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Four loop reciprocity of twist two operators in $\mathcal{N}=4$ SYM

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ABSTRACT: The complete universal anomalous dimension of twist-2 operators in $\mathcal{N}=4$ SYM has been recently conjectured at four loops in terms of maximum transcendentality combinations of harmonic sums. It reproduces the known cusp anomaly, NLO BFKL poles, and the diagrammatic result for the Konishi operator. In this paper, we prove that it passes a further deep test related to a generalized Gribov-Lipatov reciprocity. This holds for both the asymptotic Bethe Ansatz contribution [1] and the novel wrapping correction [2]. This result suggests reciprocity to be a very stable and intrinsic property of twist-2 operators.

Contents

1. Introduction and discussion

The calculation of the four loop universal anomalous dimension of $\mathcal{N} = 4$ SYM twist-2 operators, whose formula is now available thanks to the findings of [1] and the recent completion [2], is a remarkable example of the combined power of integrability and QCD-inspired Ansätze in determining a gauge theory perturbative formula. The general form of the anomalous dimension is in fact naturally split into an asymptotic part and a wrapping correction. While the all loop asymptotic Bethe Ansatz of $\mathcal{N} = 4$ SYM [3] is well-suited to correctly determine the asymptotic part, it drastically fails when wrapping corrections come into play [1], something happening at four loops for the twist-2 operators under consideration. It is a recent achievement that, in turn, such wrapping contributions can be exactly determined by exploiting the integrability of the string sigma model in $AdS_5 \times S^5$, dual to $\mathcal{N} = 4$ SYM via AdS/CFT correspondence. Namely, the identification of anomalous dimensions with energies of string states in $AdS_5 \times S^5$ and the finite size nature of the wrapping contribution allow to compute the effects of the latter as leading virtual corrections to the infinite volume limit via generalized Lüscher formulas [4].

In the case of twist-2 operators, two other ingredients, both with a QCD origin, have been crucial in providing the final exact result. On one hand, the maximum transcendentality principle [5] has made feasible the evaluation of both the asymptotic and the wrapping contribution to the spin N dependent anomalous dimension $\gamma(N)$ [1]. Furthermore, from the next-to-leading order BFKL equations [6] a prescription can be extracted for the pole structure of the analytically continued anomalous dimension. Such prescription was determinant to state the failure of Bethe equations in describing the spectrum of short operators [1], as well as the correctness of the full result including the wrapping correction [2].

In this paper we show that the four loop result of [1, 2] satisfies yet another QCD-related property, the so-called (generalized Gribov-Lipatov) *reciprocity* [7, 8, 9]. This result is not totally surprising for the asymptotic part of the anomalous dimension, at least in view of what has been already noticed in [10][2] and because of similar observations made in the last two years for a rich set of twist operators in QCD and $\mathcal{N} = 4$ SYM [11]. Instead, the fact that reciprocity holds also for the wrapping contribution is novel and remarkable. As we shall discuss below, this gives a serious argument to consider reciprocity a crucial tool for checking the correctness of any future expression of anomalous dimension for twist operators.

Reciprocity emerges in studying the large spin N behavior of the available anomalous dimensions of twist-2 operators in QCD and $\mathcal{N} = 4$ SYM. It is known that subleading terms in the large spin expansion obey (three loops) hidden relations, the Moch-Vermaseren-Vogt (MVV) constraints [12]. In QCD such relations can be related with the crossing reciprocity of Deep Inelastic Scattering (DIS) and e^+e^- annihilation. Technically, reciprocity in the twist-2 case holds for the Dokshitzer-Marchesini-Salam (DMS) evolution kernel $\widetilde{P}$ governing simultaneously the distribution and fragmentation functions [7] [3]. In the usual x-space description of DIS, the reciprocity prediction turns out to be the following simple analog of Gribov-Lipatov reciprocity

$$\widetilde{P}(x) = -x\,\widetilde{P}\left(\tfrac{1}{x}\right). \tag{1.1}$$

The kernel $\widetilde{P}$ is fully determined (at least perturbatively) by the spin dependent anoma-

[1] In the case of the wrapping correction, the maximum transcendentality principle was used only in the evaluation of the so-called purely rational contribution. For the maximum transcendentality conjecture at previous loop orders, see footnote 7.

[2] See Section 7 there.

[3] The DSM evolution kernel has recently received a nice confirmation in [13].

lous dimension $\gamma(N)$. Indeed, taking the Mellin transform of $\widetilde{P}$ we get

$$P(N) = \int_0^1 dx\, x^{N-1}\, \tilde{P}(x) = \mathbf{M}\left[\tilde{P}(x)\right], \tag{1.2}$$

and the DMS evolution equations (for a finite theory like $\mathcal{N} = 4$ SYM) predict the functional relation

$$\gamma(N) = P\left(N + \tfrac{1}{2}\gamma(N)\right). \tag{1.3}$$

The origin of the MVV relations can be traced back to the reciprocity relation Eq. (1.1) which we equivalently write as the following constraint on $P(N)$ at large N

$$N \to \infty: \qquad P(N) = \sum_{\ell \geq 0} \frac{a_\ell(\log\, J^2)}{J^{2\,\ell}}, \tag{1.4}$$

where $J^2 = N\,(N+1)$ and a_ℓ are suitable *coupling-dependent* polynomials. Of course, Eq. (1.4) implies an infinite set of constraints on the coefficients of the large N expansion of $P(N)$ organized in standard $1/N$ power series. Indeed, a generic expansion around $N = \infty$ can involve odd powers of $1/J$ forbidden in Eq. (1.4). The peculiar combination J^2 is nothing but the Casimir of the collinear subgroup $SL(2,\mathbb{R}) \subset SO(2,4)$ of the conformal group [14] and the above constraint is simply parity invariance under $J \to -J$.

It is well known, since [9], that all this can be suitably generalized to twist-L operators in $\mathcal{N} = 4$ SYM belonging to the $\mathfrak{sl}(2)$ sector where Eq. (1.4) is expected to hold with the only replacement

$$\text{twist} - L: \qquad J^2 = \left(N + \frac{L}{2}\right)\left(N + \frac{L}{2} - 1\right). \tag{1.5}$$

In the following, we shall say that a twist-L anomalous dimension $\gamma(N)$ *is reciprocity respecting (RR) iff Eqs. (1.4,1.5) hold for the associated* $P(N)$ [4].

Reciprocity is a non perturbative feature valid at all orders in the coupling constant. At weak coupling, a perturbative test requires the knowledge of the multi-loop anomalous dimensions as closed functions of N. These are currently available for various twist-2 and 3 operators [11]. Three-loop tests of reciprocity for QCD and for the universal twist 2 supermultiplet in $\mathcal{N} = 4$ SYM were discussed in [9, 7]. A four-loop test for the twist 3 anomalous dimension in the $sl(2)$ sector was performed in [15]. It is important to recall that reciprocity is expected to hold only for *minimal* anomalous dimensions of twist operators [5].

At strong coupling, the investigation of reciprocity has been naturally achieved by employing the AdS/CFT correspondence, which indicates the folded string as the configuration dual to twist-2 operators [17]. This analysis, initiated in [9] for the folded string at the classical level, has been recently extended in [18] at one loop in string perturbation theory, as well as to classical spiky strings configurations (see also [19]).

[4] Later, we'll make use of the above definition of reciprocity for a general linear combination of harmonic sums written in Mellin space, see Appendix A.

[5] The anomalous dimensions of operators with twist higher than two occupy a band [16], the lower bound of which is the *minimal* dimension for given spin and twist.

Remarkably, the large spin expansion of the classical string energy does respect MVV-like relations at one-loop [6], providing a strong indication that these relations hold not only in weak coupling (gauge theory) but also in strong coupling (string theory) perturbative expansions, something certainly expected from the convergence of planar perturbation theory.

