

Linear and nonlinear tails II: exact decay rates in spherical symmetry

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We derive the exact late-time asymptotics for small spherically symmetric solutions of nonlinear wave equations with a potential. The dominant tail is shown to result from the competition between linear and nonlinear effects.

I. INTRODUCTION

We consider linear and nonlinear wave equations with a potential term

$$\square u + \lambda V u = F(u), \quad \square = \partial_t^2 - \Delta, \quad (1)$$

in three spatial dimensions for spherically symmetric initial data

$$u(0, r) = f(r), \quad \partial_t u(0, r) = g(r), \quad r := |x| \quad (2)$$

with f, g of compact support. The spherical symmetry of the initial data is preserved in evolution so $u = u(t, r)$. We are interested in the asymptotic behaviour of $u(t, r)$ for late times $t \gg r$.

Our approach is based on the perturbative calculation which has been developed by the last three authors in concrete physical applications [1, 2, 3] and recently put on the rigorous ground by the first author in [4] (below referred to as part I). In part I the convergence of the perturbation scheme was proved in a weighted space-time L^∞ -norm which provided pointwise estimates on the solution $u(t, r)$ in the whole spacetime. Moreover, upper bounds on the errors (remainders of the perturbation series) for every perturbation order were obtained. Here, we are going to combine the qualitative global weighted- L^∞ estimates with the quantitative perturbation scheme in order to obtain precise late-time asymptotics of solutions. To this end, we first solve the linear perturbation equations analytically up to the second (nontrivial) order (in spherical symmetry this can be done explicitly) and show that our decay estimate is optimal. Then, we prove that the sum of all higher-order perturbations does not modify the dominant asymptotics, hence the second order perturbation gives the precise approximation of the tail of the solution u . Along the way, we illustrate our analytical results with numerical solutions of the initial value problem (1-2).

The basis of our analysis is given by the theorem of Strauss and Tsutaya [6], recently generalized by NS [5], which states that

$$|u(t, x)| \leq \frac{C}{(1+t+|x|)(1+|t-|x||)^{q-1}} \quad \forall (t, x) \in \mathbb{R}_+^{1+3} \quad (3)$$

with $q := \min(m-1, k, p-1)$ provided that the potential V and the initial data f, g satisfy pointwise bounds

$$|V(x)| \leq \frac{V_0}{(1+|x|)^k}, \quad k > 2, \quad (4)$$

$$|f(x)| \leq \frac{f_0}{(1+|x|)^{m-1}}, \quad |\nabla f(x)| \leq \frac{f_1}{(1+|x|)^m}, \quad |g(x)| \leq \frac{g_0}{(1+|x|)^m}, \quad m > 3, \quad (5)$$

with small V_0, f_0, f_1, g_0 and the analytic nonlinearity satisfying for $p > 1 + \sqrt{2}$

$$|F(u)| \leq F_1|u|^p, \quad |F(u) - F(v)| \leq F_2|u - v| \max(|u|, |v|)^{p-1} \quad \text{for } |u|, |v| < 1. \quad (6)$$

This is true for classical solutions [6], i.e. for $(f, g) \in \mathcal{C}^3 \times \mathcal{C}^2$, $V \in \mathcal{C}^2$ and $F \in \mathcal{C}^2$, leading to $u \in \mathcal{C}^2$ and remains true also for weak solutions, e.g. $u \in \mathcal{C}^0$, [5].

Here, for simplicity, we consider initial data of compact support so the decay rate q is determined solely by the spatial decay rate of the potential k and the leading power of the nonlinearity p . Generalization of these results to the initial data with the fall-off (5) is straightforward.

The paper is organized as follows. We first study the purely linear situation with the potential term only. Then, we repeat the calculations for the purely nonlinear case without the potential. Finally, we combine both results in the general case (1).

Notation

We use the symbol $\langle x \rangle := 1 + |x|$ to denote the spatial weighted- L^∞ norm

$$\|f\|_{L_m^\infty} := \|\langle r \rangle^m f(r)\|_{L^\infty(\mathbb{R}_+)} \quad (7)$$

We also define a space-time weighted- L^∞ norm

$$\|u\|_{L_{s,q}^\infty} := \|\langle t+r \rangle^s \langle t-r \rangle^{q-s} u(t, r)\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)}. \quad (8)$$

We will frequently use the fact that the finiteness of $\|u\|_{L_{1,q}^\infty}$ guarantees the decay of u like $1/t$ on the lightcone $t \sim r$ and like $1/t^q$ for fixed r as well as $1/r^q$ for fixed t . Note that functions with compact support in \mathbb{R}_+ belong to all spaces L_m^∞ with any $m > 0$, what we will denote by L_∞^∞ . Analogously $L_{1,\infty}^\infty$ will stand for functions that belong to $L_{1,q}^\infty$ for any q .

We introduce the following notation for solutions of the wave equations. Let I_V be a linear map from the space of initial data to the space of solutions of the wave equation (1)-(2) with $F(u) = 0$, so that $u = I_V(f, g)$. For the wave equation with a source term and zero initial data

$$\square u + Vu = F, \quad u(0, r) = 0, \quad \partial_t u(0, r) = 0, \quad (9)$$

we denote the solution by $u = L_V(F)$, where L_V is a linear map from the space of source functions to the space of solutions to the above problem. Note that, due to linearity, the solution u of a wave equation with source F and nonzero initial data f, g is the sum of these two contributions

$$u = L_V(F) + I_V(f, g). \quad (10)$$

Observe that if we put the potential term on the r.h.s. we obtain

$$\square u = -Vu + F, \quad (11)$$

which, treated as a wave equation without potential (on the l.h.s.), is formally solved by

$$u = -L_0(Vu) + L_0(F) + I_0(f, g). \quad (12)$$

Here the solution u appears on both sides what seems to make the formula useless, but it will allow us to formulate various iteration schemes, e.g.

$$u_{n+1} = -L_0(Vu_n) + L_0(F(u_n)) + I_0(f, g) \quad (13)$$

for which we will study convergence in suitable $L_{1,q}^\infty$ norms.

Finally, we define constants which arise from estimates proved in [5] and improved in [9]

$$C_m := \max\left(\frac{9}{2(m-2)}, 5\right), \quad (14)$$

$$C_{p,q} := 2 + \frac{8}{p-1} + \frac{2}{q-1}. \quad (15)$$

The latter will be referred to as a bound on the allowed strength of the potential. We wish to emphasize that this bound, although not optimal, is not arbitrarily small but finite, which is crucial in applications (like, for instance, the Regge-Wheeler equation describing waves propagating on Schwarzschild geometry).

