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β-deformation for matrix model of M-theory

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Abstract

A new class of deformation of the matrix model of M-theory is considered. The deformation is analogous to the so-called β-deformation of $D = 3 + 1$, $\mathcal{N} = 4$ super-Yang–Mills theory, which preserves the conformal symmetry. It is shown that the deformed matrix model can be considered as a matrix model of M-theory on a certain curved background in eleven-dimensional supergravity, under a scaling limit involving the deformation parameter and N (the size of the matrices). The background belongs to the so-called pp-wave type metric with a non-constant four-form flux depending linearly on transverse coordinates. Some stable solutions of the deformed model are studied, which correspond to membranes with the torus topology. In particular, it is found that apparently distinct configurations of membranes, having different winding numbers, are indistinguishable in the matrix model. Simultaneous introduction of both β-deformation and mass-deformation is also considered, and, in particular, a situation is found in which the stable membrane configuration interpolates between a torus and a sphere, depending on the values of the deformation parameters.

1. Introduction

Although M-theory [1,2] plays a crucial role in non-perturbative physics of String theory, its formulation is not yet established. The best candidate so far, the matrix model of M-theory [3,4], has fundamental unsolved problems such as the problem of $N \to \infty$ limit and the eleven-dimensional Lorentz invariance. Another important issue is the relation of the matrix model to the supergravity background. As the matrix model should contain degrees of freedom of eleven-dimensional supergravity, condensation of them should in principle yield the matrix model on

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curved backgrounds. Also, the information of the supergravity equations of motion should somehow be incorporated into the matrix model formulation of M-theory.

An attractive approach to these problems is to consider the matrix model as a regularised version [3,5] of supermembrane theory [6,7]; the large N limit can be interpreted as the renormalisation of membrane theory, and the Lorentz generators are known in membrane theory [8]. Also, the relation of supermembrane theory to the background equation of motion is well understood [6,7].

However, we are still far from the complete resolution to these issues, and it is necessary to gain more experience of and insight into the physics of membranes and the matrix model. In this paper, we will consider a new deformation of the matrix model, based on an analogy to four-dimensional gauge theory. The model rather unexpectedly turns out to be equivalent to a regularised membrane theory on a certain curved background. General motivation to study this deformation would be twofold. First, it will be useful to have explicit examples, in order to understand the general relation between the matrix model and backgrounds. Second, as the deformed model has parameters which can be controlled freely, one might expect to find tractable and interesting physics by tuning them. Indeed, we find that the deformed model has stable solutions, which correspond to membranes with torus topology.

The explicit form of our deformation is motivated by the following consideration. The matrix model of M-theory and four-dimensional $\mathcal{N}=4$ supersymmetric Yang–Mills theory (SYM) are similar in many ways; in particular, they both have the maximal supersymmetry which is highly restrictive. The $\mathcal{N}=4$ SYM has conformal symmetry as well, and gives a prime example of a fixed line of the renormalisation group flow in the theory space of four-dimensional field theory. The deformations of $\mathcal{N}=4$ SYM which preserve the conformal symmetry are interesting from this point of view, and have been studied extensively, in particular for the case where the $\mathcal{N}=1$ supersymmetry is also preserved. One class of such deformations is the β-deformation with single deformation parameter [9–11]. Recently this deformation was revisited in the context of the AdS/CFT correspondence [12], and was generalised to a deformation with three parameters where the supersymmetry is in general completely broken [13]. For field-theoretic discussion and proofs of the conformal invariance (or the scale invariance) of the β-deformed theory in general, see [9–11,14,15].

The β-deformation (including its non-supersymmetric generalisation) consists in modifying the Yukawa couplings and the quartic couplings of scalar fields by certain phase factors.[1] As the matrix model of M-theory has similar Yukawa and scalar quartic couplings, phase factors can be introduced in a similar manner. It therefore seems natural to study this deformation of the matrix model, and consider whether it has also some significance.

One of the main results of this paper is that this deformed matrix model, introduced from a rather mathematical analogy to four-dimensional theory, indeed admits an interpretation from the M-theory point of view. We shall show that this model, under a certain scaling limit involving both N (the size of matrices) and the deformation parameter, can be considered as a matrix model of M-theory on a certain curved background, and that the background solves the supergravity equations of motion. We do this by showing that the matrix model arises from regularisation of supermembrane theory on that background. The background belongs to the so-called pp-wave (or plane-wave) backgrounds and is supported by a *non-constant* four-form flux. The

[1] Actually, this prescription is only true in the leading order of $1/N$. In order to maintain the scale invariance, one in general needs to introduce $1/N$ corrections to various couplings. We will comment on this issue for the matrix model at the end of Section 4.

pp-wave background with a non-constant flux in eleven-dimensional supergravity is first considered in [16].

In [17], another deformed matrix model, the BMN matrix model, was proposed, which is characterised by mass terms for scalars and fermions, and by cubic scalar couplings. In [18,19] it was shown that this model is equivalent to regularised supermembrane theory on a supergravity background, which is also of the pp-wave type, but is supported by a constant four-form flux. This analysis for the original maximally supersymmetric BMN matrix model was later generalised to less supersymmetric models [20]. The linearised coupling between general supergravity backgrounds and the matrix model was studied in [21–23]. We also mention that a matrix model similar to ours are considered and used in [24] to understand the AdS/CFT correspondence. In [25] similar deformation for the zero-dimensional IKKT matrix model is considered, and stable solutions similar to those considered in this paper are discussed. Different analogies of the β-deformation in the M-theory context are pursued in [26,27].

The organisation of this paper is as follows. We first define our deformed matrix model in Section 2. Its supersymmetry is also considered. Section 3 is devoted to establishing the equivalence between the deformed matrix model and the regularised supermembrane theory on the background. In Section 4 we discuss some stable solutions of the model. The solutions correspond to membranes with the topology of a torus wrapped in general several times on a certain $S^1 \times S^1$. We show that some physically distinct configurations in conventional membrane theory are indistinguishable in the matrix model. In Section 5 we consider the matrix model associated with the background which involve both our deformation parameters and the BMN-like mass parameters. In particular, we find a class of models where the stable membrane configuration has the topology of either a torus or a sphere, depending on the values of the deformation parameters. We conclude in Section 6 with some discussion.

2. Deformation

In this section we describe the deformation, which is motivated by an analogy to the β-deformation (and its non-supersymmetric generalisation) of $\mathcal{N}=4$ SYM in four dimension. The deformation can be described succinctly by using the $*$-product notation [12,13] explained below.

The original matrix model is governed by the Hamiltonian

$$H = \mathrm{Tr}\left(\frac{1}{2}\left(\Pi^\alpha\right)^2 - \frac{1}{4}\left[X^\alpha, X^\beta\right]^2 + \Psi^T\gamma^\alpha[X_\alpha, \Psi]\right), \tag{1}$$

and the phase space constraints corresponding to the $U(N)$ gauge symmetry,

$$\left[X^\alpha, \Pi^\alpha\right] - 2i\Psi^T\Psi = 0, \tag{2}$$

where X^α ($\alpha = 1, \dots, 9$) are $N \times N$ Hermitian matrices and Π^α are their conjugate momenta, and Ψ^a ($a = 1, \dots, 16$) are fermionic $N \times N$ Hermitian matrices which are canonically conjugate to themselves. The relevant Dirac brackets are

$$\left\{\left(X^\alpha\right)^i{}_j, \left(\Pi^\beta\right)^k{}_l\right\}_{\mathrm{D.B.}} = \delta^{\alpha\beta}\delta^i{}_l\delta^k{}_j \quad (i, j, k, l = 1, \dots, N), \tag{3}$$

$$\left\{\left(\Psi^a\right)^i{}_j, \left(\Psi^b\right)^k{}_l\right\}_{\mathrm{D.B.}} = -\frac{i}{2}\delta^{ab}\delta^i{}_l\delta^k{}_j \quad (i, j, k, l = 1, \dots, N). \tag{4}$$

The model has a $SO(9)$ symmetry under which X transforms as a $SO(9)$ vector and Ψ as a 16-component real spinor. We choose a real and symmetric representation of 16×16 gamma matrices γ^α. Equivalently, the model is described by the action

$$S=\int \mathrm{Tr}\left(\frac{1}{2}\left(D_0X^\alpha\right)^2+\frac{1}{4}\left[X^\alpha,X^\beta\right]^2+i\Psi^T D_0\Psi-\Psi^T\gamma^\alpha\left[X^\alpha,\Psi\right]\right)dt, \tag{5}$$

where the covariant derivative is given by $D_0 f=\partial_0 f-[-iA_0,f]$.

The class of deformation we consider is in general parametrised by six parameters. Before describing the general deformation we will focus on a particular case which is parametrised by a single-parameter β as it is much easier to grasp.

We should first introduce some notations. We choose two commuting $U(1)$ charges in the "flavour" $SO(9)$ symmetry, the rotation in the 12 plane and 34 plane, and call them as $Q_{(1)}$ and $Q_{(2)}$. We define complex combinations of scalars,

$$Z=\frac{X^1+iX^2}{\sqrt{2}},\qquad W=\frac{X^3+iX^4}{\sqrt{2}}, \tag{6}$$

which have definite $Q_{(1)}$, $Q_{(2)}$ charges. We denote the $U(1)$ charges of a field f appearing in the matrix model Hamiltonian by $Q^f_{(1)}$ and $Q^f_{(2)}$; for example, $Q^Z_{(1)}=1$, $Q^Z_{(2)}=0$ and $Q^{W^\dagger}_{(2)}=-1$.

We then introduce the $*$-product by

$$f*g=e^{i\pi\beta(Q^f_{(1)}Q^g_{(2)}-Q^g_{(1)}Q^f_{(2)})}fg. \tag{7}$$

In this paper we will only consider the case where β is real. Thus the $*$-product is the usual product simply modified by a flavour-dependent phase factor.

Our deformation consists in replacing all commutators appearing in the original matrix model Hamiltonian (1), or, equivalently, in the action (5), by the $*$-commutator defined by

$$[f,g]_*=f*g-g*f. \tag{8}$$

Thus, the Hamiltonian and the action of the deformed model are

$$H=\mathrm{Tr}\left(\frac{1}{2}\left(\Pi^\alpha\right)^2-\frac{1}{4}\left(\left[X^\alpha,X^\beta\right]_*\right)^2+\Psi^T\gamma^\alpha[X_\alpha,\Psi]_*\right), \tag{9}$$

$$S=\int \mathrm{Tr}\left(\frac{1}{2}\left(D_0X^\alpha\right)^2+\frac{1}{4}\left(\left[X^\alpha,X^\beta\right]_*\right)^2+i\Psi^T D_0\Psi-\Psi^T\gamma^\alpha\left[X^\alpha,\Psi\right]_*\right)dt. \tag{10}$$

The phase space constraints (2) are unchanged. In the above formulae, relevant expressions in the bosonic potential term are

$$\begin{aligned}
&[Z,W]_*=e^{i\pi\beta}ZW-e^{-i\pi\beta}WZ,\\
&\left[Z,W^\dagger\right]_*=e^{-i\pi\beta}ZW^\dagger-e^{+i\pi\beta}W^\dagger Z,
\end{aligned} \tag{11}$$

and its complex conjugate. Other commutators such as $[X^5,Z]$ or $[Z,Z^\dagger]$ are left unchanged. For fermionic terms, we need projectors such as $(1\pm\gamma^{\bar{z}z})/2$ which pick up components with $Q^{(1)}$-charge $\pm 1/2$.[2] For example, we have

$$\begin{aligned}
Z*\Psi&=Z*\left(\frac{1+\gamma^{\bar{w}w}}{2}\Psi+\frac{1-\gamma^{\bar{w}w}}{2}\Psi\right)\\
&=e^{\frac{i\pi\beta}{2}}Z\frac{1+\gamma^{\bar{w}w}}{2}\Psi+e^{-\frac{i\pi\beta}{2}}Z\frac{1-\gamma^{\bar{w}w}}{2}\Psi,
\end{aligned} \tag{12}$$

[2] We use $\gamma^{z\bar{z}}=\frac{1}{2}(\gamma^z\gamma^{\bar{z}}-\gamma^{\bar{z}}\gamma^z)$, $\gamma^z=\frac{1}{\sqrt{2}}(\gamma^1+i\gamma^2)$ and $\gamma^{\bar{z}}=\frac{1}{\sqrt{2}}(\gamma^1-i\gamma^2)$.

which can also be written as

$$Z * \Psi = Z e^{i\pi\beta\frac{1}{2}\gamma^{\bar{w}w}}\Psi. \tag{13}$$

The generalisation of this one-parameter deformation is obtained just by extending the definition of the $*$-product to include more general $U(1)$ generators in the $SO(9)$ symmetry. There are four independent commuting $U(1)$ charges. We choose the rotations in the 12, 34, 56, 78 planes, and label them by indices $I, J, \ldots = 1, 2, 3, 4$. For each pair of $U(1)$ charges (I, J) a deformation parameter, $\beta^{(IJ)} = -\beta^{(JI)}$, can be introduced. Hence there are six independent parameters. The generalised $*$-product is now given by

$$f * g = \left(e^{i\pi\sum_{I<J}\beta^{(IJ)}(Q^f_{(I)}Q^g_{(J)} - Q^f_{(J)}Q^g_{(I)})}\right) fg. \tag{14}$$

The one-parameter deformation described before is the special case where the only non-zero deformation parameter is $\beta^{(12)} = \beta(= -\beta^{(21)})$.

