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Renormalization Group and Effective Potential in Classically Conformal Theories

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Abstract

For a classically conformal model of QCD coupled via quarks to a colorless scalar field we present an *exact* solution of the RG improved effective (Coleman-Weinberg) potential at one loop. This closed form expression allows us to discuss the range of validity of the effective potential as well as the issue of 'large logarithms' in a way different from previous such analyses. Remarkably, in all examples considered, convexity of the effective potential is restored by the RG improvement, or otherwise the potential becomes unstable. In the former case, symmetry breaking becomes unavoidable due to the appearance of an infrared barrier, which hints at a so far unsuspected link between Λ_{QCD} and the scale of electroweak symmetry breaking.

1. Introduction. The main purpose of this paper is to clarify some aspects of the Coleman-Weinberg (CW) mechanism of radiatively induced breaking of symmetry [1, 2] by means of examples, for which the renormalization group (RG) improved versions of the one-loop effective CW potential can be obtained in closed form. We believe that these results constitute further evidence in support of the scenario proposed in [3, 4, 5], according to which the CW mechanism can be implemented in the context of the full standard model with a classically conformal Lagrangian (*i.e.* without explicit mass terms in the scalar potential), provided all couplings remain bounded over a large range of energies. Our analysis reveals commonly accepted wisdom concerning possible realistic applications of the CW mechanism to be incomplete; in particular, intuition based on the textbook examples of pure ϕ^4 and

scalar QED (which is also reviewed here in a slightly new perspective) may fail when there are non-trivial cancellations in the relevant β-functions, as a consequence of which Landau poles are shifted to very large scales. As we argued in [3], this may be the ultimate reason for the stability of the weak scale *vis-à-vis* the Planck scale M_{Planck}, if the standard model couplings were to conspire in precisely such a way so as to make the model survive to very large scales (a similar scenario, but without radiative symmetry breaking, was proposed and elaborated in [6, 7]; see also [8, 9] for a different attempt to implement the CW mechanism).

RG methods to improve the effective potential have been applied and studied in previous work, and we refer readers to [10, 11, 12, 13, 14, 15] for earlier treatments and comprehensive bibliographies. Nevertheless, the Standard Model like *exact* solution which we present has not yet appeared in the literature to the best of our knowledge (a simpler case of a scalar field coupled to fermions was discussed e.g. in [16]) . All examples that we discuss support a key assertion of our previous work, namely that *the CW potential can be trusted in the range where the running couplings (expressed as functions of the classical field φ) stay small.* It is this criterion which should be used to ascertain the consistency of the CW potential, rather than the widely used 'rule of thumb', according to which certain products of the logarithm of $\langle\varphi\rangle$ and the pertinent coupling must be small. The latter requirement is deficient in that it does not take into account possible cancellations, whereas our results show that the running couplings may stay small in the presence of such cancellations despite 'large logarithms'.

Equally importantly, and rather unexpectedly, the present work sheds new light on another long-standing issue, namely the apparent clash between symmetry breaking and convexity of the effective potential. As is well known on general grounds (see e.g. [17, 18, 19, 16]) the effective potential must be a *convex function* of the classical scalar field, a condition that is generically violated by the (unimproved) one-loop expressions derived in quantum field theory. Amazingly, though still somewhat empirically, for the classically conformal theories we find that, in all examples studied, either the convexity of the potential over its domain of definition is restored by the RG improvement, or otherwise the potential develops an instability [1]. The restoration

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[1]Although these statements seem not to apply in the presence of explicit mass terms, when the CW potential only represents a small correction to the classical potential, the point, in our opinion, even then requires further study.

of convexity is mainly due to the fact that the effective potential not only exhibits an ultraviolet (UV) barrier at the location Λ_{UV} of the Landau pole, but in general also an infrared (IR) barrier Λ_{IR} which arises through the couplings of the scalar field to the other fields, and is only visible in the exact expressions. The allowed regions for φ are thus separated by a 'forbidden zone' $|\varphi| < \Lambda_{\mathrm{IR}}$. In this way, the conflict between convexity on the one hand, and the existence of non-trivial vacua with $\langle\varphi\rangle \neq 0$ is completely resolved. When the running coupling turns negative before this barrier is reached, the potential becomes unbounded from below for very small φ.