The plan of the paper is the following. In Sec. (2), we briefly introduce the kernel P and its relation with the anomalous dimension of twist-2 operators. In Sec. (3), we fully prove at a rigorous level that the kernel P is reciprocity respecting. Finally, in Sec. (4) we comment on the fine structure of the large N expansion of the four loop result. A few Appendices are devoted to technical details. In particular, App. (A) briefly recalls the basic definitions and properties of harmonic sums, App. (B) contains the detailed proof of the main theorems used in Sec. (3), and App. (C) reports very detailed large N expansions of the anomalous dimension and DMS kernel. We remark that a three loop reciprocity proof first appeared in [8] to which we are indebted for various ideas and methods.

2. The four loop twist-2 anomalous dimension and its P kernel

The twist-2 anomalous dimension is given up to four loops by [7]

$$\gamma(N) = g^2\,\gamma_1(N) + g^4\,\gamma_2(N) + g^6\,\gamma_3(N) + g^8\,\gamma_4(N) + \mathcal{O}(g^{10}), \tag{2.1}$$

where $g^2 = \frac{g_{\rm YM}^2 N}{16\,\pi^2}$ and (see Appendix A for the definition of harmonic sums $S_{a_1,\dots,a_d}$)

$$\begin{aligned}
\gamma_1(N) &= 8S_1, &(2.2)\\
\gamma_2(N) &= -16\left(S_{-3} + 2S_1\left(S_{-2} + S_2\right) + S_3 - 2S_{-2,1}\right), &(2.3)\\
\gamma_3(N) &= -64\Big(-3S_{-5} + 2S_{-3}S_2 - 2S_{-2}S_3 - S_5 - \left(2S_1^2 + S_2\right)\left(3S_{-3} + S_3 - 2S_{-2,1}\right) + \\
&\quad +6\left(S_{-4,1} + S_{-3,2} + S_{-2,3}\right) - S_1\Big(S_{-2}^2 + 4S_2S_{-2} + 2S_2^2 + 8S_{-4} + 3S_4 - 12S_{-3,1} \\
&\quad -10S_{-2,2} + 16S_{-2,1,1}\Big) - 12\left(S_{-3,1,1} + S_{-2,1,2} + S_{-2,2,1}\right) + 24S_{-2,1,1,1}\Big), &(2.4)\\
\gamma_4(N) &= \gamma_4^{ABA}(N) + \gamma_4^{\rm wrapping}(N), &(2.5)\\
S_{a_1,\dots,a_d} &\equiv S_{a_1,\dots,a_d}(N). &(2.6)
\end{aligned}$$

Above, $\gamma_4^{ABA}(N)$ is the result has been computed in [1] via the asymptotic Bethe Ansatz and can be found in Table 1 of that reference. The wrapping contribution $\gamma_4^{\rm wrapping}(N)$ has

[6] In the case of classical spiky strings [20] only partial consequences of the functional relation (1.3) but not the full reciprocity invariance (1.1) apply as discussed in [18]. However, this nicely agrees with the fact that spiky strings should correspond to an operator of twist higher than two with *non-minimal* anomalous dimension for a given spin, for which reciprocity is not expected to hold. Indeed, anomalous dimensions of twist three operators with energies close to the upper boundary of the band do not respect reciprocity as well, as seen recently in [24].

[7] Closed expressions at two loops are known from explicit field-theory calculations [29] and at three-loops from a conjecture [30] inspired from the maximum transcendentality principle [5] applied to the QCD splitting functions at three-loops [12]. Up to three loops, the same formulas can also be computed by the asymptotic Bethe ansatz [31] for fixed values of M. It is only recently that the three loop conjecture has been proved via the Baxter approach method [32].

been recently calculated in [2] and reads

$$\gamma_4^{\text{wrapping}}(N) = 256\,(S_{-5} - S_5 + 2S_{-2,-3} - 2S_{3,-2} + 2S_{4,1} - 4S_{-2,-2,1})\,S_1^2 + \\ -640\,\zeta_5\,S_1^2 - 512\,S_{-2}\,\zeta_3\,S_1^2. \tag{2.7}$$

The P kernel defined by (1.3) can be derived from the anomalous dimension by simply inverting (1.3). Expanding perturbatively P as

$$P(N) = g^2\,P_1(N) + g^4\,P_2(N) + g^6\,P_3(N) + g^8\,P_4(N) + \mathcal{O}(g^{10}), \tag{2.8}$$

we find the relations

$$P_1 = \gamma_1, \tag{2.9}$$

$$P_2 = \gamma_2 - \frac{1}{2}\gamma_1\gamma_1', \tag{2.10}$$

$$P_3 = \frac{1}{8}\gamma_1''\gamma_1^2 + \frac{1}{4}\left(\gamma_1'\right)^2\gamma_1 - \frac{1}{2}\gamma_2'\gamma_1 + \gamma_3 - \frac{1}{2}\gamma_2\gamma_1', \tag{2.11}$$

$$P_4 = -\frac{1}{48}\gamma_1^{(3)}\gamma_1^3 - \frac{3}{16}\gamma_1'\gamma_1''\gamma_1^2 + \frac{1}{8}\gamma_2''\gamma_1^2 - \frac{1}{8}\left(\gamma_1'\right)^3\gamma_1 + \frac{1}{2}\gamma_1'\gamma_2'\gamma_1 - \frac{1}{2}\gamma_3'\gamma_1 + \\ +\frac{1}{4}\gamma_2\gamma_1''\gamma_1 + \frac{1}{4}\gamma_2\left(\gamma_1'\right)^2 + \gamma_4 - \frac{1}{2}\gamma_3\gamma_1' - \frac{1}{2}\gamma_2\gamma_2'. \tag{2.12}$$

It is using the formula (A.14) for the derivatives of harmonic sums that the above expressions become explicit linear combination of products of harmonic sums. For the purpose of proving reciprocity for the P-kernel, it is however useful to rewrite it in a canonical basis, *i.e.* as linear combinations of single sums. This can be done by using the shuffle algebra relation (A.6).

For example, in the case of the anomalous dimension, we can rewrite it as

$$\gamma_1 = 8\,S_1, \tag{2.13}$$

$$\gamma_2 = 16S_{-3} + 16S_3 - 32S_{1,-2} - 32S_{1,2} - 32S_{2,1}, \tag{2.14}$$

$$\gamma_3 = 128S_{-5} + 128S_5 - 256S_{-4,1} - 128S_{-3,-2} - 64S_{-3,2} - 128S_{-2,-3} - 512S_{1,-4} + \\ -256S_{1,4} - 576S_{2,-3} - 320S_{2,3} - 128S_{3,-2} - 320S_{3,2} - 256S_{4,1} + \\ +128S_{-2,-2,1} + 128S_{-2,1,-2} + 512S_{1,-3,1} + 128S_{1,-2,-2} + 128S_{1,-2,2} + \\ +768S_{1,1,-3} + 256S_{1,1,3} + 256S_{1,2,-2} + 256S_{1,2,2} + 256S_{1,3,1} + 384S_{2,-2,1} + \\ +256S_{2,1,-2} + 256S_{2,1,2} + 256S_{2,2,1} + 256S_{3,1,1} - 512S_{1,1,-2,1}. \tag{2.15}$$

The expression of γ_4 is very long and we do not report it. We just give the canonical result for the wrapping parts leaving S_1 as a factor since it is separately RR.

$$\gamma_4^{\text{wrapping}} = 128S_1^2(2S_{-5} - 2S_5 + 4S_{-2,-3} - 4S_{3,-2} + \\ +4S_{4,1} - 8S_{-2,-2,1} - 5\zeta_5 - 4S_{-2}\zeta_3). \tag{2.16}$$

3. Proof of reciprocity

The proof that P_4 is reciprocity respecting (RR) is based on a clever rewriting in terms of special linear combinations of harmonic sums with nice properties under the (large-)J parity $J \to -J$. In the following section we shall introduce them as a preliminary step.