We group eight Hermitian scalar fields into four complex fields Z^I $(I = 1, 2, 3, 4)$ as

$$Z^1 = \frac{1}{\sqrt{2}}\left(X^1 + iX^2\right), \tag{15}$$

$$Z^2 = \frac{1}{\sqrt{2}}\left(X^3 + iX^4\right), \tag{16}$$

$$Z^3 = \frac{1}{\sqrt{2}}\left(X^5 + iX^6\right), \tag{17}$$

$$Z^4 = \frac{1}{\sqrt{2}}\left(X^7 + iX^8\right). \tag{18}$$

The generalised $*$-commutators for bosonic fields in (9) and (10) are now given by

$$\left[Z^I, Z^J\right]_* = e^{i\pi\beta^{(IJ)}} Z^I Z^J - e^{-i\pi\beta^{(IJ)}} Z^J Z^I \tag{19}$$

and

$$\left[Z^I, Z^{\dagger J}\right]_* = e^{-i\pi\beta^{(IJ)}} Z^I Z^{\dagger J} - e^{i\pi\beta^{(IJ)}} Z^{\dagger J} Z^I. \tag{20}$$

For indices I, J we will not imply the summation over repeated indices.

In general the $SO(9)$ symmetry is broken down into the $U(1)^4$ symmetry spanned by $Q_{(I)}$'s. If $\beta^{(I4)} = 0$, the matrix model can be considered as a result of dimensional reduction of the three-parameter deformation of $D = 3 + 1$, $\mathcal{N} = 4$ SYM introduced in [13]. Furthermore if $\beta^{(12)} = \beta^{(23)} = \beta^{(31)}$ the model is a dimensionally reduced form of the $D = 3 + 1$, β-deformed $\mathcal{N} = 4$ SYM with $\mathcal{N} = 1$ supersymmetry, and has corresponding supersymmetry.

The deformation in general breaks both kinematical and dynamical supersymmetry of the original matrix model. However, for special values of the deformation parameters, a part of the supersymmetry remain unbroken, provided that the supersymmetry transformation law is appropriately modified. For these special values, there exist 16-component spinors $\delta\xi$ whose $U(1)$-charges $s_I = \pm\frac{1}{2}$ satisfy

$$\sum_J \beta^{(IJ)} s_J = 0, \tag{21}$$

for all $I = 1, 2, 3, 4$. This relation is equivalent to the condition that the $*$-product between $\delta\xi$ and any field reduces to the ordinary product. This property of $\delta\xi$, which we call the $*$-neutrality, ensure the invariance of the action (10) under the modified supersymmetry transformation with $\delta\xi$ as the infinitesimal parameter. This is true for both dynamical and kinematical supersymmetry. The modified transformation law for the dynamical supersymmetry is given by

$$\delta X^\alpha = i\Psi^T \gamma^\alpha \delta\xi, \tag{22}$$

$$\delta\Psi = \left(\frac{1}{2} D_0 X^\alpha \gamma^\alpha - \frac{i}{4}\left[X^\alpha, X^\beta\right]_* \gamma^{\alpha\beta}\right)\delta\xi. \tag{23}$$

Because of the $*$-neutrality, one can move $\delta\xi$ in the variation of the action, without producing extra phase factors. Then one can show the invariance of the action, in the same manner as in the original matrix model, using the associativity of the $*$-product and the property that if the product fg is uncharged, $f * g = fg$. The transformation law for the kinematical supersymmetry is not modified and given by

$$\delta\Psi = \delta\xi \mathbf{1}, \qquad \delta X^\alpha = 0, \tag{24}$$

where $\mathbf{1}$ is the $N \times N$ unit matrix.

Let us investigate the condition on parameters $\beta^{(IJ)}$ under which $*$-neutral spinors satisfying (21) exist. The 16-component spinors can be spanned by the following basis vectors,

$$|s_1, s_2, s_3, s_4\rangle, \tag{25}$$

labelled by the eigenvalues of the four $U(1)$ charges. We abbreviate, for instance, $|\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\rangle$ by $|+, +, -, +\rangle$.

It is sufficient to consider the case where $|+, +, +, +\rangle$ is $*$-neutral, which automatically implies that $|-, -, -, -\rangle$ is also $*$-neutral. Choosing other element of the basis (25) is related by a simple redefinition of β's. For example, using $|+, +, +, -\rangle$ instead of $|+, +, +, +\rangle$ amounts to flipping the sign of $\beta^{(I4)}$, $I = 1, 2, 3$. From (21), one finds four linear equations, for six variables $\beta^{(IJ)}$. Actually, it is easy to see that only three of the equations are linearly independent, and hence β's are parametrised by three parameters. Concretely, we choose $\beta^{(12)}$, $\beta^{(23)}$, $\beta^{(31)}$, and express the others by

$$\beta^{(14)} = \beta^{(31)} - \beta^{(12)}, \qquad \beta^{(24)} = \beta^{(12)} - \beta^{(23)}, \qquad \beta^{(34)} = \beta^{(23)} - \beta^{(31)}. \tag{26}$$

Under this condition, the deformed model has the dynamical and kinematical supersymmetry, each with two-component supercharges.

To consider the case of the higher supersymmetry, there are two essentially distinct possibilities, namely, to add (a) $|+, +, +, -\rangle$ and $|-, -, -, +\rangle$, or (b) $|+, +, -, -\rangle$ and $|-, -, +, +\rangle$, as $*$-neutral spinors. After reducing eight linear equations following from (21) to independent equations, one finds that the possibility (a) leads to the single-parameter deformation with

$$\beta^{(I4)} = 0, \qquad \beta^{(12)} = \beta^{(23)} = \beta^{(31)}, \tag{27}$$

which is equivalent to the condition that the deformed model is the dimensionally reduced version of $D = 3 + 1$, β-deformed SYM with the $\mathcal{N} = 1$ supersymmetry, discussed before. The case (b) yields the condition

$$\beta^{(12)} = 0, \qquad \beta^{(34)} = 0, \qquad \beta^{(14)} = \beta^{(31)} = \beta^{(23)} = \beta^{(24)}, \tag{28}$$

which also give a single-parameter deformation. These conditions (28) cannot be made equivalent to (27) by reshuffling of the coordinates; one can show that the bosonic flavour symmetry in this case is completely broken down into $U(1)^4$ symmetry, whereas in the case of (27), the flavour $SO(3)$ symmetry (the rotation within X^7, X^8, X^9 directions) is present. If we further increase the number of $*$-neutral spinors, we only arrive at the trivial case where all β's are zero. We have thus essentially completed the classification of the supersymmetry of the β-deformed matrix model.

3. Deformed model and $D = 11$ SUGRA background

The aim of this section is to show that the deformed matrix model, described in Section 2, in a certain scaling limit, is equivalent to a regularised membrane theory on a certain curved background of eleven-dimensional supergravity. This makes a good case for considering the deformed model as the matrix model of M-theory on that background.

We will begin by a brief review, in Section 3.1, of bosonic membrane theory on flat spacetime in the lightcone gauge, and the regularisation procedure, the matrix regularisation, in order to collect necessary formulae. Then we describe, in Section 3.2, how the scaling limit naturally arises from consideration of the $*$-commutator in the light of matrix regularisation. Under this scaling limit, we then describe the continuum theory corresponding to the deformed model, in Section 3.3, and show that this continuum theory is equivalent to lightcone membrane theory on a certain background. We then show that this background solves the equations of motion of eleven-dimensional supergravity, including an overall factor. Up to this point, we will confine ourselves to the bosonic degrees of freedom of membrane theory. In Section 3.4, we analyse the fermionic degrees of freedom, and show that the fermionic sector of the deformed model precisely matches with that of regularised lightcone supermembrane theory on the background.

3.1. Review of bosonic membrane in lightcone gauge and matrix regularisation on flat spacetime

This subsection consists of a brief review of the lightcone gauge formalism (in the spirit of [28]) for the bosonic part of membrane theory on flat spacetime, and the matrix regularisation procedure which turns lightcone membrane theory into the matrix model.

The action of the bosonic part of membrane theory on flat spacetime is given by its 3-volume,

$$S = \int \mathcal{L}\, d^2\sigma\, d\tau = -T \int \sqrt{-\det(h_{ij})}\, d^2\sigma\, d\tau,$$
$$h_{ij} = \eta_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma^i} \frac{\partial x^\nu}{\partial \sigma^j} \quad (i, j = 0, 1, 2), \tag{29}$$

where $x^\mu(\sigma^0, \sigma^1, \sigma^2) = x^\mu(\tau, \sigma^1, \sigma^2)$ $(\mu = 0, 1, \ldots, 10)$ gives the parametrisation of the membrane worldvolume embedded in spacetime, and T is the membrane tension. We hereafter mostly work in the length scale in which $T = 1$. The canonical momenta

$$\mathcal{P}_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\tau x^\mu)} \tag{30}$$

satisfy the following identities (phase space constraints), which represent the reparametrisation invariance of the action,

$$\eta^{\mu\nu}\mathcal{P}_\mu \mathcal{P}_\nu + \frac{1}{2}\{x^\mu, x^\nu\}^2 = 0, \tag{31}$$

$$\mathcal{P}_\mu \frac{\partial x^\mu}{\partial \sigma^r} = 0 \quad (r = 1, 2), \tag{32}$$

where we have defined

$$\{f, g\} = \frac{\partial f}{\partial \sigma^1}\frac{\partial g}{\partial \sigma^2} - \frac{\partial f}{\partial \sigma^2}\frac{\partial g}{\partial \sigma^1}, \tag{33}$$

where f and g are functions defined on the (σ^1, σ^2)-space. We shall call this structure, analogous to the Poisson brackets (for a system with one degree of freedom), as the Lie brackets in this paper.

In the lightcone gauge, we first identify the τ coordinate with the spacetime coordinate x^+,[3]

$$\tau = x^+. \tag{34}$$

We then partially fix (σ^1, σ^2) coordinates by requiring the momentum density $\mathcal{P}_-$ to be constant in the (σ^1, σ^2) directions, or equivalently,

$$\mathcal{P}_- = \frac{P_-}{[\sigma]}, \tag{35}$$

where $P_- = \int \mathcal{P}_- \, d^2\sigma$ is the total momentum in the $-$ direction and $[\sigma] = \int d^2\sigma$ is a constant representing the total area of the base space (σ^1, σ^2).[4]

These gauge fixing conditions (34), (35) allows one to explicitly solve the phase space constraints (31), (32); we have

$$\frac{\partial x^-}{\partial \sigma^r} = -\frac{[\sigma]}{P_-}\mathcal{P}^\alpha \frac{\partial x^\alpha}{\partial \sigma^r} \quad (r = 1, 2), \tag{36}$$

$$-\mathcal{P}_+ = \frac{[\sigma]}{2(-P_-)}\left(\left(\mathcal{P}^\alpha\right)^2 + \frac{1}{2}\left\{x^\alpha, x^\beta\right\}^2\right). \tag{37}$$

Here indices α, β for transverse directions run through 1 to 9.

From (37), we obtain the Hamiltonian

$$H = -P_+ = \int (-\mathcal{P}_+)\, d^2\sigma = \int \frac{[\sigma]}{2(-P_-)}\left(\left(\mathcal{P}^\alpha\right)^2 + \frac{1}{2}\left\{x^\alpha, x^\beta\right\}^2\right) d^2\sigma. \tag{38}$$

We put the factor -1 before the lightcone component of the momentum because $-\mathcal{P}_- > 0$ and $-\mathcal{P}_+ > 0$ hold. Eq. (36) implies the integrability condition

$$\left\{x^\alpha, \mathcal{P}^\alpha\right\} = 0. \tag{39}$$

This equation acts as a phase space constraint of lightcone membrane theory, and corresponds to the residual reparametrisation invariance by area preserving diffeomorphisms acting on the (σ^1, σ^2)-space.

The bosonic sector of the original matrix model, described by the Hamiltonian (1) and the constraint (2), can be considered as a regularised version of the continuum theory described by (38) and (39). Let us recall basic relations involved in this matrix regularisation. In matrix regularisation, functions $f(\sigma^1, \sigma^2), g(\sigma^1, \sigma^2), \ldots$ are turned into $N \times N$ matrices $\hat{f} = \rho(f), \hat{g} = \rho(g), \ldots$. These matrices give discrete approximation to the corresponding functions. Some operations acting on functions have counterparts acting on the corresponding

[3] Our lightcone convention is $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^{10})$.