We recall some standard definitions of asymptotic calculus. The notation $f(t) = \mathcal{O}(g(t))$ for $t \rightarrow \infty$ means that there exist constants $C, T > 0$ such that

$$|f(t)| \leq C|g(t)| \quad (16)$$

for all $t > T$. The notation $f(t) = o(h(t))$ for $t \rightarrow \infty$ means that

$$\lim_{t \rightarrow \infty} \frac{f(t)}{h(t)} = 0. \quad (17)$$

We will also use the symbol $f(t) \cong g(t)$ for an asymptotic approximation, as a shorthand to $f(t) = g(t)[1 + o(1)]$ as $t \rightarrow \infty$. In case when we write $f(t) \cong ct^{-q}$ and the constant c may become zero, this notation should be read as $f(t) = ct^{-q} + o(t^{-q})$.

II. LINEAR CASE WITH POTENTIAL

First, we consider the linear wave equation

$$\square u + \lambda V(r)u = 0 \quad (18)$$

with initial data (2), where $f(r)$ and $g(r)$ are supported on the interval $r \in [0, R]$. We assume that $V(r) \cong V_0/r^k$ for $r \gg 1$ and $\lambda > 0$ is a small parameter, bounded by some finite constant $C_V > 0$ (which will be defined later). Moreover, we assume that the potential V and the initial data $f, \nabla f, g$ are (at least) continuous and satisfy

$$\|V\|_{L_k^\infty} = 1 \quad (19)$$

and

$$f_0 := \|f\|_{L_k^\infty}, \quad f_1 := \|\nabla f\|_{L_{k+1}^\infty}, \quad g_0 := \|g\|_{L_{k+1}^\infty} \quad (20)$$

with $f_0, f_1, g_0 < \infty$ for some $k > 2$.

A. Perturbation series

We define the perturbation series

$$u = \sum_{n=0}^{\infty} \lambda^n v_n. \quad (21)$$

Inserting (21) into equation (18) we get the following perturbation scheme

$$\square v_0 = 0, \quad (v_0, \dot{v}_0)(0) = (f, g) \quad \rightarrow \quad v_0 = I_0(f, g) \quad (22)$$

$$\square v_{n+1} = -Vv_n, \quad (v_{n+1}, \dot{v}_{n+1})(0) = (0, 0) \quad \rightarrow \quad v_{n+1} = -L_0(Vv_n). \quad (23)$$

Due to linearity of (18) it turns out that the partial sums

$$u_n := \sum_{k=0}^n \lambda^k v_k, \quad n \geq 0 \quad (24)$$

satisfy the following iteration scheme

$$u_{-1} := 0 \quad (25)$$

$$u_n := I_0(f, g) - \lambda L_0(Vu_{n-1}), \quad n \geq 0. \quad (26)$$

Then, from part I, we have the following

Theorem 1. *For f, g and V as above and any $k > 2$, the sequence u_n converges (in norm) in $L_{1,k}^{\infty}$ provided that $\lambda < C_{k,k}^{-1}$. The limit $u := \lim_{n \rightarrow \infty} u_n$ satisfies*

$$|u(t, r)| \leq \frac{C}{\langle t+r \rangle \langle t-r \rangle^{k-1}}, \quad \forall (t, r) \quad (27)$$

where a positive constant C depends only on f_0, f_1, g_0, λ and k .

From the proof of theorem 1 it follows that

$$\|v_n\|_{L_{1,k}^{\infty}} = \frac{\|u_n - u_{n-1}\|_{L_{1,k}^{\infty}}}{\lambda^n} \leq (C_{k,k})^n \|I_0(f, g)\|_{L_{1,k}^{\infty}}, \quad (28)$$

hence $v_n \in L_{1,k}^{\infty}$ for all $n \geq 0$. At the lowest order $v_0 = u_0$ we have an arbitrarily fast decay estimate, $v_0 \in L_{1,\infty}^{\infty}$, as follows from Huygens' principle. All higher-order terms $v_n (n = 1, 2, \dots)$ contain contributions from the backscattering off the potential and are only in $L_{1,k}^{\infty}$. Since $u \in L_{1,k}^{\infty}$, we expect that all u_n starting from $u_1 \in L_{1,k}^{\infty}$ predict qualitatively correct asymptotic behaviour of u .

B. Optimal decay estimate

Theorem 2. *Under the above assumptions, for $t \gg r + R$, we have*

$$v_1(t, r) \cong c_1 t^{-k}, \quad (29)$$

where the constant c_1 is given by (39).

Proof. For v_0 satisfying (22) from lemma 1 we have

$$u_0(t, r) = v_0(t, r) = \frac{h(t-r) - h(t+r)}{r}, \quad (30)$$

where h is given by (A.3). To solve equation (23) we use the Duhamel representation for the solution of the inhomogeneous equation $\square v = N(t, r)$ with zero initial data

$$v(t, r) = \frac{1}{2r} \int_0^t d\tau \int_{|t-r-\tau|}^{t+r-\tau} \rho N(\tau, \rho) d\rho. \quad (31)$$

This formula can be easily obtained by integrating out the angular variables in the standard formula $\phi = G^{ret} * N$ where $G^{ret}(t, x) = (2\pi)^{-1} \Theta(t) \delta(t^2 - |x|^2)$ is the retarded Green's function of the wave operator in 3 + 1 dimensions. It is convenient to express (31) in terms of null coordinates $\xi = \tau + \rho$ and $\eta = \tau - \rho$

$$v(t, r) = \frac{1}{4r} \int_{|t-r|}^{t+r} d\xi \int_{-\xi}^{t-r} d\eta \frac{(\xi - \eta)}{2} \tilde{N}(\xi, \eta), \quad (32)$$

where $\tilde{N}(\xi, \eta) := N\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) = N(\tau, \rho)$. Using this representation we get from (23)

$$v_1(t, r) = -\frac{1}{4r} \int_{|t-r|}^{t+r} d\xi \int_{-\xi}^{t-r} d\eta \frac{(\xi - \eta)}{2} V(\rho(\xi, \eta)) \tilde{v}_0(\xi, \eta). \quad (33)$$

Since the initial data f, g are supported on $[0, R]$, the function $h(x)$ is supported on $[-R, R]$. Then, for $t > r + R$ equation (33) simplifies to

$$v_1(t, r) = -\frac{1}{4r} \int_{-R}^{+R} d\eta h(\eta) \int_{t-r}^{t+r} d\xi V(\rho(\xi, \eta)). \quad (34)$$

