Open-closed strings: Two-dimensional extended TQFTs and Frobenius algebras

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Abstract

We study a special sort of 2-dimensional extended Topological Quantum Field Theories (TQFTs) which we call open-closed TQFTs. These are defined on open-closed cobordisms by which we mean smooth compact oriented 2-manifolds with corners that have a particular global structure in order to model the smooth topology of open and closed string world-sheets. We show that the category of open-closed TQFTs is equivalent to the category of knowledgeable Frobenius algebras. A knowledgeable Frobenius algebra $(A, C, \imath, \imath^*)$ consists of a symmetric Frobenius algebra A, a commutative Frobenius algebra C, and an algebra homomorphism $\imath\colon C \to A$ with dual $\imath^*\colon A \to C$, subject to some conditions. This result is achieved by providing a generators and relations description of the category of open-closed cobordisms. In order to prove the sufficiency of our relations, we provide a normal form for such cobordisms which is characterized by topological invariants. Starting from an arbitrary such cobordism, we construct a sequence of moves (generalized handle slides and handle cancellations) which transforms the given cobordism into the normal form. Using the generators and relations description of the category of open-closed cobordisms, we show that it is equivalent to the symmetric monoidal category freely generated by a knowledgeable Frobenius algebra. Our formalism is then generalized to the context of open-closed cobordisms with labeled free boundary components, *i.e.* to open-closed string worldsheets with D-brane labels at their free boundaries.

1 Introduction

Motivated by open string theory, boundary conformal field theory, and extended topological quantum field theory, open-closed cobordisms have been a topic of considerable interest to mathematicians and physicists. By open-closed cobordisms we mean a category $\mathbf{2Cob}^{\mathrm{ext}}$ whose objects are diffeomorphism classes of compact oriented smooth 1-manifolds, *i.e.* free unions of circles S^1 and unit intervals $I = [0, 1]$. The morphisms are certain compact oriented smooth 2-manifolds with corners. The corners of such a manifold f are required to coincide with the boundary points ∂I of the intervals. The boundary of f viewed as a topological manifold, minus the corners, consists of components that are either 'black' or 'coloured'. Each corner is required to separate a black

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component from a coloured one. The black part of the boundary coincides with the union of the source and the target object. Two such manifolds with corners are considered equivalent if they are related by an orientation preserving diffeomorphism which restricts to the identity on the black part of the boundary. An example of such an open-closed cobordisms is depicted here[3],
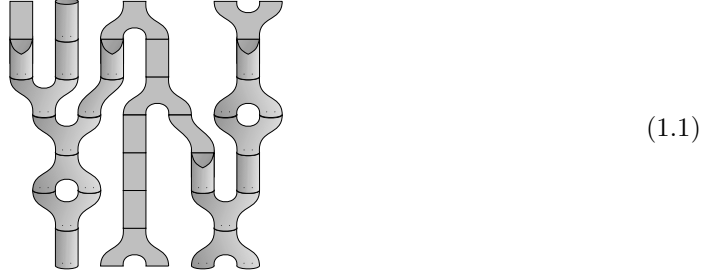
$$(1.1)$$

where the boundaries at the top and at the bottom of the diagram are the black ones. In Section 3, we present a formal definition which includes some additional technical properties. Gluing such cobordisms along their black boundaries, *i.e.* putting the building blocks of (1.1) on top of each other, is the composition of morphisms. The free union of manifolds, *i.e.* putting the building blocks of (1.1) next to each other, provides $\mathbf{2Cob}^{\text{ext}}$ with the structure of a strict symmetric monoidal category.

Open-closed cobordisms can be seen as a generalization of the conventional 2-dimensional cobordism category $\mathbf{2Cob}$. The objects of this symmetric monoidal category are diffeomorphism classes of compact oriented smooth 1-manifolds *without boundary*; the morphisms are diffeomorphism classes of compact oriented smooth cobordisms between them.

