Emergent non-commutative matter fields from Group Field Theory models of quantum spacetime

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Abstract. We offer a perspective on some recent results obtained in the context of the group field theory approach to quantum gravity, on top of reviewing them briefly. These concern a natural mechanism for the emergence of non-commutative field theories for matter directly from the GFT action, in both 3 and 4 dimensions and in both Riemannian and Lorentzian signatures. As such they represent an important step, we argue, in bridging the gap between a quantum, discrete picture of a pre-geometric spacetime and the effective continuum geometric physics of gravity and matter, using ideas and tools from field theory and condensed matter analog gravity models, applied directly at the GFT level.

1. Introduction

In this contribution we offer a perspective on some recent results [1, 2] obtained in the context of the group field theory (GFT) approach to quantum gravity [3, 4], on top of reviewing them briefly. These concern a natural mechanism for the emergence of non-commutative field theories for matter directly from the GFT action, in both 3 and 4 dimensions and in both Riemannian and Lorentzian signatures. The interest of such results is manifold. First, they show a straightforward link between spin foam/loop quantum gravity models, via GFTs, and non-commutative geometry. Second, they are, in principle, a crucial step forward in the attempt to relate this class of models with effective frameworks, like Deformed Special Relativity, that form the basis of much current Quantum Gravity Phenomenology [5]. On top of all this, in this contribution we argue that they can be naturally understood within a certain interpretative framework for GFTs and for the relation between its quantum microscopic spacetime structures and continuum spacetime physics. The perspective we present is not, of course, the only possible consistent one, nor the one shared by all the researchers working in this specific area. It is, however, a scenario that is consistent with what we presently know about GFTs, and with a recent proposal about the emergence of continuum physics from them, as put forward by the author. Moreover, it is a scenario that makes these results all the more exciting, which is, we believe, an added benefit. It is based on three main points: 1) GFTs are the most complete and fundamental definition of quantum gravity models based on spin network or simplicial gravity structures; 2) continuum spacetime physics is to be looked for in the collective "many-particle"physics of GFT quanta; 3) therefore, instead of usual LQG, spin foam or simplicial gravity techniques, one could try to adopt a condensed matter perspective on quantum spacetime, and see what condensed matter ideas and tools give when applied to the quantum spacetime system directly at the GFT level; 4) in particular, this shift in perspective and tools can be very useful for bridging the gap between a microscopic, quantum, discrete picture of a pre-geometric spacetime and the effective continuum geometric physics of gravity and matter we are accustomed and have phenomenological access to. We have exposed the above points and drawn some possible consequences of this perspective in [6]. Here we point out how the results on the emergence of non-commutative field theories from GFTs can fit in it.

1.1. Group Field Theory formulation of Quantum Gravity

Group Field Theories [3, 4] are quantum field theories on (possibly, extensions of) group manifolds characterized by a peculiar non-local coupling of fields, designed to produce, in their perturbative expansion around the vacuum, Feynman diagrams that can be put in 1 to 1 correspondence with d-dimensional simplicial complexes. The fundamental field is interpreted as a second quantized (d-1)-simplex or as a (second quantized) elementary spin network vertex. The Feynman amplitudes can be understood as simplicial gravity path integrals (although not necessarily written in terms of some gravity action) or, dually, as spin foam models, i.e. sum over histories of spin networks states. GFTs are thus at the same time a generalization of matrix models for 2d quantum gravity to higher dimensions, and a new version of the simplicial quantum gravity approaches, and a complete definition of the covariant dynamics of spin networks in loop quantum gravity. Let us be a bit more specific.

In their simplest type of models, the field is a $\mathbb{C}$-valued function of d group elements $\phi(g_1, .., g_d)$, for a group G (configuration space) being for example the $SO(d-1,1)$ Lorentz group. Each argument corresponds to one of the boundary (d-2)-faces of the (d-1)-simplex represented by the field. Additional symmetry requirements can be imposed. Typically, one imposes invariance under diagonal action of G: $\phi(g_1, ..., g_d) = \phi(g_1 g, ..., g_d g)$. Fields and action can then be expanded in modes, i.e. in group representations. Both group and representation variables can be given a straightforward interpretation (justified by their role in the corresponding amplitudes) as (pre-) geometric data, i.e. data defining a discrete geometry for the simplicial complex associated with each Feynman diagram. The details of course depend on the specific model. A GFT model is defined by a choice of action:

$$S = \frac{1}{2}\int dg_i d\tilde{g}_i\, \phi(g_i)\mathcal{K}(g_i\tilde{g}_i^{-1})\phi(\tilde{g}_i) + \frac{\lambda}{(d+1)!}\int dg_{ij}\, \phi(g_{1j})...\phi(g_{d+1j})\, \mathcal{V}(g_{ij}g_{ji}^{-1}),$$

i.e. of a kinetic and interaction functions $\mathcal{K}$ and $\mathcal{V}$. The interaction term describes the interaction of (d-1)-simplices to form a d-simplex by gluing them along their (d-2)-faces (arguments of the fields); this gives the mentioned non-local combinatorial pairing of fields. The nature of the interaction is specified by the choice of function $\mathcal{V}$. The kinetic term involves two fields each representing a given (d-1)-simplex seen from one of the two d-simplices (interaction vertices) sharing it, so that the choice of kinetic functions $\mathcal{K}$ specifies how the geometric degrees of freedom corresponding to their d (d-2)-faces are propagated from one vertex to the next. The quantum theory is defined in terms of the expansion in Feynman diagrams of the partition function:

$$Z = \int \mathcal{D}\phi\, e^{-S[\phi]} = \sum_{\Gamma} \frac{\lambda^N}{sym[\Gamma]}\, Z(\Gamma),$$

where N is the number of interaction vertices in the Feynman graph Γ, $sym[\Gamma]$ is a symmetry factor for the graph and $Z(\Gamma)$ the corresponding Feynman amplitude. By construction ([3, 4] the Feynman diagram Γ is a collection of 2-cells (faces), edges and vertices, i.e. a 2-complex, topologically dual to a d-dimensional simplicial complex. The resulting complexes/triangulations can have arbitrary topology, each corresponding to a particular *interaction process* of the fundamental building blocks of space, i.e. (D-1)-simplices. In representation space each Feynman diagram is a spin foam (a 2-complex with faces f labelled by group representations), and each Feynman amplitude defines a spin foam model. Because of the geometric interpretation for the (group-theoretic) variables these Feynman amplitudes define a sum-over-histories for discrete quantum gravity, i.e. a sum over simplicial geometries, within a sum over simplicial topologies.