3.1 Definite-parity linear combinations of harmonic sums

Let us consider the space Λ of $\mathbb{R}$-linear combinations of harmonic sums $S_{\mathbf{a}}$ with generic multi-indices

$$\mathbf{a} = (a_1, \dots, a_d), \qquad a_i \in \mathbb{Z}\backslash\{0\}, \tag{3.1}$$

where d is not fixed. This is the structure of P at any perturbative order.

For any $a \in \mathbb{Z}\backslash\{0\}$, we define the linear map $\omega_a : \Lambda \to \Lambda$ by assigning its action on elementary harmonic sums as follows

$$\omega_a(S_{b,\mathbf{c}}) = S_{a,b,\mathbf{c}} - \frac{1}{2} S_{a\wedge b,\mathbf{c}}, \tag{3.2}$$

where, for $n, m \in \mathbb{Z}\backslash\{0\}$, the wedge-product is

$$n \wedge m = \text{sign}(n)\,\text{sign}(m)\,(|n| + |m|). \tag{3.3}$$

Besides basic harmonic sums, it is also convenient to work with complementary sums $\underline{S_{\mathbf{a}}}$ which are defined in Appendix A. On the space $\underline{\Lambda}$ of their $\mathbb{R}$-linear combinations we define in a similar way a linear map $\underline{\omega_a}$.

In the spirit of [8, 15], we now introduce the following combinations of (complementary) harmonic sums

$$\begin{aligned} \Omega_a &= S_a, & \underline{\Omega_a} &= S_a = \underline{S_a}, \\ \Omega_{a,\mathbf{b}} &= \omega_a(\Omega_{\mathbf{b}}), & \underline{\Omega_{a,\mathbf{b}}} &= \underline{\omega_a}(\underline{\Omega_{\mathbf{b}}}). \end{aligned} \tag{3.4}$$

The main tool that we shall need are the following two theorems which are proved in full details and rigor in App. (B).

Theorem 1: *The subtracted complementary combination* $\underline{\widehat{\Omega}_{\mathbf{a}}}$, $\mathbf{a} = (a_1, \dots, a_d)$ *has definite parity* $\mathcal{P}_{\mathbf{a}}$ *under the (large-)*J *transformation* $J \to -J$ *and*

$$\mathcal{P}_{\mathbf{a}} = (-1)^{|a_1|+\dots+|a_d|}\,(-1)^d \prod_{i=1}^{d} \varepsilon_{a_i}. \tag{3.5}$$

Theorem 2: *The combination* $\Omega_{\mathbf{a}}$, $\mathbf{a} = (a_1, \dots, a_d)$ *with odd positive* a_i *and even negative* a_i *has positive parity* $\mathcal{P} = 1$.

Remark 1: For clarity, let us emphasize once again that a quantity has $\mathcal{P} = \pm 1$ iff its large J expansion is in inverse powers $1/J^{2n}$ ($\mathcal{P} = 1$) or $1/J^{2n+1}$ ($\mathcal{P} = -1$) with possible logarithmic enhancements, *i.e.* powers of $\log\, J^2$. Thus, in particular, a quantity is RR iff it has $\mathcal{P} = +1$.

Remark 2: Theorem 2 follows from Theorem 1 (see Appendices). In this paper we shall use Theorem 2 only, but we quote Theorem 1 as a separate result since it can be relevant in more involved situations.

Remark 3: A special case of Theorem 1 appeared in [8]. A general proof of Theorem 1 in the restricted case $\mathbf{a} = (a_1, \dots, a_\ell)$ with *positive* $a_i > 0$ and *rightmost indices* $a_\ell \neq 1$ can be found in [15]. Appendix (B) contains the proof of the general case.

Just to give an illustrative example of Theorem 2, let us consider the combination

$$\Omega_{1,-2}(N) = S_{1,-2}(N) - \frac{1}{2}S_{-3}(N), \tag{3.6}$$

which is expected to be RR. The large N expansion (for even N) is

$$\begin{aligned}\Omega_{1,-2}(N) = &\left(-\frac{1}{12}\pi^2\log N + \frac{\zeta_3}{4} - \frac{\gamma_E\pi^2}{12}\right) - \frac{\pi^2}{24N} + \frac{1}{144}\pi^2\frac{1}{N^2} + \left(-\frac{1}{4} - \frac{\pi^2}{1440}\right)\frac{1}{N^4} + \\ &+\frac{1}{2}\frac{1}{N^5} + \left(\frac{1}{4} + \frac{\pi^2}{3024}\right)\frac{1}{N^6} - 2\frac{1}{N^7} + \cdots .\end{aligned}$$

Rewriting the expansion in terms of $J^2 = N(N+1)$ we find

$$\begin{aligned}\Omega_{1,-2} = &\left(-\frac{1}{24}\pi^2\log(J^2) + \frac{\zeta_3}{4} - \frac{\gamma_E\pi^2}{12}\right) - \frac{\pi^2}{72J^2} + \left(-\frac{1}{4} + \frac{\pi^2}{360}\right)\frac{1}{J^4} + \\ &+\left(1 - \frac{\pi^2}{945}\right)\frac{1}{J^6} + \left(-\frac{11}{2} + \frac{\pi^2}{1260}\right)\frac{1}{J^8} + \cdots ,\end{aligned} \tag{3.7}$$

which is indeed invariant under $J \to -J$. One easily checks that this happens due to a cancellation of wrong $1/J^{2n+1}$ terms coming from $S_{1,-2}(N)$ and $S_{-3}(N)$. Just to give an example of combination not allowed and where such cancellations do not hold, we show

$$\begin{aligned}\Omega_{1,2} = &\left(\frac{1}{12}\pi^2\log(J^2) - \frac{3\zeta_3}{2} + \frac{\gamma\pi^2}{6}\right) + \frac{1}{J} + \frac{\pi^2}{36J^2} - \frac{11}{72}\frac{1}{J^3} + \\ &-\frac{1}{180}\pi^2\frac{1}{J^4} + \frac{823}{28800}\frac{1}{J^5} + \cdots .\end{aligned} \tag{3.8}$$

3.2 The reduction algorithm

The strategy to prove reciprocity for the kernel P is simple: For each loop order ℓ,

1. Consider in P_ℓ the sums with maximum depth, each of them, say $S_{\mathbf{a}}$, appears uniquely as the maximum depth term in $\Omega_{\mathbf{a}}$.

2. Subtract all the Ω's required to cancel these terms and keep track of this subtraction.

3. Repeat the procedure with depth decreased by one.

At the end, if the remainder is zero and if the full subtraction is composed of Ω's with the right parities, as prescribed by the above theorem, we have proved that P is reciprocity respecting.

Of course, this is *sufficient but not necessary*. If the final remainder is not zero or if we have had to subtract a wrong parity Ω combination, we cannot exclude that P is RR. However, in our case, we have found that up to four loops and including wrapping, the above algorithm works perfectly and provides a rewriting of P which is manifestly reciprocity respecting.

3.3 Example: P_ℓ for $\ell = 1, 2, 3$

At one loop, we have immediately the desired result from

$$P_1 = 8\, S_1 = 8\ \Omega_1. \tag{3.9}$$

At two-loops, written in the canonical basis, the kernel (3.10) reads

$$P_2 = 16 S_{-3} - \frac{16\pi^2 S_1}{3} - 16 S_3 - 32 S_{1,-2}. \tag{3.10}$$

The π^2 term comes from the derivatives appearing in the expression of P_2 in terms of γ_1 and γ_2. Applying the reduction algorithm, one finds

$$P_2 \;=\; -32\ \Omega_{1,-2} - 16\ \Omega_3 - \frac{16}{3}\pi^2\ \Omega_1. \tag{3.11}$$

All Ω combinations have odd-positive or even-negative indices and are thus reciprocity respecting.

