Are the spectra of geometrical operators in Loop Quantum Gravity really discrete?

Bianca Dittrich[1][*], Thomas Thiemann[1,2][†]

[1] Perimeter Institute for Theoretical Physics
31 Caroline Street North, Waterloo, ON N2L 2Y5, Canada
[2] MPI f. Gravitationsphysik, Albert-Einstein-Institut,
Am Mühlenberg 1, 14476 Potsdam, Germany

August 17, 2007

Abstract

One of the celebrated results of Loop Quantum Gravity (LQG) is the discreteness of the spectrum of geometrical operators such as length, area and volume operators. This is an indication that Planck scale geometry in LQG is discontinuous rather than smooth. However, there is no rigorous proof thereof at present, because the afore mentioned operators are not gauge invariant, they do not commute with the quantum constraints.

The relational formalism in the incarnation of Rovelli's partial and complete observables provides a possible mechanism for turning a non gauge invariant operator into a gauge invariant one.

In this paper we investigate whether the spectrum of such a physical, that is gauge invariant, observable can be predicted from the spectrum of the corresponding gauge variant observables. We will not do this in full LQG but rather consider much simpler examples where field theoretical complications are absent. We find, even in those simpler cases, that kinematical discreteness of the spectrum does not necessarily survive at the gauge invariant level. Whether or not this happens depends crucially on how the gauge invariant completion is performed.

This indicates that "fundamental discreteness at Planck scale in LQG" is far from established. To prove it, one must provide the detailed construction of gauge invariant versions of geometrical operators.

1 Introduction

Among the candidates for a theory of quantum gravity Loop Quantum Gravity (LQG) [1] has gained more and more popularity in recent years. One of the successes of LQG is the rigorous construction of (spatial) geometrical operators, such as the area, the volume and the length operator [2, 3] on the so called kinematical Hilbert space of LQG [4]. Moreover it turns out that these geometrical operators have a discrete spectrum. In particular there exists an area gap [2, 3] but apparently no volume gap [5].[1]

Another result is [6], which shows that the kinematical Hilbert space of LQG is the unique quantum representation for the holonomies and flux variables used in LQG satisfying certain

---

[*] bdittrich AT perimeterinstitute.ca

[†] Thomas.Thiemann AT aei.mpg.de

[1] There exist a volume gap on the subspace of the kinematical Hilbert space spanned by spin networks with vertex valence less or equal to four. However, on the full kinematical Hilbert space numerical evidence for an accumulation point at zero was found, see the last three references in [5].

covariance conditions with respect to the (spatial) diffeomorphism group. This shows that the discreteness of the spectra for the geometrical operators follows from a minimal set of requirements. Furthermore the volume operator is pivotal to obtain a uv–finite quantization of the (gravity and matter) Hamiltonian constraints of the theory [7].

In summary the geometrical operators play a very important role for the structure and further developement of LQG. However, so far the discreteness of their spectrum is a result at the kinematical level only. With 'kinematical level' we mean that the geometrical operators defined so far are gauge dependent, i.e. they are not invariant under spacetime diffeomorphisms. True observables have to be gauge independent, in the canonical formalism such observables are known as Dirac observables. Physical measurements are described by Dirac observables and not by kinematical observables.

The explicit construction of Dirac observables in general relativity is very difficult since it requires the solution of the dynamics of the theory. However methods to obtain Dirac observables (at least as classical phase space functions) are available [8, 9]. With these methods one can construct so–called complete observables [10], a specific type of relational observables. The main idea here is to use some set of fields T as "clocks and rods" and to express another field f (the "partial observable") in the coordinates defined by these "clocks and rods". Under a diffeomorphism both f and T transform in the same way, so that the relation between f and T does not change. More details and a proof that the resulting phase space functions are indeed Dirac observables can be found in the second reference in [8].

Indeed one way how to construct Dirac observables out of the geometrical operators mentioned above is to use matter fields in order to localize the region of which one wants to measure for instance the (spatial or space–time) volume. This is also suggested in [2]. The question would be if the Dirac observables constructed in this manner (assuming that such Dirac observables can be quantized as self–adjoint operators on a yet to be constructed physical Hilbert space) have a discrete spectrum or not. [2] argues for a discrete spectrum of the Dirac observables.

Contrary to this expectation we will show that in this note, without further assumptions, it is impossible to make any predictions about the spectra of the complete observables, even if the spectra of the partial observables and the clocks are known. We will do that by means of quite simple examples with finitely many degrees of freedom, where the physical Hilbert space can be explicitly constructed. In more complicated (field theory) examples, even the physical Hilbert space might not be unique and the spectra of observables could in principle depend on the choice of the Hilbert space.

The plan of the paper is as follows:

In the next section we will explain the method of partial and complete observables. This allows us to associate to a gauge variant function (the partial observable) a family of gauge invariant functions (the complete observable). Futhermore one has to choose a clock. We can then consider in which way the spectrum of the complete observables depends on the spectrum of the partial observable, the clock and the constraint.

In section 3 we will show with the help of concrete examples that *every possible combination of continuous and discrete spectra for the constraint, the partial observable, the clock and the complete observable can be realized.*

We will end with a discussion of the results and their meaning for LQG. In particular we will coment on earlier results within the LQG literature on the physical spectrum of geometrical operators in the context of 2+1 Lorentzian gravity [11, 12] and 3+1 Lorentz covariant gravity [13].