Several GFT models have been and are currently studied, and much more is known about GFTs in general. For all this, we refer to the literature [3, 4, 7, 8].

1.2. From GFT to continuum spacetime physics?

GFTs are thus a (tentative) class of models for quantum spacetime, and thus *not defined on any spacetime*, in which its microscopic structure is described by elementary building blocks given by spin network vertices or simplices, labelled by pre-geometric data. This description is valid and useful in the regime in which the perturbative expansion of the GFT partition function is valid, i.e. small coupling constant and few GFT quanta involved in the physical process one is considering. In this approximation a discrete spacetime *emerges* as a Feynman diagram from this "pre-spacetime" theory.

This is one of the intriguing aspects of GFTs. It is also the origin of the main open issue: how to recover a continuum spacetime, its continuum geometry and all continuum physics, including usual General Relativity and quantum field theories for matter, from such a different description of the pre-geometric regime? This is the outstanding problem faced by *all* current discrete approaches to quantum gravity[9].

The problem of the continuum involves actually several intertwined questions. Some of them are conceptual: where does a concept of geometry originate from in a non-geometric or pre-geometric framework? where to look for notions and properties of space and time in models defined in absence of any spacetime structure? Some are purely kinematical: should we expect the pre-geometric GFT data to give rise to the continuum geometry? or could an emergent macroscopic continuum geometry be totally unrelated to these microscopic and pre-geometric data? what is the correct approximation scheme to obtain a continuum spacetime from the discrete GFT structures? Some are instead dynamical: what is the physical regime, the physical conditions, in which the continuum approximation is viable, justified and/or useful? what is the physical dynamical process leading to this regime from the pre-geometric one, and how to describe it?

We have discussed in [6] our personal perspective on this problem, and the possibility that GFTs could be the right setting for tackling it, and how. We summarize it here, briefly. From the GFT point of view, the crucial issue is whether we expect the continuum approximation (as opposed to large scale, semi-classical or other a priori distinct approximations) to involve very large numbers of GFT quanta or not[1]. We opt for a positive answer, as naive reasoning would suggest (one would expect a generic continuum spacetime to be formed by zillions of Planck size building blocks, rather than few macroscopic ones). If this is the case, then we are dealing, from the GFT point of view, with a many-particle system whose microscopic theory is given by some fundamental GFT action and we are interested in its collective dynamics and states in some thermodynamic approximation. This simple thought alone suggests us to look for ideas and techniques from statistical field theory and condensed matter theory, and to try to apply/reformulate/re-interpret them in a GFT context.This also immediately suggests that the GFT formalism is the most natural setting for studying this dynamics even coming from the pure loop quantum gravity perspective or from simplicial quantum gravity. In the first case, in fact, GFTs offer a second quantized formalism for the same quantum geometric structures,

[1] In the full theory; the same question in a symmetry reduced context, for example, may have a different answer.

and quantum field theory is indeed what comes natural in condensed matter when dealing with large collections of particles/atoms. In the second case, GFTs offer an alternative non-perturbative definition of the quantum dynamics of simplicial gravity, re-interprets the usual sum over discrete geometries as a perturbative expansion around the no-spacetime vacuum, and in doing so suggest the possibility of different vacuum states and a different reformulation of the same dynamics that is better suited for studying the dynamics of (combinatorially) complicated simplicial geometries. See [6] for more details.

2. Condensed matter perspective and GFT implementation

So, what can we learn from condensed matter models in our quest for bridging the gap between microscopic quantum and discrete pre-geometry in absence of spacetime, and macroscopic continuum spacetime and associated physics, with matter fields, geometry, gauge interactions and all that?

Condensed matter theory comes in help in two main ways, one very general, one more specific.

The first: condensed matter theory is exactly the area of physics concerned with the general issue of understanding the physical behaviour of large assemblies of microscopic constituents, in their collective behaviour, of elucidating their emergent properties, and of developing the appropriate notions and mathematical tools for describing the different layers of collective organization that coexists in a given phase, as well as the transition from one phase to another. Therefore, once we realize that a classical continuum spacetime is the result of the collective properties and dynamics of large numbers of GFT quanta, it is natural to look at condensed matter theory for insights.

One could be a bit more specific (and speculative) and conjecture that continuum spacetime arises as a *fluid phase* of this large assembly of GFT quanta, maybe after a process analogous (mathematically) to condensation in GFT momentum space (Bose condensation, for example). This conjecture is also argued for in [6]. Continuum geometrodynamics, be it classical (General Relativity) or quantum could then arise from the effective GFT hydrodynamics in this fluid (or condensed) phase. This is a second conjecture put forward in [6]. Again, in order to put these ideas to test, one has to turn to condensed matter theory and to statistical field theory, i.e. we should develop statistical group field theory, and study that special condensed matter system that is a quantum spacetime. A corresponding research programme has been lied down tentatively in [6], and it involves developing first for GFTs and then applying several techniques, common in statistical field theory and condensed matter. One of them is the renormalization group, and in fact one way to study the phase structure of GFTs as well as the behaviour of GFT systems at different scales would certainly be in terms of perturbative and non-perturbative renormalization group transformations. No systematic study of GFT renormalization has been performed until now, but work is in progress from various groups. In particular, a systematic treatment of perturbative GFT renormalization for the 3d Boulatov model is under way, and its first results can be found in [10]. Another avenue would be to develop an Hamiltonian statistical mechanics for GFTs, and in turn this involves several steps: a canonical/hamiltonian GFT formalism, which made highly non-trivial by the non-local nature of these models, a precise understanding of the GFT analogue of the notion of energy, temperature, pressure, etc, and a re-interpretation of any such notion in quantum gravity terms. With these tools at hand one should analyze the phase structure of various GFT models and prove the existence of an appropriate fluid phase in some of them. Then, the hydrodynamic approximation in this phase should be developed and the final goal would be to show the emergence of General Relativity from this GFT hydrodynamic description. It is too early to report on progress on this front.

There is a second way, however, in which condensed matter theory comes into the game. It provides specific examples of systems in which the collective behaviour of the microscopic constituents in some hydrodynamic approximation gives rise to effective emergent geometries as well as matter fields. Thus it gives further support, by means of explicit examples, to the idea that continuum geometry and gravity may emerge naturally from fundamental systems which do not have a geometric or gravitational nature per se, at least in their fundamental formulation (GFTs are of this type). These are the so-called analog gravity models [11].