At three loops, the kernel P_3 is

$$\begin{aligned} P_3 \;=\; & -128 S_{-5} - \frac{32}{3}\pi^2 S_{-3} + \frac{16\pi^4 S_1}{3} + \frac{32\pi^2 S_3}{3} + 128 S_5 - 64 S_{-4,1} + \\ & -128 S_{-3,-2} - 128 S_{-2,-3} + 64 S_{1,-4} + \frac{64}{3}\pi^2 S_{1,-2} - 64 S_{1,4} + \\ & -128 S_{2,-3} + 128 S_{3,-2} - 64 S_{4,1} + 128 S_{-2,-2,1} + 128 S_{-2,1,-2} + 256 S_{1,-3,1} + \\ & +128 S_{1,-2,-2} + 256 S_{1,1,-3} + 256 S_{2,-2,1} - 512 S_{1,1,-2,1} + \\ & +64 S_2 \zeta_3 - 128 S_{1,1}\zeta_3 \end{aligned} \tag{3.12}$$

Again, π^{2n} and ζ_3 terms come from derivative of γ_k, $k = 1, 2$. The reduction algorithm gives

$$\begin{aligned} P_3 \;=\; & -512\ \Omega_{1,1,-2,1} + 128\ \Omega_{-2,-2,1} + 128\ \Omega_{-2,1,-2} + 128\ \Omega_{1,-2,-2} + 64\ \Omega_{-4,1} + \\ & +192\ \Omega_{1,-4} + 128\ \Omega_{3,-2} + 32\ \Omega_5 + \frac{64}{3}\pi^2\ \Omega_{1,-2} + \frac{32\pi^2\ \Omega_3}{3} + \\ & -128\ \Omega_{1,1}\,\zeta_3 + \frac{16\pi^4\ \Omega_1}{3} \end{aligned} \tag{3.13}$$

which is reciprocity respecting since it contains only allowed Ω terms.

3.4 The four loop ABA contribution

The reduction algorithm that we have illustrated in the 1, 2, and 3 loop cases can be applied to the four loop expression. The expression for P_4^{ABA} in the canonical basis is very long, and we do not show it. Applying the reduction algorithm one finds

$$\begin{aligned}
P_4^{ABA} = & -8192\,\Omega_{1,1,1,-2,1,1} + 6144\,\Omega_{-2,-2,1,1,1} + 6144\,\Omega_{-2,1,-2,1,1} + 4096\,\Omega_{-2,1,1,-2,1} + \\
& +6144\,\Omega_{1,-2,-2,1,1} 6144\,\Omega_{1,-2,1,-2,1} + 2048\,\Omega_{1,-2,1,1,-2} + 6144\,\Omega_{1,1,-2,-2,1} + \\
& +4096\,\Omega_{1,1,-2,1,-2} + 6144\,\Omega_{1,1,1,-2,-2} - 1024\,\Omega_{-2,-2,-2,1} - 1536\,\Omega_{-2,-2,1,-2} + \\
& -2048\,\Omega_{-2,1,-2,-2} + 1024\,\Omega_{1,-4,1,1} - 1536\,\Omega_{1,-2,-2,-2} + 3072\,\Omega_{1,1,-4,1} + \\
& +1024\,\Omega_{1,1,-2,3} + 2048\,\Omega_{1,1,1,-4} + 2048\,\Omega_{1,3,-2,1} + 1024\,\Omega_{3,-2,1,1} + 2048\,\Omega_{3,1,-2,1} + \\
& -2048\,\Omega_{-4,-2,1} - 1280\,\Omega_{-4,1,-2} - 2048\,\Omega_{-2,-4,1} - 768\,\Omega_{-2,-2,3} - 1536\,\Omega_{-2,1,-4} + \\
& -256\,\Omega_{-2,3,-2} - 2304\,\Omega_{1,-4,-2} - 1792\,\Omega_{1,-2,-4} - 2048\,\Omega_{1,1,5} - 1536\,\Omega_{1,5,1} + \\
& -1280\,\Omega_{3,-2,-2} - 1536\,\Omega_{5,1,1} - 768\,\Omega_{-6,1} - 128\,\Omega_{-4,3} + 384\,\Omega_{-2,5} - 1408\,\Omega_{1,-6} + \\
& -896\,\Omega_{3,-4} - 256\,\Omega_{5,-2} + 640\,\Omega_7 + \frac{2048}{3}\pi^2\,\Omega_{1,1,-2,1} + 1024\pi^2\,\Omega_{1,1,1,-2} + \\
& -\frac{512}{3}\pi^2\,\Omega_{-2,-2,1} - \frac{512}{3}\pi^2\,\Omega_{-2,1,-2} - \frac{512}{3}\pi^2\,\Omega_{1,-2,-2} - \frac{256}{3}\pi^2\,\Omega_{-4,1} + \\
& -256\pi^2\,\Omega_{1,-4} - \frac{512}{3}\pi^2\,\Omega_{3,-2} + 1536\zeta_3\,\Omega_{-2,1,1} + 1280\,\Omega_{1,-2,1}\zeta_3 + 1024\,\Omega_{1,1,-2}\zeta_3 \\
& +640\zeta_3\,\Omega_{1,3} + 640\,\Omega_{3,1}\zeta_3 - 320\,\Omega_{-4}\zeta_3 + \frac{1088}{15}\pi^4\,\Omega_{1,1,1} - \frac{64}{3}\pi^4\,\Omega_{1,-2} - \frac{752}{45}\pi^4\,\Omega_3 + \\
& +\Omega_{1,1}\big(-\frac{256}{3}\pi^2\zeta_3 + 2560\zeta_5\big) - \frac{256}{45}\pi^6\,\Omega_1 - \zeta_3(2\,\Omega_{-2,1} + \Omega_3).
\end{aligned} \quad (3.14)$$

This proves reciprocity of the ABA term since, once again, only allowed Ω's appear !

3.5 The four loop wrapping contribution

The wrapping contribution starts at four loops. It enters directly P_4 with no mixing with lower loop order terms. Thus, we can apply immediately the reduction algorithm without need of taking any derivative. The result is very simple. It reads

$$P_4^{\text{wrapping}} = -4\,\Omega_{-2,-2,1} - 2\,\Omega_{3,-2}, \quad (3.15)$$

and is clearly reciprocity respecting.

4. Expansions at large N and inheritance

In this final Section, we discuss the fine structure of the DMS kernel $P(N)$ at large N. The general structure of soft gluon emission governing the very large N behaviour of $\gamma(N)$ predicts the leading contribution $\gamma(N) \sim f_{\text{cusp}}(\lambda)\log N$ where the coupling dependent coefficient $f_{\text{cusp}}(\lambda)$, *a.k.a.* cusp anomaly, is expected to be universal in *both twist and flavour*. This is precisely what is observed in the various exact multiloop expressions discussed in Appendix F of [18].

This leading logarithmic behaviour is also the leading term in the function $P(N)$. Concerning the subleading terms, as remarked in [8] to which we defer for a full discussion, the function $P(N)$ obeys at three loops a very powerful additional *simplicity* constraint. Indeed, it does not contains logarithmically enhanced terms $\sim \log^n(N)/N^m$ with $n \geq m$ apart from the leading cusp logarithm.

This immediately implies that the leading logarithmic functional relation

$$\gamma(N) = f_{\rm cusp}(\lambda) \log \left(N + \tfrac{1}{2} f_{\rm cusp}(\lambda) \log N + ... \right) + ... \tag{4.1}$$

predicts correctly the maximal logarithmic terms $\log^m N/N^m$

$$\gamma(N) \sim f_{\rm cusp} \log N + \frac{f_{\rm cusp}^2}{2} \frac{\log N}{N} - \frac{f_{\rm cusp}^3}{8} \frac{\ln^2 N}{N^2} + ... \tag{4.2}$$

whose coefficients are simply proportional to $f_{\rm cusp}^{m+1}$ [15, 24, 18].