[4] The constant $[\sigma]$ depends on conventions, and cancels out in any relation between physical observables.

matrices. This correspondence, apart from the linearity of $\rho(f)$, can be summarised as follows

$$\rho(fg) \approx \frac{1}{2}\big(\rho(f)\rho(g) + \rho(g)\rho(f)\big), \tag{40}$$

$$\rho\big(\{f, g\}\big) \approx -i\frac{2\pi N}{[\sigma]}\big[\rho(f), \rho(g)\big], \tag{41}$$

$$\frac{1}{[\sigma]}\int f\, d^2\sigma \approx \frac{1}{N}\operatorname{Tr}\rho(f). \tag{42}$$

Left-hand sides and right-hand sides of these formulae are equal up to higher order corrections in $1/N$. The first relation means that multiplication of two functions corresponds to multiplication (more precisely taking one-half of the anti-commutator) of the corresponding matrices. The second relation then tells us that Lie brackets between two functions correspond to the commutator of the corresponding matrices multiplied by a factor proportional to N.[5] In particular, this relation implies that the commutator of two matrices of order unity is of order $1/N$.

After an appropriate rescaling of the matrices and the time coordinate, the matrix-regularised Hamiltonian (and the constraint) becomes identical to that of the matrix model. See Appendix B for details.

3.2. ∗-commutator and scaling limit

The first step towards the continuum version of the deformed matrix model is to find the continuum counterpart of the ∗-commutator. A scaling limit involving N and the deformation parameters β naturally arises in this consideration. This scaling limit plays an essential role in this paper.

We first focus on the single-parameter deformation. Defining $\hat{z} = \rho(z)$, $\hat{w} = \rho(w)$, we have

$$\begin{aligned}[\hat{z}, \hat{w}]_* &= e^{i\pi\beta}\hat{z}\hat{w} - e^{-i\pi\beta}\hat{w}\hat{z} \\ &\approx [\hat{z}, \hat{w}] + i2\pi\beta\frac{1}{2}(\hat{z}\hat{w} + \hat{w}\hat{z}),\end{aligned} \tag{43}$$

where we have assumed $\beta \ll 1$. The constant rescaling between $\hat{z}, \hat{w}$ and Z, W, noted at the end of Section 3.1, is without effect, as all expressions in this subsection are homogeneous in $\hat{z}$ and $\hat{w}$.

As noted below (41), the commutator term above is of order $1/N$. Hence in the regime where β is also of order $1/N$, or equivalently, if we fix βN when taking N large, the two terms in (43) are comparable and both contribute to the dynamics of membranes. Throughout this paper, we shall assume this scaling limit.[6] We stress that the deformation remains non-trivial in the $N \to \infty$ limit, albeit the scaling limit $\beta \sim 1/N$, since the commutator term already is of order $1/N$. Then,

[5] The factor before the commutator in (41) can be understood as follows. By using the well-known mathematical analogy between matrix regularisation and quantisation of a system with single degree of freedom, it corresponds to $1/(i\hbar)$ in quantum mechanics. Now, every state vector in quantum mechanics occupies the area $2\pi\hbar$ in the (x, p)-space (in the semi-classical regime). In matrix regularisation there are N independent "state vectors", and the (σ^1, σ^2)-space is divided into N parts with equal area $[\sigma]/N$. Hence $[\sigma]/N$ corresponds to $2\pi\hbar$, and $-i2\pi N/[\sigma]$ to $1/(i\hbar)$.

[6] The decomposition of the ∗-commutator into a commutator and an anti-commutator piece is possible even if $\beta \sim 1$. In this case, the commutator term, which represents the effect of the membrane tension, becomes negligible compared to the anti-commutator term. This limiting case, which might be called as the "membrane bit" regime, might also be interesting.

applying (41) and (40) to (43), we obtain

$$[\hat{z},\hat{w}]_* \approx i\frac{[\sigma]}{2\pi N}\left(\rho\big(\{z,w\}\big)+\beta N\frac{(2\pi)^2}{[\sigma]}\rho(zw)\right) \tag{44}$$

$$= i\frac{[\sigma]}{2\pi N}\rho\left(\{z,w\}+\beta N\frac{(2\pi)^2}{[\sigma]}zw\right). \tag{45}$$

Therefore, the deformation (replacing the commutator by the $*$-commutator) amounts, in the continuum theory, to replacing the Lie brackets $\{z,w\}$ as

$$\{z,w\}\longrightarrow\{z,w\}+\beta N\frac{(2\pi)^2}{[\sigma]}zw. \tag{46}$$

Similarly, the Lie brackets $\{z,\bar{w}\}$ should be replaced as

$$\{z,\bar{w}\}\longrightarrow\{z,\bar{w}\}-\beta N\frac{(2\pi)^2}{[\sigma]}z\bar{w}. \tag{47}$$

The generalisation to the six-parameter deformation is straightforward. We assume all deformation parameters $\beta^{(IJ)}$ to be of order $1/N$. Then the deformation amounts to

$$\left\{z^I,z^J\right\}\longrightarrow\left\{z^I,z^J\right\}+\beta^{(IJ)}N\frac{(2\pi)^2}{[\sigma]}z^Iz^J, \tag{48}$$

$$\left\{z^I,\overline{z^J}\right\}\longrightarrow\left\{z^I,\overline{z^J}\right\}-\beta^{(IJ)}N\frac{(2\pi)^2}{[\sigma]}z^I\overline{z^J}, \tag{49}$$

in the continuum theory. Similar replacements are necessary for the Lie brackets between scalar fields and fermionic fields, which can be derived using projectors acting on fermionic fields as in (12),

$$\left\{z^I,\psi\right\}\longrightarrow\left\{z^I,\psi\right\}+\sum_J\beta^{(IJ)}N\frac{(2\pi)^2}{[\sigma]}z^I\left(\frac{1}{2}\gamma^{\bar{J}J}\psi\right), \tag{50}$$

$$\left\{\overline{z^I},\psi\right\}\longrightarrow\left\{\overline{z^I},\psi\right\}-\sum_J\beta^{(IJ)}N\frac{(2\pi)^2}{[\sigma]}\overline{z^I}\left(\frac{1}{2}\gamma^{\bar{J}J}\psi\right). \tag{51}$$

These will be used in Section 3.4.

3.3. Background

Now that we know the continuum counterpart of the $*$-commutator, it is easy to obtain the continuum theory which gives the deformed matrix model upon matrix regularisation; one should just apply the substitution (46), (47) and their complex conjugates to the Hamiltonian of the original continuum theory (38). Relevant terms in (38) are

$$\int\frac{[\sigma]}{(-P_-)}\big(\{z,w\}\{\bar{z},\bar{w}\}+\{z,\bar{w}\}\{\bar{z},w\}\big)\,d^2\sigma, \tag{52}$$

and we get the continuum version of the deformed matrix model,

$$H = (\text{orig.}) + \int \left(2\frac{[\sigma]}{(-P_-)} \left(\beta N \frac{(2\pi)^2}{[\sigma]} \right)^2 z w \bar{z} \bar{w} + \frac{[\sigma]}{(-P_-)} \left(\beta N \frac{(2\pi)^2}{[\sigma]} \right) \right.$$
$$\left. \times \left(zw\{\bar{z}, \bar{w}\} + \bar{z}\bar{w}\{z, w\} - z\bar{w}\{\bar{z}, w\} - \bar{z}w\{z, \bar{w}\} \right) \right) d^2\sigma, \tag{53}$$

where (orig.) stands for the original Hamiltonian (38). The constraint (39) is left as it is.

Below, we shall show that this continuum theory described by the Hamiltonian (53) is identical to membrane theory on a certain background. The general action for the bosonic sector of membrane theory coupled to the metric $G_{\mu\nu}(x)$ and the three-form gauge field $A_{\mu\nu\rho}(x)$ is given by

$$S = S_1 + S_2, \tag{54}$$
$$S_1 = -T \int \sqrt{-\det(h_{ij})}\, d^2\sigma\, d\tau, \tag{55}$$
$$h_{ij} = G_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma^i} \frac{\partial x^\nu}{\partial \sigma^j} \quad (i, j = 0, 1, 2), \tag{56}$$
$$S_2 = T \int A_{\mu\nu\rho} \frac{\partial x^\mu}{\partial \sigma^0} \frac{\partial x^\nu}{\partial \sigma^1} \frac{\partial x^\rho}{\partial \sigma^2} d^3\sigma. \tag{57}$$

We have put the membrane tension $T(=1)$ in (57) to make $A_{\mu\nu\rho}$ dimensionless.

Introduction of these backgrounds affects the phase space constraint (31), (32): the flat metric is replaced by the curved metric, and the three-form gauge field shifts the momenta $\mathcal{P}_\mu$ to $\mathcal{P}_\mu - (1/2)A_{\mu\nu\rho}\{x^\nu, x^\rho\}$. Hence we have

$$G^{\mu\nu} \left(\mathcal{P}_\mu - \frac{1}{2} A_{\mu\rho\sigma} \left\{ x^\rho, x^\sigma \right\} \right) \left(\mathcal{P}_\nu - \frac{1}{2} A_{\nu\lambda\tau} \left\{ x^\lambda, x^\tau \right\} \right)$$
$$+ \frac{1}{2} G_{\mu\rho} G_{\nu\sigma} \left\{ x^\mu, x^\nu \right\} \left\{ x^\rho, x^\sigma \right\} = 0, \tag{58}$$
$$\left(\mathcal{P}_\mu - \frac{1}{2} A_{\mu\nu\rho} \left\{ x^\nu, x^\rho \right\} \right) \frac{\partial x^\mu}{\partial \sigma^r} = 0 \quad (r = 1, 2). \tag{59}$$

As we shall soon see, it is sufficient to introduce only G^{--} (or G_{++}) and $A_{+\alpha\beta}$ components of the background fields. The curved background of this type (often called the pp-wave or the plane-wave background) is particularly well suited to the lightcone gauge formalism; the lightcone membrane theory on the background can be derived simply by following the same steps as in Section 3.1, starting from (58) and (59). The resulting Hamiltonian is

$$H = -P_+ = (\text{orig.}) + \int \left(\frac{[\sigma]}{2(-P_-)} \left(\frac{-P_-}{[\sigma]} \right)^2 G^{--} - \frac{1}{2} A_{+\alpha\beta} \left\{ x^\alpha, x^\beta \right\} \right) d^2\sigma. \tag{60}$$

The constraint (39) is not affected by the introduction of the background.

The background fields can now be identified by comparing (53) and (60). The terms linear in β in (53) match with the terms containing $A_{+\alpha\beta}$ in (60), and the term quadratic in β corresponds to the term involving G^{--}; we get

$$G^{--} = 4(2\pi)^4 \alpha^2 |z|^2 |w|^2, \tag{61}$$
$$A_{+\bar{z}\bar{w}} = -(2\pi)^2 \alpha z w, \tag{62}$$
$$A_{+zw} = -(2\pi)^2 \alpha \bar{z} \bar{w}, \tag{63}$$

$$A_{+\bar{z}w} = (2\pi)^2 \alpha z \bar{w}, \tag{64}$$
$$A_{+z\bar{w}} = (2\pi)^2 \alpha \bar{z} w. \tag{65}$$

Here we have defined

$$\alpha = \frac{T}{-P_-} \beta N. \tag{66}$$

The background fields depend on the deformation parameter β only through this combination.[7] In other words, this rescaled parameter α, rather than β, is the appropriate parameter to measure the deformation of the background from flat spacetime. The gauge invariant four-form flux defined by

$$F_{\mu\nu\rho\sigma} = \partial_\mu A_{\nu\rho\sigma} \pm (\text{cyclic permutations}), \tag{67}$$

is given by

$$\begin{aligned} F_{+zw\bar{w}} &= 2(2\pi)^2 \alpha \bar{z}, \\ F_{+wz\bar{z}} &= -2(2\pi)^2 \alpha \bar{w}, \\ F_{+\bar{z}\bar{w}w} &= 2(2\pi)^2 \alpha z, \\ F_{+\bar{w}\bar{z}z} &= -2(2\pi)^2 \alpha w. \end{aligned} \tag{68}$$

Thus, the four-form flux is not constant and depends linearly on transverse coordinates.

It is crucial to see whether these background fields satisfy equations of motion of eleven-dimensional supergravity. As is well known, and as we shall explain in Section 3.4 (see in particular discussion around (85)–(87)), the κ-symmetry of the supermembrane action is related to the following equations of motion for the background

$$-R_{\mu\nu\rho}{}^{\mu} = -\frac{1}{12} F_{\nu\mu_1\mu_2\mu_3} F^{\mu_3\mu_2\mu_1}{}_{\rho} + \frac{1}{144} F_{\mu_1\mu_2\mu_3\mu_4} F^{\mu_4\mu_3\mu_2\mu_1} G_{\nu\rho}, \tag{69}$$
$$D_\mu F^{\mu\nu\rho\sigma} = 0, \tag{70}$$

where we have omitted the Chern–Simon coupling $\epsilon^{\nu\rho\sigma\mu_1\cdots\mu_5\tau_1\cdots\tau_5} F_{\mu_1\cdots\mu_5} F_{\tau_1\cdots\tau_5}$ in (70), as it vanishes trivially for our background. For our convention of the curvature tensor, see bosonic components of (C.19). These equations of motion reduce for the pp-wave background to

$$\frac{1}{2} \partial^\alpha \partial_\alpha (G^{--}) = \frac{1}{12} F_{+\alpha\beta\gamma} F_+{}^{\alpha\beta\gamma}, \tag{71}$$
$$\partial^\alpha F_{+\alpha\beta\gamma} = 0. \tag{72}$$

Our background solves these equations of motion. We wish to stress the strictness of this requirement. Not only the forms of various components of the background fields, but also the overall coefficient in (71) should be correct. This high degree of consistency is achieved without any artificial tuning of parameters. In particular, the numerical factor $\frac{1}{12}$ in the equation of motion for the metric (69) cannot be absorbed into rescaling of $A_{\mu\nu\rho}$ and $G_{\mu\nu}$, since the normalisation convention of them is already fixed by choosing the membrane action to be (54)–(57).[8]

[7] We have restored the membrane tension T to see that α has the dimension of $(\text{length})^{-2}$.