The emergence of gravity and (generically) curved geometries in analog condensed matter models takes place in the hydrodynamic regime, i.e. usually at the level of a (modified) Gross-Pitaevskii equation or of the corresponding Lagrangian/Hamiltonian, which in turn is usually obtained in the mean field theory approximation (or refined version of the same) around some background configuration of the fluid under study. The background configurations that have proven more interesting are those identifying Bose-Einstein condensates or fermionic superfluids [11]. However, the emergence of effective metric fields is more general and not confined to these rather peculiar systems. What happens is that the collective parameters describing the fluid and its dynamics in these background configurations (e.g. the density and velocity of the fluid in the laboratory frame) can be recast as the component functions of an *effective metric field*.

This would be only cosmetics if not for two further results. 1) In some very special cases and in some particular approximation the hydrodynamic equations governing the dynamic of the effective metrics, when recast in geometric terms, can also be seen to reproduce known geometrodynamic theories, at least in part, ranging from Newtonian gravity to (almost) GR; see [12]. This means that it is possible to reproduce also gravitational dynamics as emergent from systems that are not geometric in nature or form. 2) The effective dynamics of perturbations (quasi-particles, themselves collective excitations of the fundamental constituents of the fluid) around the same background configurations turns out to be given by matter field theories in curved spacetimes, whose geometry is indeed the one identified by the effective metrics obtained from the collective background parameters of the fluid.

The first type of results is so far limited to special systems, peculiar approximations, and ultimately not fully satisfactory, in the sense that it has not been possible yet to reproduce, say, the Einstein-Hilbert dynamics in any, however idealized, condensed matter system [12]. This seems to suggest that the ideas and techniques developed in this context and to this aim are certainly useful and interesting, but need to be applied to some very peculiar system representing (quantum) spacetime, if one wants to explain in this way the emergence of geometry and General Relativity in the real world. Therefore, these results certainly encourage and support the idea of applying similar ideas and techniques in Group Field Theory, but they do not provide a sure guidance on how exactly this should be done.

The second type of results, however, is much more general and applies to a very large class of systems and approximations, including systems as common as ordinary fluids (e.g. water) in everyday physical conditions [11]. It is this type of results that we focus on in this contribution, because we are able to obtain similar results in a GFT context.

The general scheme of what goes on in all these condensed matter systems (including BEC, superfluids, etc) concerning the emergence of matter field theories on effective metric spacetimes is well captured by the following "meta-model"described in [11]. Consider a system described by a single scalar field $\phi(x)$, living on a flat metric spacetime of trivial topology (a good approximation for the quantum geometry of any lab in any research institute on Earth). The field ϕ can be the effective order parameter for a BEC, the collective field encoding the velocity and density of an ordinary fluid in the hydrodynamic approximation, or whatever else. Assume that its dynamics is encoded in a lagrangian $L(\phi, \partial_\mu \phi)$ depending on the field and its partial derivatives. Let us expand the field around some classical solution ϕ_0 of the equations of motion, as: $\phi(x) = \phi_0(x) + \phi_1(x)$. Next we expand the lagrangian itself to obtain an effective action for the perturbation field $\phi_1(x)$ (we focus our attention to the kinetic term only); we obtain, generically, an effective Klein-Gordon operator on a curved metric:

$$
\begin{aligned}
S(\phi) &= S(\phi_0) + \frac{1}{2}\int \left[\frac{\partial^2 L}{\partial(\partial_\mu\phi)\partial(\partial_\nu\phi)}\mid_{\phi_0}\,\partial_\mu\phi_1\partial_\nu\phi_1 + \left(\frac{\partial^2 L}{\partial\phi\partial\phi} - \partial_\mu\frac{\partial^2 L}{\partial(\partial_\mu\phi)\partial\phi}\right)\mid_{\phi_0}\right] + (.....) = \\
&= S(\phi_0) + \frac{1}{2}\int \sqrt{-g}\,[\phi_1\Box_{\phi_0}\phi_1 - V(\phi_0)\,\phi_1^2] + (.....\textit{interactions}),
\end{aligned} \tag{1}
$$

with the operator $\Box_{\phi_0} = g^{\mu\nu}\partial_\mu\partial_\nu$, for the effective (inverse) metric $\sqrt{-g}\,g^{\mu\nu} = \frac{\partial^2 L}{\partial(\partial_\mu\phi)\partial(\partial_\nu\phi)}\mid_{\phi_0}$. It is the possible to invert to obtain the effective metric and from this the other tensors characterizing the effective spacetime geometry. Notice that both the "fundamental"field and the quasi-particle one live on a 4-dimensional spacetime of trivial topology, although endowed with a different metric in general. We refer to the literature for further details and applications of the above general result. The main point to notice here is that one generically obtains an effective spacetime geometry to which the quasi-particles couple, depending only in its precise functional form on the fundamental Lagrangian L and on the classical solution ϕ_0 chosen; they do not couple to the initial (laboratory) flat background metric.

This is the type of mechanism we want to reproduce in a Group Field Theory context. Assuming that a given GFT model (Lagrangian) describes the microscopic dynamics of a *discrete quantum spacetime*, and that some solution of the corresponding fundamental equations can be interpreted as identifying a given quantum spacetime configuration, 1) can we obtain an effective macroscopic *continuum* field theory for matter fields from it? and if so, 2) what is the effective spacetime and geometry that these emergent matter fields see?

Answering these questions means, as we have tried to emphasize, making an important step towards bridging the gap between our fundamental discrete models of spacetime, and the usual continuum description of spacetime. It also means getting closer to possible quantum gravity phenomenology, and to experimental falsifiability. Let us also notice that, while the correspondence between classical solutions of the fundamental equations and effective geometries is a priori unexpected in condensed matter systems, which are non-geometric in nature, so that they are referred to as *analog* gravity models, in the GFT case the situation is different. We have here models which are non-geometric and far from usual geometrodynamics in their formalism, but which at the same time are expected to encode quantum geometric information and indeed to determine, in particular in their classical solutions, a (quantum and therefore classical) geometry for spacetime [3], also at the continuum level, the issue being how they exactly do so. We are, in other words, far beyond a pure analogy. What we find is that it is possible to apply the same procedure to GFT models and that one can obtain rather straightforwardly effective continuum field theories for matter fields. The effective matter field theories that we obtain most easily from GFTs are quantum field theories on non-commutative spaces of Lie algebra type.