Notice that the fact that the cusp anomaly is known at all orders in the coupling via the results of [27, 28] naturally implies (*under the "simplicity" assumption for P*) a proper *prediction* for *all* coefficients of the type $\log^m N/N^m$ at *all orders* in the coupling constant, and in particular for those appearing in the large spin expansion of the energies of certain semiclassical string configurations (dual to the operators of interest). Such prediction has been checked in [18] up to one loop in the sigma model semiclassical expansion, as well as in [19] at the classical level [8].

As noticed in [18], Appendix F, the asymptotic part of the four loop anomalous dimension for twist-2 operators already revealed an exception to this "rule", being the term $\log^2 N/N^2$ not given only in terms of the cusp anomaly. Interestingly enough, the large spin expansion of the wrapping contribution of [2], which correctly does not change the leading asymptotic behavior (cusp anomaly), first contributes at order $\log^2 N/N^2$. Thus, while on the basis of (4.2) one would expect in the large spin expansion of the four loop anomalous dimension a term of the type

$$(c_{22})_4 \frac{\log^2 N}{N^2} \qquad \texttt{with} \qquad (c_{22})_4^{\rm Naive} = \left(-\tfrac{f_{\rm cusp}^3}{8}\right)_4 = 64\,\pi^2 \tag{4.3}$$

expanding (2.5) and (2.7) below one finds (see Appendix C, formulas (C.4) and (C.5))

$$(c_{22})_4^{\rm ABA} = 64\pi^2 - 128\,\zeta_3 \qquad \texttt{and} \qquad (c_{22})_4^{\rm wrapping} = -\frac{64}{3}\pi^2 - 128\,\zeta_3 \tag{4.4}$$

which summed up do not reproduce (4.3). This indicates that, in the case of the twist-2 operators and starting at four loops, the P-function ceases to be "simple" in the meaning of [8]. This is confirmed by explicitely looking at the the structure of its asymptotic expansion (see Appendix C, formula C.9), and prevents the tower of subleading logarithmic singularities $\log^m N/N^m$ to be simply inherited from the cusp anomaly.

5. Discussion

The present analysis together with the related work in [7, 9, 8, 15, 21, 22, 23, 18] leads to the following conclusions.

Reciprocity has been tested in $\mathcal{N} = 4$ SYM at weak coupling for the minimal dimension of operators of twist-2 and three for all possible flavors and at strong coupling up to

[8] In [19], a nice explanation for the relation of the $\log^m N/N^m$ coefficients to the cusp anomaly has been given in terms of the pp-wave limit for the case of spiky strings in $AdS_3 \times S^1$.

one loop in the string sigma-model calculation [18] [9]. In [10], hints were given suggesting that the asymptotic part of the four loop result for the twist-2 operators, derived from the Bethe Ansatz in [1], was presumably reciprocity respecting. In this paper we have proved this claim in full rigor, showing that reciprocity also applies to the wrapping contribution.

All this suggests that reciprocity can then be considered a *hidden* symmetry of $\mathcal{N} = 4$ SYM, intrinsic in the Asymptotic Bethe Ansatz of the theory and thus related in some unknown way with the structure built in there. Because it holds also in the presence of wrapping, it is reasonable to consider reciprocity as an important testing device for checking the correctness of any future expression of minimal anomalous dimension for twist operators, as well as for the energies of their string dual counterpart [10].

While it would be significative to *derive* reciprocity in $\mathcal{N} = 4$ SYM from first principles (and it is expected that the AdS/CFT correspondence might help in this), a reasonable attitude can be, in view of the previous point and as in the case of the integrability of the theory, to just *assume* that reciprocity holds.

This would strongly simplify any attempt to calculate further examples of multiloop anomalous dimensions, at higher loop and twist [26]. The use of both the maximum transcendentality principle *and* reciprocity drastically reduces the number of terms that have to be calculated via Bethe Ansatz and generalised Lüscher techniques, and is expected to give a fast and correct answer where other methods as the Baxter approach still need further achievements.

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[9]Reciprocity in QCD and in SYM theories with $\mathcal{N} = 0, 2$ supercharges is discussed in [9].

[10]For example, the conjecture of [25] for the coefficient of ζ_5 in the twist-2 anomalous dimension at four loops contains in principle an arbitrary rational number. A precise value was suggested in [25] on the basis of some deep physical intuition and then confirmed in [2]. That same value would be unambiguously selected by requiring the reciprocity of the conjecture.

A. Harmonic sums

A.1 Basic definitions

The basic definition of nested harmonic sums $S_{a_1,\ldots,a_n}$ is recursive

$$S_a(N) = \sum_{n=1}^{N} \frac{\varepsilon_a^n}{n^{|a|}}, \tag{A.1}$$

$$S_{a,\mathbf{b}}(N) = \sum_{n=1}^{N} \frac{\varepsilon_a^n}{n^{|a|}}\, S_{\mathbf{b}}(n), \tag{A.2}$$

where

$$\varepsilon_a = \begin{cases} +1, \text{ if } a \geq 0, \\ -1, \text{ if } a < 0 \end{cases}. \tag{A.3}$$

Given a particular sum $S_{\mathbf{a}} = S_{a_1,\ldots,a_n}$ we define

$$\text{depth } (S_{\mathbf{a}}) = n, \tag{A.4}$$

$$\text{transcendentality}(S_{\mathbf{a}}) = |\mathbf{a}| \equiv |a_1| + \cdots + |a_n|. \tag{A.5}$$

For a product of S sums, we define transcendentality to be the sum of the transcendentalities of the factors.

Product of S sums can be reduced to linear combinations of single sums by using iteratively the shuffle algebra [33] defined as follows

$$\begin{aligned} S_{a_1,\ldots,a_n}(N)\, S_{b_1,\ldots,b_m}(N) = &\sum_{\ell=1}^{N} \frac{\varepsilon_{a_1}^{\ell}}{\ell^{|a_1|}}\, S_{a_2,\ldots,a_n}(\ell)\, S_{b_1,\ldots,b_m}(\ell) + \\ &+ \sum_{\ell=1}^{N} \frac{\varepsilon_{b_1}^{\ell}}{\ell^{|b_1|}}\, S_{a_1,\ldots,a_n}(\ell)\, S_{b_2,\ldots,b_m}(\ell) + \\ &- \sum_{\ell=1}^{N} \frac{\varepsilon_{a_1}^{\ell}\, \varepsilon_{b_1}^{\ell}}{\ell^{|a_1|+|b_1|}}\, S_{a_2,\ldots,a_n}(\ell)\, S_{b_2,\ldots,b_m}(\ell). \end{aligned} \tag{A.6}$$

A.2 Complementary and subtracted sums

Let $\mathbf{a} = (a_1, \ldots, a_\ell)$ be a multi-index. For $a_1 \neq 1$, it is convenient to adopt the concise notation

$$S_{\mathbf{a}}(\infty) \equiv S_{\mathbf{a}}^*. \tag{A.7}$$

Complementary harmonic sums are defined recursively by

$$\underline{S_a} = S_a, \tag{A.8}$$

$$\underline{S_{\mathbf{a}}} = S_{\mathbf{a}} - \sum_{k=1}^{\ell-1} S_{a_1,\ldots,a_k}\, \underline{S_{a_{k+1},\ldots,a_\ell}^*}. \tag{A.9}$$

The definition is ill when $\mathbf{a}$ has some rightmost 1 indices. In this case, we treat S_1^* as a formal object in the above definition and set it to zero in the end. It can be shown that

