the limit $\mu_1, \mu_2 \rightarrow 0$

$$\lim_{\mu_1, \mu_2 \rightarrow 0} \|w\|_{H'^{2,\delta}} = 0. \tag{80}$$

Then, the sequence u_μ is Cauchy in the $H'^{2,\delta}$ norm, with $\delta \in (-1, -1/2)$. □

Note that this theorem also implies that u_0 is a strong solution in the Sobolev spaces $H'^{2,\delta}$ of equation (15) also at the origin.

6. Final comments

In this paper, we have studied the extreme limit of the Bowen–York family of initial data. We have found that the extreme solution exists and has similar properties to the extreme Kerr black hole data. It is straightforward to generalize the results presented here for more general second fundamental forms keeping the conformal flatness of the data. A more relevant and difficult generalization would involve more general background metric. In particular, it would be interesting to generalize the extreme limit for binary Kerr black hole data. A possible strategy to attack this problem is to prove, using similar techniques as those presented here, that the sequence of two non-extreme Kerr black holes constructed in [1] actually converges in the extreme limit.

As it was mentioned in the introduction, there exists a variational characterization of the extreme limit. The extreme initial data, and hence data with cylindrical ends, appear naturally as minimum of the mass in appropriate class of data. The example presented here incorporates a new class of data in which this variational characterization holds. As we said in section 2, we expect that this minimum of the mass (i.e. the extreme solution) has no horizon. Moreover, we expect that a small perturbation of an extreme solution (in particular, the extreme Bowen–York data) will always have an horizon. It would be interesting to prove or disprove this conjecture.

Acknowledgments

It is our pleasure to thank J A Valiente Kroon for useful discussions. SD is supported by CONICET (Argentina). MEGC is supported by a fellowship of CONICET (Argentina). This work was supported in part by grant PIP 6354/05 of CONICET (Argentina), grant 05/B270 of Secyt-UNC (Argentina) and the Partner Group grant of the Max Planck Institute for Gravitational Physics, Albert Einstein Institute (Germany). SD is grateful for the hospitality and support of the Max Planck Institute for Gravitational Physics, Albert Einstein Institute, where part of this work took place.

Appendix A. Weak maximum principle on unbounded domains

In this section we prove the version of the weak maximum principle used in the main part of the paper. This version applies for unbounded domains and weak solutions. Although this is certainly a standard result, we were not able to find exactly this form in the literature. The proof is similar to the proof of theorem 8.1 in [20] and lemma 5.2 in [27]. A related version of the maximum principle for classical solutions and asymptotically flat manifolds (without inner boundaries) is proved in [11].

Let B be a bounded domain in \mathbb{R}^n (in this paper we always worked in \mathbb{R}^3 , however the dimension plays no role here). We consider the following unbounded domain $\Omega = \mathbb{R}^n \setminus B$. We allow the possibility that B is empty, in this case we have $\Omega = \mathbb{R}^n$.

Theorem A.1. *Let $u \in H_{\text{loc}}^1(\Omega)$ satisfy*

$$\Delta u - au \leq 0 \quad \text{in } \Omega, \tag{A.1}$$

where $a \geq 0$ is a bounded and measurable function. We also assume that

$$u \geq 0 \quad \text{in } \partial\Omega, \tag{A.2}$$

and

$$u \rightarrow 0 \quad \text{as } r \rightarrow \infty. \tag{A.3}$$

Then, $u \geq 0$ in Ω .

Remarks: Since the function u is only weakly differentiable the inequalities in this theorem must be understood in the weak sense. In the case where $\partial\Omega$ is empty we do not have condition (A.2).

Proof. Choose $\epsilon > 0$. Define the function $v = \max\{-\epsilon - u, 0\}$. Since $u \geq 0$ on $\partial\Omega$ and $u \rightarrow 0$ as $r \rightarrow \infty$, then v has compact support in Ω . We denote by Ω^+ the support of v . We also have that $v \in H^1(\Omega)$ (see [20]). Then, we can use v as a test function for the inequality (A.1). Integrating by parts we obtain

$$\int_{\Omega} \partial v \partial u + avu \geq 0. \quad (\text{A.4})$$

Suppose that $-\epsilon - u > 0$ at some point in Ω , then the support Ω^+ of v is non-empty. Since $\partial v = -\partial u$ in Ω^+ , using (A.4) we obtain

$$0 \geq \int_{\Omega^+} avu \geq \int_{\Omega^+} |\partial u|^2, \quad (\text{A.5})$$

where in the first inequality we have used the hypothesis $a \geq 0$. Then, we conclude that u (and hence v) is constant on Ω^+ . Since v has compact support, this implies that v is zero and we get a contradiction. Hence, we have proved that $-\epsilon \leq u$ for an arbitrary, positive, ϵ . Letting $\epsilon \rightarrow 0$ we obtain the desired result. \square

Appendix B. Explicit expression of the subsolution $u_{\mu}^{-}(x)$

In this section we construct the explicit solution to the equation

$$\Delta u_{\mu}^{-}(x) = F \quad (\text{B.1})$$

where F is given by

$$F = \sin^2 \theta R(\mu, r), \quad (\text{B.2})$$

and

$$R(\mu, r) = -\frac{18J^2}{8r^6 \left(1 + \frac{m}{r} + \frac{\mu^2}{4r^2}\right)^{7/2}}. \quad (\text{B.3})$$

