Twist-three at five loops,
Bethe Ansatz and wrapping

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Abstract

We present a formula for the five-loop anomalous dimension of $\mathcal{N} = 4$ SYM twist-three operators in the $\mathfrak{sl}(2)$ sector. We obtain its asymptotic part from the Bethe Ansatz and finite volume corrections from the generalized Lüscher formalism, considering scattering processes of spin chain magnons with virtual particles that travel along the cylinder. The complete result respects the expected large spin scaling properties and passes non-trivial tests including reciprocity constraints. We analyze the pole structure and find agreement with a conjectured resummation formula. In analogy with the twist-two anomalous dimension at four-loops wrapping effects are of order $(\log^2 M/M^2)$ for large values of the spin.

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1 Introduction and discussion

Recent developments in computing finite size effects on the asymptotic spectrum of $\mathcal{N} = 4$ SYM twist-two operators are very promising in order to ultimately find the complete spectral equations of the dilatation generator.

The revealing of integrable structures on both sides of the AdS/CFT correspondence [1] gradually led to powerful tools for computing anomalous dimension of gauge invariant operators by means of the Bethe Ansatz [2]. The factorized two-body S-Matrix [3] that governs scattering processes of the spin chain particles and excitations of the $AdS_5 \times S^5$ string worldsheet is determined by the $\mathfrak{psu}(2,2|4)$ symmetry of $\mathcal{N} = 4$ up to a phase factor [4]. In order to also determine this algebraic ambiguity a crossing-like equation for the dressing phase has been derived [5]. It allows for multiple solutions [6], one of which gets singled out [7] by reconciliation with an explicit diagrammatic calculation of the four-loop anomalous dimensions of twist-two operators in the large spin limit [8]. It still remains an open problem to explicitly show the crossing invariance of the dressed asymptotic Bethe equations. For this purpose the representations of the dressing factor in [9] might prove useful.

It was shown that these equations are asymptotic in nature, and need to be corrected by wrapping effects [10]. An explicit calculation of the anomalous dimension of twist-two operators from the asymptotic Bethe Ansatz at four-loops unequivocally showed that the pole prescript by BFKL physics can not be fulfilled [11]. The Bethe Ansatz therefore does not produce the correct result at and beyond wrapping order.

However, for exactly these operators complete results have been obtained for the first time. For the simplest representative of twist-two operators, the Konishi-field, a field theoretical calculation starting from the asymptotic dilatation generator [12], finite-size corrections to the Bethe Ansatz using Lüscher formulas [13], and finally a full-fledged Feynman calculus [14] identically determined the complete anomalous dimension.

The successful application of the Lüscher formalism relies on a generalization of the Lüscher formulas [15] to both non-relativistic models as well as multi-particle states, which had been conjectured in [16, 13]. With this formalism applied to the 2d worldsheet QFT of the $AdS_5 \times S^5$ superstring it has been possible to compute the four-loop anomalous dimension of twist-two operators at general values of the spin [17]. The result passed several non-trivial tests from BFKL and reciprocity constraints [17, 18]. The leading transcendental part had been confirmed in an impressive field theory computation [19].

For the complete spectral equations of $\mathcal{N} = 4$ SYM, however, thermodynamic Bethe Ansatz methods ought to be applied, as has been initiated for string and gauge theory in [20, 21]. A Y-system, which is believed to yield anomalous dimensions of arbitrary local operators of planar $\mathcal{N} = 4$ SYM has been recently conjectured in [22].

The aim of this work is to continue the application of the Bethe Ansatz and the Lüscher formalism to the next operators in reach, namely twist-three operators. The leading wrapping contribution to the anomalous dimension of twist-three operators will appear at five-loops. In order to compute the complete five-loop anomalous dimension of the ground state we start from an ansatz based on the maximum transcendentality principle [23] for both the asymptotic and wrapping contributions. The asymptotic part can be determined from the Bethe equations after the initial ansatz has been upgraded with further constraints from reciprocity. To compute the wrapping contribution, we apply the generalized Lüscher formulas [17] to operators of twist-three.

Our result passes some important consistency tests. Its leading asymptotic behavior for

large values of the spin reproduces the universal scaling function at five-loop order. The first subleading correction coincides with the results of [24, 25]. Contributions from finite-size effects start at order $(\log^2 M/M^2)$, as in the case of twist-two operators [18]. In contrast to the latter, there is no BFKL equation for twist-three operators, and therefore no prediction for the pole structure of our result. However, we analyze the behavior of the anomalous dimension at the singular value of spin $M = -2$. Interestingly, the pole structure agrees with the conjectured resummation formula of [11], once contributions from wrapping effects are taken into account. Additionally, the wrapping correction obtained from the Lüscher formula precicesly matches with the computation from the Y-system[1] [22]. The complete result, including the wrapping contribution, is reciprocity respecting.

The main body of the paper follows to the above outlined procedure. Some basic definitions of harmonic sums are recalled in Appendix A. Appendix B contains the analysis of the asymptotic structure of the anomalous dimensions and their corresponding P-kernels up to five loops. Speculations related to a pattern in the asymptotic structure of anomalous dimensions of twist operators are also given in Appendix B.

2 Asymptotic Bethe equations

We will start our analysis with the contribution to the final result stemming from the asymptotic Bethe Ansatz equations. Twist-three operators are embedded in the $\mathfrak{sl}(2)$ sub-sector of $\mathcal{N} = 4$ SYM. They can be represented by an insertion of M covariant derivatives $\mathcal{D}$ into the protected half-BPS state $\mathrm{Tr}\,\mathcal{Z}^3$

$$\mathrm{Tr}\,\left(\mathcal{D}^{s_1}\mathcal{Z}\,\mathcal{D}^{s_2}\,\mathcal{Z}\,\mathcal{D}^{s_3}\mathcal{Z}\right) + \dots\,, \quad \text{with} \quad M = s_1 + s_2 + s_3\,. \tag{2.1}$$

Their anomalous dimensions can be obtained from a non-compact, length-three $\mathfrak{sl}(2)$ spin chain with M excitations underlying a factorized two-body scattering [3]. However, the interaction range between scattering particles increases with orders of the coupling constant in perturbation theory. If it exceeds the length of the spin chain and *wraps* around it, the S-matrix picture [3, 4] loses its meaning, as no asymptotic region can be defined any longer. For twist L operators this effect, delayed by superconformal invariance, starts at order g^{2L+4}. Nevertheless, the Bethe Ansatz does not cease to work but gives an incomplete result, which does not incorporate these corrections [11].

It was shown that the Bethe Ansatz result for twist-two operators can be completed by considering additional scattering effects with virtual particles [17]. It passes non-trivial tests with BFKL [17], as well as reciprocity [18] constraints and reproduces the correct scaling behavior at large values of the spin M [18, 7], proposing a certain confidence. We compute these wrapping effects for twist $L = 3$ in section 5.

The Bethe Ansatz equations for the operators (2.1) with our choice of the coupling constant $g^2 = \frac{g_{\mathrm{YM}}^2 N}{16\,\pi^2}$ are given by

$$\left(\frac{x_k^+}{x_k^-}\right)^3 = \prod_{\substack{j=1 \\ j \neq k}}^{M} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-}\frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+}\,\exp(2i\theta(u_k, u_j))\,, \quad \prod_{k=1}^{M}\frac{x_k^+}{x_k^-} = 1\,. \tag{2.2}$$

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[1]We thank Pedro Vieira for his support in the numerical cross-check.