3. Results on emergent effective matter

Let us now introduce and review briefly the results obtained so far.

3.1. Non-commutative field theories as group field theories

First of all we introduce the relevant class of non-commutative field theories. The basic point is the duality between Lie algebra and corresponding Lie group re-interpreted as the non-commutative version of the usual duality between coordinate and momentum space. More precisely, if we have a non-commutative spacetime of Lie algebra type $[X_\mu, X_\nu] = C_{\mu\nu}^{\lambda}X_\lambda$, the corresponding momentum space is naturally identified with the corresponding Lie group, on which non-commutative coordinates X_μ act as (Lie) derivatives. From this perspective, we understand the origin of the spacetime non-commutativity to be the curvature of the corresponding momentum space, a sort of Planck scale "co-gravity"[18]. The link with GFTs is then obvious: in momentum space the field theory on such non-commutative spacetime will be given, by definition, by some sort of group field theory. The task will then be to derive the relevant field theories for matter from interesting GFT models of quantum spacetime.

In 3 spacetime dimensions the results obtained concern a euclidean non-commutative spacetime given by the $\mathfrak{su}(2)$ Lie algebra , i.e. whose spacetime coordinates are identified with the $\mathfrak{su}(2)$ generators with $[X_i, X_j] = i\frac{1}{\kappa}\epsilon_{ijk}X_k$. Momenta are instead identified with group elements SU2) [13], acquiring a non-commutative addition property following the group composition law. A scalar field theory in momentum space is then given by a group field theory of the type:

$$S[\psi] \;=\; \frac{1}{2}\int_{\mathrm{SU}(2)} dg\psi(g)\mathcal{K}(g)\psi(g^{-1}) - \frac{\lambda}{3!}\int [dg]^3\,\psi(g_1)\psi(g_2)\psi(g_3)\delta(g_1g_2g_3), \tag{2}$$

in the 3-valent case, where the integration measure is the Haar measure on the group. The non-commutative Fourier transform [13] relates then functions on the group and non-commutative functions in the enveloping algebra of $\mathfrak{su}(2)$. It is based on the non-commutative plane waves $e_g = e^{ik_iX_i} \in \widehat{C}_\kappa(\mathfrak{su}(2))$, where k_i are local coordinates on the group manifold labeling the group element (and thus the momentum) g and bounded as $\mid k \mid\leq \kappa$ (so that it is natural to identify κ with a Planckian maximal mass/momentum scale). It reads:

$$\widehat{\phi}(X) = \int_{\mathrm{SU}(2)} dg\, e_{g(k_i)}\,\phi(g(k)), \quad X \in \mathfrak{su}(2), \quad \widehat{\phi}(X) \in \mathcal{U}(\mathfrak{su}(2)). \tag{3}$$

Strictly speaking, this construction works on SO(3), but can be extended to SU(2) in several ways [14]. Using this, the action above can be rewritten in configuration space. We will see below some simple example of such action, derived straightforwardly from a group field theory model for 3d quantum gravity. We notice that one can define a further map from elements of the enveloping algebra of $\mathfrak{su}(2)$ to functions on $\mathbb{R}^3$ (isomorphic to the same Lie algebra as a vector space) endowed with a non-commutative star product again reflecting the non-commutative composition of momenta following the rules of group multiplication. See [13, 14, 15] for details. We also notice that the Feynman amplitudes of the above scalar field action (with simple kinetic terms) can be derived from the Ponzano-Regge spin foam model coupled to point particles [16], in turn obtainable from an extended GFT formalism [17]. We will see that the GFT construction to be presented allows to bypass completely the spin foam formulation of the coupled theory.

The 4-dimensional non-commutative spacetime that is of most direct relevance for Quantum Gravity phenomenology is so-called κ-Minkowski [18]. We recall here the main features of such space and of the non-commutative field theory defined on it, referring for further details to [2] and references therein. κ-Minkowski space-time can be identified with the Lie algebra $\mathfrak{an}_3$, which is a subalgebra of $\mathfrak{so}(4,1)$. Indeed, if $J_{\mu\nu}$ are the generators of $\mathfrak{so}(4,1)$, the $\mathfrak{an}_3$ generators are:

$$X_0 = \frac{1}{\kappa}J_{40}, \quad X_k = \frac{1}{\kappa}(J_{4k} + J_{0k}), \quad k = 1, ..., 3\ , \tag{4}$$

which, once identified with the coordinates of our non-commutative space, characterize it with the commutation relations:

$$[X_0, X_k] \;= -\frac{i}{\kappa}X_k, \qquad [X_k, X_l] \;= 0, \quad k, l = 1, ..., 3. \tag{5}$$

Using this, we can then define non-commutative plane waves with the AN_3 group elements as $h(k_\mu) = h(k_0, k_i) \;\equiv\; e^{ik_0X_0}e^{ik_iX_i}$, thus identifying the coordinates on the group k_μ as the wave-vector (in turn related to the momentum). From here, a non-commutative addition of wave-vectors follows from the group multiplication of the corresponding plane waves.

Crucial for our construction, the Iwasawa decomposition relates $\mathrm{SO}(4,1)$ and AN_3 as [19]:

$$\mathrm{SO}(4,1) \;=\; \mathrm{AN}_3\,\mathrm{SO}(3,1)\,\cup\,\mathrm{AN}_3\mathcal{M}\,\mathrm{SO}(3,1), \tag{6}$$

where the two sets are disjoint and $\mathcal{M}$ is the diagonal matrix with entries $(-1,1,1,1,-1)$ in the fundamental 5d representation of $\mathrm{SO}(4,1)$. Since De Sitter space-time dS_4 can be defined as the coset $\mathrm{SO}(4,1)/\mathrm{SO}(3,1)$, an arbitrary point v on it can be uniquely obtained as:

$$v \;=\; (-)^\epsilon h(k_\mu).v^{(0)} = h(k_\mu)\mathcal{M}^\epsilon.v^{(0)}, \qquad \epsilon = 0 \text{ or } 1, \quad h \in \mathrm{AN}_3, \tag{7}$$

where we have taken a reference space-like vector $v^{(0)} \equiv (0,0,0,1) \in \mathbb{R}^4$, such that its little group is the Lorentz group $\mathrm{SO}(3,1)$ and the action of $\mathrm{SO}(4,1)$ on it sweeps the whole De Sitter space, and defined the vector $v \equiv h(k_\mu).v^{(0)}$ with coordinates:

$$v_0 \;=\; -\sinh\frac{k_0}{\kappa} + \frac{\mathbf{k}^2}{2\kappa^2}e^{k_0/\kappa} \quad v_i \;=\; -\frac{k_i}{\kappa} \quad v_4 \;=\; \cosh\frac{k_0}{\kappa} - \frac{\mathbf{k}^2}{2\kappa^2}e^{k_0/\kappa}. \tag{8}$$