The solution is constructed integrating the Green function of the Laplacian, that is

$$u_{\mu}^{-}(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{F(x')}{|x - x'|} dx'. \quad (\text{B.4})$$

We use the expansion of the Green function in terms of spherical harmonics (see, for example, [23]). The angular dependence of the source F is given by $\sin^2 \theta$, which has an expansion in terms of the following two spherical harmonics:

$$Y_{00} = \frac{1}{\sqrt{4\pi}}, \quad Y_{20} = \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1), \quad (\text{B.5})$$

namely

$$\sin^2 \theta = \frac{2}{3} \sqrt{4\pi} \left(Y_{00} - \frac{Y_{20}}{\sqrt{5}} \right). \quad (\text{B.6})$$

Hence, it follows that the angular dependence of the solution can also be expanded in terms of these two spherical harmonics. That is, u_{μ}^{-} has the form (18) where the radial functions $\chi_1(r)$

and $\chi_2(r)$ are given by the following integrals:

$$\chi_1 = \int_0^r R(r', \mu) \frac{1}{r} r'^2 dr' + \int_r^\infty R(r', \mu) \frac{1}{r'} r'^2 dr', \quad (\text{B.7})$$

$$\chi_2 = \int_0^r R(r', \mu) \frac{r'^2}{r^3} dr' + \int_r^\infty R(r', \mu) \frac{r^2}{r'^3} dr'. \quad (\text{B.8})$$

Computing these integrals, we find

$$\begin{aligned} \chi_1 = \frac{2\sqrt{\pi}J^2}{5rq^6} & \left(-8(4\mu^2 + 3q^2)(2r + \mu) \right. \\ & + \frac{(4\mu^2 + 3q^2)(16r^4 + \mu^4) + 4mr(5q^2 + 8\mu^2)(4r^2 + \mu^2)}{(r^2 + mr + \frac{\mu^2}{4})^{3/2}} \\ & \left. + \frac{+6r^2(5q^4 + 16\mu^4 + 20q^2\mu^2)}{(r^2 + mr + \frac{\mu^2}{4})^{3/2}} \right), \end{aligned} \quad (\text{B.9})$$

and

$$\begin{aligned} \chi_2 = \frac{2\sqrt{\pi}J^2}{5r^3q^6} & \left(-8(2r + \mu)(16r^4 - 8r^3\mu + 4r^2\mu^2 - 2r\mu^3 + \mu^4) \right. \\ & + \frac{256r^8 + \mu^8 + 6rm(64r^6 + \mu^6) + 96r^6(q^2 + 2\mu^2)}{(r^2 + mr + \frac{\mu^2}{4})^{3/2}} \\ & \left. + \frac{4r^3m(2\mu^2 - q^2)(4r^2 + \mu^2) + 6r^2(2\mu^6 + \mu^4q^2 + r^2q^4)}{(r^2 + mr + \frac{\mu^2}{4})^{3/2}} \right). \end{aligned} \quad (\text{B.10})$$

From these expressions we see that

$$u_\mu^- = \frac{64J^2}{5r(2m + \mu)^3} + O(r^{-2}) \quad r \rightarrow \infty, \quad (\text{B.11})$$

and

$$u_\mu^-(r=0) = \frac{4J^2(2m - \mu)^3}{5\mu q^6}. \quad (\text{B.12})$$

When $\mu = 0$ the radial functions (B.9)–(B.10) reduce to

$$\chi_1|_{\mu=0} = \frac{4J^2\sqrt{\pi}}{5q^4\sqrt{r}} \left(\frac{24r^2 + 40qr + 15q^2}{(r+q)^{3/2}} - 24\sqrt{r} \right), \quad (\text{B.13})$$

$$\chi_2|_{\mu=0} = \frac{4J^2\sqrt{\pi}}{5q^6\sqrt{r}} \left(\frac{128r^4 + 192r^3q + 48r^2q^2 - 8q^3r + 3q^4}{(r+q)^{3/2}} - 128r^{5/2} \right). \quad (\text{B.14})$$

In this case the asymptotic behaviors are given by

$$u_0^- = \frac{8J^2}{5rq^3} + O(r^{-2}) \quad r \rightarrow \infty, \quad (\text{B.15})$$

and

$$u_0^- = \frac{9J^2(17 - \cos^2\theta)}{25q^{7/2}\sqrt{r}} + O(1) \quad r \rightarrow 0. \quad (\text{B.16})$$

Finally, we mention that it is possible to construct a positive lower bound which is spherically symmetric and has the correct behavior at the origin and at infinity. Namely, from (18) we deduce

$$u_\mu^- \geq Y_{00} \left(\chi_1 - \frac{1}{5}\chi_2 \right) \geq 0. \quad (\text{B.17})$$