Next, we write the potential in the form $V(r) = r^{-k}[V_0 + w(r)]$ with $w(r) \rightarrow 0$ as $r \rightarrow \infty$ at any rate (i.e. $w(r) = o(1)$ for $r \gg 1$). Then

$$v_1(t, r) = -\frac{1}{4r} \int_{-R}^{+R} d\eta h(\eta) \int_{t-r}^{t+r} d\xi \frac{2^k}{(\xi - \eta)^k} [V_0 + w(\xi - \eta)] \equiv \hat{v}_1(t, r) + \tilde{v}_1(t, r), \quad (35)$$

where

$$\hat{v}_1(t, r) = -\frac{2^{k-2}}{r} V_0 \int_{-R}^{+R} d\eta h(\eta) \int_{t-r}^{t+r} d\xi (\xi - \eta)^{-k} \quad (36)$$

$$\tilde{v}_1(t, r) = -\frac{2^{k-2}}{r} \int_{-R}^{+R} d\eta h(\eta) \int_{t-r}^{t+r} d\xi (\xi - \eta)^{-k} w(\xi - \eta). \quad (37)$$

Using lemma 2 for $t \gg r + R$ and $k > 2$ we get

$$\widehat{v}_1(t, r) = \frac{c_1}{t^k} + \mathcal{O}\left(\frac{r + R}{t^{k+1}}\right), \quad (38)$$

with

$$c_1 = -2^{k-1}V_0 \int_{-R}^{+R} h(\eta) d\eta. \quad (39)$$

Using lemma 2 again, we get an estimate for \widetilde{v}_1

$$\begin{aligned} |\widetilde{v}_1(t, r)| &\leq \frac{2^{k-2}}{r} \int_{-R}^{+R} d\eta |h(\eta)| \sup_{t-r \leq \zeta \leq t+r} |w(\zeta - \eta)| \int_{t-r}^{t+r} d\xi (\xi - \eta)^{-k} \\ &\leq \sup_{t-r-R \leq \zeta \leq t+r+R} |w(\zeta)| \frac{2^{k-2}}{r} \int_{-R}^{+R} d\eta |h(\eta)| \int_{t-r}^{t+r} d\xi (\xi - \eta)^{-k} \\ &= \sup_{t-r-R \leq \zeta \leq t+r+R} |w(\zeta)| \left[\frac{\widetilde{c}_1}{t^k} + \mathcal{O}\left(\frac{r + R}{t^{k+1}}\right) \right], \end{aligned} \quad (40)$$

where

$$\widetilde{c}_1 = -2^{k-1} \int_{-R}^{+R} |h(\eta)| d\eta. \quad (41)$$

Note that the prefactor in (40) vanishes asymptotically for large t

$$\lim_{t \rightarrow \infty} \sup_{t-r-R \leq \zeta \leq t+r+R} |w(\zeta)| = 0, \quad (42)$$

hence, for $t \gg r + R$ we have

$$v_1(t, r) = \frac{c_1}{t^k} + \mathcal{O}\left(\frac{r + R}{t^{k+1}}\right) + o(1) \cdot \left[\frac{\widetilde{c}_1}{t^k} + \mathcal{O}\left(\frac{r + R}{t^{k+1}}\right) \right] = \frac{c_1}{t^k} + o\left(\frac{1}{t^k}\right). \quad (43)$$

If the potential behaves like $V(r) = V_0 r^{-k} + W(r)$ with $W(r) = \mathcal{O}(r^{-k-1})$ for $r \gg 1$, it follows from (38) that

$$\begin{aligned} v_1(t, r) &= \frac{c_1}{t^k} + \mathcal{O}\left(\frac{r + R}{t^{k+1}}\right) + \mathcal{O}\left(\frac{\widetilde{c}_1}{t^{k+1}}\right) + \mathcal{O}\left(\frac{r + R}{t^{k+2}}\right) \\ &= \frac{c_1}{t^k} + \mathcal{O}\left(\frac{1 + r + R}{t^{k+1}}\right), \end{aligned} \quad (44)$$

which gives the more detailed information about the sub-leading term. \square

Theorem 3. *Under the assumptions of theorem 2, for $t \gg r + R$, we have*

$$u(t, r) \cong \lambda v_1(t, r) [1 + \mathcal{O}(\lambda)], \quad (45)$$

hence

$$u(t, r) \cong Ct^{-k}, \quad C = \lambda c_1 + \mathcal{O}(\lambda^2). \quad (46)$$

Proof. Knowing that the perturbation series converges for some λ we can bound the error in the n -th order relative to the exact solution by estimating the sum of all higher order terms. For the convergent sequence u_n we get from the proof of Theorem 1 that

$$\|u - u_n\|_{L_{1,k}^\infty} \leq \frac{(C_{k,k}\lambda)^{n+1}}{1 - C_{k,k}\lambda} \|I_0(f, g)\|_{L_{1,k}^\infty}, \quad (47)$$

what provides the pointwise bound on the error

$$|u(t, r) - u_n(t, r)| \leq \frac{(C_{k,k}\lambda)^{n+1}}{1 - C_{k,k}\lambda} \cdot \frac{C_{k+1} \cdot (f_0 + f_1 + g_0)}{\langle t+r \rangle \langle t-r \rangle^{k-1}} \quad \forall t, r \geq 0. \quad (48)$$

For $n = 1$ with $u_1 = v_0 + \lambda v_1$ we have

$$|u(t, r) - v_0(t, r) - \lambda v_1(t, r)| \leq \frac{(C_{k,k}\lambda)^2}{1 - C_{k,k}\lambda} \cdot \frac{C_{k+1} \cdot (f_0 + f_1 + g_0)}{\langle t+r \rangle \langle t-r \rangle^{k-1}} =: \Delta_1(t, r). \quad (49)$$

A simple inequality (which follows immediately from Bernoulli's inequality)

$$\frac{1}{(1 - \zeta)^\sigma} \leq \frac{1}{1 - \sigma\zeta} = 1 + \frac{\sigma\zeta}{1 - \sigma\zeta} \leq 2, \quad \forall \zeta \leq 1/(2\sigma), \quad \sigma >, \quad (50)$$

implies that

$$\frac{1}{\langle t-r \rangle^q} = \frac{1}{(1+t)^q \left(1 - \frac{r}{1+t}\right)^q} \leq \frac{2}{(1+t)^q} \quad (51)$$

for $\zeta := r/(1+t) \leq 1/(2q)$, hence it holds for all $t \geq 2qr$. The error term can then be estimated by

$$\Delta_1(t, x) \leq 2(C_{k,k}\lambda)^2 \frac{C_{k+1} \cdot (f_0 + f_1 + g_0)}{(1+t)^k} \mathcal{O}\left(\frac{\lambda^2}{t^k}\right), \quad (52)$$

where we have used twice the inequality (50) for $t \geq 2(k-1)r$ and $\lambda \leq 1/(2C_{k,k})$.