In an appendix we consider an additional toy model which is geared to constructing a gauge invariant observable from a baby version of a volume operator by means of a baby version of a scalar field as discussed in [2]. We find that in contrast to the arguments spelled out there, the

kinematically discrete spectrum switches to a gauge invariant continuous one.

2 Partial and Complete Observables

Here we will explain the method of partial and complete observables. The examples we are going to discuss exhibit only one constraint, i.e. only one gauge degree of freedom. We will comment on the case with several constraints (general relativity has infinitely many constraints) lateron.

Let us start with an example, namely the relativistic particle in two space–time dimensions. Here we have the mass shell constraint

$$C = p_0^2 - p_1^2 - m^2 \tag{2.1}$$

where m is the mass and (p_0, p_1) the momentum of the particle. Now we have to choose a clock variable T and another partial observable f. Our choice is $T = x_0$ and $f = x_0 p_1 - x_1 p_0$ where (x_0, x_1) is the position vector of the particle. The complete observable $F(\tau)$, where τ is a phase space independent parameter, is defined to be the gauge invariant phase space function with the following property: On the hypersurface $T = \tau$ it should coincide with the phase space function f. Gauge invariance means here that $F(\tau)$ has to be invariant under the flow of the mass shell constraint (2.1), i.e. that $\{F(\tau), C\} = 0$, where the Poisson brackets are defined by $\{x_i, p_j\} = \delta_{ij}$.

One way to find the complete observable is to compute the flow of the clock variable T under the constraint C, that is

$$T(t) := \sum_{k=0}^{\infty} \frac{t^k}{k!} \{T, C\}_k \tag{2.2}$$

where $\{g, C\}_{k+1} = \{\{g, C\}_k, C\}$, $\{g, C\}_0 = g$ are iterated Poisson brackets. Next one has to solve the equation $T(t) \stackrel{!}{=} \tau$ for t. The solution $t_s = t_s(\tau)$ is in general also phase space dependent. For the relativistic particle we have

$$T(t) = x_0 + 2t p_0 \stackrel{!}{=} \tau , \tag{2.3}$$

hence $t_s = \frac{1}{2} p_0^{-1} (\tau - x_0)$.

Now we have to insert the solution t_s into the flow of f under the constraint C. This gives the complete observable $F(\tau)$:

$$F(\tau) = f(t_s(\tau)) = \sum_{k=0}^{\infty} \frac{(t_s(\tau))^k}{k!} \{f, C\}_k \quad . \tag{2.4}$$

Since for the relativistic particle $f(t) = (x_0 + 2t p_0) p_1 - (x_1 - 2t p_1) p_0$ we find

$$F(\tau) = \tau p_1 - (x_1 p_0 - (\tau - x_0) p_1) = 2\tau p_1 - (x_0 p_1 + x_1 p_0) \quad . \tag{2.5}$$

Indeed (2.5) coincides with f on the hypersurface $T = x^0 = \tau$ and is invariant under the flow of C.

This construction can be generalized to systems with arbitrary many (first class) constraints by introducing as many clock variables T_i as there are constraints C_i, where i is in some index set I. The complete observable $F(\tau_i)$ associated to these clock variables and a partial observable f is defined to be the gauge invariant function which coincides on the hypersurface $\{T_i = \tau_i \, \forall i \in I\}$ with the function f. For methods to compute complete observables see [8].

We want to remark that the complete observable might not always be well defined. For instance the surface $T = \tau$ might meet a gauge orbit several times which leads to multiple

solutions of the equation $T(t) \overset{!}{=} \tau$ and hence might lead to multiple valued complete observables as is discussed in the first reference in [8]. Some examples in the next section will have multiple solutions t_s of the equation $T(t) \overset{!}{=} \tau$ but in these the partial observable f is chosen such that $f(t)$ coincides on all solutions t_s and hence we will always get a well defined complete observable.

On the other hand it might happen that for a fixed parameter τ the surface $T = \tau$ does not meet a certain set of gauge orbits at all. Here we will assume that the parameter τ is always chosen such that one can define the complete observable at least on an open set of the phase space. In our examples this complete observable can always be continued analytically to a gauge invariant function defined on the whole phase space. [2]

Furthermore often we will give the complete observables modulo terms that vanish on the constraint hypersurface, since these terms are not relevant for the quantization of the observables on the physical Hilbert space.

With the relativistic particle we already have an example where the spectrum of the quatization of the partial observable $f = x_0 p_1 - x_1 p_0$ is discrete (since it generates rotations in the (x_0, x_1)–plane) and the spectrum of the quantization of the corresponding complete observable $F(\tau = 0) = -x_0 p_1 - x_1 p_0$ is continuous (since it generates boosts).

Such a change of the spectral properties can be understood in the following way:

Heuristically the quantisation $\hat{f}$ of a (smooth) phase space function f has a discrete spectrum or a continuous spectrum if the vector field χ_f associated to the phase space function f generates compact orbits or non–compact orbits respectively.

The vector field χ_f associated to f is defined by the equation

$$\{g, f\} = \chi_f(g) \tag{2.6}$$

which has to hold for every smooth function g. Here the Poisson brackets $\{\cdot, \cdot\}$ are defined via the symplectic form ω defined on the phase space in question.