References

- [1] Avila G A and Dain S 2008 The Yamabe invariant for axially symmetric two Kerr black holes initial data *Class. Quantum Grav.* **25** 225002 (arXiv:0805.2754)
- [2] Bartnik R 1986 The mass of an asymptotically flat manifold *Commun. Pure Appl. Math.* **39** 661–93
- [3] Bartnik R and Isenberg J 2004 The constraint equations *The Einstein Equations and Large Scale Behavior of Gravitational Fields* ed P T Chruściel and H Friedrich (Basle: Birkhäuser) pp 1–38 (arXiv:gr-qc/0405092)
- [4] Beig R and Murchadha N O 1994 Trapped surfaces in vacuum spacetimes *Class. Quantum Grav.* **11** 419–30
- [5] Bowen J M and York J W Jr 1980 Time-asymmetric initial data for black holes and black-hole collisions *Phys. Rev. D* **21** 2047–55
- [6] Brandt S and Brügmann B 1997 A simple construction of initial data for multiple black holes *Phys. Rev. Lett.* **78** 3606–9
- [7] Brügmann B, Gonzalez J A, Hannam M, Husa S and Spherhake U 2008 Exploring black hole superkicks *Phys. Rev. D* **77** 124047 (arXiv:0707.0135)
- [8] Campanelli M, Lousto C O, Zlochower Y and Merritt D 2007 Large merger recoils and spin flips from generic black-hole binaries *Astrophys. J.* **659** L5–8 (arXiv:gr-qc/0701164)
- [9] Cantor M 1979 A necessary and sufficient condition for York data to specify an asymptotically flat spacetime *J. Math. Phys.* **20** 1741–4
- [10] Carter B 1973 Black hole equilibrium states *Black Holes/Les Astres Occlus (École d'Été Phys. Théor., Les Houches, 1972)* (New York: Gordon and Breach) pp 57–214
- [11] Choquet-Bruhat Y, Isenberg J and York J W Jr 1999 Einstein constraint on asymptotically Euclidean manifolds *Phys. Rev. D* **61** 084034 (arXiv:gr-qc/9906095)
- [12] Cook G and York J W 1990 Apparent horizons for boosted or spinning black holes *Phys. Rev. D* **41** 1077–85
- [13] Cook G B 2001 Initial data for numerical relativity *Living Rev. Rel.* **3** (2000) 5 (electronic) <http://www.livingreviews.org/lrr-2000-5>
- [14] Dain S 2006 Angular momentum–mass inequality for axisymmetric black holes *Phys. Rev. Lett.* **96** 101101 (arXiv:gr-qc/0511101)
- [15] Dain S 2008 The inequality between mass and angular momentum for axially symmetric black holes *Int. J. Mod. Phys. D* **17** 519–23 (arXiv:0707.3118 [gr-qc])
- [16] Dain S 2008 Proof of the angular momentum–mass inequality for axisymmetric black holes *J. Diff. Geom.* **79** 33–67 (arXiv:gr-qc/0606105)
- [17] Dain S and Friedrich H 2001 Asymptotically flat initial data with prescribed regularity *Commun. Math. Phys.* **222** 569–609 (arXiv:gr-qc/0102047)
- [18] Dain S, Lousto C O and Takahashi R 2002 New conformally flat initial data for spinning black holes *Phys. Rev. D* **65** 104038 (arXiv:gr-qc/0201062)
- [19] Dain S, Lousto C O and Zlochower Y 2008 Extra-large remnant recoil velocities and spins from near-extremal-Bowen–York-spin black-hole binaries *Phys. Rev. D* **78** 024039 (arXiv:0803.0351)
- [20] Gilbarg D and Trudinger N S 2001 *Elliptic Partial Differential Equations of Second Order* (Berlin: Springer) (reprint of the 1998 edition)
- [21] González J A, Hannam M, Spherhake U, Brügmann B and Husa S 2007 Supermassive recoil velocities for binary black-hole mergers with antialigned spins *Phys. Rev. Lett.* **98** 231101
- [22] Herrmann F, Hinder I, Shoemaker D, Laguna P and Matzner R A 2007 Gravitational recoil from spinning binary black hole mergers *Astrophys. J.* **661** 430–6 (arXiv:gr-qc/0701143)
- [23] Jackson J D 1999 *Classical Electrodynamics* 3rd edn (New York: Wiley)
- [24] Koppitz M *et al* 2007 Getting a kick from equal-mass binary black hole mergers *Phys. Rev. Lett.* **99** 041102 (arXiv:gr-qc/0701163)
- [25] Lang S 1983 *Real Analysis* 2nd edn (Advanced Book Program) (Reading, MA: Addison-Wesley)
- [26] Lovelace G, Owen R, Pfeiffer H P and Chu T 2008 Binary-black-hole initial data with nearly-extremal spins *Phys. Rev. D* **78** 084017 (arXiv:0805.4192)
- [27] Maxwell D 2006 Rough solutions of the Einstein constraint equations *J. Reine Angew. Math.* **590** 1–29 (arXiv:gr-qc/040508)
- [28] Valiente Kroon J A 2004 Asymptotic expansions of the Cotton–York tensor on slices of stationary spacetimes *Class. Quantum Grav.* **21** 3237–50 (arXiv:gr-qc/0402033)