In the explicit examples we shall see that the Landau pole Λ_{UV} can be shifted to very large values. By contrast, in semi-realistic models involving the strong interactions, the IR barrier Λ_{IR} is unmovable because its value is in essence dictated by the known IR properties of α_s. *If Λ_{IR} exists, we must have $\langle\varphi\rangle \neq 0$, and symmetry breaking becomes unavoidable!* To be sure, our QCD-like example is not yet fully realistic in that the minimum is too close to Λ_{IR}, whereas the scale of electroweak symmetry breaking is more than a hundred times larger than Λ_{QCD}. Nevertheless, we find it most remarkable how the strong interactions – so far not thought to play any role in this context – may intervene to enforce breaking of electroweak symmetry (and conformal invariance). Readers may recall that symmetry breaking in the standard model is conventionally implemented by means of an explicit mass term $m^2\varphi^2$ — leaving us with the question why nature should prefer a negative value of m^2 over the equally consistent positive value!

For simplicity we consider only a single real scalar field with couplings to different non-scalar fields; this has the advantage that the full result takes a rather simple form (cf. (1) and (7) below) which is completely determined by the β-function. The multi-field case, where such simplifications are presumably absent, will require separate study. After explaining some general features of the RG improvement procedure in section 2, we first consider scalar QED in section 3, not only recovering known results, but also to expose an IR instability that has gone unnoticed so far. In section 4, we turn to our main example, QCD coupled to a colorless scalar field via Yukawa interactions. This example incorporates some essential features of the model investigated in [3, 5] in that the various running couplings keep each other under control over a large range of energies so as to ensure the survival of the theory up to some large scale. Finally, we present some numerical results and discuss the normalization of couplings in terms of physical parameters.

2. Generalities. In a classically conformally invariant theory [2] with one real scalar field that couples to any number of fermions and/or gauge fields, the effective CW potential is generally of the form [3]

$$W_{\text{eff}}(\varphi) = \varphi^4 f(L, g) \tag{1}$$

where

$$L \equiv \ln \frac{\varphi^2}{v^2} \tag{2}$$

and v is some mass scale required by regularization. The letter g in (1) stands for a collection of coupling constants $\{g_1, g_2, \dots\}$ corresponding to the quartic scalar self-coupling and various (dimensionless) couplings to and among other fields (fermions, gauge fields) in the theory. Below we will also use letters u, x, y, z from the end of the alphabet to denote convenient combinations of these couplings, as they occur in the β-functions.

The quantity v is the only dimensionful parameter in the theory, but has no physical significance in itself; eventually, it should thus be replaced by a more physical parameter, such as the vacuum expectation value $\langle \varphi \rangle$. When one uses dimensional regularization (as we do), v enters via the replacement

$$\int \frac{d^4k}{(2\pi)^4} \quad \longrightarrow \quad \frac{1}{v^{2\epsilon}} \int \frac{d^{4+2\epsilon}k}{(2\pi)^{4+2\epsilon}} \tag{3}$$

which is to be performed in all divergent integrals. The parameter v breaks conformal invariance explicitly, and the question is then whether this breaking persists after renormalization, in which case the classical conformal symmetry is broken by anomalies. For the renormalization we employ the prescription of [4], according to which the local part of the effective action is to be kept conformally invariant throughout the regularization procedure: in this way, the structure of the anomalous Ward identity is preserved at every step (as originally suggested in [21]).

The function f must be determined from a perturbative expansion [1, 2]. A main problem then is to assess the reliability of such approximations, and to ascertain whether extrema found by minimizing the perturbative potential

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[2]See e.g. the recent article [20] for a comprehensive review of conformal invariance in field theory and quantum field theory.

[3]We generally write W_{eff} for the RG improved, and V_{eff} for the unimproved effective potential at one loop.

are within the perturbative range or not. On general grounds, the dimensionless function f must satisfy the RG equation (see e.g. [10])

$$\left[-2\frac{\partial}{\partial L} + \sum \beta_i(g)\frac{\partial}{\partial g_i}\right] f(L, g_j) = 0 \tag{4}$$

where the β-functions $\beta_j(g) \equiv \beta_j(g_1, g_2, \dots)$ depend on the theory under consideration. We note that a perturbative analysis of these equations was already begun in [14, 15], where, however, no closed form solutions were presented. Unlike those authors we will be using *one-particle irreducible* (1PI) β-functions here. This means that the RG improved effective potential is here expressed as a function of the *renormalized* scalar field, so there are no extra factors involving the anomalous scaling in (1), in contradistinction to the expressions given in [10, 11, 12].