From Huygens' principle for (22) with initial data of compact support it follows that $v_0(t, r) = 0$ for $t > r + R$, hence for every $r \geq 0$ and sufficiently large $t > \max[r + R, 2(k-1)r]$ we have

$$|u(t, r) - \lambda v_1(t, r)| \leq |u(t, r) - v_0(t, r) - \lambda v_1(t, r)| + |v_0(t, r)| = \mathcal{O}\left(\frac{\lambda^2}{t^k}\right), \quad (53)$$

and

$$\begin{aligned} \left|u(t, r) - \lambda \frac{c_1}{t^k}\right| &\leq |u(t, r) - \lambda v_1(t, r)| + \lambda \left|v_1(t, r) - \frac{c_1}{t^k}\right| \\ &= \mathcal{O}\left(\frac{\lambda^2}{t^k}\right) + o\left(\frac{\lambda}{t^k}\right), \end{aligned} \quad (54)$$

where we have used the result of theorem 2, eq. (38). Therefore

$$u(t, r) \cong \frac{C}{t^k}, \quad C = \lambda c_1 + \mathcal{O}(\lambda^2). \quad (55)$$

□

This gives the precise quantitative information about the late-time tail of $u(t, r)$ and shows that the estimate in theorem 1 is optimal (for $t \gg r$) (see Table 1 and Fig. 1 for the numerical verification).

		$\lambda V_0 = 10^{-1}$		$\lambda V_0 = 10^{-3}$		V_0
		Theory	Numerics	Theory	Numerics	
$k = 3$	Exponent	3.0	2.9996	3.0	3.0000	0.3485
	Amplitude	-3.5449×10^{-1}	-3.0429×10^{-1}	-3.5449×10^{-3}	-3.5394×10^{-3}	
$k = 4$	Exponent	4.0	4.00001	4.0	4.00000	0.2339
	Amplitude	-7.0898×10^{-1}	-6.6885×10^{-1}	-7.0898×10^{-3}	-7.0856×10^{-3}	
$k = 5$	Exponent	5.0	5.00000	5.0	5.00000	0.1560
	Amplitude	-1.4179	-1.3745	-1.4179×10^{-2}	-1.4175×10^{-2}	

TABLE I: Linear case with a potential term $\lambda V(r) = \lambda V_0 \frac{\tanh^{k+2}(r)}{r^k}$. The values of V_0 follow from the normalization condition (19). The results are obtained for the initial data of the form: $f(r) = 0$, $g(r) = 4(r^2 - 1) \exp(-r^2)$, which corresponds to $h(z) = z^2 \exp(-z^2)$ (see A.3). The number at the Theory-Amplitude entry gives the value of λc_1 with c_1 defined in (39). Note that for $\lambda V_0 = 10^{-1}$ the values of λ are actually greater than the convergence radius of perturbation series obtained in theorem 1, but still, the formula (55) seems to work very well.

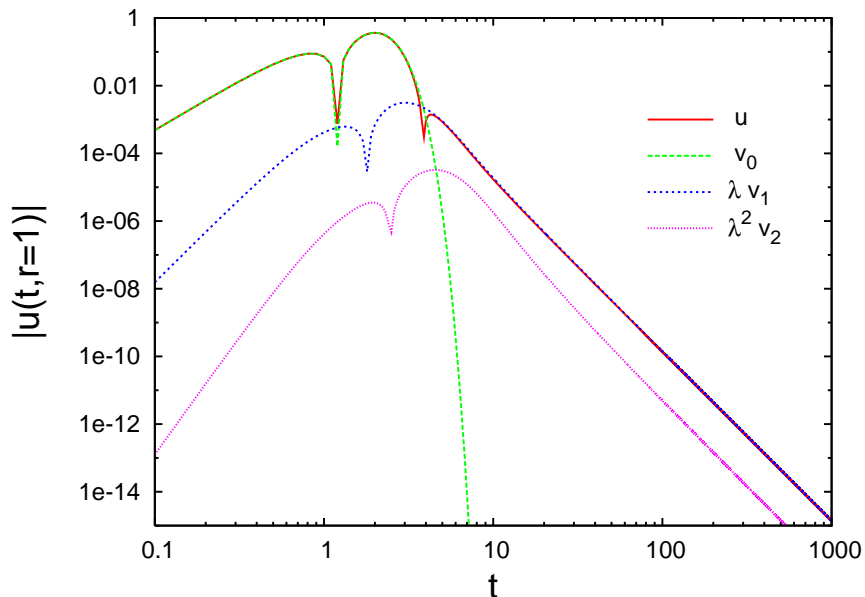


FIG. 1: We plot (on log-log scale) the numerical solution $u(t, r = 1)$ of equation (18) for $\lambda V(r) = 0.1 \tanh^7 r / r^5$ (this corresponds to $\lambda = 0.64$). The initial data are: $f(r) = 0$, $g(r) = 4(r^2 - 1) \exp(-r^2)$, which corresponds to $h(z) = z^2 \exp(-z^2)$ (see A.3). The first three terms in the perturbation expansion (21) are superimposed. In agreement with theorem 3, the tail is perfectly approximated by λv_1 (cf. Table 1).

III. NONLINEAR CASE WITHOUT A POTENTIAL TERM

Now, we consider the nonlinear wave equation of the form

$$\square u = F(u) \quad (56)$$

with initial data (f, g) supported on the interval $r \in [0, R]$ and satisfying (20) with $f_0, f_1, g_0 < \varepsilon$. The nonlinear term obeys $|F(u)| \leq F_1 |u|^p$ for $|u| < 1$ and $|F(u) - F(v)| \leq F_2 |u - v|$

$v|\max(|u|, |v|)^{p-1}$. The second condition is satisfied e.g. for $F(u) = u^p$ with $F_2 = p$ or for $F \in \mathcal{C}^1$ such that $|F'(u)| \leq F_2|u|^{p-1}$ for $|u| < 1$.