Assume that the equation $T = \tau$ defines a good gauge fixing, i.e. that the surface $T = \tau$ meets each gauge orbit once and only once. As it is explained in [8, 9] in this case the following holds: The space of all complete observables with this choice of clock variable T can be mapped via a symplectomorphism onto the reduced phase space obtained with the gauge fixing $T = \tau$ and equipped with the Dirac bracket $\{\cdot, \cdot\}^D$. This map is given by mapping the complete observable associated to f to the gauge restriction of f, i.e. the restriction of the function f to the reduced phase space $\{C = 0, T = \tau\}$.

Hence considering complete observables and the symplectic flows they generate is equivalent to considering the reduced phase space $\{C = 0, T = \tau\}$ with its induced symplectic form, which is given by the Dirac bracket. That is a complete observables associated to a function f generates compact orbits if and only if this is the case for the symplectic flow of the gauge restriction of f defined via the Dirac bracket. This symplectic flow preserves in particular the reduced phase space $\{C = 0, T = \tau\}$. In fact the vector fields χ_f^D associated to the gauge restriction of f via the Dirac bracket can be understood as the vector fields χ_f projected to the hypersurface $\{C = 0, T = \tau\}$.

Now whether the orbits generated by the vector field χ_f are compact or not is a global property of the vector field and it might change under the projection onto the reduced phase space. Also the global properties of the reduced phase space and how the reduced phase space is embedded into the full phase space is important. This explains the different spectra for the partial observable f and the complete observable $F(\tau)$. Moreover this argument shows that it is very difficult to predict which kind of spectrum the complete observables will have, since it requires control of the global features of the reduced phase space.

---

[2]This convention is neccessary for instance for the examples with the rotation generator as constraint. Note that with this convention it may happen that the complete observable $F(\tau)$ assumes negative values whereas the partial observable f is non–negative.

In the next section we will give examples for all possible combinations of discrete (i.e. pure point) or continuous (i.e. absolutely continous) spectra for the (quantized) constraint, the clock variable T, the partial observable f and the complete observable $F(\tau)$.

3 The examples

In our examples we will use the phase space $\mathbb{R}^2 \times \mathbb{R}^2$ with canonical coordinates (x_1, p_1, x_2, p_2) and Poisson brackets $\{x_i, p_j\} = \delta_{ij}$ (where δ_{ij} is the Kronecker symbol).

We choose the space of square integrable functions $L_2(\mathbb{R}^2)$ as the kinematical Hilbert space. The quantized configuration coordinate functions $\hat{x}_j$ act as multiplication operators $\hat{x}_j = x_j$ and the quantized momenta as derivative operators $\hat{p}_j = -i\hbar\partial_j$.

We will start our considerations with the constraint $\hat{C}_d = \hat{x}_1\hat{p}_2 - \hat{x}_2\hat{p}_1$, that is the generator of rotations on $\mathbb{R}^2$. The constraint operator has a discrete spectrum $\{\hbar z, z \in \mathbb{Z}\}$ (indicated by the index d). We will assume that the partial and complete observables have either an entirely discrete (i.e. pure point) or entirely continuous (i.e. absolutely continuous) spectrum and that the spectral type of the complete observable does not depend on the particular value of the parameter τ. Then there exist eight combinations for the spectra of the clock variable $\hat{T}$ the partial observable $\hat{f}$ and the complete observable $\hat{F}(\tau)$. We will indicate these combinations by (d,i,j,k) with i,j,k = \{c,d\} for the spectrum i of the clock variable $\hat{T}$, the spectrum j of the partial observable $\hat{f}$ and the spectrum k of the complete observable $\hat{F}(\tau)$.

The construction of the physical Hilbert space for the constraint $\hat{C}_d$ is straightforward.

The constraint has a pure point spectrum and moreover the point zero is included in this spectrum so that we have solutions ψ_{phys} to the constraint equation $\hat{C}_d\psi_{phys} = 0$ that are elements of the kinematical Hilbert space. A physical inner product is given by the restriction of the kinematical inner product to these solutions.

The gauge invariant functions we will encounter in our examples of complete observables are given by $r^2 := x_1^2 + x_2^2$ and $h := x_1^2 + x_2^2 + p_1^2 + p_2^2$. The corresponding quantizations have continuous spectrum and discrete spectrum respectively. Details on the physical Hilbert space and the algebra of Dirac observables can be found in the appendix A.

We will begin the discussion of the examples with the case (d,d,d,d). Here we choose as clock variable and as partial observable

$$T = x_1^2 + p_1^2 \quad , \qquad f = x_2^2 + p_2^2 \quad , \tag{3.1}$$

that is the harmonic oscillators in the x_1- and x_2-coordinates respectively. The corresponding quantum operators have obviously discrete spectrum.

To find the complete observable we have first to solve the equation

$$T(t) := \sum_{k=0}^{\infty} \frac{t^k}{k!}\{T, C_d\}_k \overset{!}{=} \tau \tag{3.2}$$

for t and then to insert the solutions of this equation into $f(t)$. However there is a shorter way: Note that

$$
\begin{aligned}
T(t) &= x_1^2\cos^2(t) + x_2^2\sin^2(t) - 2x_1x_2\cos(t)\sin(t) + \\
&\quad p_1^2\cos^2(t) + p_2^2\sin^2(t) - 2p_1p_2\cos(t)\sin(t) \quad .
\end{aligned} \tag{3.3}
$$