The general solution of (4) is obtained by first solving the system of ordinary differential equations for the running couplings, viz. [4]

$$2\frac{\mathrm{d}}{\mathrm{d}L}\hat{g}_j(L) = \beta_j\big(\hat{g}(L)\big) \tag{5}$$

where the initial values $\hat{g}_j(0)$ are the input parameters from the classical Lagrangian. Given a solution (5), the partial differential equation (4) is solved by

$$f(L, g_j) \equiv F\big(\hat{g}_1(L), \hat{g}_2(L), \dots\big) \tag{6}$$

where F is an a priori *arbitrary* function, which can be determined by matching it with the perturbative (loop) expansion of the effective action. The precise choice of F together with the choice of β-functions fixes the renormalization scheme (recall that the β-functions are renormalization scheme dependent in higher loop order). We take

$$f\big(\hat{g}(L)\big) \equiv \hat{g}_1(L) \tag{7}$$

where g_1 is the scalar self-coupling. At any order in the loop expansion, this is the most natural choice because the effective potential (1) then reduces to the classical potential in the limit $\hbar \to 0$ (we recall that each factor of L is accompanied by a factor of $\hbar$, but we will usually set $\hbar = 1$). Other schemes

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[4]For notational clarity we always put hats on the L-dependent running couplings.

can differ from this one by higher powers of the (running) coupling constants on the r.h.s. of (7), such that

$$F\big(\hat{g}(L)\big) = \hat{g}_1(L) + \sum_{i,j} \alpha_{ij}\hat{g}_i(L)\hat{g}_j(L) + \ldots \tag{8}$$

corresponding to a non-linear redefinition of the coupling constants. Obviously the above expression for the RG improved effective potential can be trusted as long as the running couplings $\hat{g}_j(L)$ remain small (in which case the higher order corrections in (8) are likewise small). Imposing smallness of the running couplings in turn determines an admissible range of values for L, and thereby for the field φ itself. In realistic applications we will seek to ensure by judicious choice of the couplings that this admissible range extends beyond the Planck mass for positive L, but we usually will also encounter a *lower* bound for negative L. With the replacement of the one-loop effective potentials by RG improved potentials, there is no more need to consider running coupling constants. Rather, the energy scale is now replaced by the classical field φ itself, or more precisely, the ratio φ/v. Let us also recall that (1) corresponds to a resummation of the perturbation series where only leading logarithms are kept.

To explain the main point in a nutshell, let us consider the textbook example of pure (*i.e.* massless) ϕ^4 theory, for which the one-loop effective potential is [1]

$$V_{\mathrm{eff}}(\varphi) = \frac{g}{4}\varphi^4 + \frac{g^2}{256\pi^2}\varphi^4 L \tag{9}$$

The RG improved potential is found by first solving the RG equation for the scalar self-coupling

$$2\frac{\mathrm{d}\hat{y}}{\mathrm{d}L} = \frac{1}{8}\hat{y}^2 \qquad \text{with} \quad \hat{y}(L) \equiv \frac{\hat{g}(L)}{4\pi^2} \tag{10}$$

The well known solution is [22]

$$\hat{y}(L) = \frac{y_0}{1 - (1/16)y_0 L} \tag{11}$$

exhibiting the famous Landau pole at $\varphi = \Lambda_{\mathrm{UV}} \equiv v \exp(8/y_0)$. From (1) and (7), the RG improved potential is therefore given by

$$W_{\mathrm{eff}}(\varphi, g_0) = \frac{\pi^2 y_0 \varphi^4}{1 - (1/16)y_0 L} = \pi^2 y_0 \varphi^4 \Big[1 + (1/16)y_0 L + \ldots\Big] \tag{12}$$