A. Perturbation series

In order to construct a well-defined perturbation scheme to all orders we have to assume additionally that $F(u)$ is analytic at $u = 0$ and its Taylor series starts at power $p \geq 3$

$$F(u) = u^p \sum_{n=0}^{\infty} b_n u^n, \quad b_0 \neq 0 \quad (57)$$

Then, for small initial data

$$(u, \dot{u})(0) = (\varepsilon f, \varepsilon g) \quad (58)$$

we introduce the perturbation series for the solution of (56)

$$u = \sum_{n=1}^{\infty} \varepsilon^n v_n. \quad (59)$$

Inserting this series into (56) and collecting terms according to powers of ε we obtain the following perturbation scheme

$$\square v_1 = 0, \quad (v_1, \dot{v}_1)(0) = (f, g) \quad \rightarrow \quad v_1 = I_0(f, g) \quad (60)$$

$$\square v_{n+1} = F_n(v_1, \dots, v_n), \quad (v_{n+1}, \dot{v}_{n+1})(0) = (0, 0) \quad \rightarrow \quad v_{n+1} = L_0(F_n(v_1, \dots, v_n)), \quad (61)$$

for $n \geq 1$, where F_n result from collecting the nonlinear terms with the same powers of ε

$$F_n(v_1, \dots, v_n) \sum_k a_k^n v_1^{\alpha_k^{n,1}} \cdots v_n^{\alpha_k^{n,n}}, \quad (62)$$

where $\alpha_k^{n,m} \in \mathbb{N}$ satisfy $\sum_{m=1}^n m \alpha_k^{n,m} = n + 1$ and $\sum_{m=1}^n \alpha_k^{n,m} \geq p$ for every n, k . The coefficients a_k^n are functions of b_m only (see [4] for the explicit formula).

We call this expansion the “zero background” case because the zero-order term v_0 is absent. If a v_0 term was present in the series above (i.e. the summation started at $n = 0$), we would have an additional equation $\square v_0 = F(v_0)$ which is genuinely nonlinear (in contrast to the above system of linear wave equations with source terms). Its solution v_0 represents a “background” around which the perturbations v_n are calculated.

From part I, we have the following

Theorem 4. *With f, g and $F(u)$ as above for any $p \geq 3$ and sufficiently small ε the series defined in (59)-(61) converges (in norm) in $L_{1,p-1}^\infty$ to the solution of equation (56) with initial data (58).*

Since the introduction of the auxiliary parameter ε in the perturbation series expansion serves only to generate a system of linear equations equivalent to the original nonlinear equation, we can eventually remove the parameter ε and assume that the initial data are such that f_0, f_1, g_0 are sufficiently small. Then, solving the system (60)-(61) and summing up the convergent series $\sum_{n=1}^{\infty} v_n u$ we obtain a solution of the nonlinear wave equation (56).

B. Optimal decay estimate

The perturbation scheme (60)-(61) can be written as

$$v_1 = I_0(f, g) \tag{63}$$

$$v_2 = v_3 = \dots = v_{p-1} = 0 \tag{64}$$

$$v_p = L_0(F_{p-1}(v_1, \dots, v_{p-1})) = b_0 L_0((v_1)^p) \tag{65}$$

$$v_{n+1} = L_0(F_n(v_1, \dots, v_n)), \quad n \geq p. \tag{66}$$

We have $v_1 = I_0(f, g) \in L_{1,\infty}^\infty$ and $v_n \in L_{1,p-1}^\infty$ for $n \geq 2$.

Theorem 5. *Under the above assumptions, for $t \gg r + R$, we have*

$$v_p(t, r) \cong d_p t^{-(p-1)}, \tag{67}$$

where the constant d_p is given by (72).

Proof. In analogy with equations (30-32) we have from lemma 1

$$v_1(t, r) = \frac{h(t-r) - h(t+r)}{r} \tag{68}$$

and from (65)

$$v_p(t, r) = \frac{1}{8r} \int_{|t-r|}^{t+r} d\xi \int_{-\xi}^{t-r} d\eta (\xi - \eta) f_0(v_1(\eta, \xi))^p. \tag{69}$$

As before, interchanging the order of integration we get for $t > r + R$

$$v_p(t, r) = \frac{2^{p-3} f_0}{r} \int_{-R}^{+R} d\eta (h(\eta))^p \int_{t-r}^{t+r} d\xi (\xi - \eta)^{-p+1}. \tag{70}$$

Using lemma 2 we get for $t \gg r + R$ and $p \geq 3$

$$v_p(t, r) = \frac{d_p}{t^{p-1}} + \mathcal{O}\left(\frac{r+R}{t^p}\right), \tag{71}$$

where

$$d_p = 2^{p-2} b_0 \int_{-R}^{+R} d\eta (h(\eta))^p. \tag{72}$$

□

Now, we will show that v_p dominates the perturbation series for large times and small ε and has the same decay rate as the full solution u of the nonlinear wave equation (see Table 2 and Fig. 2 for the numerical verification).

Theorem 6. *Under the assumptions of theorem 5, for small ε and $t \gg r + R$, we have*

$$u(t, r) \cong \varepsilon^p v_p(t, r) [1 + \mathcal{O}(\varepsilon)], \tag{73}$$

hence

$$u(t, r) \cong D t^{-p+1}, \quad D = d_p \varepsilon^p + \mathcal{O}(\varepsilon^{p+1}). \tag{74}$$

Proof. We need to show that $\varepsilon I_0(f, g)$ and $\varepsilon^{n+1} L_0(F_n(v_1, \dots, v_n))$ for $n \geq p$ are small relative to $\varepsilon^p d_p(x) t^{-(p-1)}$. As before, for $v_1 = I_0(f, g) \in L_{1,\infty}^\infty$ Huygens' principle and compact support of the initial data imply that $v_1(t, r) = 0$ for sufficiently large t (and fixed r). From the convergence proof for the perturbation series we know that there exist constants $M, \rho > 0$ such that $\|v_n\|_{L_{1,p-1}^\infty} \leq M \rho^n$ for all $n \geq 1$. Hence, for sufficiently small $\varepsilon < 1/\rho$ we can estimate the remainder of the perturbation series

$$\left\| \sum_{m=p+1}^{\infty} \varepsilon^m v_m \right\|_{L_{1,p-1}^\infty} \leq M \sum_{m=p+1}^{\infty} \varepsilon^m \rho^m \leq \frac{M \varepsilon^{p+1} \rho^{p+1}}{1 - \varepsilon \rho} \leq C \varepsilon^{p+1}. \quad (75)$$

This implies that for $t \gg r$

$$\left| \sum_{m=p+1}^{\infty} \varepsilon^m v_m(t, r) \right| \leq \frac{C \varepsilon^{p+1}}{\langle t+r \rangle \langle t-r \rangle^{p-2}} = \mathcal{O}\left(\frac{\varepsilon^{p+1}}{t^{p-1}}\right), \quad (76)$$

so

$$|u(t, r) - \varepsilon^p v_p(t, r)| \leq |\varepsilon v_1(t, r)| + \left| \sum_{m=p+1}^{\infty} \varepsilon^m v_m \right| = \mathcal{O}\left(\frac{\varepsilon^{p+1}}{t^{p-1}}\right). \quad (77)$$