[8] The special rescaling of the background fields $G'_{\mu\nu} = \lambda G_{\mu\nu}$ and $A'_{\mu\nu\rho} = \lambda^{3/2} A_{\mu\nu\rho}$ only changes the action by an overall factor, which can be absorbed into a redefinition of the tension T. Hence this rescaling should not and does not change the physics of the background. In particular, it does not affect the equation of motion (69). We have fixed this rescaling by choosing the components of the metric other than G_{++} to be equal to those of the flat spacetime metric.

For the general six-parameter deformation, we should apply the substitution (48), (49) to the Hamiltonian (38), and again compare it to the expression (60). By using the parameter $\alpha^{(IJ)}$ for the continuum theory defined by

$$\alpha^{(IJ)} = \frac{T}{-P_-}\beta^{(IJ)}N, \tag{73}$$

the background thus identified is given by

$$G^{--} = 2(2\pi)^4 \sum_I \sum_J \left(\alpha^{(IJ)}\right)^2 \left|z^I\right|^2 \left|z^J\right|^2, \tag{74}$$

$$A_{+I\bar{J}} = (2\pi)^2 \alpha^{(IJ)} \overline{z^I} z^J, \tag{75}$$

$$A_{+IJ} = -(2\pi)^2 \alpha^{(IJ)} \overline{z^I}\,\overline{z^J}, \tag{76}$$

$$A_{+\bar{I}\bar{J}} = -(2\pi)^2 \alpha^{(IJ)} z^I z^J, \tag{77}$$

with the four-form flux

$$F_{+IJ\bar{J}} = 2(2\pi)^2 \alpha^{(IJ)} \overline{z^I}, \tag{78}$$

$$F_{+\bar{I}\bar{J}J} = 2(2\pi)^2 \alpha^{(IJ)} z^I. \tag{79}$$

These background fields again solve the equations of motion (71), (72).

3.4. Fermionic sector

So far, we have focused on bosonic degrees of freedom, and have identified the background (74)–(79). We will now consider the fermionic sector of supermembrane theory propagating on this background, and show that it precisely reproduces the fermionic sector of the deformed matrix model; the prescription for the deformation, introduced in Section 2, is consistent with the eleven-dimensional physics, even including the fermionic sector.

Let us first recall some of the basic properties of supermembrane theory on curved backgrounds [6,7]. For brevity, we will frequently refer the reader to Appendix C for explicit formulae. There is a 32-component fermionic field on the membrane worldvolume, $\theta^a(\sigma^0, \sigma^1, \sigma^2)$, $a = 1, \dots, 32$. We write x^μ and $\theta^a = x^a$ collectively as x^A, $A = (\mu, a)$, which can be considered as coordinates on a superspace. The full action describing the supermembranes on general curved backgrounds is given by

$$S = S_1 + S_2, \tag{80}$$

$$S_1 = -\int \sqrt{-\det(h_{ij})}\, d^2\sigma\, d\tau, \tag{81}$$

$$h_{ij} = \eta_{\hat{\mu}\hat{\nu}} \pi_i{}^{\hat{\mu}} \pi_j{}^{\hat{\nu}} \quad (i, j = 0, 1, 2), \tag{82}$$

$$\pi_i{}^{\hat{\mu}} = \partial_i x^A E_A{}^{\hat{\mu}}, \tag{83}$$

$$S_2 = \int \partial_2 x^C \partial_1 x^B \partial_0 x^A A_{ABC}\, d^3\sigma. \tag{84}$$

Only in this section, we distinguish tangent-space indices $\hat{A} = (\hat{\mu}, \hat{a})$, $\hat{B} = (\hat{\nu}, \hat{b}), \dots$ from curved-space indices $A, B, \dots$. The background superfields in this action are (a part of) the supervielbein $E_A{}^{\hat{\mu}}(x^\nu, \theta^a)$ and the three-form gauge potential $A_{ABC}(x^\mu, \theta^a)$. If we take $\theta = 0$,

$E_\nu{}^{\hat{\mu}}$ and $A_{\mu\nu\rho}$ reduce to the bosonic component fields of supergravity, the elfbein and the three-form gauge field. For our background, the gravitino field is not present.

A key feature of supermembrane theory is the κ-symmetry (C.9), (C.10), which is the local fermionic symmetry with 16 anti-commuting parameters. The action is κ-symmetric if the following constraints on the superspace torsion and the field strength tensor are satisfied,[9]

$$T_{\hat{a}\hat{b}}{}^{\hat{\mu}} = -2i\Gamma^{\hat{\mu}}{}_{\hat{a}\hat{b}}, \tag{85}$$

$$F_{\hat{\mu}\hat{\nu}\hat{a}\hat{b}} = 2i\Gamma_{\hat{\mu}\hat{\nu}\hat{a}\hat{b}}, \tag{86}$$

$$0 = T_{\hat{\mu}\hat{\nu}}{}^{\hat{\rho}} = T_{\hat{a}\hat{\mu}}{}^{\hat{\nu}} = F_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{b}} = F_{\hat{\mu}\hat{a}\hat{b}\hat{c}} = F_{\hat{a}\hat{b}\hat{c}\hat{d}}. \tag{87}$$

Our conventions for the 32×32 gamma matrices $\Gamma^{\hat{\mu}}$, the superspace torsion T, and the four-form field strength F are summarised in Appendix C, (C.1)–(C.6), (C.8), (C.15)–(C.18).

These constraints are equivalent to the fundamental equations in the superspace formulation of eleven-dimensional supergravity [31,32]. In this way, the information of the equation of motion of eleven-dimensional supergravity is incorporated in supermembrane theory. The component formulation of supergravity, which was used in Section 3.3, is related to the superspace formulation in the following way. The equations of motion for the component supergravity fields are derived in [31,32] from the superspace constraints (85)–(87) by successive applications of Bianchi identities (C.21)–(C.22) fixing, in particular, the numerical coefficients in (69). Thus, the superspace formulation implies the component formulation. It is believed that the converse is also true: given component fields satisfying the component equations of motion, it is widely assumed that superfields exist which satisfy the conditions, (a) their lowest non-trivial components coincide with the given component fields, (b) they satisfy all of the superspace constraints (85)–(87), order by order in the θ-expansion. Although this property is not proven, we shall also assume this here, as it is highly unlikely that the two formulations are not equivalent, because of the strong restriction from the local supersymmetry.

The full construction of the superfields from given component fields is also technically hard in general, and is so far achieved only for special cases with a high degree of symmetry (see e.g. [33]). Instead of constructing the full superfields for our background, we will directly obtain the Hamiltonian in the lightcone gauge, just by assuming the existence of the full superfields. This is possible because, for the pp-wave background, the gauge fixing condition for the κ-symmetry,

$$\left(\Gamma^{\hat{+}}\right)^{\hat{a}}{}_{\hat{b}}\theta^{\hat{b}} = 0, \quad \theta^{\hat{a}} = \delta^{\hat{a}}_b\theta^b, \tag{88}$$

drastically reduces the number of relevant terms appearing in the Hamilton formalism, as we shall explain below. This type of argument is used for example in [34,35] for type IIB string theory. Our treatment more closely follows that in [36]. See [20] for the application to supermembrane theory on the pp-wave background with a constant flux.

We start by observing that, in the θ-expansion of a superfield, any pair of two θ's can be written in terms of fundamental bi-spinors $\Gamma^{\hat{\mu}_1\cdots\hat{\mu}_n}{}_{\hat{a}\hat{b}}\theta^{\hat{a}}\theta^{\hat{b}}$ with $n = 1, 2, 5$. Under the condition (88),

[9] One might wonder why these constraints include (through the definition of the torsion) superfields which are not contained in the action, i.e. the vielbein with spinor tangent index, $E_A{}^{\hat{a}}$, and the connection, $\Omega_{A\hat{B}}{}^{\hat{C}}$. An answer to this question is that these extra superfields act as kinds of integration constants: the action described by $E_A{}^{\hat{\mu}}$ and A_{ABC} is κ-symmetric when $E_A{}^{\hat{a}}$ and $\Omega_{A\hat{B}}{}^{\hat{C}}$ exist such that, together with given $E_A{}^{\hat{\mu}}$ and A_{ABC}, (85)–(87) are satisfied. Similar issues for superstring theory are discussed in [29]. We also remark that in [6,7] apparently weaker constraints are given, which are equivalent to (85)–(87) by a field redefinition [7,30].

only non-zero bi-spinors are those with single upper $\hat{-}$ index with arbitrary number of transverse $SO(9)$ vector indices, $\Gamma^{\hat{-}\hat{\alpha}_1\cdots\hat{\alpha}_{n-1}}{}_{\hat{a}\hat{b}}\theta^{\hat{a}}\theta^{\hat{b}}$.[10]

Another necessary ingredient is a gauge fixing for the background superfields [37], similar to the normal coordinates in Riemannian geometry. The gauge transformations for the backgrounds, namely, the general coordinate transformation on the superspace, the local Lorentz transformation, and the gauge transformation for the three-form field, are (partially) fixed by imposing the conditions (C.24)–(C.26). In this gauge, it is possible to formulate an algorithm which iteratively calculates higher order terms in the θ-expansion, based on a part (but not all) of the constraints and Bianchi identities. It is known that, as a result, the coefficients in the θ-expansion in this gauge are expressed in terms of $R_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^{\hat{\sigma}}$, $F_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}$, $\Omega_{\hat{\mu}\hat{\nu}}{}^{\hat{\rho}}$, $E_{\mu}{}^{\hat{\nu}}$, and their covariant derivatives in the bosonic directions, evaluated at $\theta=0$. The vector indices of the fundamental bi-spinors should be contracted with these structures, except for the indices of the original superfield.

However, on the pp-wave background (which does not depend on x^+), these expressions are trivial except for $E_+{}^{\hat{-}}|_{\theta=0}=\frac{1}{2}G^{--}$, $\Omega_{+\alpha+}|_{\theta=0}=\frac{1}{2}\partial_\alpha G^{--}$, $R_{+\alpha+\beta}|_{\theta=0}=-\frac{1}{2}\partial_\alpha\partial_\beta G^{--}$, and their covariant derivatives in the transverse directions. Thus, there are no lower $\hat{-}$ indices to match the upper $\hat{-}$ indices coming from the bi-spinors. Hence, only a few terms in the θ-expansion of various superfields can survive in the lightcone gauge formulation of supermembrane theory on the pp-wave background. In fact, one can show, with the help of some dimensional analysis, that there are only three relevant terms, except for the purely bosonic ones already treated in Section 3.3. These relevant terms are the $(\theta)^1$-part (linear in θ's) of $A_{\mu\nu a}$ and $E_a{}^{\hat{\mu}}$, and the $(\theta)^2$-part of $E_+{}^{\hat{-}}$. The first two terms exist for flat spacetime and are unchanged for our background. They are respectively responsible for (the commutator part of) the Yukawa couplings and the Dirac brackets for the fermionic variables in the matrix model. The third term vanishes for flat spacetime, and is the only new contribution from fermionic fields, appearing in the curved background. We shall see below that this term also contributes to the Yukawa couplings of the matrix model, and deform the commutators into $*$-commutators. Other terms either vanish by themselves or do not appear in the lightcone formalism, due to the relations, $\frac{\partial x^+}{\partial\sigma^r}=0$ $(r=1,2)$, and $\gamma^+\frac{\partial\theta}{\partial\sigma^i}=0$ $(i=0,1,2)$.

Explicit expressions for these three terms can be calculated using the gauge fixing condition [37]

$$\theta^b\partial_b A_{\mu\nu a}\big|_{\theta=0}=\theta^b\frac{1}{2}F_{ba\mu\nu}\big|_{\theta=0}=i\theta^{\hat{b}}\Gamma_{\mu\nu\hat{b}\hat{c}}\delta_a^{\hat{c}}, \tag{89}$$

$$\theta^b\partial_b E_a{}^{\hat{\mu}}\big|_{\theta=0}=-i\theta^{\hat{b}}\gamma^{\hat{\mu}}{}_{\hat{b}\hat{c}}\delta_a^{\hat{c}}, \tag{90}$$

$$\begin{aligned}&\frac{1}{2!}\theta^b\theta^a\partial_a\partial_b E_\nu{}^{\hat{\mu}}\big|_{\theta=0}\\&\quad=\frac{i}{36}\theta^{\hat{b}}\theta^{\hat{c}}\left(\Gamma^{\hat{\mu}}\left(F_{\nu\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3}\Gamma^{\hat{\sigma}_3\hat{\sigma}_2\hat{\sigma}_1}+\frac{1}{8}F_{\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3\hat{\sigma}_4}\Gamma^{\hat{\sigma}_4\hat{\sigma}_3\hat{\sigma}_2\hat{\sigma}_1}{}_\nu\right)\bigg|_{\theta=0}\right)_{\hat{b}\hat{c}}.\end{aligned} \tag{91}$$

These expressions are also derived using another method, the method of the gauge completion, in [38,39].