Hence we have for solutions t_s of the equation $T(t) = \tau$

$$
\begin{aligned}
2x_1x_2\cos(t_s)\sin(t_s) + 2p_1p_2\cos(t_s)\sin(t_s) &= x_1^2\cos^2(t_s) + x_2^2\sin^2(t_s) + \\
&\quad p_1^2\cos^2(t_s) + p_2^2\sin^2(t_s) - \tau \quad .
\end{aligned} \tag{3.4}
$$

5

We use this equation in

$$
\begin{aligned}
F(\tau) = f(t_s) \;=\;& 2x_1 x_2 \cos(t_s)\sin(t_s) + 2p_1 p_2 \cos(t_s)\sin(t_s) + \\
& x_2^2 \cos^2(t_s) + x_1^2 \sin^2(t_s) + p_2^2 \cos^2(t_s) + p_1 \sin^2(t_s)
\end{aligned}
\tag{3.5}
$$

and find

$$
F(\tau) = x_1^2 + x_2^2 + p_1^2 + p_2^2 - \tau
\tag{3.6}
$$

that is the sum of two harmonic oscillator Hamiltonians minus a constant τ. The corresponding quantum operator has a discrete spectrum.

Hence we found an example for the case (d,d,d,d). Examples for all the other cases (d,i,j,k) can be found in a similar manner. The examples are summarized in table (3.7).

Case	T	f	$F(\tau)$
(d,d,d,d)	$x_1^2 + p_1^2$	$x_2^2 + p_2^2$	$x_1^2 + p_1^2 + x_2^2 + p_2^2 - \tau$
(d,c,d,d)	$(1-\omega^2)x_1^2 + (1-\omega'^2)x_2^2$	$p_1^2 + \omega^2 x_1^2 + p_2^2 + \omega'^2 x_2^2$	$x_1^2 + p_1^2 + x_2^2 + p_2^2 - \tau$
(d,d,c,d)	$p_1^2 + \omega^2 x_1^2 + p_2^2 + \omega'^2 x_2^2$	$(1-\omega^2)x_1^2 + (1-\omega'^2)x_2^2$	$x_1^2 + p_1^2 + x_2^2 + p_2^2 - \tau$
(d,c,c,d)	$x_1^2 + p_2^2$	$x_2^2 + p_1^2$	$x_1^2 + p_1^2 + x_2^2 + p_2^2 - \tau$
(d,d,d,c)	$(p_1^2 + \omega^2 x_1^2) - (p_2^2 + \omega'^2 x_2^2)$	$-(p_1^2 + \omega'^2 x_1^2) + (p_2^2 + \omega^2 x_2^2)$	$(\omega^2 - \omega'^2)(x_1^2 + x_2^2) - \tau$
(d,d,c,c)	$p_1^2 + x_1^2$	$p_1^2 - x_2^2$	$-x_1^2 - x_2^2 + \tau$
(d,c,d,c)	$p_1^2 - x_2^2$	$p_1^2 + x_1^2$	$x_1^2 + x_2^2 + \tau$
(d,c,c,c)	x_1^2	x_2^2	$x_1^2 + x_2^2 - \tau$

$$\tag{3.7}$$

We will now turn to a constraint with a continuous spectrum. We use the same phase space and the same kinematical Hilbert space as before, namely $\mathbb{R}^2 \times \mathbb{R}^2$ and $L_2(\mathbb{R}^2)$ respectively. We will work with the momentum constraint $C_c = p_1$, whose quantization $\hat{C}_c = -i\hbar\partial_1$ has a continuous spectrum.

The Dirac observable algebra for this constraint is spanned by x_2 and p_2, i.e. it is given by phase space functions which do not depend on x_1 or p_1. Physical wave functions are wave functions which do not depend on x_1. A physical inner product can be defined by omitting the integration over x_2 in the kinematical inner product. Hence we are left with a physical Hilbert space $L_2(\mathbb{R}, dx_2)$ which carries the usual representation of the basic Dirac observables $\hat{x}_2$ and $\hat{p}_2$.

The calculation of the complete observables are straightforward. In the following table (3.8) we have listed the examples for all possible cases (c,i,j,k) where again i,j,k can take values d or c and indicate a discrete or continuous spectrum respectively of the corresponding quantum observable.

Case	T	f	$F(\tau)$
(c,d,d,d)	$x_1^2 + p_1^2$	$x_1^2 + p_1^2 + x_2^2 + p_2^2$	$x_2^2 + p_2^2 + \tau$
(c,c,d,d)	x_1^2	$x_1^2 + p_1^2 + x_2^2 + p_2^2$	$x_2^2 + p_2^2 + \tau$
(c,d,c,d)	$x_1^2 + p_1^2 + x_2^2 + p_2^2$	x_1^2	$-x_2^2 - p_2^2 + \tau$
(c,c,c,d)	$x_1 + p_2^2$	$x_1 - x_2^2$	$-x_2^2 - p_2^2 + \tau$
(c,d,d,c)	$x_1^2 + p_1^2$	$(x_1 p_2 - x_2 p_1)^2$	τp_2^2
(c,d,c,c)	$x_1^2 + p_1^2 + x_2^2 + p_2^2$	$x_1^2 + x_2^2$	$-p_2^2 + \tau$
(c,c,d,c)	$x_1^2 + x_2^2$	$x_1^2 + p_1^2 + x_2^2 + p_2^2$	$p_2^2 + \tau$
(c,c,c,c)	x_1	$x_1 + x_2$	$x_2 + \tau$

$$(3.8)$$

Consider in particular the cases (c,c,d,d) and (c,c,d,c) which can be generalized to

$$
\begin{aligned}
T &= x_1^2 + \gamma x_2^2 \\
f &= x_1^2 + p_1^2 + x_2^2 + p_2^2 \\
F(\tau) &\simeq (1 - \gamma)x_2^2 + p_2^2 + \tau
\end{aligned}
$$

$$(3.9)$$

with $0 \leq \gamma < 1$ or $\gamma \geq 1$ for a discrete spectrum of the complete observable or a continuous spectrum of the complete observable respectively. The symbol $\simeq$ indicates that the last equation in (3.9) holds modulo terms which vanish on the constraint hypersurface. Hence we have an example where just changing the clock variable T leads to the change of the spectrum of the complete observable.