From theorem 5, for $t \gg r + R$ we have $v_p = d_p t^{-(p-1)} + \mathcal{O}(t^{-p})$, hence

$$\left| u(t, r) - \frac{d_p \varepsilon^p}{t^{p-1}} \right| = \mathcal{O}\left(\frac{\varepsilon^{p+1}}{t^{p-1}}\right) + \mathcal{O}\left(\frac{\varepsilon^p}{t^p}\right), \quad (78)$$

which finally gives

$$u(t, r) \cong D t^{-p+1}, \quad D = d_p \varepsilon^p + \mathcal{O}(\varepsilon^{p+1}). \quad (79)$$

□

		$\varepsilon = 1$		$\varepsilon = 10^{-1}$	
		Theory	Numerics	Theory	Numerics
$p = 3$	Exponent	2.0	2.0009	2.0	2.0008
	Amplitude	0.1421	0.1265	0.1421×10^{-4}	0.1427×10^{-4}
$p = 4$	Exponent	3.0	3.0013	3.0	3.0012
	Amplitude	9.0873×10^{-2}	8.4433×10^{-2}	9.0873×10^{-6}	9.1631×10^{-6}
$p = 5$	Exponent	4.0	4.0015	4.0	4.0015
	Amplitude	5.9925×10^{-2}	6.1192×10^{-2}	5.9925×10^{-7}	6.0597×10^{-7}

TABLE II: Nonlinear case: $F(u) = u^p$, without a potential term. The initial data are the same as for Table 1 and Fig. 1. The number at the Theory-Amplitude entry gives the value of $\varepsilon^p d_p$, with d_p defined in (72) (with $b_0 = 1$).

IV. NONLINEAR CASE WITH A POTENTIAL TERM

Finally, let us consider the full nonlinear wave equation (1) with a potential with initial data (f, g) supported on the interval $r \in [0, R]$ and satisfying (20) with $f_0, f_1, g_0 < \varepsilon$. The nonlinear term $F(u)$ is the same as in the previous section.

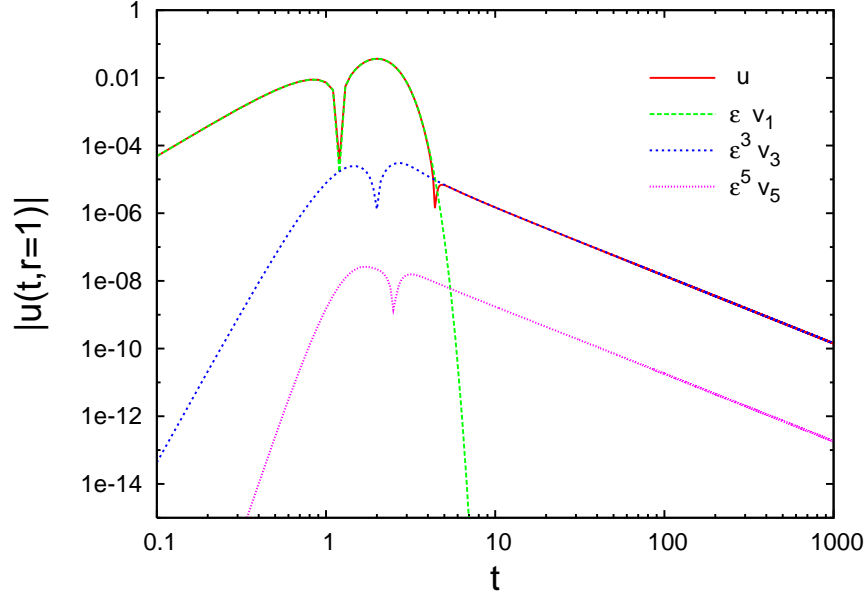


FIG. 2: We plot (on log-log scale) the numerical solution $u(t, r = 1)$ of equation (56) with $F(u) = u^3$. The initial data are the same as in Fig. 1 and $\varepsilon = 0.1$. The first three terms in the perturbation expansion (57) are superimposed. In agreement with theorem 5, the tail is perfectly approximated by $\varepsilon^3 v_3$ (cf. Table 2).

A. Perturbation series

Defining the perturbation expansion for the nonlinear wave equation with a potential (1)

$$u = \sum_{n=1}^{\infty} \varepsilon^n v_n, \quad (80)$$

we encounter the problem of two scales which are given by parameters λ (measuring the strength of the potential) and ε (measuring the strength of the initial data). Since these parameters play only an auxiliary role in generating the perturbation scheme, we make a convenient choice and assign to λ a scale of some power of ε , say $\lambda = \tilde{\lambda} \varepsilon^a$ with $a \in \mathbb{N}_+$.

Then, the power series (80) inserted into the wave equation (1) gives

$$v_{-n} := 0, \quad n \geq 0, \quad (81)$$

$$v_1 := I_0(f, g), \quad (82)$$

$$v_{n+1} := -\tilde{\lambda} L_0(V v_{n+1-a}) + L_0(F_n(v_1, \dots, v_n)), \quad n \geq 1. \quad (83)$$

In the following we choose $a := p - 1$ because then the lowest-order nontrivial term, v_p (all lower-order terms with $1 < n < p$ vanish), contains contributions both from V and F and gives a good approximation to u , as will be shown below.

In this case, from part I, we also have a convergence result

Theorem 7. *With f, g, V and $F(u)$ as above for any $k > 2$, $p \geq 3$, $\lambda < C_{q,k}^{-1}$ and sufficiently small ε the series defined in (80)-(83) converges (in norm) in $L_{1,q}^{\infty}$ for $q = \min(p - 1, k)$ to the solution of the equation (1) with initial data (58).*

B. Optimal decay estimate

For $a = p - 1$ the system (81)-(83) takes the form

$$v_{-n} := 0, \quad n \geq 0 \tag{84}$$

$$v_1 = I_0(f, g) \tag{85}$$

$$v_2 = v_3 = \dots = v_{p-1} = 0 \tag{86}$$

$$v_p = -\tilde{\lambda}L_0(Vv_1) + L_0(F_{p-1}(v_1, \dots, v_{p-1})) = -\tilde{\lambda}L_0(Vv_1) + b_0L_0((v_1)^p) \tag{87}$$

$$v_{n+1} = -\tilde{\lambda}L_0(Vv_{n-p+2}) + L_0(F_n(v_1, \dots, v_n)), \quad n \geq p. \tag{88}$$

Theorem 8. *Under the above assumptions, for $t \gg r + R$, we have*

$$v_p(t, r) \cong d_p t^{-q}, \quad q := \min(k, p - 1), \tag{89}$$

where the constant e_p is defined in (97).