The study of open-closed cobordisms plays an important role in conformal field theory if one is interested in boundary conditions, and open-closed cobordisms have a natural string theoretic interpretation. The intervals in the black boundaries are interpreted as open strings, the circles as closed strings, and the open-closed cobordisms as string worldsheets. Here we consider only the underlying smooth manifolds, but not any additional conformal or complex structure. Additional labels at the coloured boundaries are interpreted as D-branes or boundary conditions on the open strings.

An open-closed Topological Quantum Field Theory (TQFT), which we formally define in Section 4 below, is a symmetric monoidal functor $\mathbf{2Cob}^{\text{ext}} \to \mathcal{C}$ into a symmetric monoidal category $\mathcal{C}$. If $\mathcal{C}$ is the category of vector spaces over a fixed field k, then the open-closed TQFT assigns vector spaces to the 1-manifolds I and S^1, it assigns tensor products to free unions of these manifolds, and k-linear maps to open-closed cobordisms.

Such an open-closed TQFT can be seen as an extension of the notion of a 2-dimensional TQFT [2] which is a symmetric monoidal functor $\mathbf{2Cob} \to \mathcal{C}$. We refer to this conventional notion of 2-dimensional TQFT as a *closed TQFT* and to the morphisms of $\mathbf{2Cob}$ as *closed cobordisms*. For the classic results on 2-dimensional closed TQFTs, we recommend the original works [3–5] and the book [6].

The most powerful results on closed TQFTs crucially depend on results from Morse theory. Morse theory provides a generators and relations description of the category $\mathbf{2Cob}$. First, any compact cobordism Σ can be obtained from gluing a finite number of elementary cobordisms along their boundaries. In order to see this, one chooses a Morse function $f\colon \Sigma \to \mathbb{R}$ such that all critical points have distinct critical values and considers the pre-images $f^{-1}([x_0 - \varepsilon, x_0 + \varepsilon]) \subseteq \Sigma$ of intervals that contain precisely one critical value $x_0 \in \mathbb{R}$. Each such pre-image is the free union of one of the elementary cobordisms,


$$(1.2)$$

---

[3]In order to get a feeling for these diagrams, the reader might wish to verify that this cobordism is diffeomorphic to the one depicted in Figure 1 of [1].

with zero or more cylinders over S^1. The different elementary cobordisms (1.2) are precisely the Morse data that characterize the critical points, and the way they are glued corresponds to the handle decomposition associated with f. The Morse data of (1.2) provide the *generators* for the morphisms of **2Cob**. Our diagrams, for example (1.1), are organized in such a way that the vertical axis of the drawing plane serves as a Morse function, and the cobordisms are composed of building blocks that contain at most one critical point.

Second, given two Morse functions $f_1, f_2 \colon \Sigma \to \mathbb{R}$, the handle decompositions associated with f_1 and f_2 are related by a finite sequence of *moves*, *i.e.* handle slides and handle cancellations. This means that there are diffeomorphisms such as,

$$\cong \tag{1.3}$$

which provide us with the *relations* of **2Cob**. When we explicitly construct the diffeomorphism that relates two handle decompositions of some manifold, we call these diffeomorphisms *moves*. The example (1.3) corresponds to a cancellation of a 1-handle and a 2-handle. Below is an example of sliding a 1-handle past another 1-handle.

$$\cong \tag{1.4}$$
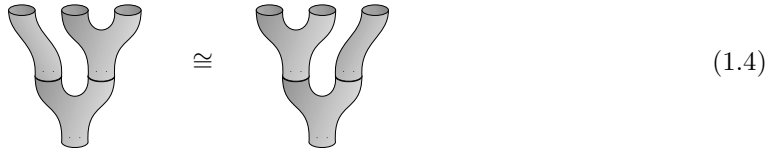
Whereas it is not too difficult to construct by brute-force a set of diffeomorphisms between manifolds such as those in (1.3) and (1.4), *i.e.* to show that a set of relations is necessary, it is much harder to show that they are also sufficient, *i.e.* that any two handle decompositions are related by a finite sequence of moves such as (1.3) and (1.4). In order to establish this result, one strategy is to prove that there exists a *normal form* for the morphisms of **2Cob** which is characterized by topological invariants, and then to show that the relations suffice in order to transform an arbitrary handle decomposition into this normal form. For closed cobordisms, this is done in detail in [4, 6].