The lightcone gauge formulation of supermembrane theory on our background can now be derived, in a way similar to the bosonic theory. The starting point is phase space constraints,

[10] In general, for any two spinors ξ,η satisfying $\Gamma^{\hat{+}}\xi=0$, $\Gamma^{\hat{+}}\eta=0$, the expression $\Gamma^{\hat{\mu}_1\cdots\hat{\mu}_n}{}_{\hat{a}\hat{b}}\xi^{\hat{a}}\eta^{\hat{b}}$ vanishes except for $\Gamma^{\hat{-}\hat{\alpha}_1\cdots\hat{\alpha}_{n-1}}{}_{\hat{a}\hat{b}}\xi^{\hat{a}}\eta^{\hat{b}}$.

(C.34), (C.35), which are supersymmetric generalisations of (58), (59), and a new constraint (C.36), which solves canonical momenta of θ completely in terms of other variables. We shall skip most of the intermediate steps and just present the result of this analysis in the following.

Firstly, the Dirac brackets can be calculated in the standard way, using the condition (88) and the constraint (C.36), with the help of (90),

$$\sqrt{2}\frac{(-P_-)}{[\sigma]}\left\{\psi^a(\sigma'),\psi^b(\sigma'')\right\}_{\mathrm{D.B.}}=-\frac{i}{2}\delta^{ab}\delta^2(\sigma'-\sigma''). \tag{92}$$

For the supersymmetric case, $\partial_r x^-$ also has the contribution from the fermionic coordinates, in addition to the right-hand side of (36). Consequently, the constraint corresponding to the area preserving diffeomorphism becomes, using (90),

$$0=\left\{\mathcal{P}^\alpha,x^\alpha\right\}+\frac{-P_-}{[\sigma]}i\sqrt{2}\left\{\psi^T,\psi\right\}. \tag{93}$$

We now focus on the Hamiltonian. The right-hand side of (91) reduces to

$$-\frac{\sqrt{2}i}{24}F_{+\hat\alpha_1\hat\alpha_2\hat\alpha_3}\psi^T\gamma^{\hat\alpha_3\hat\alpha_2\hat\alpha_1}\psi, \tag{94}$$

where ψ is the 16-component spinor, defined in (C.7), which is a part of the 32-component spinor θ surviving the lightcone gauge condition (88). This term contributes to the lightcone Hamiltonian in a similar manner as the first term in (60), through the relation $G^{--}=2E_+{}^{\hat{-}}$. Another contribution to the Hamiltonian comes from (89), in a similar way to the second term in (60),

$$\int A_{+\nu a}\left\{x^\nu,\theta^a\right\}d^2\sigma=\int\sqrt{2}i\psi^T\gamma^\alpha\left\{x^\alpha,\psi\right\}d^2\sigma. \tag{95}$$

We note that $A_{+\nu a}$ is anti-commuting.

We thus obtain the full Hamiltonian for supermembranes propagating on our background,

$$\begin{aligned}H={}&(\text{bosonic})+\int\Bigg(\sqrt{2}i\psi^T\gamma^\alpha\left\{x^\alpha,\psi\right\}\\&-\frac{\sqrt{2}i}{2}(2\pi)^2\frac{-P_-}{[\sigma]}\sum_{I,J}\alpha^{(IJ)}\left(\bar z^I\psi^T\gamma^{\bar JJI}\psi+z^I\psi^T\gamma^{J\bar J\bar I}\psi\right)\Bigg)d^2\sigma,\end{aligned} \tag{96}$$

where (bosonic) stands for the purely bosonic part of the Hamiltonian for our background.

We should compare this with the continuum theory corresponding to the deformed matrix model (9), in the regime $\beta N\sim 1$. As we have seen for the bosonic sector, this continuum theory can be obtained by deforming the Lie brackets of the original continuum theory for flat spacetime. For the fermionic sector, the Hamiltonian for flat spacetime is given by the first term in the integrand of (96), and the relevant substitution is (50), (51). We get

$$\begin{aligned}H={}&(\text{bosonic})+\int\Bigg(\sqrt{2}i\psi^T\gamma^\alpha\left\{x^\alpha,\psi\right\}+\sqrt{2}i\psi^T\sum_{I,J}\bigg(\gamma^{\bar I}\beta^{(IJ)}N\frac{(2\pi)^2}{[\sigma]}z^I\left(\frac{1}{2}\gamma^{\bar JJ}\psi\right)\\&-\gamma^I\beta^{(IJ)}N\frac{(2\pi)^2}{[\sigma]}\overline{z^I}\left(\frac{1}{2}\gamma^{\bar JJ}\psi\right)\bigg)\Bigg)d^2\sigma.\end{aligned} \tag{97}$$

This expression precisely matches with (96), under the definition (73). Thus, we have shown that the deformed matrix model (in the scaling limit) is equivalent to matrix-regularised supermembrane theory on our background.

4. Stable solution

In this section, we consider some stable solutions in the deformed model. They correspond to membranes with torus topology. We also discuss some of their properties. In particular, we shall show that two apparently distinct configurations of membranes, labelled by different winding numbers, are actually equivalent in the matrix model. We also consider classical flat directions associated with the solutions, and discuss quantum corrections to them, using an analogy to the four-dimensional theory.

We shall focus on the single-parameter deformation for simplicity.[11] We first observe that every zero-energy configuration is a (marginally) stable configuration, since the potential term is always non-negative. Therefore, if one has a configuration in which every $*$-commutators vanish, the configuration is stable. This is similar to the situation in the original matrix model where a configuration with commuting (or simultaneously diagonalisable) X^α is a stable solution. For simplicity, we set all coordinates other than Z and W to zero. Then the vanishing of all $*$-commutators amounts simply to

$$[Z,W]_* = 0, \qquad \left[Z,W^\dagger\right]_* = 0. \tag{98}$$

If the deformation parameter β takes one of the special values, $\beta = \frac{n}{N}$, where n is an integer, this equation can be solved explicitly[12] by using the so-called clock, shift matrices defined by

$$h_1 = \begin{pmatrix} 1 & & & 0 \\ & e^{i\frac{2\pi}{N}} & & \\ & & \ddots & \\ 0 & & & e^{i\frac{2\pi}{N}(N-1)} \end{pmatrix}, \qquad h_2 = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & 0 & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}, \tag{99}$$

which satisfy

$$h_1 h_2 = e^{-i\frac{2\pi}{N}} h_2 h_1. \tag{100}$$

In the simplest case, $\beta = \frac{1}{N}$,

$$Z = a h_1, \qquad W = b h_2, \tag{101}$$

is a solution to (98) where a, b are arbitrary parameters.

As is well known, the matrices h_1, h_2 play a basic role in matrix regularisation of membranes with torus topology [41]. A function on a torus can be represented by a function defined on $[0,2\pi] \times [0,2\pi]$, periodic in both σ^1, σ^2 directions. In this convention, the matrices h_1, h_2 correspond to the functions

$$e^{i\sigma^1}, \quad e^{i\sigma^2}, \tag{102}$$

[11] By considering the general deformation it should be possible to construct higher-dimensional analogues of the stable solutions considered here.

[12] An four-dimensional analogue of this class of solutions is first discussed in [40]. See also [12,25].

respectively. All periodic functions can be generated from them, and the function $e^{i(m_1\sigma^1+m_2\sigma^2)}$ corresponds to

$$e^{i\frac{\pi}{N}m_1m_2}h_1^{m_1}h_2^{m_2}, \tag{103}$$

where the extra phase factor ensures the correct behaviour under the complex conjugation. Thus, the stable solution (101) corresponds to a configuration in membrane theory

$$z=a'e^{i\sigma^1}, \qquad w=b'e^{i\sigma^2}, \tag{104}$$

where arbitrary constants a', b' are given by $a'=(2\pi T)^{-\frac{1}{3}}a$, $b'=(2\pi T)^{-\frac{1}{3}}b$, using (B.8). This configuration describes a membrane with torus topology, which are embedded into four-dimensional space (parametrised by x^1, x^2, x^3, x^4) with a simple $S^1\times S^1$ shape. It is easy to check that this is a solution to continuum membrane theory (60) on our background, noting the relation $[\sigma]=(2\pi)^2$.

In general, for $\beta=\frac{n}{N}$ with any integer n, the following matrices are solutions to Eq. (98),

$$Z=ae^{i\frac{\pi}{N}l_1l_2}h_1^{l_1}h_2^{l_2}, \qquad W=be^{i\frac{\pi}{N}m_1m_2}h_1^{m_1}h_2^{m_2}, \tag{105}$$

when four integers l_1, l_2, m_1, m_2 satisfy

$$l_1m_2-l_2m_1=n. \tag{106}$$

The matrices (105) correspond, in the continuum theory, to

$$z=a'e^{i(l_1\sigma^1+l_2\sigma^2)}, \qquad w=b'e^{i(m_1\sigma^1+m_2\sigma^2)}, \tag{107}$$

describing a membrane which is in general wrapped on the same $S^1\times S^1$ several times.

For given n, some of the membrane solutions (107) describe the same object in a different parametrisation, e.g. $z=a'e^{i\sigma^1}, w=b'e^{i(\sigma^2+\sigma^1)}$ and $z=a'e^{i\sigma^1}$, $w=b'e^{i\sigma^2}$. The corresponding matrix solutions are equivalent by some unitary transformation, as it should be. On the other hand, some of these membrane configurations are physically distinct, in the conventional membrane picture. For example, for $n=2$, we consider the two configurations, (a) $z=a'e^{i2\sigma^1}$, $w=b'e^{i\sigma^2}$ and (b) $z=a'e^{i\sigma^1}$, $w=b'e^{i2\sigma^2}$. They have different winding numbers: the case (a) corresponds to a membrane wrapped twice around the circle in the z-plane (with radius a') and once around that in the w-plane (with radius b'), and (b) corresponds to a membrane wrapped once around the circle in the z-plane and twice around that in the w-plane. In conventional formulation of membrane theory, although they have the same energy, they are distinct objects. In particular, they have different spectrums for the fluctuations around them (except for the special case $a'=b'$). In (a), the allowed wavelength of the fluctuations around the configuration is $\frac{4\pi a'}{l}$ in the σ^1 direction and $\frac{2\pi b'}{m}$ in the σ^2 direction with integers l, m; because of the double wrapping, the fluctuation in the s^1 direction allows excitation with doubled wavelength $4\pi a'$. For (b) the allowed wavelengths are $\frac{2\pi a'}{l}$ and $\frac{4\pi b'}{m}$.

However, one can show that the corresponding matrices

$$\text{(a):}\quad Z=a(h_1)^2, \qquad W=bh_2, \tag{108}$$

$$\text{(b):}\quad Z=ah_1, \qquad W=b(h_2)^2, \tag{109}$$

are related by a similarity transformation $X^\alpha \to U X^\alpha U^{-1}$, with a unitary matrix U.[13] Therefore, in the matrix model, these two configurations should be considered physically equivalent.[14]

In order to further understand this remarkable phenomena, it is natural to focus on the fluctuation spectrums around these configurations in the matrix model, since, as is explained above, in the conventional membrane picture, they distinguish the two configurations. We have computed the fluctuation spectrums around these configurations, and found that they are the same (as a matter of course) and labelled by the "wavelengths" $\frac{4\pi a'}{l}$ and $\frac{4\pi b'}{m}$; in contrast to the membrane analysis, the largest "wavelengths" are doubled both in the σ^1 and σ^2 directions. This is technically a consequence of the appearance of a sin-function instead of a linear function which occurs in general for a discretised system. The details will be presented elsewhere.

One possible interpretation to this remarkable degeneracy is the following. It has long been suspected that the membranes in M-theory are non-Abelian objects, similar to D-branes. For recent interesting developments, see e.g. [43,44]. This non-Abelian nature might be making the concept of winding numbers ill-defined. For D-branes, one can argue that, for example, at least the distinction is vague between (i) two coinciding D-branes, each of them wrapping a circle once and (ii) single D-brane wrapping the circle twice. Starting from (i), we know that the coordinates are described by 2×2 matrices X, and the natural boundary condition for them, representing the wrapping, is $X(0) = UX(2\pi)U^{-1}$ with unitary matrix U. If $U = 1$ this gives the usual two singly wrapped D-branes. However if one take $U = \sigma^1$, where σ^1 here represents the Pauli matrix, this boundary condition describes single object wrapped around the circle twice. The use of this type of boundary conditions plays an essential role in the matrix string proposal [45–47].