Examples with more than one constraint can be easily constructed by taking "a direct sum" of the above examples for one constraint: For instance consider the constraints $C_1 = p_1$ and $C_2 = p_3$ on a phase space $\mathbb{R}^4 \times \mathbb{R}^4$. Choose as partial observables

$$
\begin{aligned}
T_1 &= x_1^2 \\
T_2 &= x_3^2 \\
f &= x_1^2 + p_1^2 + x_2^2 + p_2^2 + x_3^2 + p_3^2 + x_4^2 + p_4^2 \quad .
\end{aligned}
$$

$$(3.10)$$

The corresponding complete observable is $F(\tau_1, \tau_2) \simeq x_2^2 + p_2^2 + x_4^2 + p_4^2 + \tau_1 + \tau_2$. Hence we have an example with two constraints and two clock variables with continuous spectrum and a partial observable f and a complete observable $F(\tau_1, \tau_2)$ with discrete spectrum.

4 Discussion

The examples show that it is very difficult to make any predictions about the spectra of physical observables even if the spectra of gauge variant observables are known to which the physical observables are associated in a certain sense (e.g. here through the method of partial and complete observables). The examples considered here are very simple and do not include further complications one would expect for the geometrical operators in LQG. For instance one might consider space–time geometrical operators rather than spatial ones.

Among earlier work on these issues we select the following three: In [11] the authors consider 2+1 Lorentzian gravity without matter and construct a length operator within the Hamiltonian theory. Since the authors do not require that the spatial slices be spacelike, the length operator measures the length of timelike or spacelike curves. The spacelike spectrum is continuous while the timelike one is discrete corresponding to the coexistence of continuous and discrete spectrum of the Casimir for SO(1,2). However, the spectrum is at the kinematical level only and it is

unknown what happens when one implements the dynamics. Of more relevance to our paper is [12] which concerns Riemannian 2+1 gravity with point particles. Now the length operator can be made into a Dirac observable and the spectrum is discrete both for the kinematical and physical length observable! However, it is not clear whether a similar construction will also work in 3+1 dimensions, and the analysis is done for the Riemannian case only. Finally there is a body of literture called "Lorentz covariant gravity" in 3+1 dimensions, see for instance [13], where one keeps the connection a Lorentz connection at the price of the connections not to Poisson commute. Hence one does not have any (connection) representation and thus rigorous spectral theory cannot be performed.

Our heuristic argument of how one could understand the change from discrete to continuous spectra and vice versa shows that the global features of the kinematical and reduced phase space play an important role. However it is very hard to get complete control over the reduced phase space of general relativity. It might be therefore complicated to give any conditions that would ensure the discreteness of the spectra for the complete observables.

As a more general lesson these examples show that physical observables might have very unexpected properties. Here we considered the spectra of physical observables. Other properties are for instance commutation relations of physical observables, see for instance the third reference in [8]. Moreover there are indications that considering physical observables might lead to a notion of non–commutative space–time [14].

Although we cannot say yet whether the true physical geometrical operators of LQG will have discrete spectra or not without spelling out additional details, we want to emphasize that their kinematical versions are still important. For instance, the kinematical volume operator enters the definition of the Hamiltonian constraint and surely many other Dirac observables which are gauge invariant aggregates built out of gauge variant operators.

A   Example: Physical Hilbert space

Here we will shortly discuss the representation of the Dirac observable algebra on the physical Hilbert space for the constraint $\hat{C}_d = \hat{x}_1\hat{p}_2 - \hat{x}_2\hat{p}_1$. The Dirac observable algebra is spanned by

$$\hat{d} = \tfrac{1}{2}(\hat{x}_i\hat{p}_i + \hat{p}_i\hat{x}_i) = \hat{x}_i\hat{p}_i - i\hbar \qquad \hat{e}^+ = \hat{x}_i\hat{x}_i \qquad \hat{e}^- = \hat{p}_i\hat{p}_i \qquad (A.1)$$

where we sum over repeated indices. With this operator ordering for $\hat{d}$ the observables (A.1) represent the $sl(2,\mathbb{R})$-algebra:

$$[\hat{d}, \hat{e}^{\pm}] = \mp 2i\hbar e^{\pm} \qquad\qquad [\hat{e}^+, \hat{e}^-] = 4i\hbar\hat{d} \quad . \qquad (A.2)$$