The sign $(-)^\epsilon$ corresponds to the two components of the Iwasawa decomposition. We then introduce the set $\mathrm{AN}_3^c \equiv \mathrm{AN}_3 \cup \mathrm{AN}_3\mathcal{M}$, such that the Iwasawa decomposition reads $\mathrm{SO}(4,1) = \mathrm{AN}_3^c\,\mathrm{SO}(3,1)$ and that AN_3^c is isomorphic to the full de Sitter space. Actually, one can check that AN_3^c is itself a group. A crucial point is that the component v_4 of the above vector is left invariant by the action of the Lorentz group $\mathrm{SO}(3,1)$. This suggests to use this function of the "momentum"k_μ as a new (deformed) invariant energy-momentum (dispersion) relation, in the construction of a deformed version of particle dynamics and field theory on κ-Minkowski spacetime. This is the basis for much current QG phenomenology [5].

Finally, we will need an integration measure on AN_3 in order to define a Fourier transform. The group AN_3 is provided with two invariant Haar measures: $\int dh_L = \int d^4k_\mu$, $\int dh_R = \int e^{+3k_0/\kappa}\,d^4k_\mu$, which are respectively invariant under the left and right action of the group. The left invariant measure can be obtained from the 5d parametrization used above as:

$$\kappa^4 \int \delta(v_A v^A - 1)\theta(v_0 + v_4)\,d^5v_A \;=\; \int d^4k_\mu \;=\; \int dh_L, \tag{9}$$

so it is the natural measure on AN_3 inherited from the Haar measure on $\mathrm{SO}(4,1)$. However, this measure is not Lorentz invariant, due to the restriction $v_+ > 0$. To get a Lorentz invariant measure, we write the same measure as a measure on $\mathrm{AN}_3^c \equiv \mathrm{AN}_3 \cup \mathrm{AN}_3\mathcal{M} \sim dS$:

$$\int dh_L \equiv \int_{\mathrm{AN}_3} dh_L^+ + \int_{\mathrm{AN}_3\mathcal{M}} dh_L^- = \int \delta(v_A v^A - 1)d^5v. \tag{10}$$

Another way to obtain a Lorentz invariant measure is to consider the elliptic de Sitter space $dS/\mathbb{Z}_2$ where we identify $v_A \leftrightarrow -v_A$, which amounts to identifying the group elements $h(k_\mu) \leftrightarrow h(k_\mu)\mathcal{M}$. This space is indeed isomorphic to AN_3 as a manifold. One way to achieve nicely this restriction at the field theory level is to consider only fields on De Sitter space (or on AN_3^c) which are however invariant under the parity transformation $v_A \leftrightarrow -v_A$ [15].

For the free real scalar field $\phi : G \to \mathbb{R}$, we define the action

$$\text{ß}(\phi) = \int dh\,\phi(h)\,\mathcal{K}(h)\,\phi(h), \quad \forall h \in G, \tag{11}$$

where dh is the left invariant measure. We then interpret $G = \mathrm{AN}_3, \mathrm{AN}_3^c$ as the momentum space. We demand $\mathcal{K}(h)$ to be a function on G invariant under the Lorentz transformations, which suggests to use some function $\mathcal{K}(h) = f(v_4(h))$. Two common choices are

$$\mathcal{K}_1(h) = (\kappa^2 - \pi_4(h)) - m^2, \quad \mathcal{K}_2(h) = \kappa^2 - (\pi_4(h))^2 - m^2, \quad \pi_4 = \kappa v_4. \tag{12}$$

The above action is then Lorentz invariant if we choose a Lorentz invariant measure, i.e. dh_L in the case of either generic fields on AN_3^c or symmetric fields on AN_3.

Finally, the following generalized Fourier transform relates functions on the group $\mathcal{C}(G)$ and elements of the enveloping algebra $\mathcal{U}(\mathfrak{an}_3)$, i.e. non-commutative fields on the non-commutative spacetime $\mathfrak{an}_3$, i.e. on κ-Minkowski. For $G = \mathrm{AN}_3^c, \mathrm{AN}_3$, respectively:

$$\widehat{\phi}(X) = \int_{\mathrm{AN}_3} dh_L^+ \, h(k_\mu)\, \phi^+(k) + \int_{\mathrm{AN}_3\mathcal{M}} dh_L^- \, h(k_\mu)\, \phi^-(k), \quad X \in \mathfrak{an}_3, \;\; \widehat{\phi}(X) \in \mathcal{U}(\mathfrak{an}_3)$$
$$\widehat{\phi}(X) = \int_{\mathrm{AN}_3} dh_L \, h(k_\mu)\, \phi(k), \quad X \in \mathfrak{an}_3, \;\; \widehat{\phi}(X) \in \mathcal{U}(\mathfrak{an}_3)$$

where we used the non-abelian plane-wave $h(k_\mu)$ [13, 15]. The group field theory action on G can now be rewritten as a non-commutative field theory on κ-Minkowski (in the AN_3 case)

$$ß(\phi) = \int dh_L \, \phi(h)\, \mathcal{K}(h)\, \phi(h) = \int d^4X \, \left(\partial_\mu \widehat{\phi}(X) \partial^\mu \widehat{\phi}(X) + m^2 \widehat{\phi}^2(X)\right). \tag{13}$$

The Poincaré symmetries are naturally deformed in order to be consistent with the non-trivial commutation relations of the κ-Minkowski coordinates [15].