The spectral parameters $x^\pm$ are defined in terms of the rapidities u by [26]

$$x^\pm(u) = x(u \pm i/2), \quad x(u) = \frac{u}{2}\left(1 + \sqrt{1 - 4g^2/u^2}\right). \tag{2.3}$$

The phase shift $\theta(u_k, u_j)$ acquired by two particles with rapidities u_k, u_j passing each other is given to five-loop order by the expansion [7]

$$\theta(u_k, u_j) = \left(4\zeta_3 g^6 - 40\zeta_5 g^8\right)(q_2(u_k)q_3(u_j) - q_3(u_k)q_2(u_j)) + \mathcal{O}(g^{10}), \tag{2.4}$$

where the $q_r(u)$ correspond to the conserved magnon charges [26]

$$q_r(u) = \frac{i}{r-1}\left(\frac{1}{(x^+(u))^{r-1}} - \frac{1}{(x^-(u))^{r-1}}\right). \tag{2.5}$$

From the solution to (2.2) given in terms of the Bethe roots $x_k^\pm$ one computes the anomalous dimension by

$$\gamma^{\mathrm{ABA}}(g) = 2g^2 \sum_{k=1}^{M} q_2(u_k). \tag{2.6}$$

Due to the length $L = 3$ of the operators (2.1), wrapping effects are expected to contribute at order g^{10}, such that the anomalous dimension is written perturbatively as

$$\begin{aligned}
\gamma(M) &= g^2\,\gamma_2(M) + g^4\,\gamma_4(M) + g^6\,\gamma_6(M) + g^8\,\gamma_8(M) \\
&+ g^{10}\left(\gamma_{10}^{\mathrm{ABA}}(M) + \gamma_{10}^{\mathrm{wrapping}}(M)\right) + \dots,
\end{aligned} \tag{2.7}$$

where we tacitly assumed that to order g^8 the complete result is identical to γ^{ABA} and therefore dropped its index.

At one-loop the Bethe roots u_k are given by zeros of the Wilson polynomial [11]

$$P_M(u) = {}_4F_3\left(\begin{array}{c} -\frac{M}{2},\ \frac{M}{2}+1,\ \frac{1}{2}+iu,\ \frac{1}{2}-iu \\ 1,\ 1,\ 1 \end{array}\bigg| 1\right). \tag{2.8}$$

Closed expressions for the corrections to the Bethe roots to three-loop order have also been obtained in [27] from the Baxter approach [28]. However, it is currently unclear if the asymptotic Baxter equation [28] reproduces the same result as the Bethe Ansatz at and beyond wrapping order.

In order to obtain closed expressions for the anomalous dimension we will therefore solve (2.2) perturbatively for fixed values of the spin M and match the coefficients in an appropriate ansatz which assumes the maximum transcendentality principle [23]. Up to four loops, these expressions have been derived in [11] and [29]. They are given by

$$\gamma_2 = 8\,S_1 \tag{2.9}$$

$$\gamma_4 = -16\,S_1\,S_2 - 8\,S_3 \tag{2.10}$$

$$\gamma_6 = 32\,S_1\,S_2^2 + 48\,S_3\,S_2 + 16\,S_1\,S_4 + 40\,S_5 - 32\,S_{2,3} + 64\,S_1\,S_{3,1} + 32\,S_{4,1} - 64\,S_{3,1,1} \tag{2.11}$$

$$\begin{aligned}
\gamma_8 =\ & 8\,S_7 + 112\,S_{1,6} + 240\,S_{2,5} - 80\,S_{3,4} - 464\,S_{4,3} - 336\,S_{5,2} - 80\,S_{6,1} - 640\,S_{1,1,5} \\
& -512\,S_{1,2,4} + 384\,S_{1,3,3} + 512\,S_{1,4,2} - 512\,S_{2,1,4} + 320\,S_{2,2,3} + 640\,S_{2,3,2} + 64\,S_{2,4,1} \\
& +384\,S_{3,1,3} + 704\,S_{3,2,2} + 384\,S_{3,3,1} + 576\,S_{4,1,2} + 576\,S_{4,2,1} + 384\,S_{5,1,1} + 1280\,S_{1,1,1,4} \\
& -256\,S_{1,1,3,2} + 512\,S_{1,1,4,1} - 384\,S_{1,2,2,2} + 256\,S_{1,2,3,1} - 384\,S_{1,3,1,2} - 384\,S_{1,3,2,1} \\
& -384\,S_{1,4,1,1} - 384\,S_{2,1,2,2} + 256\,S_{2,1,3,1} - 384\,S_{2,2,1,2} - 384\,S_{2,2,2,1} - 384\,S_{2,3,1,1} \\
& -384\,S_{3,1,1,2} - 384\,S_{3,1,2,1} - 384\,S_{3,2,1,1} - 384\,S_{4,1,1,1} - 1024\,S_{1,1,1,3,1} - 128\,S_1\,S_3\,\zeta_3.
\end{aligned} \tag{2.12}$$

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All sums are evaluated at argument $M/2$ and only positive indices appear. At four loop-order, the dressing phase of the Bethe equations starts to contribute to (2.12) with a term proportional to ζ_3.

To determine the five-loop result in the same fashion implies a tremendous computational effort in view of the necessary precision [2]. We have obtained rational values for the anomalous dimension up to $M \approx 200$, which is however too far from the requirement to fit the coefficients of the corresponding ansatz vector of constant degree of transcendentality. In order to find a closed form for γ_{10}^{ABA} in terms of nested harmonic sums further constraints to reduce the number of coefficients are needed.

3   Parity invariance of $\gamma(M)$

The multi-loop anomalous dimension $\gamma(M)$ is conjectured to obey a powerful constraint known as generalized Gribov-Lipatov *reciprocity*. This constraint, arising in the QCD context, has been presented in [31, 32] as a special (space-time symmetric) reformulation of the parton distribution functions evolution equation, and approached in [33] from the point of view of the large M expansion. In particular, in [33] such analysis has been generalised to anomalous dimensions of operators of arbitrary twist L, and reciprocity has been dubbed *parity invariance* in the sense clarified below. Reciprocity has been checked in various multi-loop calculations of weakly coupled $\mathcal{N} = 4$ gauge theory [34, 35, 36, 37, 29, 18].

The reciprocity or parity invariance condition is easily expressed in terms of the P-function (kernel), depending on the Lorentz spin M, which is in one-to-one correspondence, at least perturbatively, with the anomalous dimension $\gamma(M)$ as follows from [31, 33, 32]

$$\gamma(M) = P\left(M + \tfrac{1}{2}\gamma(M)\right). \tag{3.1}$$

The parity invariance condition is a constraint that arises in the large M expansion of $P(M)$, which is expected to take the following form

$$P(M) = \sum_{\ell \geq 0} \frac{a_\ell(\log J^2)}{J^{2\ell}}, \qquad J^2 = \frac{M}{2}\left(\frac{M}{2} + 1\right), \tag{3.2}$$

where the a_ℓ are *coupling-dependent* polynomials. Eq. (3.2) implies an infinite set of constraints on the coefficients of the large M expansion of $P(M)$ organized in a standard $1/M$ power series. The name *parity-invariance* is related to the absence of terms of the form $1/J^{2n+1}$, odd under $J \to -J$.

In the following we will use the constraint Eq. (3.2) as a guiding principle in order to obtain the five-loop expression γ_{10}^{ABA}. To this aim, we need to express it as a more practical test such that it can be applied to any proposed combination of harmonic sums. This task can be performed with a basic result of [18], which we recall in following.

3.1   Harmonic combinations with definite parity

The notation for complementary harmonic sums $\underline{S}_{\mathbf{a}}$ is recalled in Appendix A. Let us introduce the map ω_a, $a \in \mathbb{N}$, which acts linearly on linear combinations of harmonic sums

$$\omega_a(S_{b,\mathbf{c}}) = S_{a,b,\mathbf{c}} - \frac{1}{2}\, S_{a+b,\mathbf{c}}. \tag{3.3}$$

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[2]See [30] for the five-loop anomalous dimension of a different class of operators (the field strength operators $\text{Tr}\,\mathcal{F}^L$), determined as a closed function of their length L.