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This expression exhibits the very same Landau pole as the scalar self-coupling, and in this way limits the range of trustability to the values $y_0 L < 16$, or equivalently $|\varphi| < \Lambda_{\text{UV}}$ (there is no IR barrier, so φ can become arbitrarily small). Because there is no non-trivial minimum of W_{eff} in this range we recover the well-known result that the symmetry breaking minimum in the unimproved potential V_{eff} (9) is spurious. The existence of a spurious minimum $\langle \varphi \rangle \neq 0$ in (9) is related to the fact that the function (9) is *not convex*. The RG improved version (12), on the other hand, *is* convex, and therefore the minimum is moved back to $\langle L \rangle = -\infty$, that is, $\langle \varphi \rangle = 0$. As we will see in our 'real world' example (and also in scalar QED), the latter possibility is precluded by an IR barrier, which entails $\langle L \rangle > -\infty$.

3. Massless scalar QED revisited. Next we turn to massless scalar QED, which describes the coupling of one complex scalar field to electromagnetism. Although this is generally thought to be well understood, we here encounter a surprise in the guise of an IR instability. Introducing

$$y = \frac{g}{4\pi^2}, \quad u = \frac{e^2}{4\pi^2} \tag{13}$$

for the scalar self-coupling and the electromagnetic coupling, respectively, we have the system of equations

$$2\frac{\mathrm{d}\hat{y}}{\mathrm{d}L} = 2a\hat{y}^2 + 2b\hat{u}^2 \quad , \qquad 2\frac{\mathrm{d}\hat{u}}{\mathrm{d}L} = 4c\hat{u}\hat{y} \tag{14}$$

where the functions on the right hand side are the appropriate 1PI one-loop β functions [22] [5]. In order not to distract from the main point, we do not specify the (positive) numerical coefficients a, b, c, but set $c = a$ to arrive at completely explicit formulas. However, there is no problem of principle with substituting the actual values from the scalar QED Lagrangian and performing a numerical analysis of the equations (14): the qualitative features will be the same.

The solution satisfying the initial conditions

$$\hat{y}(0) = y_0, \quad \hat{u}(0) = u_0 \tag{15}$$

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[5]Note that there is no contribution $\propto \hat{u}\hat{y}$ on the r.h.s. of the first equation, as was already pointed out in [1].

is given explicitly by

$$
\begin{aligned}
\hat{u}(L) &= \frac{u_0}{(1 - ay_0 L)^2 - abu_0^2 L^2} \\
\hat{y}(L) &= \frac{y_0 + (bu_0^2 - ay_0^2)L}{(1 - ay_0 L)^2 - abu_0^2 L^2}
\end{aligned}
\tag{16}
$$

As we explained, $\hat{y}(L, y_0, u_0)$ then also solves the RG equation (4)

$$
\left[-2\frac{\partial}{\partial L} + (2ay^2 + 2bu^2)\frac{\partial}{\partial y} + 4au^2\frac{\partial}{\partial u} \right] f(L, y, u) = 0
\tag{17}
$$

with the proper classical limit $f(0, y, u) = y$. Hence, from (1) and (7),

$$
W_{\text{eff}}(\varphi, y, u) = \pi^2 \varphi^4 \frac{y + (bu^2 - ay^2)L}{(1 - ayL)^2 - abu^2 L^2}
\tag{18}
$$

from which we read off the well known (unimproved) one-loop effective potential by expanding in powers of L:

$$
V_{\text{eff}}(\varphi, y, u) = \pi^2 \varphi^4 \big[y + (ay^2 + bu^2)L \big]
\tag{19}
$$

Let us first recall the standard treatment of (19), as explained in [1]. By taking y and u small, with the additional relation $y = \alpha u^2$, and $\alpha = \mathcal{O}(1)$, one easily checks that the minimum occurs at $\langle L \rangle \approx -1/2 - \alpha/b$. This is $\mathcal{O}(1)$, independently of the input value of the coupling u. Because $u\langle L \rangle$ is thus small, one concludes that the symmetry breaking minimum is not spurious, unlike for pure ϕ^4.