$\underline{S_{\mathbf{a}}}^* < \infty$ in all cases and hence it is meaningful to define subtracted complementary sums as

$$\underline{\widehat{S}_{\mathbf{a}}} = \underline{S_{\mathbf{a}}} - \underline{S^*_{\mathbf{a}}}. \tag{A.10}$$

Explicitely,

$$\underline{\widehat{S}_{\mathbf{a}}}(N) = (-1)^\ell \sum_{n_1=N+1}^{\infty} \frac{\varepsilon_{a_1}^{n_1}}{n_1^{|a_1|}} \sum_{n_2=n_1+1}^{\infty} \frac{\varepsilon_{a_2}^{n_2}}{n_2^{|a_2|}} \cdots \sum_{n_\ell=n_{\ell-1}+1}^{\infty} \frac{\varepsilon_{a_\ell}^{n_\ell}}{n_\ell^{|a_\ell|}}. \tag{A.11}$$

A.3 Derivatives of harmonic sums

Given the fact that a generic sum has the asymptotic expansion

$$S_{\mathbf{a}}(N) = \sum_{\ell=0}^{\infty} \frac{P_\ell(\log\ N)}{N^\ell} + (-1)^N \sum_{\ell=0}^{\infty} \frac{Q_\ell(\log\ N)}{N^\ell}, \tag{A.12}$$

we want to define $S'_{\mathbf{a}}(N)$ as a combination of harmonic sums such that their asymptotic expansion is

$$S'_{\mathbf{a}}(N) = \frac{d}{dN} \sum_{\ell=0}^{\infty} \frac{P_\ell(\log\ N)}{N^\ell} + (-1)^N \frac{d}{dN} \sum_{\ell=0}^{\infty} \frac{Q_\ell(\log\ N)}{N^\ell}, \tag{A.13}$$

This remark is in order to explain how we treat the $(-1)^N$ factor. For sums with only positive indices, this derivative is just the ordinary derivative. Indeed one can shows that apart from the $(-1)^N$, the sums are smooth functions of N (finite sum trick).

After these preliminary remarks, the master formula for derivatives is

$$S'_{a_1,\dots,a_\ell} = -\sum_{k=1}^{\ell} |a_k|\, \widehat{S}_{a_1,\dots,a_k\wedge 1,\dots,a_\ell} + \sum_{k=2}^{\ell} |a_k| \sum_{p=1}^{k-1} \underline{\widehat{S}_{a_1,\dots,a_p}}\, S^*_{a_{p+1},\dots,a_k\wedge 1,\dots,a_\ell} \tag{A.14}$$

which reads more explicitly

$$\begin{aligned} S'_{a_1,\dots,a_\ell} = & -\sum_{k=1}^{\ell} |a_k|\, \widehat{S}_{a_1,\dots,a_k\wedge 1,\dots,a_\ell} + \\ & + |a_2|\, \underline{\widehat{S}_{a_1}}\, S^*_{a_2\wedge 1,a_3,a_4,\dots} + \\ & + |a_3|\, (\underline{\widehat{S}_{a_1}}\, S^*_{a_2,a_3\wedge 1,a_4,\dots} + \underline{\widehat{S}_{a_1,a_2}}\, S^*_{a_3\wedge 1,a_4,\dots}) + \\ & + |a_4|\, (\underline{\widehat{S}_{a_1}}\, S^*_{a_2,a_3 a_4\wedge 1,\dots} + \underline{\widehat{S}_{a_1,a_2}}\, S^*_{a_3,a_4\wedge 1,\dots} + \underline{\widehat{S}_{a_1,a_2,a_3}}\, S^*_{a_4\wedge 1,\dots}) + \cdots \end{aligned} \tag{A.15}$$

A.4 Mellin transforms

Let $\mathbf{a} = \{a_1, \dots, a_\ell\}$ be a multi-index with the important restriction that there are no rightmost indices different from 1, $a_\ell \neq 1$.

Defining recursively the functions $G(x)$ via

$$G_{a_1,\dots,a_\ell}(x) = \frac{\varepsilon_{a_1}^N}{\Gamma(|a_1|)} \int_x^1 \frac{dy}{y-\varepsilon_{a_2}\dots\varepsilon_{a_\ell}} \ln^{|a_1|-1}\frac{y}{x}\; G_{a_2,\dots,a_\ell}(y) \tag{A.16}$$

$$\dots \qquad \dots$$

$$G_{a_{\ell-1},a_\ell}(v) = \frac{\varepsilon_{a_{\ell-1}}^N}{\Gamma(|a_{\ell-1}|)} \int_v^1 \frac{dw}{w-\varepsilon_{a_\ell}} \ln^{|a_{\ell-1}|-1}\frac{w}{v}\; G_{a_\ell}(w)$$

$$G_{a_\ell}(w) = \frac{\varepsilon_{a_\ell}^N}{\Gamma(|a_\ell|)} \ln^{|a_\ell|-1}\frac{1}{w} \tag{A.17}$$

the Mellin transform of the subtracted sums of (A.11) reads then

$$\underline{\widehat{S}_{\mathbf{a}}}(N) = \mathbf{M}\left[\frac{x}{x-\varepsilon_{a_1}\dots\varepsilon_{a_\ell}}\; G_{a_1,\dots,a_\ell}(x)\right] \tag{A.18}$$

For example, for three indices it is

$$\underline{\widehat{S}_{a,b,c}}(N) = \frac{(\varepsilon_a\varepsilon_b\varepsilon_c)^N}{\Gamma(|a|)\Gamma(|b|)\Gamma(|c|)}\; \mathbf{M}\left[\frac{x}{x-\varepsilon_a\varepsilon_b\varepsilon_c}\int_x^1 \frac{dy}{y-\varepsilon_b\varepsilon_c}\ln^{|a|-1}\frac{y}{x}\int_y^1 \frac{dz}{z-\varepsilon_c}\ln^{|b|-1}\frac{z}{y}\;\ln^{|c|-1}\frac{1}{z}\right] \tag{A.19}$$

For our purpose, it is important to notice that the function G in (A.18) satisfies the property

$$\begin{aligned} G_{a_1,\dots,a_\ell}\left(\tfrac{1}{x}\right) &= (-1)^{\sum_{i=1}^{\ell}(|a_i|-1)}\Big\{G_{a_1,\dots,a_\ell}(x) - \textstyle\sum_{k=1}^{\ell-1} G_{a_1,\dots,a_k\wedge a_{k+1},\dots,a_\ell}(x) \\ &+\Big[\sum_{k=1}^{\ell-1} G_{a_1,\dots,a_{k-1}\wedge a_k\wedge a_{k+1},\dots,a_\ell}(x) + \sum_{k=1}^{\ell-2} G_{a_1,\dots,a_{k-1}\wedge a_k,a_{k+1}\wedge a_{k+2},\dots,a_\ell}(x)\Big] \\ &-\Big[\sum_{k=1}^{\ell-1} G_{a_1,\dots,a_{k-2}\wedge a_{k-1}\wedge a_k\wedge a_{k+1},a_{k+2},\dots,a_\ell}(x) + \dots\Big] + \dots + (-1)^{\ell-1} G_{a_1\wedge a_2\wedge\dots\wedge a_\ell}(x)\Big\} \end{aligned} \tag{A.20}$$

Above, the sign of each contribution is determined by $(-1)^{n_\mathrm{w}}$, with n_w is the number of the wedge-products in the G-functions appearing in that piece. For example, for three indices it is

$$G_{a,b,c}\left(\tfrac{1}{x}\right) = (-1)^{|a|+|b|+|c|-1}\;\left[G_{a,b,c}(x) - G_{a\wedge b,c}(x) - G_{a,b\wedge c}(x) + G_{a\wedge b\wedge c}(x)\right] \tag{A.21}$$