We also introduce a complementary map $\underline{\omega}_a$, acting in a similar way on complementary sums

$$\underline{\omega}_a(\underline{S}_{b,\mathbf{c}}) = \underline{S}_{a,b,\mathbf{c}} - \frac{1}{2}\,\underline{S}_{a+b,\mathbf{c}}. \tag{3.4}$$

Finally, let us introduce the combinations Ω

$$\Omega_a = S_a, \tag{3.5}$$

$$\Omega_{a,\mathbf{b}} = \omega_a\,(\Omega_\mathbf{b}), \tag{3.6}$$

and the analogous complementary combinations $\underline{\Omega}_\mathbf{a}$. It is of course possible to change the basis from $S_\mathbf{a}$ to $\underline{\Omega}_\mathbf{a}$. For example, up to a degree of transcendentality three, we have

$$\underline{\Omega}_1 = S_1\,,\quad \underline{\Omega}_2 = S_2\,,\quad \underline{\Omega}_{1,1} = S_{1,1} - \frac{S_2}{2}\,,\quad \underline{\Omega}_3 = S_3\,,\quad \underline{\Omega}_{2,1} = S_{2,1} - \frac{S_3}{2}\,,$$

$$\underline{\Omega}_{1,2} = -\frac{1}{6}\pi^2 S_1 - \frac{S_3}{2} + S_{1,2}\,,\quad \underline{\Omega}_{1,1,1} = \frac{S_3}{4} - \frac{S_{1,2}}{2} - \frac{S_{2,1}}{2} + S_{1,1,1}\,, \tag{3.7}$$

which can be inverted in order to obtain

$$S_2 = \underline{\Omega}_2\,,\quad S_{1,1} = \frac{\underline{\Omega}_2}{2} + \underline{\Omega}_{1,1}\,,\quad S_3 = \underline{\Omega}_3\,,\quad S_{2,1} = \frac{\underline{\Omega}_3}{2} + \underline{\Omega}_{2,1}\,,$$

$$S_{1,2} = \frac{\pi^2\underline{\Omega}_1}{6} + \frac{\underline{\Omega}_3}{2} + \underline{\Omega}_{1,2}\,,\quad S_{1,1,1} = \frac{\pi^2\,\underline{\Omega}_1}{12} + \frac{\underline{\Omega}_3}{4} + \frac{\underline{\Omega}_{1,2}}{2} + \frac{\underline{\Omega}_{2,1}}{2} + \underline{\Omega}_{1,1,1}\,. \tag{3.8}$$

The crucial result is then given by the following theorem [18].
Theorem:
(a) The combination $\underline{\Omega}_{a_1,\dots,a_d}$ with positive $\{a_i\}$ is parity-even iff

$$(-1)^{a_1+\cdots+a_d} = (-1)^d. \tag{3.9}$$

(b) If this condition is not satisfied, the expansion of $\underline{\Omega}_{a_1,\dots,a_d}$ is parity-odd, with the (trivial) exception of the leading constant term.
(c) The combination $\Omega_{a_1,\dots,a_d}$ with positive odd $\{a_i\}$ is parity-even.

From this theorem we deduce the following
Theorem (parity-invariance test): *a specific linear combination of harmonic sums is parity invariant iff it does not contain parity-odd terms when transformed from the $S_\mathbf{a}$ basis to the $\underline{\Omega}_\mathbf{a}$ basis.*
To see how this test can be used let us consider an illustrative example, the two-loop anomalous dimension. One starts with the following ansatz of transcendentality three

$$\gamma_4 = a_1 S_3 + a_2\,S_{1,2} + a_3\,S_{2,1} + a_4\,S_{1,1,1}\,, \tag{3.10}$$

with all sums evaluated at $M/2$. The corresponding P_4-kernel, derived by inverting formula (3.1) and replacing the perturbative expansion (2.7), reads in a canonical basis

$$P_4 = \gamma_4 - \frac{1}{4}\gamma_2\gamma_2' \equiv (a_1 - 16)S_3 + (a_2 + 16)S_{1,2} + (a_3 + 16)S_{2,1} - 16\,\zeta_2\,S_1 + a_4 S_{1,1,1}\,, \tag{3.11}$$

where we used the one-loop result (2.9). Writing (3.11) in terms of the $\underline{\Omega}$ basis one finds

$$P_4 = c_1\,\underline{\Omega}_1 + c_3\,\underline{\Omega}_3 + c_{1,2}\,\underline{\Omega}_{1,2} + c_{2,1}\,\underline{\Omega}_{2,1} + c_{1,1,1}\,\underline{\Omega}_{1,1,1} + \text{const}\,, \tag{3.12}$$

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where the c_i are linear combinations of the coefficients a_i. The combinations $\underline{\Omega_1}$, $\underline{\Omega_3}$, $\underline{\Omega_{1,1,1}}$ are all reciprocity respecting, according to the above theorem. Imposing reciprocity on $\overline{P_4}$ implies the vanishing of the coefficients of those $\underline{\Omega}$ with wrong parity, namely

$$c_{1,2} = a_2 + 16 + \frac{a_4}{2} = 0, \qquad\qquad c_{2,1} = a_3 + 16 + \frac{a_4}{2} = 0. \qquad (3.13)$$

This leads to the conditions $a_3 = a_2$ and $a_4 = -2(16 + a_2)$, that are indeed satisfied by the expression in (2.10). Thus, reciprocity has determined 2 of the 4 unknown coefficients in the initial ansatz for the anomalous dimension [3].

4 Determination of γ_{10}^{ABA}

The strategy to derive the asymptotic part of the anomalous dimension at $n = 5$ loops incorporates a combined use of the maximum transcendentality principle, reciprocity and Bethe equations.

The starting point is to write γ_{10}^{ABA} as a linear combination of harmonic sums of transcendentality $\tau = 2n - 1 = 9$. For a given τ, basic combinatorics leads to the fact that there are $2^{\tau-1}$ linearly independent harmonic sums with positive indices. This means that there are in principle 256 terms which potentially contribute to the anomalous dimension.

From the numerical solution of the asymptotic Bethe equations it is possible to obtain a long list of rational values for $\gamma_{10}^{\text{ABA}}(M)$ for fixed values of M. The list-length is smaller than 256 due to rather hard computational limitations. However, these limitations can be overcome by means of parity-invariance.

To constrain the 256 unknown coefficients via reciprocity one has to impose parity in the sense of Eq. (3.2) on the five-loop contribution P_{10} to the kernel P defined in Eq. (3.1). This contribution can be derived from the anomalous dimension by simply inverting Eq. (3.1) and taking into account the perturbative expansion Eq. (2.7). Finally, we apply the previous parity-invariance test and obtain a large set of linear constraints on the unknown coefficients. The total number of constraints from Bethe equations and parity-invariance is now larger than 256 and we find an over-determined set of linear equations, which is solvable. The final result is given in Table 1, in which terms multiplied by ζ_3 and ζ_5 are directly induced from the dressing factor. As is the case for lower-loop orders, only positive indices appear in the participating harmonic sums. The result that we have obtained by the above stated methods can be checked as follows:

1. *Scaling function (cusp anomaly).* A consistency check of the formula presented in Table 1 is given by its leading asymptotic behavior, namely

$$\gamma_{10}^{\text{ABA}}(M) \sim 32\Big(\frac{887}{14175}\pi^8 + \frac{4}{3}\pi^2\,\zeta_3^2 + 40\,\zeta_3\,\zeta_5 \Big) \log M, \quad \text{for} \quad M \to \infty. \qquad (4.1)$$

It coincides with the five-loop contribution in the weak coupling perturbative expansion of the integral equation obtained in [7], which is believed to describe the universal scaling function. Put differently, we confirm its universality [38, 39] for twist-three at five loops.

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[3]The coefficient a_4 has only been kept to show the exact number of constraints coming from reciprocity. It could have been set to zero from the beginning because at large M the term $S_{1,1,1} \sim \log^3 M$ is not compatible with the universal leading logarithmic behavior (cusp anomaly).