Inspection of the RG improved potential (18) reveals a different story, however. Evidently, that expression blows up when the denominator in (17) vanishes, that is, for

$$
2\ln\left[\frac{\Lambda_{\text{UV}}}{v}\right] \approx +\frac{1}{\sqrt{ab}} \cdot \frac{1}{u} \quad , \qquad 2\ln\left[\frac{\Lambda_{\text{IR}}}{v}\right] \approx -\frac{1}{\sqrt{ab}} \cdot \frac{1}{u}
\tag{20}
$$

The upper (positive) value is the usual Landau pole, whereas the lower value Λ_{IR} represents an IR barrier, disallowing small values of φ. However, the numerator in (18) becomes negative for $L < -\alpha/b$, which is *to the right* of the IR barrier (since u is small). In other words, before φ reaches Λ_{IR}, the potential (18) *turns negative* and becomes unbounded from below. Hence,

though inside the region of applicability of perturbation theory, the symmetry breaking vacuum is at best *meta-stable*.

This pathology appears to be generic: inspection of (14) shows that the source of the problem is the strict positivity of $\hat{y}'(L)$. The only way to avoid the IR instability is to push $\ln(\Lambda_{IR}/v)$ back to $-\infty$, as may happen for special choices of the input parameters[6], but then we are back to (12) and there is no symmetry breaking! The conventional analysis based on (19) cannot distinguish between these possibilities, as it is insensitive to the value of c. When the RG improved potential W_{eff} is unstable, it also fails to be convex, exemplifying the claimed link between instability and lack of convexity. The only way out, then, is to look for a system where the β-functions exhibit cancellations between different couplings, which is our next example.

4. QCD with one colorless scalar. Our main example is closely modeled on [3, 5], and thus incorporates features of the Standard Model. The Lagrangian

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + i\bar{q}\gamma^\mu D_\mu q + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + g_Y\phi\bar{q}q - \frac{g}{4}\phi^4 \qquad (21)$$

is classically conformally invariant, and depends on three couplings, the gauge coupling g_s, the Yukawa coupling g_Y and the scalar self-coupling g. The (real) scalar field ϕ is not charged under Yang-Mills $SU(N)$ (hence colorless), but couples to color-charged quarks via the Yukawa term, much like the standard model Higgs. The main advantage of the model (21) is that the one-loop RG equations can again be solved exactly. This enables us to exhibit new phenomena that are not visible in the usual perturbative expansion, and which require a minimum of *three* independent couplings. An important feature is that the one-loop β-functions for the scalar and for the fermions both have two contributions of opposite sign, so that both couplings can remain stationary over a large range of energies with suitable initial conditions.

The 1PI one-loop β-function equations are now given by the system

$$2\frac{d\hat{y}}{dL} = a\hat{y}^2 - b\hat{x}^2 \;\;,\;\;\; 2\frac{d\hat{x}}{dL} = 2c\hat{x}^2 - 4e\hat{x}\hat{z} \;\;,\;\;\; 2\frac{d\hat{z}}{dL} = -2e\hat{z}^2 \qquad (22)$$

for the functions $\hat{y} \equiv \hat{y}(L)$, $\hat{x} \equiv \hat{x}(L)$ and $\hat{z} \equiv \hat{z}(L)$, where

$$x \equiv \frac{g_Y}{4\pi^2} \;\;,\;\;\; y \equiv \frac{g}{4\pi^2} \;\;,\;\;\; z \equiv \frac{g_s}{4\pi^2} \equiv \frac{\alpha_s}{\pi} \qquad (23)$$

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[6] Such as, for instance, $c = \frac{1}{2}(a+b)$ and $y_0 = u_0$, in which case we can take $\hat{y}(L) = \hat{u}(L)$.

As before we leave the (positive) numerical coefficients a, b, c, e unspecified, but have chosen the coefficient in the last equation of (22) so as to arrive at reasonably simple and fully explicit formulas (there are some other choices for which explicit solutions can be found; otherwise, one can resort to numerical analysis). The above set of equations mimics some features of the corresponding equations encountered in more realistic examples, such as the one in [3]: for the scalar, one contribution comes from the scalar self-interaction and the other from the Yukawa coupling (which is large for the top quark). The one-loop β function for Yukawa coupling likewise has two contributions of opposite sign, from the Yukawa interaction and from the gauge interactions, respectively. The one-loop β function for gauge interactions has only one contribution of negative sign (asymptotic freedom). The whole system of equations admits a stable solution with suitable initial values, because asymptotic freedom keeps the YM coupling under control, which in turn slows down the running of the Yukawa coupling, which in turn keeps the scalar self-coupling under control. This cascade of mutual cancellations is the mechanism invoked in [3, 4] to avoid Landau poles or instabilities between the small scales and the Planck scale. Furthermore, our toy model shows that indeed the one-loop solution of RG equations for the effective potential has a minimum for a field much below the value of the field where the Landau pole occurs.