In order to describe open-closed cobordisms using generators and relations, one would need a generalization of Morse theory for manifolds with corners. Such a generalization of Morse theory can be used in order to find the generators of **2Cob**[ext], and brute force can be used to establish the necessity of certain relations. However, we are not aware of any abstract theorem that would guarantee the sufficiency of these relations.

The first main result of this article is a normal form for open-closed cobordisms with an induction proof that the relations suffice in order to transform any handle decomposition into the normal form. As a consequence, for any two diffeomorphic open-closed cobordisms whose handle decompositions are given, we explicitly construct a diffeomorphism relating the two by constructing the corresponding sequence of moves.

The description of **2Cob**[ext] in terms of generators and relations has emerged over the last couple of years from consistency conditions in boundary conformal field theory, going back to the work of Cardy and Lewellen [7, 8], Lazaroiu [9], and Moore and Segal, see, for example [10, 11], and these results have been known to the experts for some time. The first result of this article which we claim is new, is the normal form and our induction proof that the relations are sufficient.

This, in turn, implies the following result in Morse theory for our sort of compact 2-manifolds with corners which has so far not been available by other means: The handle decompositions associated with any two Morse functions on the same manifold are related by a finite sequence of handle slides and handle cancellations.

Once a description of **2Cob**[ext] in terms of generators and relations is available, it is possible to find an algebraic characterization for the symmetric monoidal category of open-closed TQFTs.

Whereas the category of closed TQFTs is equivalent as a symmetric monoidal category to the category of commutative Frobenius algebras [4], we prove that the category of open-closed TQFTs is equivalent as a symmetric monoidal category to the category of *knowledgeable Frobenius algebras*. We define knowledgeable Frobenius algebras in Section 2 precisely for this purpose. A knowledgeable Frobenius algebra $(A, C, \imath, \imath^*)$ consists of a symmetric Frobenius algebra A, a commutative Frobenius algebra C, and an algebra homomorphism $\imath \colon C \to A$ with dual $\imath^* \colon A \to C$, subject to some conditions. This is the second main result of the present article. The structure that emerges is consistent with the results of Moore and Segal [10].

The algebraic structures relevant to boundary conformal field theory have been studied by Fuchs and Schweigert [12]. In a series of papers, for example [13], Fuchs, Runkel, and Schweigert study Frobenius algebra objects in ribbon categories. Topologically, this corresponds to a situation in which the surfaces are embedded in some 3-manifold and studied up to ambient isotopy. In the present article, in contrast, we consider Frobenius algebra objects in a symmetric monoidal category, and our 2-manifolds are considered equivalent as soon as they are diffeomorphic (as abstract manifolds).

Various extensions of open-closed topological field theories have also been studied. Baas, Cohen, and Ramírez have extended the symmetric monoidal category of open-closed cobordisms to a symmetric monoidal 2-category whose 2-morphisms are certain diffeomorphisms of the open-closed cobordisms [1]. This work extends the work of Tillmann who defined a symmetric monoidal 2-category extending the closed cobordism category [14]. She used this 2-category to introduce an infinite loop space structure on the plus construction of the stable mapping class group of closed cobordisms [15]. Using a similar construction as Tillmann's, Baas, Cohen, and Ramírez have defined an infinite loop space structure on the plus construction of the stable mapping class group of open-closed cobordisms, showing that infinite loop space structures are a valuable tool in studying the mapping class group. Other results on purely open cobordisms and their higher-categorical structures can be found in [16, 17].