Asymptotic five-loop anomalous dimension of twist-three operators

$\gamma_{\mathbf{10}}^{\mathbf{ABA}} = 136S_9 + 368S_{1,8} + 2832S_{2,7} + 4272S_{3,6} + 848S_{4,5} - 3024S_{5,4} - 2736S_{6,3} - 1168S_{7,2}$
$-496S_{8,1} - 5376S_{1,1,7} - 12352S_{1,2,6} - 8832S_{1,3,5} + 1600S_{1,4,4} + 3968S_{1,5,3} - 64S_{1,6,2}$
$-1344S_{1,7,1} - 12352S_{2,1,6} - 13760S_{2,2,5} - 2112S_{2,3,4} + 4288S_{2,4,3} - 960S_{2,5,2} - 5440S_{2,6,1}$
$-9088S_{3,1,5} - 2432S_{3,2,4} + 5120S_{3,3,3} + 2688S_{3,4,2} - 4160S_{3,5,1} + 1280S_{4,1,4} + 5824S_{4,2,3}$
$+6400S_{4,3,2} + 2112S_{4,4,1} + 5120S_{5,1,3} + 6208S_{5,2,2} + 5312S_{5,3,1} + 3904S_{6,1,2} + 3904S_{6,2,1}$
$+1728S_{7,1,1} + 21504S_{1,1,1,6} + 22784S_{1,1,2,5} + 5632S_{1,1,3,4} - 1280S_{1,1,4,3} + 6912S_{1,1,5,2}$
$+11520S_{1,1,6,1} + 22784S_{1,2,1,5} + 9088S_{1,2,2,4} - 1024S_{1,2,3,3} + 6784S_{1,2,4,2} + 17152S_{1,2,5,1}$
$+5504S_{1,3,1,4} - 3456S_{1,3,2,3} - 1536S_{1,3,3,2} + 7680S_{1,3,4,1} - 4480S_{1,4,1,3} - 6272S_{1,4,2,2}$
$-3584S_{1,4,3,1} - 3840S_{1,5,1,2} - 3840S_{1,5,2,1} + 768S_{1,6,1,1} + 22784S_{2,1,1,5} + 9088S_{2,1,2,4}$
$-1024S_{2,1,3,3} + 6784S_{2,1,4,2} + 17152S_{2,1,5,1} + 9088S_{2,2,1,4} - 2688\,S_{2,2,2,3} + 640\,S_{2,2,3,2}$
$+13440S_{2,2,4,1} - 3456S_{2,3,1,3} - 7040S_{2,3,2,2} - 768S_{2,3,3,1} - 4480S_{2,4,1,2} - 4480S_{2,4,2,1}$
$+2816S_{2,5,1,1} + 6272S_{3,1,1,4} - 2944S_{3,1,2,3} - 1536S_{3,1,3,2} + 7936S_{3,1,4,1} - 2944S_{3,2,1,3}$
$-7296S_{3,2,2,2} - 768S_{3,2,3,1} - 6656S_{3,3,1,2} - 6656S_{3,3,2,1} - 1024S_{3,4,1,1} - 3968S_{4,1,1,3}$
$-6528S_{4,1,2,2} - 3584S_{4,1,3,1} - 6528S_{4,2,1,2} - 6528S_{4,2,2,1} - 4864S_{4,3,1,1} - 5376S_{5,1,1,2}$
$-5376S_{5,1,2,1} - 5376S_{5,2,1,1} - 4608S_{6,1,1,1} - 32768S_{1,1,1,1,5} - 10240S_{1,1,1,2,4} - 3072S_{1,1,1,3,3}$
$-17920S_{1,1,1,4,2} - 30720S_{1,1,1,5,1} - 10240S_{1,1,2,1,4} - 8704S_{1,1,2,3,2} - 24064S_{1,1,2,4,1}$
$+1024S_{1,1,3,1,3} + 2560S_{1,1,3,2,2} - 4096S_{1,1,3,3,1} - 512S_{1,1,4,1,2} - 512S_{1,1,4,2,1} - 10240S_{1,1,5,1,1}$
$-10240S_{1,2,1,1,4} - 8704S_{1,2,1,3,2} - 24064S_{1,2,1,4,1} + 3072S_{1,2,2,2,2} - 6656S_{1,2,2,3,1}$
$+512S_{1,2,3,1,2} + 512S_{1,2,3,2,1} - 10752S_{1,2,4,1,1} + 1024S_{1,3,1,1,3} + 3072S_{1,3,1,2,2} - 3584S_{1,3,1,3,1}$
$+3072S_{1,3,2,1,2} + 3072S_{1,3,2,2,1} - 2560S_{1,3,3,1,1} + 3072S_{1,4,1,1,2} + 3072S_{1,4,1,2,1} + 3072S_{1,4,2,1,1}$
$+3072S_{1,5,1,1,1} - 10240S_{2,1,1,1,4} - 8704S_{2,1,1,3,2} - 24064S_{2,1,1,4,1} + 3072S_{2,1,2,2,2}$
$-6656S_{2,1,2,3,1} + 512S_{2,1,3,1,2} + 512S_{2,1,3,2,1} - 10752S_{2,1,4,1,1} + 3072S_{2,2,1,2,2} - 6656S_{2,2,1,3,1}$
$+3072S_{2,2,2,1,2} + 3072S_{2,2,2,2,1} - 5632S_{2,2,3,1,1} + 3072S_{2,3,1,1,2} + 3072S_{2,3,1,2,1} + 3072S_{2,3,2,1,1}$
$+3072S_{2,4,1,1,1} + 3072S_{3,1,1,2,2} - 4096S_{3,1,1,3,1} + 3072S_{3,1,2,1,2} + 3072S_{3,1,2,2,1} - 2560S_{3,1,3,1,1}$
$+3072S_{3,2,1,1,2} + 3072S_{3,2,1,2,1} + 3072S_{3,2,2,1,1} + 4608S_{3,3,1,1,1} + 3072S_{4,1,1,1,2} + 3072S_{4,1,1,2,1}$
$+3072S_{4,1,2,1,1} + 3072S_{4,2,1,1,1} + 3072S_{5,1,1,1,1} + 16384S_{1,1,1,1,3,2} + 32768S_{1,1,1,1,4,1}$
$+8192S_{1,1,1,2,3,1} + 4096S_{1,1,1,3,1,2} + 4096S_{1,1,1,3,2,1} + 20480S_{1,1,1,4,1,1} + 8192\,S_{1,1,2,1,3,1}$
$+12288S_{1,1,2,3,1,1} + 8192S_{1,2,1,1,3,1} + 12288S_{1,2,1,3,1,1} + 8192S_{2,1,1,1,3,1} + 12288S_{2,1,1,3,1,1}$
$-16384S_{1,1,1,1,3,1,1} + \zeta_{\mathbf{3}}\left(896S_6 - 2304S_{1,5} - 1792S_{2,4} - 768S_{3,3} - 1792S_{4,2} - 2304S_{5,1}\right.$
$+2560S_{1,1,4} + 512S_{1,2,3} + 1536S_{1,3,2} + 3584S_{1,4,1} + 512S_{2,1,3} + 1536S_{2,3,1} + 512S_{3,1,2}$
$\left.+512S_{3,2,1} + 2560S_{4,1,1} - 2048S_{1,1,3,1} - 2048S_{1,3,1,1}\right) + 1280\,\zeta_{\mathbf{5}}\left(S_{1,3} + S_{3,1} - S_4\right)$

Table 1: The result for the five-loop asymptotic dimension $\gamma_{10}^{\mathrm{ABA}}\left(\frac{M}{2}\right)$, written in the canonical basis.

2. *Virtual scaling function.* A further confirmation can be found from the evaluation of the first finite-order correction to the asymptotic behavior (4.1). In general this quantity is twist-dependend and thus non-universal [40]. However, it has been shown that this dependence is only linear and an all-loop integral equation can be written [24] (see also [41, 25]). The large M expansion of Table 1 leads to the $\mathcal{O}(1/M^0)$ value

$$B_3^{(5)} = \frac{2048}{945}\pi^6\zeta_3 + 64\zeta_3^3 + \frac{8}{45}\pi^4\zeta_5 - \frac{440}{3}\pi^2\zeta_7 - 7448\zeta_9 \,. \tag{4.2}$$

It coincides with the expression written explicitly in [25]. Further interesting observations on the other subleading terms in the asymptotic expansion of $\gamma_{10}^{\mathrm{ABA}}$ will be discussed in Appendix B.

3. *BFKL-like poles.* An indirect indication of the correctness of the result emerges by looking at its analytical continuation to complex values of the spin. In particular, the structure of the expansion around $M = -2$ will be presented in Section 6.