Inserting the asymptotic freedom result (see e.g. [22])

$$\hat{z}(L) = \frac{z_0}{1 + e z_0 L} \tag{24}$$

into the second equation for the Yukawa coupling we find

$$\hat{x}(L) = \frac{1}{1 + e z_0 L} \cdot \frac{x_0}{1 + (e z_0 - c x_0) L} \tag{25}$$

which for $e = 0$ reduces to the standard result $\hat{x}(L) = x_0/(1 - c x_0 L)$. Evidently, the behavior of $\hat{x}(L)$ depends crucially on the sign of $(e z_0 - c x_0)$: if this parameter is positive, there is only an IR pole, but no Landau pole, and the Yukawa coupling would become asymptotically free.

Finally, substituting these results into the equation for the scalar self coupling we also obtain an explicit solution for $\hat{y}(L)$. Defining

$$A \equiv \sqrt{c^2 + ab} \quad (> 0) \tag{26}$$

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and the function

$$h(L) := \frac{2Ax_0(1 + ez_0L)^{-A/c}}{(1 + ez_0L)^{-A/c} - \left(1 - \frac{2Ax_0}{ay_0 + 2ez_0 + (A-c)x_0}\right)(1 + (ez_0 - cx_0)L)^{-A/c}} \tag{27}$$

we have

$$\hat{y}(L) = \frac{h(L) + (c - A)x_0 - 2ez_0\big[1 + (ez_0 - cx_0)L\big]}{a\big[1 + (ez_0 - cx_0)L\big](1 + ez_0L)} \tag{28}$$

As before, we can now read off the solution of

$$\left[-2\frac{\partial}{\partial L} + (ay^2 - bx^2)\frac{\partial}{\partial y} + (2cx^2 - 4exz)\frac{\partial}{\partial x} - 2ez^2\frac{\partial}{\partial z}\right]f(L, y, x, z) = 0 \tag{29}$$

satisfying $f(0, y, x, z) = y$ to obtain the RG improved effective potential

$$W_{\text{eff}}(L, x, y, z) = \pi^2\varphi^4\frac{h(L) + (c - A)x - 2ez\big[1 + (ez - cx)L\big]}{a\big[1 + (ez - cx)L\big](1 + ezL)} \tag{30}$$

5. Numerics and normalization. As the functional form of (30) is somewhat cumbersome, we have investigated it numerically for a variety of values of the input parameters. In all cases, we find that (30) is either convex or unstable, in agreement with our general claim. Concentrating on the first case for its obvious physical interest, a typical set of values is

$$\begin{aligned} a &= 0.5 , & b &= 2 , & c &= 0.8 , & e &= 1 \\ x_0 &= 0.6 , & y_0 &= 0.2 , & z_0 &= 0.9 \end{aligned} \tag{31}$$

For these parameters, we display the running coupling $\hat{y}(L)$ and the effective potential W_{eff} in Fig. 1 and Fig. 2, respectively; note that the scales for L in the two figures are very different. As one can see, $\hat{y}(L)$ stays well behaved up to very large values $L \sim 200$. We thus see that the smallness of $\hat{y}(L)$ does not necessarily require the product $\hat{y}(0)L$ to be small (near the Landau pole, we have $\hat{y}(0)L \sim 20$), showing that the approximation can be trusted in spite of 'large logarithms'. On the IR side, we have a pole at $\ln(\Lambda_{\text{IR}}/v) \sim -1.11$, while the minimum is located at $\langle L \rangle \sim -0.15$, where $\hat{g}(\langle L \rangle) = 0.196$. This 'closeness' of the minimum and the IR barrier is the only non-realistic feature of our model. Our curves not only put in evidence the convexity of W_{eff} but also show that $\hat{g}_1(L)$ is very flat over a large range of values for L.