To obtain (A.20), one uses recursively the result

$$\frac{\varepsilon_{a_1}}{\Gamma(a_1)}\int_x^1 \frac{dy}{y}\ln^{|a_1|-1}\frac{y}{x}\, G_{a_2,\dots,a_\ell}(y) = G_{a_1\wedge a_2,a_3,\dots,a_\ell}(x)\,. \tag{A.22}$$

B. Technical proofs

B.1 Proof of Theorem 1, no rightmost unit indices

It is possible to proceed iteratively starting from combinations $\underline{\widehat{\Omega}_a}(N)$ with one index. At each step we only focus on $\underline{\widehat{\Omega}}$ combinations with maximal number of indices, the iterative procedure ensures in fact that for the remainder the theorem has been already proved. The strategy is to write the $\underline{\widehat{\Omega}}$ in terms of their Mellin transforms exploiting (A.18) and use reciprocity in x-space via Eq. (1.1). For this purpose we use the notation of Appendix A and introduce the functions $\Gamma(x)$, whose relation with the $\Omega(N)$ functions is exactly as the one of the functions $G(x)$ with the subtracted sums $\underline{\widehat{S}}(N)$. Our derivation mimicks the analogous construction described in Sec. (2.2.1) of [15] generalizing it to the signed case.

For technical reasons, we first consider $\underline{\widehat{\Omega}_{\mathbf{a}}}$ in the case where the rightmost index in the multi-index $\mathbf{a}$ is not 1. This is necessary since we want to use the Mellin transform described in App. A.4 which are valid under this limitation. This is not a problem at depth 1 since it is well known that S_1 is parity-even. At depth larger than one, we shall discuss at the end how this limitation can be overcome. So, let us assume for the moment that $\mathbf{a} = (a_1, \dots, a_\ell)$ with $a_\ell \neq 1$.

For one index,

$$\underline{\widehat{\Omega}_a}(N) \equiv \underline{\widehat{S}_a}(N) = \mathbf{M}\left[\frac{x}{x-\varepsilon_a}\, G_a(x)\right] \equiv \mathbf{M}\left[\frac{x}{x-\varepsilon_a}\, \Gamma_a(x)\right] \tag{B.1}$$

The l.h.s. has parity $\mathcal{P} = \pm 1$ iff

$$\Gamma_a(x) = \mathcal{P}\, \varepsilon_a\, \Gamma\left(\tfrac{1}{x}\right) \tag{B.2}$$

Using (A.20) it is easy to see that

$$\varepsilon_a\, \Gamma_a\left(\tfrac{1}{x}\right) = (-1)^{|a|-1}\, \varepsilon_a\, \Gamma_a(x) \tag{B.3}$$

Thus,

$$\mathcal{P} = (-1)^{|a|-1}\, \varepsilon_a, \tag{B.4}$$

in agreement with Theorem 1. The generalisation to ℓ indices is straightforward. Using the notation $\varepsilon_i \equiv \varepsilon_{a_i}$, it is

$$\underline{\widehat{\Omega}_{a_1,\dots,a_\ell}}(N) = \mathbf{M}\left[\frac{x}{x-\varepsilon_1 \dots \varepsilon_\ell}\Gamma_{a_1,\dots,a_\ell}(x)\right] \tag{B.5}$$

where

$$\begin{aligned}
&\Gamma_{a_1,\dots,a_\ell}(x) = G_{a_1,\dots,a_\ell}(x) - \tfrac{1}{2}\sum_{k=1}^{\ell} G_{a_1,\dots,a_k \wedge a_{k+1}}(x) \\
&+ \left(-\tfrac{1}{2}\right)^2 \left[\sum_{k=1}^{\ell-1} G_{a_1,\dots,a_{k-1}\wedge a_k \wedge a_{k+1},\dots,a_\ell}(x) + \sum_{k=1}^{\ell-2} G_{a_1,\dots,a_{k-1}\wedge a_k, a_{k+1}\wedge a_{k+2},\dots,a_\ell}(x)\right] \\
&+ \dots + \left(-\tfrac{1}{2}\right)^{\ell-1} G_{a_1 \wedge \dots \wedge a_\ell}(x)\,,
\end{aligned} \tag{B.6}$$

which is nothing but the general form of Eq. (2.17) in [15]. The l.h.s. has parity $\mathcal{P}$ iff

$$\Gamma_{a_1,\dots,a_\ell}(x) = \mathcal{P}\,\varepsilon_1\dots\varepsilon_\ell\,\Gamma_{a_1,\dots,a_\ell}\left(\tfrac{1}{x}\right). \tag{B.7}$$

Using the formula (A.20) for each of the G-functions evaluated in $1/x$ appearing in the right-hand-side of (B.7), one can see that

$$\mathcal{P} = (-1)^{\sum_{i=1}^{\ell}(|a_i|-1)}\,\varepsilon_1\dots\varepsilon_\ell, \tag{B.8}$$

again in agreement with Theorem 1 which is then proved for all $\mathbf{a} = (a_1,\dots,a_\ell)$ with $a_\ell \neq 1$.

B.2 Proof of Theorem 1, extension to general $\mathbf{a}$

To conclude, let us now define the number $u_\mathbf{a}$ of rightmost 1 indices as

$$u_\mathbf{a} = \max_k \{1 \le k \le \ell \mid a_\ell = a_{\ell-1} = \cdots = a_{\ell-k+1} = 1\}. \tag{B.9}$$

We have the identity

$$\begin{aligned} S_1\,\underline{\widehat{\Omega}_\mathbf{a}} = {} & \underline{\Omega_{1,a_1,\dots,a_d}} + \underline{\Omega_{a_1,1,a_2,\dots,a_d}} + \cdots + \underline{\Omega_{a_1,\dots,a_d,1}} + \\ & -\frac{1}{4}\,\underline{\Omega_{a_1\wedge a_2\wedge 1,a_3,\dots,a_d}} - \frac{1}{4}\,\underline{\Omega_{a_1,a_2\wedge a_3\wedge 1,a_4,\dots,a_d}} + \cdots + \\ & -\frac{1}{4}\,\underline{\Omega_{a_1,\dots,a_{d-2},a_{d-1}\wedge a_d\wedge 1}}. \end{aligned} \tag{B.10}$$

This can be written as

$$\underline{\Omega_{\mathbf{a},1}} = S_1\,\underline{\widehat{\Omega}_\mathbf{a}} + \sum_{\mathbf{b}\in\mathcal{B}} \underline{\Omega_\mathbf{b}}, \tag{B.11}$$

where each multi-index $\mathbf{b} \in \mathcal{B}$ obeys

$$\mathcal{P}_\mathbf{b} = \mathcal{P}_\mathbf{a}, \qquad u_\mathbf{b} \le u_\mathbf{a}. \tag{B.12}$$

Thus, by induction over $u_\mathbf{a}$ and using the above proof of Theorem 1 for the initial case $u_\mathbf{a} = 0$, we get the proof of Theorem 1 in the general $u_\mathbf{a} \ge 0$ case.

B.3 Proof of Theorem 2

We start from the combinatorial identity

$$\Omega_{a_1,\dots,a_\ell}(N) = \sum_{k=1}^{\ell} \underline{\widehat{\Omega}_{a_1,\dots,a_k}}(N)\,\Omega_{a_{k+1},\dots,a_\ell}(\infty) + \Omega_{a_1,\dots,a_\ell}(\infty). \tag{B.13}$$

Suppose now that all even a_i are negative and all odd a_i are positive. Then $(-1)^{|a_i|} = -\mathrm{sign}(a_i)$ and it follows that for any sub-multi-index $(a_1,\dots,a_k)$ we have

$$(-1)^{\sum_{i=1}^{k}(|a_k|-1)} \prod_{i=1}^{k} \mathrm{sign}(a_i) = (-1)^k \prod_{i=1}^{k}(-1) = 1. \tag{B.14}$$

Thus, from Theorem 1, all terms in the r.h.s. of Eq. (B.13) have $\mathcal{P} = +1$ and Theorem 2 is proved.