4. *Dressing self-consistency.* The dressing induced terms in $\gamma_{10}^{\mathrm{ABA}}$ are separately parity invariant. The dressing factor starts to contribute at four loops. It was observed that at this order terms of the anomalous dimensions of twist-two and -three operators coming from the dressing factor are reciprocity respecting *separately* [36, 34, 18]. The analysis of P_{10} in the case of $\gamma_{10}^{\mathrm{ABA}}$ confirms this feature. The five-loop term proportional to ζ_3 is reciprocity respecting if combined with the corresponding four-loop ζ_3-term. The ζ_5-term at five loops is reciprocity respecting separately (see formula B.16). This seems to indicate a perturbative pattern for the reciprocity of terms that are dressing-induced. Terms proportional to transcendental sums ζ_i, which newly appear at a given loop order should automatically be reciprocity respecting. Terms proportional to transcendental sums that are also present at lower-loop orders are invariant under (3.2) when combined altogether.

5. *Additional structural properties.* All coefficients of the harmonic sums are integers, likewise to the lower loop orders. Also, P_{10} turns out to be a combination of allowed parity-even combinations of type Ω, a condition being stronger than the general parity invariance.

5   The wrapping contribution

In this section we evaluate the leading wrapping correction to the asymptotic anomalous dimension of twist-three operators.

The Lüscher type formula for multi-particle states was conjectured in [13] and successfully applied to the Konishi-operator in [13] as well as twist-two operators of general spin in [17]. It consists of two parts. One describes the modification of the particle quantization condition due to the finite volume (which will not contribute at leading order), while the second comes from the propagation of virtual particles around the cylinder and is given by

$$\Delta E(L) = -\sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \mathrm{STr}_{a_1} \left[S_{a_1 a}^{a_2 a}(q, p_1) S_{a_2 a}^{a_3 a}(q, p_2) \dots S_{a_M a}^{a_1 a}(q, p_M) \right] e^{-\tilde{\epsilon}_{a_1}(q)L} \,. \tag{5.1}$$

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This formula applies to a M-particle state of identical particles of type a, whose consecutive self-scatterings preserve the state and determine their momenta p_i via the ABA equations. The matrix $S_{ba}^{ca}(q,p)$ describes how a virtual particle of type b with momentum q scatters on a real particle of type a with momentum p. The exponential factor can be interpreted as the propagator of the virtual particle.

For twist-three operators the momenta of the particles are determined by the ABA equations in terms of the rapidities u by

$$u(p) = \frac{1}{2} \cot \frac{p}{2} \sqrt{1 + 16g^2 \sin^2 \frac{p}{2}} \ .$$

At one-loop the rapidities are given by roots of the Baxter-Q function $P_M(u)$ in (2.8). As this is an even polynomial of order M, we can repeat the derivation of [17], which leads to a result similar to the twist-two case. However, we have to take into account two differences. The first one is that the length is equal to $L = 3$, which renders the exponential part to be of the form

$$e^{-\tilde{\epsilon}_Q(q)L} = \frac{4^L g^{2L}}{(q^2 + Q^2)^L} \overset{L=3}{=} \frac{64g^6}{(q^2 + Q^2)^3} \ .$$

Additionally, the one-loop energy of twist-three operators differs from the the twist-two one. It is given by

$$\sum_{k=1}^{M} \frac{16}{1 + 4u_k^2} = 8S_1\left(\tfrac{M}{2}\right) \ .$$

In the end, we can write the wrapping correction in a very elegant way as

$$\Delta\gamma = -64g^{10} S_1\left(\frac{M}{2}\right)^2 \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{T_M(q,Q)^2}{R_M(q,Q)} \frac{64}{(q^2 + Q^2)^3} \ , \tag{5.2}$$

where R_M and T_M are functions given by the same expressions that are valid in the case of twist-two operators

$$\begin{aligned}
R_M(q,Q) &= P_M\left(\tfrac{1}{2}(q - i(Q-1))\right) P_M\left(\tfrac{1}{2}(q + i(Q-1))\right) \\
&\quad \times P_M\left(\tfrac{1}{2}(q + i(Q+1))\right) P_M\left(\tfrac{1}{2}(q - i(Q+1))\right) , \\
T_M(q,Q) &= \sum_{j=0}^{Q-1} \left[\frac{1}{2j - iq - Q} - \frac{1}{2(j+1) - iq - Q} \right] P_M\left(\tfrac{1}{2}(q - i(Q-1)) + ij\right) \ .
\end{aligned}$$

In order to obtain the wrapping contribution we calculated (5.2) for all even values of M up to $M = 40$. Assuming the maximal transcendentality principle, we expect the wrapping correction to have the following structure

$$\gamma^{\text{wrapping}}(M) = S_1\left(\tfrac{M}{2}\right)^2 \left(C_0(M)\zeta_7 + C_2(M)\zeta_5 + C_4(M)\zeta_3 + C_7(M) \right), \tag{5.3}$$

where the coefficients $C_n(M)$ have a degree of transcendentality n. We used the fact that S_1^2 is factored out in the Lüscher formula (5.2). Likewise to the case of the asymptotic Bethe Ansatz we are looking for coefficients that are linear combination of harmonic sums with positive indices. For a given degree of transcendentality n, there are 2^{n-1} independent sums. Thus, in order to obtain C_0, C_2 and C_4 it is sufficient to know $\gamma^{\text{wrapping}}(M)$ to values of

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$M = 2$, $M = 4$ and $M = 16$, respectively, such that they can be determined from the results we computed. However, to unequivocally fix C_7 it is necessary to know $\gamma^{\text{wrapping}}(M)$ to values of the spin $M = 128$ which is far from our reach. Nevertheless, we can assume, as a natural refinement of the maximal transcendentality principle, that the coefficients of the harmonic sums entering C_7 are integers. With this assumptions a result is easily found [4]. The final result with all harmonic sums being of argument $M/2$ is given by

$$\gamma^{\text{wrapping}}(M) = -64\, g^{10}\, S_1^2 \big(35\zeta_7 - 40 S_2 \zeta_5 + (-8 S_4 + 16 S_{2,2})\zeta_3$$
$$+ 2 S_7 - 4 S_{2,5} - 2 S_{3,4} - 4 S_{4,3} - 2 S_{6,1} + 8 S_{2,2,3} + 4 S_{3,3,1}\big). \qquad (5.4)$$

For fixed values of M this result matches exactly numerical evaluations of the proposed Y-system [22].

Wrapping corrections, by their nature, should not modify the leading asymptotic behavior (4.1). The result (5.4) confirms this expectation, since the factor $S_1^2 \sim \log^2 M$ multiplies a linear combination of harmonic sums, which have a leading asymptotic behavior $\sim 1/M^2$. The first wrapping contribution to the asymptotic behavior therefore only enters at order $(\log^2 M / M^2)$ (see Appendix B, formula (B.6)),

$$\gamma^{\text{wrapping}}(M) \sim -\left(768\zeta_3 - \frac{16\pi^4}{15}\right) \frac{\log^2 M}{M^2}, \quad \text{for} \quad M \to \infty. \qquad (5.5)$$

Thus, for large values of the spin wrapping corrections are of the same order as in the case of twist-two operators [18]. Further similarities with the asymptotic expansion of twist-two operators are discussed in Appendix B.

In the previous section we stated that the asymptotic part given in Table 1 *is* reciprocity invariant. Hence, for the complete anomalous dimension to be reciprocity invariant, (5.4) has to satisfy this property *separately*. Writing (5.4) in terms of $\underline{\Omega}$ and Ω

$$\gamma^{\text{wrapping}}(M) = -64\, g^{10}\, \Omega_1^2 \left(35\,\zeta_7 + 4\,\Omega_{3,3,1} + 8\,\underline{\Omega_{2,2,3}} + 24\,\zeta_3\,\underline{\Omega_{2,2}} - \Omega_7\right), \qquad (5.6)$$

one checks that this is indeed true, since according to the theorem in section 3.1 the appearing structures are all parity-invariant.

We conclude this section giving the prediction that our conjecture given in Table 1 and (5.4), together with the formulas in (2.9)-(2.12), give for the five-loop anomalous dimension of the simplest twist-three operator with even spin ($M = 2$)

$$\gamma(2) = 8\,g^2 - 24\,g^4 + 136\,g^6 - (920 + 128\,\zeta_3)\,g^8 + 8(833 + 144\zeta_3 + 480\zeta_5 - 280\zeta_7)\,g^{10} + \mathcal{O}(g^{12}). \quad (5.7)$$

6 Analytic continuation

As already mentioned, no direct checks of the consistency of the multi-loop anomalous dimension (2.7) from its pole structure are possible.