Proof. v_p defined in equation (87) is a sum of two contributions, from the potential and from the nonlinear term,

$$v_p(t, r) \equiv v_p^{pot}(t, r) + v_p^{non}(t, r), \tag{90}$$

where

$$v_p^{pot}(t, r) := -\tilde{\lambda}L_0(Vv_1), \quad v_p^{non}(t, r) := b_0L_0(v_1^p). \tag{91}$$

From theorem 2 we have

$$v_p^{pot}(t, r) = \frac{c_p}{t^k} + o\left(\frac{1}{t^k}\right) \tag{92}$$

with

$$c_p = -2^{k-1}\tilde{\lambda}V_0 \int_{-R}^{+R} h(\eta) d\eta, \tag{93}$$

and from theorem 5 we have

$$v_p^{non}(t, r) = \frac{d_p}{t^{p-1}} + \mathcal{O}\left(\frac{r+R}{t^p}\right) \tag{94}$$

with

$$d_p = 2^{p-2}b_0 \int_{-R}^{+R} d\eta (h(\eta))^p. \tag{95}$$

Depending on whether $k < p - 1$ or $k > p - 1$, the linear $v_p^{pot}(t, r)$ or the nonlinear $v_p^{non}(t, r)$ contribution to the tail is dominant, respectively. In the special case $k = p - 1$ we have

$$v_p(t, r) = \frac{c_p + d_p}{t^{p-1}} + o\left(\frac{1}{t^{p-1}}\right). \tag{96}$$

Thus, the constant in (89) is given by

$$e_p = \begin{cases} c_p & \text{if } k < p - 1 \\ c_p + d_p & \text{if } k = p - 1, \\ d_p & \text{if } k > p - 1. \end{cases} \tag{97}$$

□

Now, we will show that v_p dominates the perturbation series for large times and small ε and has the same decay rate as the full solution u of the nonlinear wave equation with the potential (see Fig. 3 for the numerical verification).

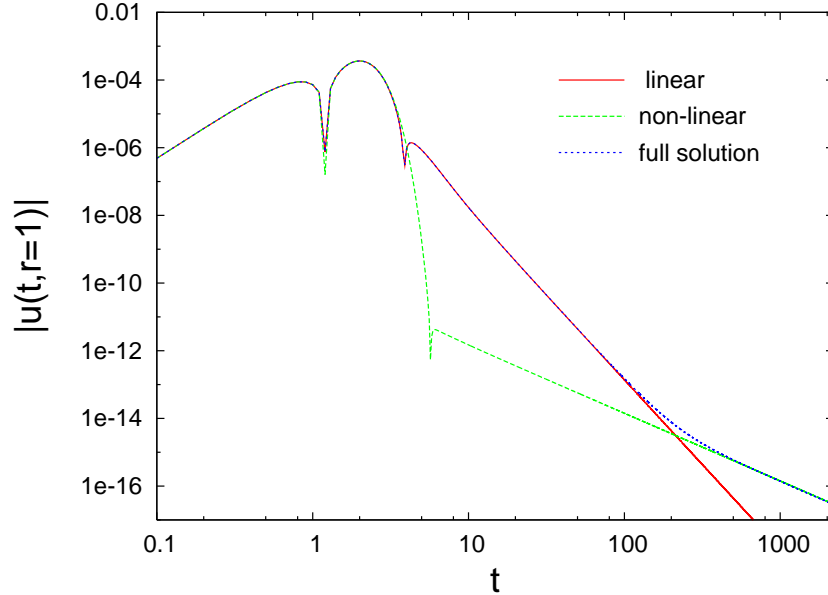


FIG. 3: We plot (on log-log scale) the numerical solution $u(t, r = 1)$ of equation (1) with $F(u) = u^3$. The potential λV and the initial data $(\varepsilon f, \varepsilon g)$ are the same as in Fig. 1 with $\lambda = 0.64$ and $\varepsilon = 0.001$. Superimposed are solutions with the nonlinearity or the potential switched off. The crossover from the linear tail $\sim t^{-5}$ (for intermediate times) to the final nonlinear tail $\sim t^{-2}$ is clearly seen.

Theorem 9. *Under the assumptions of theorem 8, for small ε and $t \gg r + R$, we have*

$$u(t, r) \cong \varepsilon^p v_p(t, r) [1 + \mathcal{O}(\varepsilon)], \quad (98)$$

hence

$$u(t, r) \cong E t^{-q}, \quad q := \min(k, p - 1), \quad E = e_p \varepsilon^p + \mathcal{O}(\varepsilon^{p+1}). \quad (99)$$

Proof. We can repeat the reasoning from the proof of theorem 6 where we used the fact that the perturbation series $\sum_{n=1}^{\infty} \varepsilon^n v_n$ is convergent. Here, theorem 7 guarantees convergence in $L_{1,q}^{\infty}$ with $q := \min(k, p - 1)$. Analogously, we obtain for $t \gg r$

$$\left| \sum_{m=p+1}^{\infty} \varepsilon^m v_m(t, r) \right| \leq \frac{C \varepsilon^{p+1}}{\langle t+r \rangle \langle t-r \rangle^{q-1}} = \mathcal{O}\left(\frac{\varepsilon^{p+1}}{t^q}\right), \quad (100)$$

so (again, $v_1(t, r)$ vanishes for $t \gg r$ by Huygens' principle)

$$|u(t, r) - \varepsilon^p v_p(t, r)| \leq |\varepsilon v_1(t, r)| + \left| \sum_{m=p+1}^{\infty} \varepsilon^m v_m \right| = \mathcal{O}\left(\frac{\varepsilon^{p+1}}{t^q}\right). \quad (101)$$

From theorem 8 we have $v_p = e_p t^{-q} + o(t^{-q})$ for $t \gg r + R$, so we get

$$\left| u(t, r) - \frac{e_p \varepsilon^p}{t^q} \right| = \mathcal{O}\left(\frac{\varepsilon^{p+1}}{t^q}\right) + o\left(\frac{\varepsilon^p}{t^q}\right), \quad (102)$$

which gives

$$u(t, r) \cong Et^{-q}, \quad E = e_p \varepsilon^p + \mathcal{O}(\varepsilon^{p+1}). \quad (103)$$

□

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APPENDIX: LEMMAS

Lemma 1. *The solution of the free wave equation*

$$\square u = 0 \quad (A.1)$$

with spherically symmetric initial data $u(0, r) = f(r)$, $\partial_t u(0, r) = g(r)$ has the form

$$u(t, r) = \frac{h(t-r) - h(t+r)}{r}, \quad (A.2)$$

where

$$h(r) = -\frac{r}{2}f(r) + \frac{1}{2} \int_r^\infty r'g(r')dr', \quad (A.3)$$

which is defined for all $r \in \mathbb{R}$ by the extension $f(-r) := f(r)$, $g(-r) := g(r)$. When f and g have compact support then h has also compact support on \mathbb{R} .