3.2. 3d case

The group field theory we start from is the Boulatov model for 3d quantum gravity [3]. We consider a real field $\phi : \mathrm{SU}(2)^3 \to \mathbb{R}$ invariant under the diagonal right action of SU(2):

$$\phi(g_1, g_2, g_3) = \phi(g_1 g, g_2 g, g_3 g), \quad \forall g \in \mathrm{SU}(2). \tag{14}$$

The action for this 3d group field theory involves a trivial propagator and the tetrahedral vertex:

$$S_{3d}[\phi] \;=\; \frac{1}{2}\int [dg]^3 \phi(g_1,g_2,g_3)\phi(g_3,g_2,g_1) - \frac{\lambda}{4!}\int [dg]^6 \phi(g_1,g_2,g_3)\phi(g_3,g_4,g_5)\phi(g_5,g_2,g_6)\phi(g_6,g_4,g_1). \tag{15}$$

The Feynman diagrams of the theory are then, by construction, 3d triangulations, while the corresponding Feynman amplitudes are given by the Ponzano-Regge spin foam model [3].

Now [1] we look at two-dimensional variations of the ϕ-field around classical solutions of the corresponding equations of motion:

$$\phi(g_3, g_2, g_1) \;=\; \frac{\lambda}{3!}\int dg_4 dg_5 dg_6 \phi(g_3,g_4,g_5)\phi(g_5,g_2,g_6)\phi(g_6,g_4,g_1). \tag{16}$$

Calling $\phi^{(0)}$ a generic solution to this equation, we look at field variations $\delta\phi(g_1, g_2, g_3) \equiv \psi(g_1 g_3^{-1})$ which do not depend on the group element g_2. We consider a specific class of classical solutions, named "flat" solutions (they can be interpreted as quantum flat space on some a priori non-trivial topology):

$$\phi^{(0)}(g_1, g_2, g_3) \;=\; \sqrt{\frac{3!}{\lambda}} \int dg \; \delta(g_1 g) F(g_2 g) \delta(g_3 g), \quad F : G \to \mathbb{R}. \tag{17}$$

As shown in [1], this ansatz gives solutions to the field equations as soon as $\int F^2 = 1$.

This leads to an effective action for the 2d variations ψ:

$$S_{eff}[\psi] \;=\; \frac{1}{2}\int \psi(g)\mathcal{K}(g)\psi(g^{-1}) - \frac{\mu}{3!}\int [dg]^3 \, \psi(g_1)\psi(g_2)\psi(g_3)\delta(g_1 g_2 g_3) - \frac{\lambda}{4!}\int [dg]^4 \, \psi(g_1)..\psi(g_4)\delta(g_1..g_4), \tag{18}$$

with the kinetic term and the 3-valent coupling given in term of F:

$$\mathcal{K}(g) \;=\; 1-2\left(\int F\right)^2 - \int dh F(h)F(hg), \qquad \frac{\mu}{3!} \;=\; \sqrt{\frac{\lambda}{3!}}\int F.$$

with $F(g)$ assumed to be invariant under conjugation $F(g) = F(hgh^{-1})$.

Such an action defines a non-commutative quantum field theory invariant under the quantum double of SU(2) (a quantum deformation of the Poincaré group) [16, 13, 1, 20, 14].

Being an invariant function, F can be expanded in group characters:

$$F(g) = \sum_{j\in\mathbb{N}/2} F_j\chi_j(g), \qquad F_0 = \int F, \quad F_j = \int dg\, F(g)\chi_j(g), \tag{19}$$

where the F_j's are the Fourier coefficients of the Peter-Weyl decomposition in irreducible representations of SU(2), labelled by $j \in \mathbb{N}/2$. The kinetic term reads then:

$$\mathcal{K}(g) = 1 - 3F_0^2 - \sum_{j\geq 0}\frac{F_j^2}{d_j}\chi_j(g) = \sum_{j\geq 0} F_j^2\left(1-\frac{\chi_j(g)}{d_j}\right) - 2F_0^2 \;\equiv\; Q^2(g) - M^2. \tag{20}$$

It is easy to check that $Q^2(g) \geq 0$ with $Q(\mathbb{I}) = 0$. We interpret this term as the generalized "Laplacian" of the theory while the 0-mode F_0 defines the mass $M^2 \equiv 2F_0^2$.

If we choose the simple solution (other choices will give more complicated kinetic terms)

$$F(g) = a + b\chi_1(g), \qquad \int F = a^2 + b^2 = 1, \tag{21}$$

we obtain

$$\mathcal{K}(g) = \frac{4}{3}(1-a^2)\,\vec{p}^{\,2} - 2a^2. \tag{22}$$

3.3. 4d case

Let us consider a general 4d GFT related to topological BF quantum field theories, i.e. whose Feynman expansion leads to amplitudes that can be interpreted as discrete BF path integrals for gauge group $\mathcal{G}$. This is given by the following action:

$$\begin{aligned} S_{4d} \;&=\; \frac{1}{2}\int [dg]^4\, \phi(g_1,g_2,g_3,g_4)\phi(g_4,g_3,g_2,g_1) \\ &-\frac{\lambda}{5!}\int [dg]^{10}\phi(g_1,g_2,g_3,g_4)\phi(g_4,g_5,g_6,g_7)\phi(g_7,g_3,g_8,g_9)\phi(g_9,g_6,g_2,g_{10})\phi(g_{10},g_8,g_5,g_1), \end{aligned} \tag{23}$$

where the field is again required to be gauge-invariant, $\phi(g_1,g_2,g_3,g_4) = \phi(g_1g,g_2g,g_3g,g_4g)$ for any $g \in \mathcal{G}$. The relevant group for our construction will be SO(4,1), which requires some regularization to avoid divergencies due to its non-compact nature.

We generalize to 4d the "flat solution" ansatz of the 3d group field theory as [1]:

$$\phi^{(0)}(g_i) \;\equiv\; \sqrt[3]{\frac{4!}{\lambda}}\int dg\, \delta(g_1g)F(g_2g)\widetilde{F}(g_3g)\delta(g_4g), \tag{24}$$

with $(\int F\widetilde{F})^3 = 1$. The effective action around such background is [2]:

$$\begin{aligned} S_{eff}[\psi] \;&=\; \frac{1}{2}\int \psi(g)\psi(g^{-1})\mathcal{K}(g) \\ &-\sqrt[3]{\frac{\lambda}{4!}}\int F\int\widetilde{F}\int \psi(g_1)..\psi(g_3)\,\delta(g_1..g_3)\left[\int F\int \widetilde{F} + \int dhF(hg_3)\widetilde{F}(h)\right] \\ &-\left(\sqrt[3]{\frac{\lambda}{4!}}\right)^2\int F\int\widetilde{F}\int \psi(g_1)..\psi(g_4)\,\delta(g_1..g_4) - \frac{\lambda}{5!}\int \psi(g_1)..\psi(g_5)\,\delta(g_1..g_5), \end{aligned} \tag{25}$$

with the kinetic operator given by:

$$\mathcal{K}(g) \,=\, \left[1-2\left(\int F\int\widetilde{F}\right)^2\int F\widetilde{F}-2\int F\int\widetilde{F}\int dhF(hg)\widetilde{F}(h)\int dhF(h)\widetilde{F}(hg)\right]. \qquad (26)$$