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[4] As is usual in such kind of conjectures, there is a powerful numerical test that can be applied to any guesswork. Typically, one is able to compute spin dependent expressions like $C_7(M)$ up to a reasonable maximum value of M in exact (rational) form. On the other hand, numerical values can be obtained with a very high number of digits for quite larger values of M. Thus, given a conjectured expression obtained from data up to M_{max} one can always test it beyond that limit with a precision of several hundreds of digits. These kinds of tests are always passed by the expressions we derive in this paper.

However, it is worth to analyze the behavior of the anomalous dimension to four-loops (2.9)-(2.12) and the five-loop part given in Table 1 and (5.4) at the singularity nearest to the origin, $M = -2$. Thus, we need the small ω expansion of general nested harmonic sums of the form

$$S_{a_1,\ldots,a_d}(-1 + \omega), \qquad a_i \in \mathbb{N}. \tag{6.1}$$

The analytic continuation we need can be obtained by observing that from the definition of harmonic sums it follows that

$$S_{a,\mathbf{b}}(-1 + \omega) = S_{a,\mathbf{b}}(\omega) - \frac{1}{\omega^a} S_{\mathbf{b}}(\omega). \tag{6.2}$$

This simple identity allows us to proceed by trivially expanding the *r.h.s* around $\omega = 0$. This is a straightforward task, once one makes use of the general formula for the derivatives of nested harmonic sums [18], and takes into account that $S_{\mathbf{a}}(0) = 0$.

The expansion of the n-th loop anomalous dimension γ_n has the general NNLO form

$$\gamma_n = a_n \, \omega^{1-2\,n} + b_n \, \zeta_2 \, \omega^{3-2\,n} + c_n \, \zeta_3 \, \omega^{4-2\,n} + \cdots, \qquad a_n, b_n, c_n \in \mathbb{Q}. \tag{6.3}$$

Up to four loops, the explicit formulas for the two highest terms (NLO) of the analytic continuation are given in [11][5]. We recall them here for convenience, also adding the NNLO contribution

$$\gamma_2 = -\frac{8}{\omega} + 8\,\zeta_2\omega - 8\,\zeta_3\omega^2 + \ldots, \quad \gamma_4 = -\frac{8}{\omega^3} + \frac{16\,\zeta_2}{\omega} + 16\,\zeta_3 + \ldots, \tag{6.4}$$

$$\gamma_6 = -\frac{8}{\omega^5} + \frac{48\,\zeta_2}{\omega^3} + \frac{48\,\zeta_3}{\omega^2} + \ldots, \quad \gamma_8 = -\frac{8}{\omega^7} + \frac{80\,\zeta_2}{\omega^5} + \frac{80\,\zeta_3}{\omega^4} + \ldots. \tag{6.5}$$

In [11], an all-loop resummation at NLO was proposed[6] and conjectured to be valid for the asymptotic γ^{ABA} part.

At five loops, we have found for the contributions of the γ^{ABA} and $\gamma^{\mathrm{wrapping}}$ parts, respectively, the expressions

$$\gamma_{10}^{\mathrm{ABA}} = -\frac{136}{\omega^9} + \frac{496\zeta_2}{\omega^7} - \frac{784\zeta_3}{\omega^6} + \ldots, \quad \gamma_{10}^{\mathrm{wrapping}} = \frac{128}{\omega^9} - \frac{384\zeta_2}{\omega^7} - \frac{128\zeta_3}{\omega^6} + \ldots. \tag{6.6}$$

The analytical continuation of the complete five-loop anomalous dimension is thus given by

$$\gamma_{10} = -\frac{8}{\omega^9} + \frac{112\zeta_2}{\omega^7} - \frac{912\zeta_3}{\omega^6} + \cdots. \tag{6.7}$$

Interestingly enough, only the above formula for the *complete* anomalous dimension matches the proposed resummation *exactly*. The latter can therefore be rewritten as

$$\gamma_{\mathrm{NLO}} = -8\frac{g^2}{\omega} \left(\frac{1}{1-t} - \zeta_2 \frac{1+3\,t^2}{(1-t)^2} \,\omega^2 \right), \qquad t = \frac{g^2}{\omega^2},, \tag{6.8}$$

with the equality valid in a perturbative sense. It is obviously tempting to extend such a conjecture to NNLO, trying to resum the poles that appear in (6.3) with ζ_3 as a coefficient. However, the five data-points available (one for each loop) hardly allow for a genuine resummation. In fact, the NNLO term in the above expression is likely to contain enough more

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[5] See Eq. (5.12) there.
[6] See Eq. (5.14) there.

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terms to be at least in a one-to-one correspondence with the available constraints. Nevertheless, we find intriguing that the following *simple* parameterization can be given, valid at five loops,

$$\gamma_{\text{NNLO}} = -8\,\frac{g^2}{\omega}\left(\frac{1}{1-t} - \zeta_2\,\frac{1+3\,t^2}{(1-t)^2}\,\omega^2 + \zeta_3\,\frac{1-5\,t+3\,t^2+t^3+128\,t^4}{(1-t)^3}\,\omega^3\right). \tag{6.9}$$

Acknowledgments

Appendix A: Harmonic sums

In this Appendix we recall some useful formulas for harmonic sums *with positive indices*. The generalization to the case of arbitrary sign for the indices is treated in many references, for example [42] (see also Appendix A of [18]).

The basic definition of nested harmonic sums $S_{a_1,\dots,a_n}$ is recursive

$$S_a(N) = \sum_{n=1}^{N}\frac{1}{n^a}, \qquad S_{a,\mathbf{b}}(N) = \sum_{n=1}^{N}\frac{1}{n^a}\,S_{\mathbf{b}}(n), \tag{A.1}$$

Given a particular sum $S_{\mathbf{a}} = S_{a_1,\dots,a_n}$ we define

$$\text{depth}\,(S_{\mathbf{a}}) = n, \tag{A.2}$$

$$\text{transcendentality}(S_{\mathbf{a}}) = \mathbf{a} \equiv a_1 + \cdots + a_n. \tag{A.3}$$

For a product of S sums, we define transcendentality to be the sum of the transcendentalities of the factors.

Complementary harmonic sums are defined recursively by

$$\underline{S_a} = S_a, \tag{A.4}$$

$$\underline{S_{\mathbf{a}}} = S_{\mathbf{a}} - \sum_{k=1}^{\ell-1} S_{a_1,\dots,a_k}\,\underline{S_{a_{k+1},\dots,a_\ell}}(\infty), \tag{A.5}$$

This definition is valid when the rightmost index of $\mathbf{a}$ is not 1. Otherwise, the above recursive definition leads to a polynomial in the formal quantity $S_1(\infty)$. In this case our *definition* of $\underline{S_{\mathbf{a}}}$ prescribes to set $S_1(\infty) \to 0$ in the end.

Appendix B: Analysis of the asymptotic structure of γ and P

Here we analyze the first few orders of the large M expansion of the twist-three anomalous dimension up to five-loops and its corresponding kernel P [7].