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Let us also comment on the question of how to normalize the couplings. Our choice (7) corresponds to normalizing all couplings at a fixed but arbitrary value $\varphi = v$ where $L = 0$, with the values $\hat{g}_j(0)$ as the input parameters from the classical Lagrangian. However, because the normalization parameter v has no physical significance in itself, and having found a non-trivial minimum at $\varphi = \langle\varphi\rangle$, one would like to change this normalization *a posteriori* to one where v is traded for the actual value of φ at the minimum (as is usually done for ϕ^4 and scalar QED). This is desirable in view of the fact that all physical quantities (masses, effective couplings) are obtained as derivatives of the effective potential at the minimum. [7]

Quite generally, having determined $\langle\varphi\rangle$, we can evolve $\hat{g}_1(L)$ from $\hat{g}_1(0)$ to $\hat{g}_1(\langle L\rangle)$, and try to fix the latter 'backwards' to some given value by varying the input parameters. Alternatively, we can define the scalar self-coupling as the fourth derivative of W_{eff} at the minimum. Thanks to (1) and (7), we can work out the relation between these two quantities explicitly:

$$\frac{1}{24\pi^2}\frac{\mathrm{d}^4 W_{\text{eff}}(\varphi, L)}{\mathrm{d}\varphi^4}\bigg|_{\varphi=\langle\varphi\rangle} = \hat{g}_1(\langle L\rangle) + \left\{\frac{25}{6}\frac{\mathrm{d}\hat{g}_1(L)}{\mathrm{d}L} + \frac{35}{6}\frac{\mathrm{d}^2\hat{g}_1(L)}{\mathrm{d}^2 L} + \right.$$
$$\left. + \frac{10}{3}\frac{\mathrm{d}^3\hat{g}_1(L)}{\mathrm{d}^3 L} + \frac{2}{3}\frac{\mathrm{d}^4\hat{g}_1(L)}{\mathrm{d}^4 L}\right\}\bigg|_{L=\langle L\rangle} \tag{32}$$

Because, at the minimum, we have $(2\hat{g}_1 + \hat{g}_1')|_{L=\langle L\rangle} = 0$, the difference could be appreciable, but there are cancellations since $\hat{g}_1' < 0$ while $\hat{g}_1'' > 4\hat{g}_1$ by convexity. Indeed, with the values (31) we find that the l.h.s ~ 0.156, to be compared with the value quoted above, $\hat{g}_1(\langle L\rangle) = 0.196$. We have checked for a range of input couplings that the difference between the two numbers does remain rather small.

Similar comments apply to the other couplings. Taking the Yukawa interaction as an example, the RG improved version of the corresponding term in the effective action takes the form

$$\Gamma_Y(\varphi, q, \bar{q}) = \hat{g}_Y(L)\varphi\bar{q}q \tag{33}$$

As before, we can evolve this coupling from $\hat{g}_Y(0)$ to $\hat{g}_Y(\langle L\rangle)$ and compare

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[7]Clearly, it makes no sense to normalize the couplings at $\varphi = 0$, not only because the fourth derivative diverges at $\varphi = 0$ for the unimproved effective potential V_{eff}, but also because this value is outside the domain of definition of W_{eff} if there are IR barriers.

with the relevant derivative. This gives

$$
\begin{aligned}
m_q &= \langle\varphi\rangle\, \hat{g}_Y(\langle L\rangle) \\
\frac{\mathrm{d}}{\mathrm{d}\varphi}\Big(\varphi\hat{g}_Y(L)\Big)\Big|_{\varphi=\langle\varphi\rangle} &= \hat{g}_Y(\langle L\rangle) + 2\frac{\mathrm{d}\hat{g}_Y(L)}{\mathrm{d}\varphi}\Big|_{L=\langle L\rangle}
\end{aligned}
\tag{34}
$$

respectively, for the fermion mass and the effective Yukawa coupling at the minimum. Again, we see that the difference between the latter and $\hat{g}_Y(\langle L\rangle)$ can be small provided $\hat{g}'_Y$ is small there. These considerations justify (to some extent) the approximation used [3, 5], where we effectively defined the couplings in terms of derivatives at the minimum of V_{eff}, and took those values as the input parameters at $\varphi = \langle\varphi\rangle$ to analyze the evolution of couplings.