Another interesting aspect of these solutions concerns the parameters a, b of them. They are the radii of the two circles in the z, w-plane, and can take arbitrary values. Thus, they parametrise the flat directions of the classical potential. The flat directions exist because, in the lightcone gauge the force coming from the membrane tension is given by the usual double commutator term, which is proportional to the cubic power of the coordinates; this can be balanced by the force from the quartic potential from the metric (74), which also has the cubic dependence on the transverse coordinates. One can exploit these flat directions to construct solutions which corresponds to two (or more) membranes having different a, b, by arranging the solutions corresponding to each membranes into block-diagonal matrices. So far our discussion has been concrete. Before concluding this section, we would like to discuss, somewhat speculatively, the quantum effect to the classical flat directions, using the analogy to four-dimensional β-deformed $\mathcal{N} = 4$ SYM, as this might give a very challenging application of our deformed model. In four-dimensional $\mathcal{N} = 4$ SYM, the quantum corrections do not break the classical scale invariance, because of the cancellation between bosonic and fermionic contributions, which is presumably a consequence of the $\mathcal{N} = 4$ supersymmetry. The β-deformed SYM, while breaking the supersymmetry (either down to $\mathcal{N} = 1$ or completely), preserves the cancellation between fermionic and bosonic contributions. This suggests that, although our deformed matrix model in general is not supersymmetric, the bosonic and fermionic contributions to the classical flat directions might cancel each other. Furthermore, it is known that, for four-dimensional β-deformed SYM, one has to introduce $1/N$-corrections to various couplings, in addition to phase factors from $*$-products, in order to retain the boson–fermion cancellation. One might expect that similar $1/N$ corrections

[13] We here assume N to be odd for simplicity. When N is even one also has to take into account configurations corresponding to two coinciding singly wrapped membranes.

[14] Similar degeneracy is also noted in [42].

are necessary to make the classical flat directions flat even at the quantum level. If this is the case, this would give a new approach to the long standing problem of the large N limit of the matrix model; by requiring the cancellation of the quantum corrections to the flat directions, one might obtain information about the behaviour of the couplings at large N (the $1/N$ corrections). We hope to report progress in this direction in the future.

5. Beta deformation with mass term

A deformation of the matrix model with mass terms and cubic couplings was introduced in [17]. This deformed model corresponds to membrane theory on a pp-wave background with a constant four-form flux [18,19]. General non-supersymmetric models are studied from the membrane theory point of view in [20]. It is natural to try to simultaneously introduce the β-deformation considered in this paper and the mass deformation. This can be achieved, as we shall see below, exploiting the linearity of the equation of motion (72) for the three-form gauge field on the pp-wave background.

We consider the pp-wave background with the four-form flux which is a linear superposition of a constant part $f_{\alpha\beta\gamma}$ and the linear (in transverse coordinates) part $F^{(1)}$ for the β-deformation identified in (78), (79),

$$F_{+\alpha\beta\gamma} = f_{\alpha\beta\gamma} + F^{(1)}{}_{+\alpha\beta\gamma}. \tag{110}$$

This solves the equation of motion for the gauge field (72).

The equation of motion for the metric (71) becomes

$$\frac{1}{2}\partial^\alpha\partial_\alpha G^{--} = \frac{1}{12}\Bigg(f_{\alpha\beta\gamma}f^{\alpha\beta\gamma} + 12\sum_{I,J}\big(f_{\bar{I}\bar{J}J}F^{(1)}_{+IJ\bar{J}} + f_{IJ\bar{J}}F^{(1)}_{+\bar{I}\bar{J}J}\big) + F^{(1)}{}_{+\alpha\beta\gamma}F^{(1)}{}_{+}{}^{\alpha\beta\gamma}\Bigg). \tag{111}$$

This equation can be solved by the ansatz

$$G^{--} = \mu_{\alpha\beta}x^\alpha x^\beta + \kappa_{\alpha\beta\gamma}x^\alpha x^\beta x^\gamma + G^{--}_{(4)}, \tag{112}$$

where $G^{--}_{(4)}$ is the metric, quartic in coordinates, given in (74). The quadratic term, where $\mu_{\alpha\beta}$ satisfies

$$\mu_\alpha{}^\alpha = \frac{1}{12}f_{\alpha\beta\gamma}f^{\alpha\beta\gamma}, \tag{113}$$

is just the metric associated with the mass-deformation. Finally, the cubic term, where $\kappa_{\alpha\beta\gamma}$ satisfies

$$\sum_J 2(2\pi)^2\alpha^{(IJ)}f_{IJ\bar{J}} = 3\Bigg(\sum_J 2\kappa_{IJ\bar{J}} + \kappa_{I99}\Bigg), \tag{114}$$

$$\sum_J 2(2\pi)^2\alpha^{(IJ)}f_{\bar{I}\bar{J}J} = 3\Bigg(\sum_J 2\kappa_{\bar{I}J\bar{J}} + \kappa_{\bar{I}99}\Bigg), \tag{115}$$

is the only essentially new term for the matrix model which is simultaneously β-deformed and mass-deformed. Here we are using the notation in which the nine real transverse coordinates are decomposed into four complex coordinates labelled by I, its complex conjugates labelled by $\bar{I}$, and the one real direction x^9.

We see that there are ambiguities in $\kappa_{\alpha\beta\gamma}$: one can add arbitrary traceless pieces to it. Instead of considering the most general possibility, we shall concentrate on a deformed model having a more or less simpler form, setting $\kappa_{I99}=0$, $\kappa_{\bar{I}99}=0$. Later in this section, we will also exploit this ambiguity to construct a particularly tractable version of the deformed matrix model.

The appearance of the cubic term might also be expected from the following consideration. The constant part of the flux contributes the term,

$$\int \frac{1}{6} f_{\alpha\beta\gamma} x^\gamma \left\{x^\alpha, x^\beta\right\} d^2\sigma, \tag{116}$$

to the Hamiltonian of the continuum theory. This term gives rise to the term

$$C_{\alpha\beta\gamma} i \operatorname{Tr} X^\alpha \left[X^\beta, X^\gamma\right] \tag{117}$$

in the Hamiltonian of the matrix model. Here, $C_{\alpha\beta\gamma}$ and $f_{\alpha\beta\gamma}$ are related by

$$C_{\alpha\beta\gamma} = -\frac{1}{6}\frac{-P_-}{N}(2\pi T)^{-\frac{2}{3}} f_{\alpha\beta\gamma}, \tag{118}$$

which follows from (40)–(42) and the rescaling relation (B.8). Now, a natural guess about the deformation of the matrix model in the present case is to replace the commutators by the $*$-commutators not only in the Yukawa and quartic scalar couplings, but also in the cubic scalar couplings (117). This means, in the continuum theory, to replace the term (116) using the substitution rules (48), (49). We would then have cubic terms in the Hamiltonian.

However, although the form of the cubic terms are correct, the over-all factor thus obtained turns out to be wrong (i.e. not consistent with the equation of motion (71)) by a factor of $\frac{2}{3}$. One simple way (which is not so elegant) to resolve this issue is to introduce a new $*'$-product defined by

$$f *' g = e^{i\frac{3}{2}\pi\beta(Q^f_{(1)}Q^g_{(2)} - Q^g_{(1)}Q^f_{(2)})} fg, \tag{119}$$

and use it to deform the cubic couplings, while using the original $*$-product for the quartic and Yukawa couplings.

Under this prescription, the deformed matrix model is given by

$$\begin{aligned} H = \operatorname{Tr}\bigg(& \frac{1}{2}\left(\Pi^\alpha\right)^2 - \frac{1}{4}\left[X^\alpha, X^\beta\right]_*^2 + \Psi^T\gamma^\alpha[X_\alpha, \Psi]_* \\ & + M_{\alpha\beta}X^\alpha X^\beta + C_{\alpha\beta\gamma} i X^\alpha\left[X^\beta, X^\gamma\right]_{*'} + \frac{i}{4}C_{\alpha\beta\gamma}\Psi^T\gamma^{\gamma\beta\alpha}\Psi\bigg), \end{aligned} \tag{120}$$

where

$$M_{\alpha\beta} = \frac{1}{2}\left(\frac{-P_-}{N}\right)^2 (2\pi T)^{-\frac{4}{3}}\mu_{\alpha\beta}. \tag{121}$$

It is easy to derive the fermionic term above, since the fermionic contribution to the continuum Hamiltonian is simply given by the sum of contributions for the purely mass-deformed case and for the purely β-deformed case. This can be easily seen from the argument in Section 3.4, in particular from (94).

It is known that, in the mass-deformed model, one finds a stable solution corresponding to a spherical membrane via the ansatz,

$$\left[X^\alpha, X^\beta\right] \propto i\epsilon^{\alpha\beta}{}_\gamma X^\gamma \quad (\alpha, \beta, \gamma = 1, 2, 3), \tag{122}$$

which corresponds to the ansatz,

$$\left\{x^\alpha, x^\beta\right\} = \frac{a}{[\sigma]}\epsilon^{\alpha\beta}{}_\gamma x^\gamma \quad (\alpha, \beta, \gamma = 1, 2, 3), \tag{123}$$

for the continuum theory. Here, $\epsilon^{\alpha\beta\gamma}$ is the totally anti-symmetric tensor in three dimension; it vanishes if any of the indices take other values than $1, 2, 3$. The parameter a is related to the flux $f_{\alpha\beta\gamma}$; see (126) below. The appearance of $[\sigma]$ in the above equation can be understood by considering the rescaling of the σ-coordinates.

On the other hand, in the β-deformed model, one finds stable solutions which correspond to membranes with the torus topology, as has been shown in Section 4. It is therefore natural to expect that, for a model having both parameter β and parameter a for the mass-deformation, the stable solution would have the topology of a torus or a sphere, depending on which of the two parameters is dominating. We note that a very similar phenomenon is studied in [42]. There, an one-parameter non-commutative algebra which interpolates between the non-commutative sphere and the non-commutative torus is constructed, together with its explicit representations; the information about the topology is encoded in the eigenvalue distributions in a manner proposed in [48].

In general, it seems difficult to study the stable solutions analytically. However, we have found that by tuning the parameters κ_{I99}, $\kappa_{\bar{I}99}$ and μ_{99} one can obtain a particularly tractable class of the deformed models, where one can explicitly demonstrate the interpolation between a sphere and a torus.

The basic idea is to write a part of the Hamiltonian in the form,

$$\frac{[\sigma]}{2(-P_-)}\frac{T^2}{2}\left(\left\{x^\alpha, x^\beta\right\}_* - \frac{a}{[\sigma]}\epsilon^{\alpha\beta}{}_\gamma x^\gamma\right)^2. \tag{124}$$

For simplicity we consider the simplest β-deformation, and we have denoted by $\{x^\alpha, x^\beta\}_*$ the continuum counterpart of the $*$-commutator, namely the right-hand side of (46), (47). As is well known, the advantage of writing the Hamiltonian in the above form is that the zero-energy solution can be found by solving the first order equation,

$$\left\{x^\alpha, x^\beta\right\}_* = \frac{a}{[\sigma]}\epsilon^{\alpha\beta}{}_\gamma x^\gamma, \tag{125}$$

which is a β-deformed version of (123). From (124), one can read off the background $\mu_{\alpha\beta}$, $f_{\alpha\beta\gamma}$, $\kappa_{\alpha\beta\gamma}$, by comparing it to (60). In particular, we have the relation between the parameter a and $f_{\alpha\beta\gamma}$,

$$f_{+\alpha\beta\gamma} = -\frac{3aT}{-P_-}\epsilon_{\alpha\beta\gamma}, \tag{126}$$

which also justifies the appearance of $[\sigma]$ in (123)–(125). Unfortunately, the backgrounds thus read off do not satisfy the equations of motion (113)–(115) by themselves. However, one can introduce extra backgrounds, κ_{I99}, $\kappa_{\bar{I}99}$, μ_{99} such that the equations of motion are satisfied. The additional terms in the Hamiltonian introduced by this tuning do not affect the stable solutions if we set $x^9 = 0$.

We have thus shown that there is a class of deformed matrix models for which Eq. (125) gives the stable solutions (with $x^9 = 0$). Using the complex coordinates and setting unimportant scalars to zero, (125) becomes

$$\{z, w\}_* = \frac{a}{[\sigma]} \frac{i}{\sqrt{2}} z, \tag{127}$$

$$\{z, \bar{w}\}_* = \frac{a}{[\sigma]} \frac{i}{\sqrt{2}} z, \tag{128}$$

$$\{z, \bar{z}\} = \frac{a}{[\sigma]} \left(-\frac{i}{\sqrt{2}}\right)(w + \bar{w}), \tag{129}$$

$$\{w, \bar{w}\} = 0 \tag{130}$$

with the help of

$$\epsilon_{z\bar{z}w} = \frac{i}{\sqrt{2}}, \qquad \epsilon_{z\bar{z}\bar{w}} = \frac{i}{\sqrt{2}}. \tag{131}$$

These equations reduce to those for the pure β-deformation if $a = 0$, in which $z \sim e^{i\sigma^1}$, $w \sim e^{i\sigma^2}$ is a solution (for $\beta N = 1$). If $\beta = 0$, it reduce to those for the pure mass-deformation, and then a sphere embedded in x^1, x^2, x^3 is a solution. A natural ansatz for the general case, which interpolates between these two solutions is

$$z = r(\sigma^2) e^{i\sigma^1}, \tag{132}$$

$$w = w(\sigma^2). \tag{133}$$

Eqs. (127)–(130) then become ordinary differential equations,

$$irw' + \beta N \frac{(2\pi)^2}{[\sigma]} rw = \frac{a}{[\sigma]} \frac{i}{\sqrt{2}} r, \tag{134}$$

$$(r^2)' = -\frac{a}{\sqrt{2}[\sigma]}(w + \bar{w}), \tag{135}$$

where we have abbreviated $f' = \frac{\partial f}{\partial \sigma^2}$. From (134), we get

$$w = C e^{i\beta N \frac{(2\pi)^2}{[\sigma]} \sigma^2} + i \frac{a}{(2\pi)^2 \sqrt{2} \beta N} \tag{136}$$

where C is an integration constant. Substituting this to (135), we get

$$r^2 = D + |C| \frac{\sqrt{2}a}{(2\pi)^2 \beta N} \cos\left(\beta N \frac{(2\pi)^2}{[\sigma]} \sigma^2\right), \tag{137}$$

where D is another integration constant and we have chosen the phase of C appropriately by shifting σ^2.