Proof. In spherical symmetry the wave equation (A.1) can be written as

$$\partial_\xi \partial_\eta (ru) = 0 \quad (A.4)$$

where $\xi = t+r$ and $\eta = t-r$. Its most general solution has the form

$$ru(t, r) = h_-(\eta) + h_+(\xi) = h_-(t-r) + h_+(t+r). \quad (A.5)$$

We require that $u(t, r)$ be finite at $r = 0$ what implies

$$0 = h_-(t) + h_+(t) \quad \Rightarrow \quad h(t) := h_-(t) = -h_+(t). \quad (A.6)$$

From the initial conditions we get

$$f(r) = u(0, r) = \frac{h(-r) - h(r)}{r} \quad (A.7)$$

$$g(r) = \partial_t u(0, r) = \frac{h'(-r) - h'(r)}{r}. \quad (A.8)$$

We can write

$$h(r) = \frac{1}{2}[h(r) + h(-r)] + \frac{1}{2}[h(r) - h(-r)] \equiv \frac{1}{2}S(r) + \frac{1}{2}A(r), \quad (A.9)$$

where we have introduced a symmetric function $S(r) := h(r) + h(-r)$ and an anti-symmetric function $A(r) := h(r) - h(-r)$. The solutions for $A(r)$ and $S'(r) = h'(r) - h'(-r)$ can be immediately read off from the initial conditions (A.7, A.8):

$$A(r) = -rf(r) \quad (\text{A.10})$$

$$S'(r) = -rg(r). \quad (\text{A.11})$$

We see that the extension of f and g on all $r \in \mathbb{R}$ defined by $f(-r) := f(r)$, $g(-r) := g(r)$ is the consistency condition for eqs. (A.10, A.11). Integrating (A.11) we get

$$S(r) - S(0) = - \int_0^r r'g(r')dr'. \quad (\text{A.12})$$

We use the freedom of choosing the integration constant and set

$$S(0) := \int_0^\infty r'g(r')dr', \quad (\text{A.13})$$

what gives (A.3). With this choice we obtain $h(r)$ compactly supported on \mathbb{R} if $f(r)$ and $g(r)$ are compactly supported. To see this, assume $f(x) = g(x) = 0$ for $|x| > R$ and consider $r > R$. The function $h(r)$ is obviously zero from (A.3). For negative arguments

$$h(-r) = \frac{r}{2} \underbrace{f(-r)}_{=0} + \frac{1}{2} \int_{-r}^\infty r'g(r')dr' = \frac{1}{2} \int_{-R}^R r'g(r')dr' = 0, \quad (\text{A.14})$$

because the integrand $r'g(r')$ is an odd function. Thus, $\text{supp } h \subset [-R, +R]$. \square

Lemma 2. *Let $\alpha > 1$. Then*

$$\int_{-R}^{+R} h(\eta) d\eta \int_{t-r}^{t+r} \frac{d\xi}{(\xi - \eta)^\alpha} = \frac{2r}{t^\alpha} \int_{-R}^{+R} h(\eta) d\eta + \mathcal{O}\left(\frac{r(r+R)}{t^{\alpha+1}}\right) \quad (\text{A.15})$$

for $t > 2\alpha(r+R)$ and all $r \geq 0$.

Proof. Consider first the inner integral for $\eta \in [-R, R]$

$$\begin{aligned} I(t, r, \eta) &:= \int_{t-r}^{t+r} \frac{d\xi}{(\xi - \eta)^\alpha} = \int_{-r}^{+r} \frac{dy}{(t - \eta + y)^\alpha} = \frac{1}{t^\alpha} \int_{-r}^{+r} \frac{dy}{\left(1 + \frac{y-\eta}{t}\right)^\alpha} \\ &= \frac{2r}{t^\alpha} + \frac{1}{t^\alpha} \int_{-r}^{+r} \left[\frac{1}{\left(1 + \frac{y-\eta}{t}\right)^\alpha} - 1 \right] dy \equiv \frac{2r}{t^\alpha} + \frac{1}{t^\alpha} \delta(t, r, \eta). \end{aligned} \quad (\text{A.16})$$

We have

$$|\delta(t, r, \eta)| \leq \int_{-r}^{+r} \left| \frac{1}{\left(1 + \frac{y-\eta}{t}\right)^\alpha} - 1 \right| dy \quad (\text{A.17})$$

and the integrand J can be estimated by

$$J := \left| \frac{1}{\left(1 + \frac{y-\eta}{t}\right)^\alpha} - 1 \right| \leq \frac{1}{\left(1 - \frac{r+R}{t}\right)^\alpha} - 1, \quad (\text{A.18})$$

what can be shown as follows. Having in mind that $-(r + R) \leq y + \eta \leq r + R$ we find for $y - \eta < 0$

$$J = \frac{1}{\left(1 + \frac{y-\eta}{t}\right)^\alpha} - 1 \leq \frac{1}{\left(1 - \frac{r+R}{t}\right)^\alpha} - 1 \equiv J_1 \quad (\text{A.19})$$

and for $y - \eta \geq 0$

$$J = 1 - \frac{1}{\left(1 + \frac{y-\eta}{t}\right)^\alpha} \leq 1 - \frac{1}{\left(1 + \frac{r+R}{t}\right)^\alpha} \equiv J_2. \quad (\text{A.20})$$

By simple algebra one can easily show that $J_2 \leq J_1$ what gives (A.18). Further, by a version of Bernoulli's inequality,

$$J \leq \frac{1}{1 - \alpha \frac{r+R}{t}} - 1 \frac{\alpha \frac{r+R}{t}}{1 - \alpha \frac{r+R}{t}}. \quad (\text{A.21})$$

Then

$$|\delta(t, r, \eta)| \leq \frac{\alpha \frac{r+R}{t}}{1 - \alpha \frac{r+R}{t}} 2r \leq 4\alpha \frac{r(r+R)}{t} \quad (\text{A.22})$$

for $t \geq 2\alpha(r + R)$. Finally,

$$\begin{aligned} \int_{-R}^{+R} h(\eta) d\eta \int_{t-r}^{t+r} \frac{d\xi}{(\xi - \eta)^\alpha} &= \int_{-R}^{+R} h(\eta) d\eta \left[\frac{2r}{t^\alpha} + \frac{\delta(t, r, \eta)}{t^\alpha} \right] \\ &= \frac{2r}{t^\alpha} \int_{-R}^{+R} h(\eta) d\eta + \mathcal{O}\left(\frac{r(r+R)}{t^{\alpha+1}}\right). \end{aligned} \quad (\text{A.23})$$

□

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