A simpler special case of the classical solution above is obtained choosing $\tilde{F}(g)=\delta(g)$ while keeping F arbitrary but with $F(\mathbb{I})=1$. Calling $c\equiv\int F$, the effective action becomes:

$$\begin{aligned} S_{eff}[\psi] \;&=\; \frac{1}{2}\int\psi(g)\psi(g^{-1})\left[1-2c^2-2cF(g)F(g^{-1})\right]-c\left(\sqrt[3]{\frac{\lambda}{4!}}\right)\int\psi(g_1)..\psi(g_3)\,\delta(g_1..g_3)\,[c+F(g_3)] \\ &\quad -c\left(\sqrt[3]{\frac{\lambda}{4!}}\right)^2\int\psi(g_1)..\psi(g_4)\,\delta(g_1..g_4)-\frac{\lambda}{5!}\int\psi(g_1)..\psi(g_5)\,\delta(g_1..g_5). \qquad (27)\end{aligned}$$

In order to make contact with deformed special relativity, we now specialize this construction to one that gives an effective field theory based on the momentum group manifold AN_3.

We start then, as anticipated, from the group field theory describing SO(4,1) BF-theory.

From the quantum gravity perspective, there are several reasons of interest in this model: 1) the McDowell-Mansouri formulation (as well as related ones [21]) defines 4d gravity with cosmological constant as a BF-theory for SO(4,1) plus a potential term which breaks the gauge symmetry from SO(4,1) down to the Lorentz group SO(3,1); this suggests to try to define Quantum Gravity in the spin foam context as a perturbation of a topological spin foam model for $SO(4,1)$ BF theory. These ideas could also be implemented directly at the GFT level, and the starting point would necessarily be a GFT for $SO(4,1)$ of the type we use here. 2) we expect [3, 4] any classical solution of this GFT model to represent quantum De Sitter space on some given topology, and such configurations would be physically relevant also in the non-topological case. 3) the spinfoam/GFT model for SO(4,1) BF-theory seems the correct arena to build a spin foam model for 4d quantum gravity plus particles on De Sitter space [22], treating them as topological curvature defects for an $SO(4,1)$ connection, similarly to the 3d case [16].

Following the above procedure we naturally obtain an effective field theory living on SO(4,1). We want then to obtain from it an effective theory on $AN_3^{(c)}$. We choose:

$$F(g) \,=\, \alpha(v_4(g)+a)\vartheta(g), \qquad \tilde{F}(g) \,=\, \delta(g). \qquad (28)$$

The function v_4 is defined as matrix element of g in the fundamental (non-unitary) five-dimensional representation of $SO(4,1)$, $v_4(g) = \langle v^{(0)}|g|v^{(0)}\rangle$, where $v^{(0)} = (0,0,0,0,1)$ is, as previously, the vector invariant under the $SO(3,1)$ Lorentz subgroup. $\vartheta(g)$ is a cut-off function providing a regularization of F, so that it becomes an L^1 function. Assuming that $\vartheta(\mathbb{I})=1$, we require $\alpha=(a+1)^{-1}$ in order for the normalization condition to be satisfied.

Then we can derive the effective action around such classical solutions for 2d field variations:

$$\begin{aligned} S_{eff}[\psi] \;&=\; \frac{1}{2}\int\psi(g)\psi(g^{-1})\left[1-2c^2-\frac{2c\vartheta^2(g)(a+v_4(g))^2}{(a+1)^2}\right]-c\left(\frac{\lambda}{4!}\right)^{\frac{1}{3}}\int\psi(g_1)..\psi(g_3)\,\delta(g_1..g_3)\,[c+F(g_3)] \\ &\quad -c\left(\frac{\lambda}{4!}\right)^{\frac{2}{3}}\int\psi(g_1)..\psi(g_4)\,\delta(g_1..g_4)-\frac{\lambda}{5!}\int\psi(g_1)..\psi(g_5)\,\delta(g_1..g_5), \qquad (29)\end{aligned}$$

where $c=\int F$. Thus the last issue to address in order to properly define this action is to compute the integral of F; this can be done, and we refer to [2] for details.

We recognize the correct kinetic term for a DSR field theory. However, the effective matter field is still defined on a $SO(4,1)$ momentum manifold. The only remaining issue is therefore to understand the "localization" process of the field ψ to AN_3^c.

The kinetic term does not show any dependence on the Lorentz sector. This suggests that the $\mathrm{SO}(3,1)$ degrees of freedom are non-dynamical and that the restriction of the field ψ to AN_3^c group elements defines the complete dynamics of the theory. This would be trivially true if not for the fact that the interaction term depend also on the Lorentz degrees of freedom. One way to make this manifest is, for example, to assume that the perturbation field ψ has a product structure $\psi(g) = \tilde{\psi}(h)\Psi(\Lambda)$. The only contribution to the kinetic term from the Lorentz sector is a constant multiplicative term $\int_{SO(3,1)} d\Lambda \Psi(\Lambda)\Psi(\Lambda)$. Therefore we get an exactly DSR-like and κ-Poincaré invariant free field theory. On the other hand, the vertex term couples Lorentz and AN_3 degrees of freedom; thus the κ-Poincaré symmetry is broken and the pure DSR-like form lost. The above also shows that, if we were to choose the dependence of the perturbation field on the Lorentz sector to be trivial, i.e. $\Psi(\Lambda) \equiv 1$, and thus to *start* from a perturbation field defined only on the AN_3 subgroup, we would indeed obtain a DSR field theory, but with an interaction term that would be more complicated that a simple polinomial interaction, due to to the integrations over the Lorentz group. Of course, it would be a possible DSR field theory nevertheless. Still, because of the form of the kinetic term, we believe a reduction to the AN_3 sector to happen dynamically, or that a proper canonical analysis would show that the $\mathrm{SO}(3,1)$ modes are pure gauge and can thus drop from the action altogether. Anyway, notice that a restricted theory obtained from the above and living on AN_3^c only is dynamically stable. In fact, if we consider only excitations of the field in AN_3^c, we will never obtain excitations in $\mathrm{SO}(3,1)$ due to momentum conservation $\delta(g_1..g_n)$ since AN_3^c is a subgroup.