For $D > |C| \frac{\sqrt{2}a}{(2\pi)^2 \beta N}$, the right-hand side is always positive. One can take the range of σ^2 as $-\pi < \sigma^2 < \pi$ without loss of generality. Then $[\sigma]$ equals to $(2\pi)^2$, and we see that βN should be an integer because of the regularity of the solution. The solution has the topology of $S^1 \times S^1$, i.e. the torus.

For $D < |C| \frac{\sqrt{2}a}{(2\pi)^2 \beta N}$, the right-hand side can become negative, whereas the left-hand side is, by definition, always positive. This implies that the range of σ^2 should be restricted to

$-l < \sigma^2 < l$, and at the point $\sigma^2 = \pm l$ the radius r should vanish. Noting that $[\sigma] = 4\pi l$, we obtain

$$0 = D + |C| \frac{\sqrt{2}a}{(2\pi)^2 \beta N} \cos(\beta N \pi). \tag{138}$$

In this case, the solution has the topology of a sphere.

To summarise, for the general case where βN is not an integer, the solution has the topology of a sphere, and the integration constants C, D are related by Eq. (138), as a consequence of the boundary condition. For the special case where βN is an integer, the solution has the topology of a torus, and the parameters C, D are only restricted by the inequality

$$D > |C| \frac{\sqrt{2}a}{(2\pi)^2 \beta N}. \tag{139}$$

Thus the dimension of the space of stable solutions enhances at these special points.

One can also construct solutions to the corresponding matrix equations

$$\left[X^\alpha, X^\beta\right]_* = i \frac{1}{N} T^{\frac{1}{3}} (2\pi)^{-\frac{2}{3}} a \epsilon^{\alpha\beta}{}_{\gamma} X^\gamma, \tag{140}$$

by using the ansatz that W is diagonal, and only non-zero elements of Z are those adjacent to diagonal elements. The form of the matrix solutions is very similar to those given in [42]. Detailed formulae will be given elsewhere.

6. Conclusion

In this paper, we have considered a class of deformation for the matrix model of M-theory. The form of the deformation, which consists in modifying the Yukawa and quartic scalar couplings by distributing flavour-dependent phase factors, is motivated from a similar deformation of $\mathcal{N} = 4$ SYM in four dimension. In four dimension, this deformation has significance that it preserves the conformal invariance of the original $\mathcal{N} = 4$ SYM. We have found that for the matrix model of M-theory, the deformation is also special in that it admits M-theory interpretation: the deformed model can be considered as the matrix model of M-theory on a certain curved background, since it is equivalent to a regularised version of supermembrane theory on that background.

It is remarkable that the deformation, introduced from a rather mathematical analogy to four-dimensional field theory, has a natural eleven-dimensional interpretation. Indeed, this interpretation requires strong consistency, as the identified background should satisfy the supergravity equations of motion. We have verified that they are indeed satisfied including an over-all factor, without any artificial tuning of parameters. One might say that, somehow, the β-deformation "knows" the eleven-dimensional supergravity. It is hard to believe that this high degree of consistency is a mere coincidence. It would be fascinating if one could find a framework to understand this consistency in a natural fashion.

In general, pp-wave backgrounds arise as a result of a limiting procedure called the Penrose limit; in order to obtain a physical interpretation of our background, it might be useful to consider what backgrounds would reduce to our pp-wave background under the Penrose limit.

The deformed model also seems to contain interesting physics. It has stable solutions, which corresponds to toric membranes with the simple $S^1 \times S^1$ shape, for some particular values of the deformation parameter. The solution has classical flat directions, which correspond to the radii of two circles. To consider quantum corrections to these flat directions is an interesting problem. Also, we have found that some configurations which are physically distinct in conventional membrane theory should be considered as the same object in the matrix model. We have argued that this might be the reflection of the non-Abelian nature of membranes. We have also studied the β-deformed model with the mass terms and found that, for a particular class of models, there are stable membrane configurations which interpolates between a membrane with the topology of a torus and a membrane with the topology of a sphere.

Finally, we wish to raise a few directions one might pursue concerning our deformed matrix model. It will be interesting to relate the model to ten-dimensional type IIA string theory, by compactifying the x^9 direction, which is not touched even for the most general deformation. Alternatively, one might compactify the x^+ direction; the stable solutions discussed in Section 4 will be wrapped in the x^+ direction, and hence can be considered as a string worldsheet with the topology of a torus. This might allow one to interpret the stable solutions as saddle points of the path integral of suitably Euclideanised type IIA string theory on a curved background.

The pp-wave background considered in this paper is with the metric and the four-form flux respectively given by quartic and linear polynomials of the transverse coordinates. It would be interesting to consider the generalisation of our matrix model which corresponds to similar pp-wave metrics associated with more general higher-order polynomials.

One can ask many questions about this model, other than those already mentioned, such as the classification of BPS states, scattering of various objects, in particular the gravitons. We hope that this model would serve as a good place to further explore the physics of membranes and the matrix model.

Note added

After completing this manuscript, I learned of a forthcoming article [49]. In [49], the authors extended the pp-wave background considered in this paper (which has four-form flux depending linearly on the transverse coordinates) to more general pp-wave backgrounds, guided by the supersymmetry. I thank N. Kim and J. Plefka for communications.

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Appendix A. Notations and conventions

In this appendix we summarise notations and conventions used in this paper. See also Appendix C for some of the notations and conventions which are specific to Section 3.4.

Our signature of the metric is

$$\eta_{\mu\nu} = \mathrm{diag}(- + \cdots +). \tag{A.1}$$

The meaning and the range of various indices are as follows:

$\mu, \nu, \ldots$:	vector indices for the spacetime; run through $0, \ldots, 10$,
$\alpha, \beta, \ldots$:	indices for transverse directions; run through $1, \ldots, 9$,
$a, b, \ldots$:	spinor indices; usually run through $1, \ldots, 32$ in Section 3.4; runthrough $1, \ldots, 16$ when explicitly stated,
$i, j, \ldots$:	either $U(N)$ matrix indices which run through $1, \ldots, N$, or the worldvolume vector indices which run through $0, 1, 2$,
$r, s, \ldots$:	parametrise the worldvolume spacelike coordinates; run through $1, 2$,
$A, B, \ldots$:	only used in Section 3.4; collectively denotes $\mu, \nu, \ldots$ indices and $a, b, \ldots$ indices; $A = (\mu, a)$.

In Section 3.4, we distinguish the tangent space indices $\hat{A}, \hat{B}, \hat{\mu}, \hat{\nu}, \hat{a}, \hat{b}, \hat{+}, \hat{-}, \ldots$ from the curved space indices $A, B, \mu, \nu, a, b, +, -, \ldots$.

We use the 16×16 real and symmetric $SO(9)$ gamma matrices,

$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2\delta^{\alpha\beta}. \tag{A.2}$$

We use

$$\gamma^{\alpha_1 \cdots \alpha_n} = \frac{1}{n!}\big(\gamma^{\alpha_1} \cdots \gamma^{\alpha_n} \pm (n! - 1 \text{ permutations})\big). \tag{A.3}$$

We define the total area of the base space of a x^+-slice of membranes, parametrised by σ^1, σ^2, as

$$[\sigma] = \int d^2\sigma. \tag{A.4}$$

Our lightcone conventions are

$$v^\pm = \frac{v^0 \pm v^{10}}{\sqrt{2}}, \tag{A.5}$$

$$\eta_{+-} = -1, \qquad \eta_{-+} = -1, \tag{A.6}$$

$$w_\pm = -w^\mp = -\frac{w^0 \mp w^{10}}{\sqrt{2}}. \tag{A.7}$$

Appendix B. Details about rescaling

We describe here the rescaling of dynamical variables and the time coordinate, necessary to bring the matrix-regularised supermembrane theory (on flat spacetime) into the normalised matrix model form (1), (2). This rescaling is not affected by the introduction of the deformation.

The Dirac brackets (or the Poisson brackets) of continuum membrane theory are given by, for bosonic variables,

$$\big\{x^\alpha(\sigma'), \mathcal{P}^\beta(\sigma'')\big\}_{\text{D.B.}} = \delta^{\alpha\beta}\delta^2(\sigma' - \sigma''). \tag{B.1}$$

The Dirac brackets between the matrices corresponding to x and $\mathcal{P}$, $\rho(x) = \hat{x}$ and $\rho(\mathcal{P}) = \hat{\mathcal{P}}$, are given by

$$\big\{(\hat{x}^\alpha)^i{}_j, (\hat{\mathcal{P}}^\beta)^k{}_l\big\}_{\text{D.B.}} = \frac{N}{[\sigma]}\delta^{\alpha\beta}\delta^i_l\delta^k_j \quad (i, j, k, l = 1, \ldots, N). \tag{B.2}$$

The factor $\frac{N}{[\sigma]}$ arises because (i) integration over σ'' in (B.1) corresponds to taking partial trace in (B.2), say contracting indices i and j, multiplied by a factor $\frac{[\sigma]}{N}$, because of (42), and (ii) the function on (σ_1, σ_2)-space taking the constant value 1 corresponds to identity matrix, as can be seen from (40). Alternatively, one can derive (B.2) from the general relation between the Hamilton formalism and the variational principle on the phase space $\delta \int (p\dot{q} - H(q,p))\, dt = 0$, with the help of the relation $\int \mathcal{P}\dot{x}\, d^2\sigma \approx \frac{[\sigma]}{N} \operatorname{Tr} \hat{\mathcal{P}}\dot{\hat{x}}$.

Similarly, Dirac brackets for the fermionic variables in the continuum theory (92),

$$\sqrt{2}\frac{(-P_-)}{[\sigma]}\left\{\psi^a(\sigma'), \psi^b(\sigma'')\right\}_{\text{D.B.}} = -\frac{i}{2}\delta^{ab}\delta^2(\sigma' - \sigma''),$$

become

$$\sqrt{2}\frac{(-P_-)}{N}\left\{\left(\hat{\psi}^a\right)^i_j, \left(\hat{\psi}^b\right)^k_l\right\}_{\text{D.B.}} = -\frac{i}{2}\delta^{ab}\delta^i_l\delta^k_j, \tag{B.3}$$

after regularisation, where $a, b = 1, \ldots, 16$.

By applying (40)–(42) to the continuum Hamiltonian,

$$H = \int \left(\frac{[\sigma]}{2(-P_-)} \left(\left(\mathcal{P}^\alpha\right)^2 + \frac{1}{2}\left\{x^\alpha, x^\beta\right\}^2 \right) + \sqrt{2} i \psi^T \gamma^\alpha \left\{x^\alpha, \psi\right\} \right) d^2\sigma, \tag{B.4}$$

we get the Hamiltonian for regularised theory,

$$\begin{aligned} H = \frac{[\sigma]}{N} \operatorname{Tr} \Bigg(& \frac{[\sigma]}{2(-P_-)} \left(\left(\hat{\mathcal{P}}^\alpha\right)^2 + \frac{1}{2}\left(\frac{2\pi N}{i[\sigma]} \left[\hat{x}^\alpha, \hat{x}^\beta\right] \right)^2 \right) \\ & + i\sqrt{2}\hat{\psi}^T \gamma^\alpha \left(\frac{2\pi N}{i[\sigma]} \left[\hat{x}^\alpha, \hat{\psi}\right] \right) \Bigg). \end{aligned} \tag{B.5}$$

Similarly the constraint (93),

$$0 = \left\{\mathcal{P}^\alpha, x^\alpha\right\} + \frac{-P_-}{[\sigma]} i\sqrt{2}\left\{\psi^T, \psi\right\},$$

becomes

$$0 = \frac{2\pi N}{i[\sigma]}\left[\hat{\mathcal{P}}^\alpha, \hat{x}^\alpha\right] + \frac{-P_-}{[\sigma]} i\sqrt{2}\frac{2\pi N}{i[\sigma]} 2\hat{\psi}^T\hat{\psi}. \tag{B.6}$$

These relations (B.5), (B.6), (B.2), (B.3) can be brought into the form (1)–(4) by using the rescaling

$$d\tau = \frac{-P_-}{N}(2\pi)^{-\frac{2}{3}} T^{-\frac{2}{3}}\, dt, \tag{B.7}$$

$$\hat{x}^\alpha = (2\pi)^{-\frac{1}{3}} T^{-\frac{1}{3}} X^\alpha, \tag{B.8}$$

$$\hat{\mathcal{P}}^\alpha = \frac{N}{[\sigma]}(2\pi)^{\frac{1}{3}} T^{\frac{1}{3}} \Pi^\alpha, \tag{B.9}$$

$$\Psi = 2^{\frac{1}{4}}\sqrt{\frac{-P_-}{N}}\hat{\psi}, \tag{B.10}$$

where we have restored the membrane tension T.