The present article is structured as follows: In Section 2, we define the notion of a knowledgeable Frobenius algebra and introduce the symmetric monoidal category **K-Frob**$(\mathcal{C})$ of knowledgeable Frobenius algebras in a symmetric monoidal category $\mathcal{C}$. We provide an abstract description in terms of generators and relations of this category by defining a category **Th**(**K-Frob**), called the *theory of knowledgeable Frobenius algebras*, and by showing that the category of symmetric monoidal functors and monoidal natural transformations **Th**(**K-Frob**) $\to \mathcal{C}$ is equivalent as a symmetric monoidal category to **K-Frob**$(\mathcal{C})$. In Section 3, we introduce the category **2Cob**$^{\mathrm{ext}}$ of open-closed cobordisms. We present a normal form for such cobordisms and characterize the category in terms of generators and relations. In Section 4, we define open-closed TQFTs as symmetric monoidal functors **2Cob**$^{\mathrm{ext}} \to \mathcal{C}$ into some symmetric monoidal category $\mathcal{C}$. We show that the category **2Cob**$^{\mathrm{ext}}$ is equivalent as a symmetric monoidal category to **Th**(**K-Frob**) which in turn implies that the category of open-closed TQFTs in $\mathcal{C}$ is equivalent as a symmetric monoidal category to the category of knowledgeable Frobenius algebras **K-Frob**$(\mathcal{C})$. In Section 5, we generalize our results to the case of labeled free boundaries. Section 6 contains a summary and an outlook on open problems.

2   Knowledgeable Frobenius algebras

2.1   Definitions

In this section, we define the notion of a knowledgeable Frobenius algebra. We consider these Frobenius algebras not only in the symmetric monoidal category **Vect**$_k$ of vector spaces over some fixed field k, but in any generic symmetric monoidal category. Other examples include the symmetric monoidal categories of Abelian groups, graded-vector spaces, and chain complexes.

Definition 2.1. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \tau)$ be a symmetric monoidal category with tensor product $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, unit object $\mathbb{1} \in |\mathcal{C}|$, associativity constraint $\alpha_{X,Y,Z} \colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$,

left- and right-unit constraints $\lambda_X \colon \mathbb{1} \otimes X \to X$ and $\rho_X \colon X \otimes \mathbb{1} \to X$, and the symmetric braiding $\tau_{X,Y} \colon X \otimes Y \to Y \otimes X$, for objects X, Y, Z of $\mathcal{C}$ (in symbols $X, Y, Z \in |\mathcal{C}|$).

1. An *algebra object* (A, μ, η) in $\mathcal{C}$ consists of an object A and morphisms $\mu \colon A \otimes A \to A$ and $\eta \colon \mathbb{1} \to A$ of $\mathcal{C}$ such that:

$$
\begin{array}{ccc}
(A \otimes A) \otimes A & \xrightarrow{\ \alpha_{A,A,A}\ } & A \otimes (A \otimes A) \\
{\scriptstyle \mu \otimes \mathrm{id}_A}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_A \otimes \mu} \\
A \otimes A & & A \otimes A \\
& \mu \searrow \ \ \swarrow \mu & \\
& A &
\end{array}
\tag{2.1}
$$

and

$$
\begin{array}{ccccc}
\mathbb{1} \otimes A & \xrightarrow{\ \eta \otimes \mathrm{id}_A\ } & A \otimes A & \xleftarrow{\ \mathrm{id}_A \otimes \eta\ } & A \otimes \mathbb{1} \\
& \lambda_A \searrow & \downarrow{\scriptstyle \mu} & \swarrow \rho_A & \\
& & A & &
\end{array}
\tag{2.2}
$$

commute.

2. A *coalgebra object* (A, Δ, ε) in $\mathcal{C}$ consists of an object A and morphisms $\Delta \colon A \to A \otimes A$ and $\varepsilon \colon A \to \mathbb{1}$ of $\mathcal{C}$ such that:

$$
\begin{array}{ccc}
& A & \\
\Delta \swarrow & & \searrow \Delta \\
A \otimes A & & A \otimes A \\
{\scriptstyle \Delta \otimes \mathrm{id}_A}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_A \otimes \Delta} \\
(A \otimes A) \otimes A & \xrightarrow{\ \alpha_{A,A,A}\ } & A \otimes (A \otimes A)
\end{array}
\tag{2.3}
$$

and

$$
\begin{array}{ccccc}
& & A & & \\
\lambda_A^{-1} \swarrow & \downarrow{\scriptstyle \Delta} & & \searrow \rho_A^{-1} & \\
\mathbb{1} \otimes A & \xleftarrow{\ \varepsilon \otimes \mathrm{id}_A\ } & A \otimes A & \xrightarrow{\ \mathrm{id}_A \otimes \varepsilon\ } & A \otimes \mathbb{1}
\end{array}
\tag{2.4}
$$

commute.