The constraint has a pure point spectrum and moreover the point zero is included in this spectrum so that we have solutions ψ_{phys} to the constraint equation $\hat{C}_d\psi_{phys} = 0$ that are elements of the kinematical Hilbert space. A physical inner product is given by the restriction of the kinematical inner product to these solutions. Explicitly we can change to polar coordinates $r \in \mathbb{R}_+, \theta \in [0, 2\pi)$:

$$x_1 = r\cos\theta \quad x_2 = r\sin\theta \qquad (A.3)$$

so that the constraint operator $\hat{C}_d$ and the Dirac observable algebra (A.1) become

$$\begin{aligned}
\hat{C}_d &= -i\hbar(x_1\partial_2 - x_2\partial_1) = -i\hbar\partial_\theta \\
\hat{d} &= -i\hbar(x_1\partial_1 + x_2\partial_2) = -i\hbar(r\partial_r + 1) \\
\hat{e}^+ &= (x_1^2 + x_2^2) = r^2 \\
\hat{e}^- &= -\hbar^2(\partial_1^2 + \partial^2) = -\hbar^2(\partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2)
\end{aligned} \qquad (A.4)$$

on the kinematical Hilbert space $L_2(rdr\,d\theta, \mathbb{R}_+ \times [0, 2\pi))$ .The physical states are the states which do not depend on the angular coordinate θ and the physical inner product can be written as

$$(\psi_{phys}, \phi_{phys})_{phys} = \int_{\mathbb{R}_+} \overline{\psi_{phys}(r)}\phi_{phys}(r)\, rdr \quad . \tag{A.5}$$

The representation (A.4) of the Dirac observable algebra (A.1) is unitary and irreducible. It is equivalent to the representation $D_{1/2}^+$ (see for instance [15]) fom the discrete series of representations of $sl(2, \mathbb{R})$. In this representation the operators $\hat{e}^+$ and $\hat{e}^+ + \hat{e}^-$ (which is the two–dimensional harmonic oscillator restricted to states with vanishing angular momentum) have absolutely continuous and pure point spectrum respectively. Theses Dirac observables appear as complete observables in our examples.

B   Baby Version of LQG Geometrical Operators

The examples considered so far could be taken as irrelevant for the situation in LQG for two reasons: First, the configuration space of the models was non compact while for the geometry part of LQG the configuration space is compact. Secondly, in all the examples considered it so happened that f, T did not Poisson commute whenever the spectrum changed. However, the idea of [2] was to use for T scalar matter while f is a geometry function, hence f, T do Poisson commute there.

In [16] we find an example based on $sl(2, R)$ where a continuous kinematical spectrum switches to a discrete gauge invariant one. However, there the configuration space is still non compact and furthermore the switch happens in the wrong direction. Thus we now explicitly display an example where both of the above mentioned properties that are missing in our models are satisfied and still the switch occurs.

Consider the phase space $T^*(S^1 \times R)$ of a particle moving on a cylinder. Denote by A the angle configuration variable ("connection") and by E its conjugate momentum ("flux"). Recall that all geometrical operators of LQG are compound operators built from fluxes and they inherit their spectral discreteness from that of the flux operators. Denote by Φ the axis configuration variable ("scalar field") and by Π its conjugate momentum. Of course, A is not a globally defined function and we must switch to the "holonomy" $h = \exp(iA)$.

We impose as constraint that the particle must spiral around the cylinder with period 2π, that is,

$$C = A - (\Phi - 2\pi[\frac{\Phi}{2\pi}]) \tag{B.1}$$

where [.] denotes the Gauss bracket. In the form (B.1) the constraint is not differentiable, hence we pass to the equivalent version

$$C = \exp(i[A - \Phi]) - 1 = 0 \quad . \tag{B.2}$$

In this form the constraint is not real valued but that will not pose any problems (alternatively work with $C = \cos(A - \Phi) - 1$).

As clock we choose $T = \Pi$ and as partial observable $f = E$. A straightforward calculation reveals that the corresponding complete observable is given by

$$F = E + \Pi - \tau \tag{B.3}$$

which is essentially the total momentum of the particle.

The kinemtical Hilbert space is given by $\mathcal{H}_{kin} = L_2(S^1, dA/2\pi) \otimes L_2(R, d\Phi)$. Operators are represented as $\hat{\Pi} = id/d\Phi$, $\hat{E} = id/dA = -h \; d/dh$ while Φ, A (or rather h) become multiplication operators. Clearly, $\hat{\Pi}, \hat{E}, \hat{\Phi}$ are self adjoint while $\hat{h}$ is unitary. The spectrum of $\hat{E}$ is discrete and takes integer values with eigenfunctions given by $T_n = h^n$ ("spin network functions").

To compute the physical Hilbert space we define the rigging map[3]

$$\eta(\psi) := \delta(\hat{C})\psi \quad .$$ (B.4)

The δ–distribution is given by

$$\delta(\hat{C}) = \delta_R(A - (\Phi - 2\pi[\frac{\Phi}{2\pi}])) = \sum_n \delta_R(\Phi, A + 2\pi n) = \sum_n \exp(in[A - \Phi]) = \delta_{S^1}(A - \Phi) \text{ (B.5)}$$

where in the last step we have applied a Poisson resummation. The physical inner product on the solutions $\eta(\psi)$ is given by

$$< \eta(\psi), \eta(\psi') >_{phys} = < \psi, \delta(\hat{C})\psi' >_{kin} = < \tilde{\psi}, \tilde{\psi}' >_{L_2(R, d\Phi)}$$ (B.6)

where $\tilde{\psi}(\Phi) = \psi(A = \Phi, \Phi)$ which is well defined because ψ is periodic in A.

One can also get this by making the Fourier expansion

$$\psi(A, \Phi) = \sum_n T_n(A) \; \psi_n(\Phi)$$ (B.7)

and solve $\exp(i[\hat{A} - \hat{\Phi}])\psi = \psi$ for the coefficients ψ_n.