6. Discussion. Much of our discussion was based on exact solutions of the β-function equations, but the extension of the present considerations to more realistic examples is straightforward. In particular, the standard model with one Higgs doublet falls into the class of models investigated here, since there remains only one real scalar field after absorption of three scalar degrees of freedom into the massive vector vector bosons. Although closed form solutions of the RG equations are no longer available in this case, we can solve numerically for the running couplings and determine the RG improved effective potential from the general formulas (1) and (7). The latter can be analyzed numerically along the lines described here. As already pointed out, the only non-realistic feature here is the closeness of the minimum to the IR barrier Λ_{IR}, which appears to be a generic feature of the one-field case. In realistic applications, on the other hand, we would have to arrange $\Lambda_{\text{IR}} \sim \Lambda_{QCD} = \mathcal{O}(1\,\text{GeV})$ and $\langle\varphi\rangle = \mathcal{O}(200\,\text{GeV})$. This may indicate the need for extra scalar fields (which are anyhow needed for the inclusion of massive neutrinos [6, 3, 7, 5, 8]).

Somewhat ironically, our analysis shows that scalar QED, often cited as *the* showcase example of the CW mechanism, suffers from a pathology. Unlike QCD, where there is an IR divergence of the coupling signaling the onset of a strongly coupled phase, the IR instability cannot be cured by keeping the same theory but switching to an alternative desription in terms of strongly coupled degrees of freedom (analogous to mesons and baryons). It would also be interesting to work out the consequences of the present results for cosmology, and in particular for models of scalar field inflation, where the effective potential (rather than the classical potential) should play a decisive

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role. We would not be surprised if our results can be used to rule out many of the currently popular 'designer potentials' for inflation.

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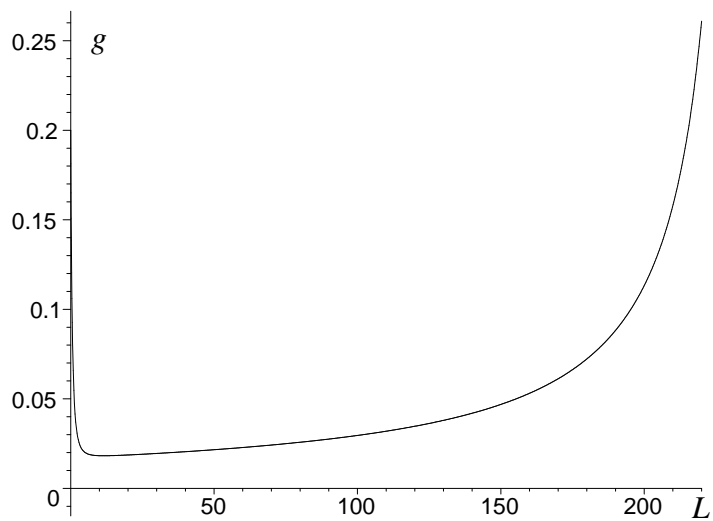
Fig. 1. The scalar self-coupling (28) for (31).
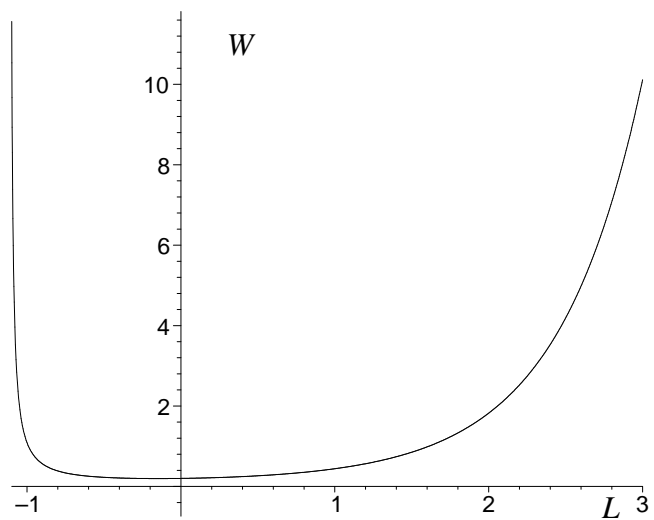
Fig. 2. The RG improved effective potential $W_{\mathrm{eff}}(L)$ (30) for (31).

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