The expansions of (2.9)-(2.12) to $\mathcal{O}(1/m^{-3})$ are given by

$$\gamma_2 = 8 \log \bar{m} + \frac{4}{m} - \frac{2}{3\,m^2} + \mathcal{O}\Big(\frac{1}{m^4}\Big)\,, \tag{B.1}$$

$$\gamma_4 = -\frac{8}{3}\pi^2 \log \bar{m} - 8\zeta_3 + \frac{1}{m}\Big[16 \log \bar{m} - \frac{4\pi^2}{3}\Big] - \frac{1}{m^2}\Big[8 \log \bar{m} - \frac{2\pi^2}{9} - 12\Big]$$
$$+ \frac{1}{m^3}\Big[\frac{8}{3}\log \bar{m} - \frac{28}{3}\Big] + \mathcal{O}\Big(\frac{1}{m^4}\Big)\,, \tag{B.2}$$

$$\gamma_6 = \frac{88}{45}\pi^4 \log \bar{m} - 8\zeta_5 + \frac{8}{3}\pi^2 \zeta_3 - \frac{1}{m}\Big[\frac{32}{3}\pi^2 \log \bar{m} + 16\zeta_3 - \frac{44\pi^4}{45}\Big]$$
$$- \frac{1}{m^2}\Big[16 \log^2 \bar{m} - \Big(32 + \frac{16\pi^2}{3}\Big) \log \bar{m} - 8 + \frac{20\pi^2}{3} + \frac{22\pi^4}{135} - 8\zeta_3\Big]$$
$$+ \frac{1}{m^3}\Big[16 \log^2 \bar{m} - \Big(64 + \frac{16\pi^2}{9}\Big) \log \bar{m} + 16 + \frac{44\pi^2}{9} - \frac{8\zeta_3}{3}\Big] + \mathcal{O}\Big(\frac{1}{m^4}\Big)\,, \tag{B.3}$$

$$\gamma_8 = -\Big(\frac{584\pi^6}{315} + 64\zeta_3^2\Big) \log \bar{m} - \frac{32}{15}\pi^4 \zeta_3 + \frac{8}{3}\pi^2 \zeta_5 + 440\zeta_7 + \frac{1}{m}\Big[\frac{48\pi^4}{5} \log \bar{m} - \frac{292\pi^6}{315} + \frac{32}{3}\pi^2 \zeta_3$$
$$- 32\zeta_3^2 - 16\zeta_5\Big] + \frac{1}{m^2}\Big[16\pi^2 \log^2 \bar{m} - \Big(64 + 32\pi^2 + \frac{24\pi^4}{5} - 64\zeta_3\Big) \log \bar{m} - 128 - \frac{8\pi^2}{3}$$
$$+ \frac{88\pi^4}{15} + \frac{146\pi^6}{945} - 32\zeta_3 - \frac{16}{3}\pi^2 \zeta_3 + \frac{16\zeta_3^2}{3} + 8\zeta_5\Big] + \frac{1}{m^3}\Big[\frac{64}{3} \log^3 \bar{m} - (96 + 16\pi^2) \log^2 \bar{m}$$
$$+ \Big(96 + \frac{176\pi^2}{3} + \frac{8\pi^4}{5} - 64\zeta_3\Big) \log \bar{m} + 112 - \frac{56\pi^2}{3} - \frac{64\pi^4}{15} + 80\zeta_3 + \frac{16}{9}\pi^2 \zeta_3 - \frac{8\zeta_5}{3}\Big]$$
$$+ \mathcal{O}\Big(\frac{1}{m^4}\Big)\,, \tag{B.4}$$

where $m = \frac{M}{2}$ and $\bar{m} = m \exp \gamma_{\rm E}$.

---

[7]In the case of higher twist $L > 2$, anomalous dimensions occupy a band [43]. In this paper we have considered the *minimal* anomalous dimension, see [44] for an asymptotic study of the full spectrum up to three loops.

At five loops, the large M expansion of Table 1 and (5.4) leads to

$$
\gamma_{10}^{\mathrm{ABA}} = \Big(\frac{28384\pi^8}{14175} + \frac{128}{3}\pi^2\zeta_3^2 + 1280\zeta_3\zeta_5 \Big) \log \bar{m} + \frac{2048}{945}\pi^6\zeta_3 + 64\zeta_3^3 + \frac{8}{45}\pi^4\zeta_5 - \frac{440}{3}\pi^2\zeta_7
$$

$$
- 7448\zeta_9 - \frac{1}{m}\Big[\Big(\frac{9472\pi^6}{945} + 256\zeta_3^2 \Big) \log \bar{m} - \frac{14192\pi^8}{14175} + \frac{448}{45}\pi^4\zeta_3 - \frac{64}{3}\pi^2\zeta_3^2 - \frac{32}{3}\pi^2\zeta_5
$$

$$
- 640\zeta_3\zeta_5 - 880\zeta_7 \Big] + \frac{1}{m^2}\Big[\Big(256 - \frac{272\pi^4}{15} - 128\zeta_3 \Big) \log^2 \bar{m} + \Big(1280 + \frac{128\pi^2}{3}
$$

$$
+ \frac{496\pi^4}{15} + \frac{4736\pi^6}{945} - 128\zeta_3 - \frac{160}{3}\pi^2\zeta_3 + 128\zeta_3^2 - 288\zeta_5 \Big) \log \bar{m} + 1920 + \frac{128\pi^2}{3}
$$

$$
+ \frac{64\pi^4}{45} - \frac{128\pi^6}{21} - \frac{7096\pi^8}{42525} + 64\zeta_3 + 32\pi^2\zeta_3 + \frac{224}{45}\pi^4\zeta_3 - 208\zeta_3^2 - \frac{32}{9}\pi^2\zeta_3^2 - 32\zeta_5
$$

$$
- \frac{16}{3}\pi^2\zeta_5 - \frac{320}{3}\zeta_3\zeta_5 - 440\zeta_7 \Big] - \frac{1}{m^3}\Big[\frac{256\pi^2}{9} \log^3 \bar{m} - \Big(128\pi^2
$$

$$
+ \frac{272\pi^4}{15} - 64\zeta_3 \Big) \log^2 \bar{m} + \Big(768 + \frac{320\pi^2}{3} + \frac{2752\pi^4}{45} + \frac{4736\pi^6}{2835} - 320\zeta_3 - \frac{160}{3}\pi^2\zeta_3
$$

$$
+ \frac{128\zeta_3^2}{3} - 288\zeta_5 \Big) \log \bar{m} + 1536 + 32\pi^2 - \frac{904\pi^4}{45} - \frac{1792\pi^6}{405} + 160\zeta_3
$$

$$
+ \frac{208}{3}\pi^2\zeta_3 + \frac{224}{135}\pi^4\zeta_3 - \frac{496\zeta_3^2}{3} + 96\zeta_5 - \frac{16}{9}\pi^2\zeta_5 - \frac{440\zeta_7}{3} \Big] + \mathcal{O}\Big(\frac{1}{m^4} \Big), \tag{B.5}
$$

$$
\gamma^{\mathrm{wrapping}} = -\Big(768\zeta_3 - \frac{16\pi^4}{15} \Big)\frac{\log^2 \bar{m}}{m^2} + \Big(768\zeta_3 - \frac{16\pi^4}{15} \Big)(\log^2 \bar{m} - \log \bar{m})\frac{1}{m^3} + \mathcal{O}\Big(\frac{1}{m^4} \Big) \tag{B.6}
$$

As expected in the case of minimal anomalous dimension for operators of twist $L \leq 3$, logarithmic enhancements in the asymptotic expansions of γ are always positive in power [8].

Notice that, when expressed in terms of the variable $M = 2m$, the maximal logarithmic terms $\log^p m/m^p$ in the expansions up to four loops, formulas (B.1)-(B.4), are compatible with a resummation of type

$$
\gamma(M) = f(g) \log \Big(M + \tfrac{1}{2} f(g) \log M + \dots \Big) + \dots . \tag{B.7}
$$

According to this, their coefficients are simply proportional to f^{m+1}

$$
\gamma(M) \sim f \log M + \frac{f^2}{2}\frac{\log M}{M} - \frac{f^3}{8}\frac{\log^2 M}{M^2} + \dots \tag{B.8}
$$

where f is the universal scaling function, whose weak coupling expansion to five-loop order can be found in [7]. At five loops, the pattern (B.8) is broken by the term $\log^2 M/M^2$ in the expansion (B.5) above [9]. Interestingly, it is precisely at this order in the large M expansion that wrapping corrections start to contribute. Explicitly, while on the basis of (B.8) one

---

[8]Terms with negative powers of the logarithm appear for twist $L \leq 3$, but for *non-minimal* anomalous dimensions [45]. For twist $L > 3$ terms $\sim \frac{1}{\log^p M}$ are present both for large L and M [46] and for finite twist, see a similar discussion in [47] and reference therein. For a general method to derive higher order terms in the $1/M$ expansion at fixed L see [48].