C. Asymptotic expansions of γ and P

We report here the first few orders for the large N expansions of the twist-2 anomalous dimension and of its kernel P.

Expanding formulas (2.2-2.4) one gets

$$\gamma_1 = 8\log\bar{N} + \frac{4}{N} - \frac{2}{3}\frac{1}{N^2} + \mathcal{O}\Big(\frac{1}{N^4}\Big) \tag{C.1}$$

$$\begin{aligned}\gamma_2 =& -\frac{8}{3}\pi^2\log\bar{N} - 24\zeta_3 + \Big(32\log\bar{N} - \frac{4\pi^2}{3}\Big)\frac{1}{N} - \big(16\log\bar{N} - \frac{2\pi^2}{9} - 24\big)\frac{1}{N^2} + \\ &+\Big(\frac{16}{3}\log\bar{N} - \frac{56}{3}\Big)\frac{1}{N^3} + \mathcal{O}\Big(\frac{1}{N^4}\Big)\end{aligned} \tag{C.2}$$

$$\begin{aligned}\gamma_3 =& \frac{88}{45}\pi^4\log\bar{N} + 160\zeta_5 + \frac{16}{3}\pi^2\zeta_3 - \Big(\frac{64}{3}\pi^2\log\bar{N} + 96\zeta_3 - \frac{44\pi^4}{45}\Big)\frac{1}{N} + \\ &-\Big(64\log^2\bar{N} - (\frac{16}{3}\pi^2 + 128)\log\bar{N} - 48\zeta_3 + \frac{22\pi^4}{135} + \frac{32\pi^2}{3}\Big)\frac{1}{N^2} + \\ &+\Big(64\log^2\bar{N} + (\frac{16}{9}\pi^2 - 256)\log\bar{N} - 16\zeta_3 + \frac{40\pi^2}{9} + 96\Big)\frac{1}{N^3} + \mathcal{O}(\frac{1}{N^4})\end{aligned} \tag{C.3}$$

where $\bar{N} = N\,e^{\gamma_E}$.

At four loops, the large N expansion of Table 1 in [1] and (2.7) leads to [11, 12]

$$\begin{aligned}\gamma_4^{\mathrm{ABA}} =& -16\Big(\frac{73}{630}\pi^6 + 4\zeta_3^2\Big)\log\bar{N} - 1400\zeta_7 - \frac{80}{3}\pi^2\zeta_5 - \frac{56}{15}\pi^4\zeta_3 \\ &+\Big(\frac{96}{5}\pi^4\log\bar{N} + 640\zeta_5 - 32\zeta_3^2 + \frac{160}{3}\pi^2\zeta_3 - \frac{292\pi^6}{315}\Big)\frac{1}{N} \\ &+\Big((64\pi^2 - 128\zeta_3)\log^2\bar{N} + (448\zeta_3 - \frac{32}{15}\pi^4 - 128\pi^2)\log\bar{N} \\ &\quad - 320\zeta_5 + \frac{16\zeta_3^2}{3} - \frac{32}{3}\pi^2\zeta_3 - 384\zeta_3 + \frac{146\pi^6}{945} + \frac{136\pi^4}{15}\Big)\frac{1}{N^2} \\ &+\Big(\frac{512}{3}\log^3\bar{N} + (128\zeta_3 - \frac{64}{3}\pi^2 - 768)\log^2\bar{N} \\ &\quad - (576\zeta_3 + \frac{64}{15}\pi^4 - \frac{512}{3}\pi^2 - 512)\log\bar{N} \\ &\quad + \frac{320\zeta_5}{3} - \frac{64}{9}\pi^2\zeta_3 + 800\zeta_3 - \frac{32\pi^4}{15} - \frac{224\pi^2}{3}\Big)\frac{1}{N^3} + \mathcal{O}\Big(\frac{1}{N^4}\Big)\end{aligned} \tag{C.4}$$

$$\gamma_4^{\mathtt{wrapping}} = -\Big(\frac{64}{3}\pi^2 + 128\zeta_3\Big)\frac{\log^2\bar{N}}{N^2} + \Big(\frac{64}{3}\pi^2 + 128\zeta_3\Big)\Big(\log^2\bar{N} - \log\bar{N}\Big)\frac{1}{N^3} + \mathcal{O}\Big(\frac{1}{N^4}\Big) \tag{C.5}$$

Expanding formulas (3.9), (3.10) and (3.12) one obtains the large N expansion of the

[11] The simple structure of the expansion for $\gamma_4^{\mathtt{wrapping}}$ is lost at higher orders in $1/N$.

[12] The asymptotic next-to-leading constant term is in agreement with [34].

kernel P up to three loops, that reads

$$P_1 = 8\log\bar N + \frac{4}{N} - \frac{2}{3}\Big(\frac{1}{N}\Big)^2 + \mathcal{O}\Big(\frac{1}{N}\Big)^4 \tag{C.6}$$

$$P_2 = -\frac{8}{3}\pi^2\log\bar N - 24\zeta_3 - \frac{4\pi^2}{3N} + \Big(8+\frac{2\pi^2}{9}\Big)\frac{1}{N^2} - \frac{8}{N^3} + \mathcal{O}\Big(\frac{1}{N}\Big)^4 \tag{C.7}$$

$$P_3 = \frac{88}{45}\pi^4\log\bar N + 160\zeta_5 + \frac{16}{3}\pi^2\zeta_3 + \frac{44\pi^4}{45\,N} - \left(\frac{16}{3}\pi^2\log\bar N + \frac{22\pi^4}{135}\right)\frac{1}{N^2} + \\ +\Big(\frac{16}{3}\pi^2\log\bar N - \frac{8\pi^2}{3}\Big)\frac{1}{N^3} + \mathcal{O}\Big(\frac{1}{N}\Big)^4 \tag{C.8}$$

Notice that, in contrast with the series (C.1-C.3) for the anomalous dimension, where the number of logarithms increases with the power of the $1/N$ suppression, the kernel appears to be *linear* in $\log N$ and, in particular, there are no maximally enhanced terms of the form $(\log(N)/N)^k$.

This "simplicity" feature is lost at four loops. Expanding (3.14) and (3.15) and summing them together one finds

$$P_4 = -16\Big(\frac{73}{630}\pi^6 + 4\zeta_3^2\Big)\log\bar N - 1400\zeta_7 - \frac{80}{3}\pi^2\zeta_5 - \frac{56}{15}\pi^4\zeta_3 - \Big(\frac{292\pi^6}{315} + 32\zeta_3^2\Big)\frac{1}{N} + \\ -\Big(\Big(256\zeta_3 + \frac{64}{3}\pi^2\Big)\log^2\bar N - \Big(64\zeta_3 + \frac{112}{15}\pi^4\Big)\log\bar N + \frac{8\pi^4}{15} - 16\pi^2\zeta_3 - \frac{16\zeta_3^2}{3} - \frac{146\pi^6}{945}\Big)\frac{1}{N^2} + \\ +\Big(\Big(256\zeta_3 + \frac{64}{3}\pi^2\Big)\log^2\bar N - \Big(320\zeta_3 + \frac{112}{15}\pi^4 + \frac{64}{3}\pi^2\Big)\log\bar N - 16\pi^2\zeta_3 + 32\zeta_3 + \frac{64\pi^4}{15}\Big)\frac{1}{N^3} + \mathcal{O}\Big(\frac{1}{N}\Big)^4 \tag{C.9}$$

In particular, at order $1/N^2$ a $\log^2 N$ appears, which is responsible for the formula (4.4) discussed in Section 4.

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