It is physically completely obvious that $\hat{F}$ has an absolutely continuous spectrum. To see this explicitly, suppose that $\eta(\psi)$ is an eigenvector of $\hat{F}$, that is,

$$\hat{F}\eta(\psi) = \eta(\hat{F}\psi) = \lambda\eta(\psi) \quad .$$ (B.8)

Since we may assume without loss of generality that ψ depends trivially on A, this is equivalent to the equation

$$\hat{\Pi}\psi = [\lambda + \tau]\psi \;\; \Rightarrow \;\; \psi(\Phi) = c \; \exp(-i[\lambda + \tau]\Phi)$$ (B.9)

for some constant c. However, in order to be normalisable in the physical inner product it follows $c = 0$. Thus $\hat{F}$ has no pure point spectrum and since the physical Hilbert space is equivalent to $L_2(R, dx)$ on which the observable $\hat{F}$ is essentially represented as id/dx it follows that $\hat{F}$ has absolutely continuous spectrum as claimed. This example illustrates nicely the heuristic argument given in section 2: The flow associated to the partial observable $f = E$ is compact on the kinematical phase space, since it integrates to circles. Under the projection to the (gauge fixed and) constraint hypersurface the flow is mapped to non–compact spirals, leading to a continuous spectrum of the associated Dirac observable.

Acknowledgments

---

[3]The rigging map maps kinematical wave functions to elements of the physical Hilbert space and is used to define the physical inner product. For more details see for instance the last reference of [1].

References

[1] A. Ashtekar, "An Introduction to Loop Quantum Gravity Through Cosmology," arXiv:gr-qc/0702030.

A. Ashtekar and J. Lewandowski, "Background independent quantum gravity: A status report," Class. Quant. Grav. **21** (2004) R53 [arXiv:gr-qc/0404018].

A. Perez, "Introduction to loop quantum gravity and spin foams," arXiv:gr-qc/0409061.

C. Rovelli, "Quantum Gravity", Cambridge University Press, (Cambridge 2004)

L. Smolin, "How far are we from the quantum theory of gravity?," arXiv:hep-th/0303185.

L. Smolin, "An invitation to loop quantum gravity," arXiv:hep-th/0408048.

T. Thiemann, "Loop quantum gravity: An inside view," arXiv:hep-th/0608210.

T. Thiemann, "Lectures on loop quantum gravity," Lect. Notes Phys. **631** (2003) 41 [arXiv:gr-qc/0210094].

T. Thiemann, "Introduction to modern canonical quantum general relativity,", Cambridge University Press, (Cambridge 2007) [arXiv:gr-qc/0110034].

[2] C. Rovelli and L. Smolin, "Discreteness of area and volume in quantum gravity," Nucl. Phys. B **442** (1995) 593 [Erratum-ibid. B **456** (1995) 753] [arXiv:gr-qc/9411005].

[3] A. Ashtekar and J. Lewandowski, "Quantum theory of geometry. I: Area operators," Class. Quant. Grav. **14** (1997) A55 [arXiv:gr-qc/9602046].

A. Ashtekar and J. Lewandowski, "Quantum theory of geometry. II: Volume operators," Adv. Theor. Math. Phys. **1** (1998) 388 [arXiv:gr-qc/9711031].

A. Ashtekar, A. Corichi and J. A. Zapata, "Quantum theory of geometry. III: Non-commutativity of Riemannian structures," Class. Quant. Grav. **15** (1998) 2955 [arXiv:gr-qc/9806041].

T. Thiemann, "A length operator for canonical quantum gravity," J. Math. Phys. **39** (1998) 3372 [arXiv:gr-qc/9606092].

[4] A. Ashtekar and C. J. Isham, "Representations of the holonomy algebras of gravity and non-Abelian gauge theories," Class. Quant. Grav. **9** (1992) 1433 [arXiv:hep-th/9202053].

A. Ashtekar and J. Lewandowski, "Projective Techniques And Functional Integration For Gauge Theories," J. Math. Phys. **36** (1995) 2170 [arXiv:gr-qc/9411046].

[5] T. Thiemann, "Closed formula for the matrix elements of the volume operator in canonical quantum gravity," J. Math. Phys. **39** (1998) 3347 [arXiv:gr-qc/9606091].

R. Loll, "Simplifying the spectral analysis of the volume operator," Nucl. Phys. B **500** (1997) 405 [arXiv:gr-qc/9706038].

J. Brunnemann and T. Thiemann, "Simplification of the spectral analysis of the volume operator in loop quantum gravity," Class. Quant. Grav. **23** (2006) 1289 [arXiv:gr-qc/0405060].

J. Brunnemann and D. Rideout, "Spectral analysis of the volume operator in loop quantum gravity," arXiv:gr-qc/0612147.

J. Brunnemann and D. Rideout, "Properties of the Volume Operator in Loop Quantum Gravity I: Results," arXiv:0706.0469 [gr-qc].

J. Brunnemann and D. Rideout, "Properties of the Volume Operator in Loop Quantum Gravity II: Detailed Presentation," arXiv:0706.0382 [gr-qc].

[6] H. Sahlmann, "Some comments on the representation theory of the algebra underlying loop quantum gravity," arXiv:gr-qc/0207111.

H. Sahlmann, "When do measures on the space of connections support the triad operators of loop quantum gravity?," arXiv:gr-qc/0207112.