[9]Further maximal logarithmic terms as $\log^3 M/M^3, \log^4 M/M^4$ continue to obey the rule as dictated by further orders in (B.8). Coefficients of terms $\log^p M/M^p$ with $p > 4$ are absent in the expansion, as checked up to $1/M^{25}$. This is again consistent with (B.8), being such coefficients of the form $\frac{f^6}{160}, -\frac{f^7}{384}, \dots$, they would contribute starting at 6 loops.

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would expect at five loops a term of type

$$(c_{22})_5 \frac{\log^2 M}{M^2} \qquad \text{with} \qquad (c_{22})_5^{\text{naive}} = \left(-\frac{f^3}{8}\right)_5 = -\frac{1024}{15}\pi^4 = -6144\,\zeta_4\ , \qquad (B.9)$$

reexpressing (B.5) and (B.6) in terms of M one finds

$$(c_{22})_5^{\text{ABA}} = 1024 - 512\zeta_3 - 6528\zeta_4 \qquad \text{and} \qquad (c_{22})_5^{\text{wrapping}} = -3072\,\zeta_3 + 384\,\zeta_4\ . \quad (B.10)$$

The sum of these terms does clearly not reproduce (B.9). This is analogous to the case of twist two operators [18] [10].

It is interesting to notice that the structure, lost at higher orders in $1/m$, of the first terms in the expansion for γ^{wrapping} is also present in its twist-two analogue. The appearance of an overall coefficient multiplying the $1/m^2$ and $1/m^3$ terms was already noticed in formula (C.5) of [18]. Another analogy between the leading asymptotic behavior of twist-2 and twist-3 wrapping contributions is their negative sign and their common pattern $\sim c_n\,\zeta_n + c_{n+1}\,\zeta_{n+1}$, where n coincides with the twist.

Concerning the P-kernel, the logarithmic structure to four loops is remarkably *simpler* than the one of the corresponding anomalous dimension, as it is only *linear* in $\log M$ [36]. In particular, there are no maximally enhanced terms of the form $(\log M/M)^k$. As discussed in [33, 32], this feature of P translates into the chance of a resummation of type (B.7). This asymptotic structure changes at five loops.

The P function, derived by inverting formula (3.1), reads in terms of $m = \frac{M}{2}$ ($\partial \equiv \partial_m$) to five-loops

$$\mathcal{P}(m) = \sum_{k=1}^{\infty} \frac{1}{k!}\left(-\tfrac{1}{4}\partial\right)^{k-1}[\gamma(m)]^k = \gamma - \frac{1}{8}\,(\gamma^2)' + \frac{1}{96}\,(\gamma^3)'' - \frac{1}{1536}\,(\gamma^4)''' + \frac{1}{30720}\,(\gamma^5)'''' + \cdots .$$

Replacing γ by the perturbative expansion (2.7) we can formally write at five loops

$$P_{10} = \gamma_{10} - \frac{1}{4}(\gamma_4\gamma_6 + \gamma_2\gamma_8)' + \frac{1}{32}(\gamma_2\gamma_4^2 + \gamma_2^2\gamma_6)'' - \frac{1}{384}(\gamma_2^3\gamma_4)''' + \frac{1}{30720}(\gamma_2^5)'''' . \quad (B.11)$$

Expanded at large M, including the wrapping contribution, this becomes

$$P_{10} = \left(\frac{28384\pi^8}{14175} + \frac{128}{3}\pi^2\zeta_3^2 + 1280\zeta_3\zeta_5\right)\log\bar{m} + \frac{2048}{945}\pi^6\zeta_3 + 64\zeta_3^3 + \frac{8}{45}\pi^4\zeta_5 - \frac{440}{3}\pi^2\zeta_7$$

$$- 7448\zeta_9 + \frac{1}{m}\left[\frac{14192\pi^8}{14175} + \frac{64}{3}\pi^2\zeta_3^2 + 640\zeta_3\zeta_5\right] + \frac{1}{m^2}\left[\left(256 - 896\zeta_3\right)\log^2\bar{m} + \right.$$

$$+ \left(1280 + \frac{128\pi^2}{3} - \frac{16\pi^4}{15} - 128\zeta_3 - \frac{64}{3}\pi^2\zeta_3 - 320\zeta_5\right)\log\bar{m} + 1920 + \frac{128\pi^2}{3} + \frac{64\pi^4}{45} +$$

$$- \frac{1024\pi^6}{945} - \frac{7096\pi^8}{42525} + 64\zeta_3 - 64\zeta_3^2 - \frac{32}{9}\pi^2\zeta_3^2 - \frac{320}{3}\zeta_3\zeta_5\right]$$

$$- \frac{1}{m^3}\left[(256 - 896\zeta_3)\log^2\bar{m} + \left(1024 + \frac{128\pi^2}{3} - \frac{16\pi^4}{15} + 768\zeta_3 - \frac{64}{3}\pi^2\zeta_3 - 320\zeta_5\right)\log\bar{m}\right.$$

$$+ 1280 + \frac{64\pi^2}{3} + \frac{88\pi^4}{45} - \frac{1024\pi^6}{945} + 128\zeta_3 + \frac{32}{3}\pi^2\zeta_3 - 64\zeta_3^2 + 160\zeta_5\right] + \mathcal{O}\left(\frac{1}{m^4}\right). \quad (B.12)$$

---

[10] One difference is however that the degree of transcendentality of the asymptotic and wrapping contributions, that differs in (B.10), is the same in the twist-two case.

The "simplicity" feature is lost, because at order $1/m^2$ a term $\log^2 m/m^2$ appears, which is responsible for the above formula (B.10).

We recall that the consequences (B.7) and (B.8) of the simplicity of the P function *and* the knowledge of f to presumably all loops [7] allow in principle an all-loop prediction for such maximal logarithmic terms, whose coefficients should be simply proportional to f^{m+1}. Indeed, such *inheritance* has been checked at strong coupling in [47] up to one-loop in the semiclassical sigma model expansion, as well as in [49] at the classical level. An independent strong coupling confirmation of (B.7) for twist-two operators has recently been given in [24]. To clarify if and how the difference in the simplicity of the P at weak and strong coupling works, further orders in the semiclassical sigma model expansion would be needed.

We conclude the appendix by reporting the separate contributions to P_{10} coming from the dressing factor, which obey the property described in point 4 of Section 4. They read

$$
\begin{aligned}
P_{10}^{(\zeta_3)} \equiv \gamma_{10}^{(\zeta_3)} - \frac{1}{4}(\gamma_2\,\gamma_8^{(\zeta_3)})' = {} & 896 S_6 - 2304 S_{1,5} - 1792 S_{2,4} - 768 S_{3,3} - 1792 S_{4,2} - 2304 S_{5,1} \\
& + 2560 S_{1,1,4} + 512 S_{1,2,3} + 1536 S_{1,3,2} + 3584 S_{1,4,1} + 512 S_{2,1,3} + 1536 S_{2,3,1} + 512 S_{3,1,2} \\
& + 512 S_{3,2,1} + 2560 S_{4,1,1} - 2048 S_{1,1,3,1} - 2048 S_{1,3,1,1} + 512 \zeta_2 S_1 S_3 - 512 S_1 S_2 S_3 \\
& + 768 \zeta_4 S_1^2 - 768 S_1^2 S_4 \,, \hspace{4cm} \text{(B.13)} \\
P_{10}^{(\zeta_5)} \equiv \gamma_{10}^{(\zeta_5)} = {} & 1280\, S_1\, S_3 \,. \hspace{6.5cm} \text{(B.14)}
\end{aligned}
$$

In the first line we included the dressing-induced contribution at four loops, which is proportional to ζ_3. These contributions can be expressed in terms of parity invariant combinations (see Theorem (c) in Section 3.1) as

$$
P_{10}^{(\zeta_3)} = 256(8\,\Omega_1\,\Omega_{1,1,3} - 4\Omega_1^2\,\Omega_{1,3} + \Omega_{3,3} + S_1\,\Omega_5 + 2\zeta_2\Omega_1\,\Omega_3 + 3\zeta_4\,\Omega_1^2)\,, \qquad \text{(B.15)}
$$

$$
P_{10}^{(\zeta_5)} = 1280\,\Omega_1\,\Omega_3. \hspace{8.5cm} \text{(B.16)}
$$

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