Then, restricting ourselves to group elements $h_i \in \mathrm{AN}_3^c$, we have the field theory:

$$S_{final}[\psi] = \frac{1}{2}\int \psi(h)\psi(h^{-1})\left[1 - 2c^2 - 2cv_4(h)^2\vartheta(h)^2\right] - c\left(\frac{\lambda}{4!}\right)^{\frac{1}{3}} \int \psi(h_1)..\psi(h_3)\,\delta(h_1..h_3)\,[c + v_4(h_3)\vartheta(h_3)]$$
$$-c\left(\frac{\lambda}{4!}\right)^{\frac{2}{3}} \int \psi(h_1)..\psi(h_4)\,\delta(h_1..h_4) - \frac{\lambda}{5!}\int \psi(h_1)..\psi(h_5)\,\delta(h_1..h_5) \quad ,$$

with implicit left-invariant measure on AN_3^c. We have thus derived a DSR scalar field theory with a κ-deformed Poincaré symmetry from the GFT for $\mathrm{SO}(4,1)$ topological BF-theory.

For other possible strategies leading to the same result, and for more details on the above one, see [2]. In particular, notice that we could have directly started from a BF-like GFT action for the group AN_3. Following the same procedure, and with appropriate regularization, we would have obtained easily a DSR field theory of the type we want. What would be less clear, in this case, and this is why we have not focused on this simplified setting, is the link between the initial theory and known classical or quantum formulations of gravity.

4. Conclusions

In this contribution, we have briefly reviewed recent results concerning the relation between non-commutative matter field theories, characterized by a configuration space with a Lie algebra structure and a momentum space being a group manifold, and a symmetry group given by quantum deformations of the Poincaré group, and group field theories, which are candidate models for the microscopic description of quantum spacetime and of its dynamics, merging the insights of canonical loop quantum gravity, covariant spin foam models and simplicial gravity approaches. In particular, we have seen how the former emerge naturally from the later, in a way reminiscent of the emergence of field theories for quasi-particles from hydrodynamics in condensed matter theory and analog gravity models. We have motivated this line of research as a step forward in a research programme trying to bridge the gap between group field theories as fundamental descriptions of a quantum, discrete spacetime at the Planck scale, and the continuum spacetime physics of gravity and matter at macroscopic scales, using ideas and methods from quantum and statistical field theory and condensed matter physics, applied directly at the GFT level. To be sure: 1) there are many technical and physical aspects of the procedure used to derive non-commutative matter field theories from GFTs that need further analysis; 2) one important difference with respect to analog gravity models should be understood: while quasi-particle dynamics on effective metrics is there obtained from the hydrodynamics of the fundamental system, in our GFT context we have applied a similar procedure directly to the fundamental microscopic field theory; one could speculate that this is why we have obtained *non-commutative* field theories for the emergent matter, as opposed to ordinary field theries; 3) the same results have motivations and relevance that are independent from the perspective we have adopted here to present them. These relate to formal aspects of the relation between non-commutative geometry and group field theory, and to the importance of these non-commutative models (and of the general DSR idea) to Quantum Gravity phenomenology [2]. For all this, we refer to the literature [14, 13, 15, 5]. However, we believe that these interesting results fit well in the more general scheme we have outlined, on the one hand supporting the picture of GFTs as fundamental theories of quantum gravity, rather than just auxiliary mathematical tools, and on the other hand acquiring, from this more general perspective, an additional reason of interest.

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[1] Fairbairn W, Livine E 2007, *Class. Quant. Grav.* **24**, 5277, gr-qc/0702125
[2] Girelli F, Livine E, Oriti D 2009, *4d deformed special relativity from group field theories*, arXiv: 0903.3475 (gr-qc)
[3] Oriti D 2006, gr-qc/0607032; Oriti D 2005, in Fauser B, Tolksdorf J and Zeidler E (editors) 2007 *Mathematical and Physical Aspects of Quantum Gravity*, Birkhaeuser, Basel, gr-qc/0512103;
[4] Freidel L 2005 *Int. J. Theor. Phys.* **44** 1769, hep-th/0505016;
[5] Amelino-Camelia G 2005, Lect. Notes Phys. 669, 59-100, gr-qc/0412136;
[6] Oriti D 2007, Proceedings of Science, [arXiv:0710.3276]
[7] Rovelli C 2004 *Quantum Gravity*, Cambridge University Press, Cambridge;
[8] Perez A 2003 *Class. Quant. Grav.* **20** R43, gr-qc/0301113;
[9] Oriti D (editor) 2009 *Approaches to Quantum Gravity*, Cambridge University Press, Cambridge
[10] Freidel L, Gurau R, Oriti D, *GFT renormalization - 3d case: power counting of divergences*, to appear
[11] Barcelo C, S. Liberati S, Visser M 2005, *Living Rev. Rel.* **8** 12, arXiv: gr-qc/0505065;
[12] Volovik G E 2001 *Phys. Rept.* **351** 195, gr-qc/0005091; Volovik G E 2006, gr-qc/0612134;
[13] Freidel L, Majid S 2006, arXiv: hep-th/0601004
[14] Livine E 2008, [arXiv:0811.1462]
[15] Freidel L, Kowalski-Glikman J, Nowak S 2007, arXiv:0706.3658 [hep-th].
[16] Freidel L, Livine E 2006, *Class.Quant.Grav.* **23**, 2021-2062 [arXiv:hep-th/0502106].
[17] Oriti D, Ryan J 2006, *Class. Quant. Grav.* **23**, 6543, gr-qc/0602010;
[18] Majid S 2000, *Foundations of quantum group theory.* Cambridge University Press, Cambridge
[19] Vilenkin N, Klimyk A, *Representations of Lie Groups*, Springer Ed.
[20] Joung E, Mourad J, Noui K 2008, arXiv:0806.4121
[21] Freidel L, Starodubtsev A 2005, arXiv:hep-th/0501191;
[22] Kowalski-Glikman J, Starodubtsev A 2008, *Phys.Rev.* D **78**, 084039, arXiv:0808.2613