Appendix C. Details for the fermionic part

We compile in this appendix some of the detailed formulae and conventions for supermembrane theory, used in Section 3.4.

We use the following anti-commutation relation for the 32×32 $SO(1,10)$ gamma matrices,

$$\Gamma^{\hat{\mu}}\Gamma^{\hat{\nu}} + \Gamma^{\hat{\nu}}\Gamma^{\hat{\mu}} = 2\eta^{\hat{\mu}\hat{\nu}}. \tag{C.1}$$

We use the representation in which all gamma matrices are real, and $\Gamma^{\hat{0}}$ is anti-Hermitian and other gamma matrices are Hermitian. More explicitly, we use

$$\Gamma^{\hat{0}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes 1_{16}, \tag{C.2}$$

$$\Gamma^{\hat{\alpha}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \gamma^{\alpha}, \tag{C.3}$$

$$\Gamma^{\hat{10}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes 1_{16}, \tag{C.4}$$

where γ^{α} are the 16×16 real and symmetric $SO(9)$ gamma matrices, and 1_{16} denotes the 16×16 unit matrix. For lightcone directions, we have

$$\Gamma^{\hat{+}} = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \otimes 1_{16}, \tag{C.5}$$

$$\Gamma^{\hat{-}} = \begin{pmatrix} 0 & 0 \\ -\sqrt{2} & 0 \end{pmatrix} \otimes 1_{16}, \tag{C.6}$$

so that, under the lightcone gauge condition, $\Gamma^{+}\theta = 0$, 32-component spinor θ reduces to the 16-component $SO(9)$ spinor ψ as

$$\theta = \begin{pmatrix} \psi \\ 0 \end{pmatrix}. \tag{C.7}$$

The gamma matrices defined above have one upper and one lower indices. We use gamma matrices with two lower indices,

$$\Gamma^{\hat{\mu}_1\cdots\hat{\mu}_n}{}_{\hat{a}\hat{b}} = \mathcal{C}_{\hat{a}\hat{c}}\Gamma^{\hat{\mu}_1\cdots\hat{\mu}_n\,\hat{c}}{}_{\hat{b}}, \tag{C.8}$$

where $\mathcal{C}$ is the charge conjugation matrix, which is proportional to $\Gamma^{\hat{0}}$ in our representation. We choose the convention, $\mathcal{C} = \Gamma^{\hat{0}}$.

The transformation law of the κ-symmetry is given by the following ansatz,[15]

$$\delta x^A E_A{}^{\hat{\mu}} = 0, \tag{C.9}$$

$$\Gamma^{\hat{a}}{}_{\hat{b}}\delta x^A E_A{}^{\hat{b}} = \delta x^A E_A{}^{\hat{a}}, \tag{C.10}$$

where $\Gamma^2 = 1$ is defined by

$$\Gamma^{\hat{a}}{}_{\hat{b}} = \sqrt{-\det h_{ij}}\,\pi^{0\hat{\mu}}\pi^{1\hat{\nu}}\pi^{2\hat{\rho}}(\Gamma_{\hat{\mu}\hat{\nu}\hat{\rho}})^{\hat{a}}{}_{\hat{b}}, \quad \pi^{i\mu} = h^{ij}\pi_j{}^{\hat{\mu}}. \tag{C.11}$$

[15] We could introduce a minus sign in (C.10). It can be absorbed by flipping the orientation on the worldvolume for a particular configuration of membranes, without any change of the theory in total.

The covariant derivative only acts on the tangent indices and our convention is such that

$$D_A v^{\hat{B}} = \partial_A v^{\hat{B}} - \Omega_{A\hat{C}}{}^{\hat{B}} v^{\hat{C}} (-1)^{\hat{C}\hat{B}}. \tag{C.12}$$

Here, indices in the exponent of (-1) are to be substituted by 0 if they are bosonic and by 1 if they are fermionic. These factors are necessary in order to maintain the correct transformation property of superspace tensors. In superspace formulation of supergravity, the gauge symmetry acting on the tangent space is taken to be the local Lorentz symmetry. As a consequence, the connection satisfies

$$\Omega_{A\hat{b}}{}^{\hat{a}} = \frac{1}{4}\left(\Gamma^{\hat{\mu}\hat{\nu}}\right)^{\hat{a}}{}_{\hat{b}} \Omega_{A\hat{\nu}\hat{\mu}}, \tag{C.13}$$

$$\Omega_{A\hat{b}}{}^{\hat{\mu}} = 0, \qquad \Omega_{A\hat{\mu}}{}^{\hat{b}} = 0. \tag{C.14}$$

The derivatives by anti-commuting variables are usual left derivatives.

The torsion tensor $T_{AB}{}^{\hat{C}}$ is defined by

$$T_{AB}{}^{\hat{C}} = \sum_{AB} D_A E_B{}^{\hat{C}} = \sum_{AB}\left(\partial_A E_B{}^{\hat{C}} - \Omega_{A\hat{D}}{}^{\hat{C}} E_B{}^{\hat{D}} (-1)^{(\hat{D}+\hat{C})(B+\hat{D})}\right), \tag{C.15}$$

where $\sum_{AB}$ stands for the graded anti-symmetric summation over all independent permutations of indices A and B. We transform two lower indices of the torsion tensor into tangent space indices via the relation,

$$T_{AB}{}^{\hat{C}} = E_B{}^{\hat{E}} E_A{}^{\hat{D}} T_{\hat{D}\hat{E}}{}^{\hat{C}} (-1)^{A(B+\hat{E})}. \tag{C.16}$$

Similarly, the four-form field strength tensor is defined, together with their tangent space components, as

$$F_{A_1A_2A_3A_4} = \sum_{A_1A_2A_3A_4} \partial_{A_1} A_{A_2A_3A_4}, \tag{C.17}$$

$$\begin{aligned} F_{A_1A_2A_3A_4} = {} & E_{A_4}{}^{\hat{B}_4} E_{A_3}{}^{\hat{B}_3} E_{A_2}{}^{\hat{B}_2} E_{A_1}{}^{\hat{B}_1} F_{\hat{B}_1\hat{B}_2\hat{B}_3\hat{B}_4} \\ & \times (-1)^{(A_2+\hat{B}_2)A_1 + (A_3+\hat{B}_3)(A_1+A_2) + (A_4+\hat{B}_4)(A_1+A_2+A_3)}. \end{aligned} \tag{C.18}$$

The torsion and the field strength satisfy the Bianchi identities, as a result of the (anti-)commutativity of the differential operators. Defining the curvature by

$$R_{AB\hat{C}}{}^{\hat{D}} = -\sum_{AB}\left(\partial_A \Omega_{B\hat{C}}{}^{\hat{D}} + \Omega_{A\hat{C}}{}^{\hat{E}} \Omega_{B\hat{E}}{}^{\hat{D}} (-1)^{\hat{E}B}\right), \tag{C.19}$$

$$R_{AB\hat{C}}{}^{\hat{D}} = E_B{}^{\hat{F}} E_A{}^{\hat{E}} R_{\hat{E}\hat{F}\hat{C}}{}^{\hat{D}} (-1)^{(B+\hat{F})A}, \tag{C.20}$$

the Bianchi identities read

$$\sum_{\hat{A}\hat{B}\hat{C}} -D_{\hat{A}} T_{\hat{B}\hat{C}}{}^{\hat{D}} - T_{\hat{A}\hat{B}}{}^{\hat{E}} T_{\hat{E}\hat{C}}{}^{\hat{D}} = \sum_{\hat{A}\hat{B}\hat{C}} -R_{\hat{A}\hat{B}\hat{C}}{}^{\hat{D}}, \tag{C.21}$$

$$\sum_{\hat{A}\hat{B}\hat{C}\hat{D}\hat{E}} \left(D_{\hat{A}} F_{\hat{B}\hat{C}\hat{D}\hat{E}} + T_{\hat{A}\hat{B}}{}^{\hat{F}} F_{\hat{F}\hat{C}\hat{D}\hat{E}}\right) = 0. \tag{C.22}$$

By applying the Bianchi identities to the superspace constraints (85)–(87), one obtains many relations between components of the torsion, the curvature, and the field strength. For example, one can show that $T_{\hat{a}\hat{b}}{}^{\hat{c}}$ vanishes. Of particular importance is the relation,

$$T_{\hat{\mu}\hat{b}}{}^{\hat{a}} = -\frac{1}{36}\left(F_{\hat{\mu}\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3}\Gamma^{\hat{\sigma}_3\hat{\sigma}_2\hat{\sigma}_1} + \frac{1}{8}F_{\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3\hat{\sigma}_4}\Gamma^{\hat{\sigma}_4\hat{\sigma}_3\hat{\sigma}_2\hat{\sigma}_1}{}_{\hat{\mu}}\right)^{\hat{a}}{}_{\hat{b}}. \tag{C.23}$$

It is easy to check the numerical coefficients in the above expression, using vector–vector–vector–spinor–spinor components of (C.22).

The gauge symmetry acting on the background fields, namely, the general coordinate invariance and the local Lorentz transformation, and the gauge transformation of the three form gauge fields, are fixed by the conditions

$$\theta^b\theta^a\left(\partial_a E_b{}^{\hat{A}} - \partial_b E_a{}^{\hat{A}}\right) = 0, \tag{C.24}$$

$$\theta^a \Omega_{a\hat{\mu}\hat{\nu}} = 0, \tag{C.25}$$

$$\theta^a A_{aAB} = 0, \tag{C.26}$$

respectively.

Apart from this, one in general sets

$$E_b{}^{\hat{a}}\big|_{\theta=0} = \delta^{\hat{a}}{}_b, \tag{C.27}$$

$$E_a{}^{\hat{\mu}}\big|_{\theta=0} = 0, \tag{C.28}$$

by using the gauge symmetries. This makes the correspondence between the $\theta = 0$ part of the superfields and the component supergravity fields simple.

We shall briefly outline the derivation of the expression for the $(\theta)^2$ part of the vector–vector component of the supervielbein (91). The condition (C.25) implies

$$\Omega_{a\hat{\mu}\hat{\nu}}|_{\theta=0} = 0, \qquad (\partial_b \Omega_{a\hat{\mu}\hat{\nu}} - \partial_a \Omega_{b\hat{\mu}\hat{\nu}})|_{\theta=0} = 0, \quad \ldots. \tag{C.29}$$

Using these relations and the definition of the torsion (C.15), we obtain

$$\frac{1}{2}\theta^b\theta^a\left(\partial_a\partial_b E_\nu{}^{\hat{\mu}}\right)\big|_{\theta=0} = \frac{1}{2}\theta^b\theta^a\left(\partial_a\left(T_{b\nu}{}^{\hat{\mu}} + \partial_\nu E_b{}^{\hat{\mu}}\right)\right)\big|_{\theta=0}. \tag{C.30}$$

One can show that the following contribution from the first term in the brackets above is the only non-vanishing term under the superspace constraints (85)–(87),

$$-\frac{1}{2}\theta^b\theta^a\left(\left(\partial_a E_\nu{}^{\hat{d}}\right)E_b{}^{\hat{c}}T_{\hat{c}\hat{d}}{}^{\hat{\mu}}\right)\big|_{\theta=0}. \tag{C.31}$$

By manipulating $\partial_a E_\nu{}^{\hat{d}}|_{\theta=0}$ in a similar manner to the manipulation in (C.30), we obtain (91).

The starting point to construct the lightcone gauge formalism for supermembrane theory is the set of phase space constraints. We denote the canonical momenta of x^μ and θ^a by $\mathcal{P}_\mu$ and $\mathcal{P}_a$. It is convenient to define $\tilde{\mathcal{P}}$'s, which are the contributions to the momenta from the S_1 in the action (80)–(84), by

$$\tilde{\mathcal{P}}_\mu = \mathcal{P}_\mu + \frac{1}{2}\left\{x^C, x^B\right\}A_{\mu BC}, \tag{C.32}$$

$$\tilde{\mathcal{P}}_a = \mathcal{P}_a - \frac{1}{2}\left\{x^C, x^B\right\}A_{BCa}. \tag{C.33}$$

In terms of them, the phase space constraints are expressed as

$$\tilde{\mathcal{P}}_\mu\tilde{\mathcal{P}}_\nu G^{\mu\nu} + (h_{11}h_{22} - h_{12}h_{21}) = 0, \tag{C.34}$$

$$\tilde{\mathcal{P}}_\mu \partial_r x^\mu + \tilde{\mathcal{P}}_a \partial_r x^a = 0, \tag{C.35}$$

$$\tilde{\mathcal{P}}_a = -E_a{}^{\hat{\nu}} e_{\hat{\mu}}{}^\mu \tilde{\mathcal{P}}_\mu, \tag{C.36}$$

where $e_{\hat{\nu}}{}^\mu$ is defined by

$$e_{\hat{\nu}}{}^\mu E_\mu{}^{\hat{\rho}} = \delta_{\hat{\nu}}{}^{\hat{\rho}}. \tag{C.37}$$

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