H. Sahlmann and T. Thiemann, "On the superselection theory of the Weyl algebra for diffeomorphism invariant quantum gauge theories," arXiv:gr-qc/0302090.

H. Sahlmann and T. Thiemann, "Irreducibility of the Ashtekar-Isham-Lewandowski representation," Class. Quant. Grav. **23** (2006) 4453 [arXiv:gr-qc/0303074].

C. Fleischhack, "Representations of the Weyl Algebra in Quantum Geometry," [arXiv:math-ph/0407006v2]

J. Lewandowski, A. Okolow, H. Sahlmann and T. Thiemann, "Uniqueness of diffeomorphism invariant states on holonomy-flux algebras," arXiv:gr-qc/0504147.

[7] T. Thiemann, "Anomaly-free formulation of non-perturbative, four-dimensional Lorentzian quantum gravity," Phys. Lett. B **380** (1996) 257 [arXiv:gr-qc/9606088].

T. Thiemann, "Quantum spin dynamics (QSD)," Class. Quant. Grav. **15** (1998) 839 [arXiv:gr-qc/9606089].

T. Thiemann, "Quantum spin dynamics (QSD) II," Class. Quant. Grav. **15** (1998) 875 [arXiv:gr-qc/9606090].

T. Thiemann, "QSD III: Quantum constraint algebra and physical scalar product in quantum general relativity," Class. Quant. Grav. **15** (1998) 1207 [arXiv:gr-qc/9705017].

T. Thiemann, "QSD IV: 2+1 Euclidean quantum gravity as a model to test 3+1 Lorentzian quantum gravity," Class. Quant. Grav. **15** (1998) 1249 [arXiv:gr-qc/9705018].

T. Thiemann, "QSD V: Quantum gravity as the natural regulator of matter quantum field theories," Class. Quant. Grav. **15** (1998) 1281 [arXiv:gr-qc/9705019].

T. Thiemann, "QSD VI: Quantum Poincare algebra and a quantum positivity of energy theorem for canonical quantum gravity," Class. Quant. Grav. **15** (1998) 1463 [arXiv:gr-qc/9705020].

[8] B. Dittrich, "Partial and complete observables for Hamiltonian constrained systems," to appear in Gen. Rel. Grav. [arXiv:gr-qc/0411013].

B. Dittrich, "Partial and Complete Observables for Canonical General Relativity," Class. Quant. Grav. **23** (2006) 6155 [arXiv:gr-qc/0507106].

B. Dittrich and J. Tambornino, "A perturbative approach to Dirac observables and their space-time algebra," Class. Quant. Grav. **24** (2007) 757 [arXiv:gr-qc/0610060].

B. Dittrich and J. Tambornino, "Gauge invariant perturbations around symmetry reduced sectors of general relativity: Applications to cosmology," to appear in Class. Quant. Grav. [arXiv:gr-qc/0702093].

[9] T. Thiemann, "Reduced phase space quantization and Dirac observables," Class. Quant. Grav. **23** (2006) 1163 [arXiv:gr-qc/0411031].

[10] C. Rovelli, "Quantum Mechanics without Time: A Model," Phys. Rev. D **42** (1990) 2638.

C. Rovelli, "Time in Quantum Mechanics: Physics beyond the Schroedinger Regime," Phys. Rev. D **43** (1991) 442.

C. Rovelli, "Is There Incompatibility Between the Ways Time is Treated in General Relativity and in Standard Quantum Mechanics", in *Conceptional Problems in Quantum Gravity*, ed. by A. Ashtekar and J. Stachel (Birkhäuser, Boston, 1991), p. 126

C. Rovelli, "What is observable in classical and quantum gravity?," Class. Quant. Grav. **8** (1991) 297.

C. Rovelli, "Partial observables," Phys. Rev. D **65** (2002) 124013 [arXiv:gr-qc/0110035].

[11] L. Freidel, E. R. Livine and C. Rovelli, "Spectra of length and area in (2+1) Lorentzian loop quantum gravity," Class. Quant. Grav. **20** (2003) 1463 [arXiv:gr-qc/0212077].

[12] K. Noui and A. Perez, "Three dimensional loop quantum gravity: Coupling to point particles," Class. Quant. Grav. **22** (2005) 4489 [arXiv:gr-qc/0402111].

[13] S. Alexandrov and Z. Kadar, "Timelike surfaces in Lorentz covariant loop gravity and spin foam models," Class. Quant. Grav. **22** (2005) 3491 [arXiv:gr-qc/0501093].

[14] R. Brunetti and K. Fredenhagen, "When does a detector click?," arXiv:quant-ph/0103144.

R. Brunetti and K. Fredenhagen, "Remarks on time-energy uncertainty relations," Rev. Math. Phys. **14** (2002) 897 [arXiv:quant-ph/0207048].

K. Fredenhagen, "Noncommutative spacetime, and quantum coordinates of an event," in *Quantum Theory and Symmetries IV*, ed. by V.K. Dobrev, (Heron Press, Sofia, 2006)

[15] D. Basu and K.W. Wolf, "The Clebsch–Gordon coefficients of the three–dimensional Lorentz algebra in the parabolic basis", J. Math. Phys. **24** (1983), 478

[16] R. Gambini and R. A. Porto, "Relational time in generally covariant quantum systems: Four models," Phys. Rev. D **63** (2001) 105014 [arXiv:gr-qc/0101057].