3. A *homomorphism of algebras* $f \colon A \to A'$ between two algebra objects (A, μ, η) and (A', μ', η') in $\mathcal{C}$ is a morphism f of $\mathcal{C}$ such that:

$$
\begin{array}{ccc}
A \otimes A & \xrightarrow{\ \mu\ } & A \\
{\scriptstyle f \otimes f}\downarrow & & \downarrow{\scriptstyle f} \\
A' \otimes A' & \xrightarrow{\ \mu'\ } & A'
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\mathbb{1} & \xrightarrow{\ \eta\ } & A \\
& \eta' \searrow & \downarrow{\scriptstyle f} \\
& & A'
\end{array}
\tag{2.5}
$$

commute.

4. A *homomorphism of coalgebras* $f\colon A \to A'$ between two coalgebra objects (A, Δ, ε) and $(A', \Delta', \varepsilon')$ in $\mathcal{C}$ is a morphism f of $\mathcal{C}$ such that:

$$
\begin{array}{ccc}
A & \xrightarrow{\ \Delta\ } & A \otimes A \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \otimes f} \\
A' & \xrightarrow[\ \Delta'\ ]{} & A' \otimes A'
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
A & & \\
{\scriptstyle f}\downarrow & \searrow^{\ \varepsilon} & \\
A' & \xrightarrow[\ \varepsilon'\ ]{} & \mathbb{1}
\end{array}
\tag{2.6}
$$

commute.

Definition 2.2. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \tau)$ be a symmetric monoidal category.

1. A *Frobenius algebra object* $(A, \mu, \eta, \Delta, \varepsilon)$ in $\mathcal{C}$ consists of an object A and of morphisms μ, η, Δ, ε of $\mathcal{C}$ such that:

   (a) (A, μ, η) is an algebra object in $\mathcal{C}$,

   (b) (A, Δ, ε) is a coalgebra object in $\mathcal{C}$, and

   (c) the following compatibility condition, called the *Frobenius relation*, holds,

$$
\begin{array}{ccccc}
& & A \otimes A & & \\
& {\scriptstyle \Delta \otimes \mathrm{id}_A}\swarrow & {\scriptstyle \mu}\downarrow & \searrow{\scriptstyle \mathrm{id}_A \otimes \Delta} & \\
(A \otimes A) \otimes A & & A & & A \otimes (A \otimes A) \\
{\scriptstyle \alpha_{A,A,A}}\downarrow & & {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \alpha^{-1}_{A,A,A}} \\
A \otimes (A \otimes A) & & & & (A \otimes A) \otimes A \\
& {\scriptstyle \mathrm{id}_A \otimes \mu}\searrow & \downarrow & \swarrow{\scriptstyle \mu \otimes \mathrm{id}_A} & \\
& & A \otimes A & &
\end{array}
\tag{2.7}
$$

2. A Frobenius algebra object $(A, \mu, \eta, \Delta, \varepsilon)$ in $\mathcal{C}$ is called *symmetric* if:

$$
\varepsilon \circ \mu = \varepsilon \circ \mu \circ \tau.
\tag{2.8}
$$

It is called *commutative* if:

$$
\mu = \mu \circ \tau.
\tag{2.9}
$$

3. Let $(A, \mu, \eta, \Delta, \varepsilon)$ and $(A', \mu', \eta', \Delta', \varepsilon')$ be Frobenius algebra objects in $\mathcal{C}$. A *homomorphism of Frobenius algebras* $f\colon A \to A'$ is a morphism f of $\mathcal{C}$ which is both a homomorphism of algebra objects and a homomorphism of coalgebra objects.

Notice that for any Frobenius algebra object $(A, \mu, \eta, \Delta, \varepsilon)$ in $\mathcal{C}$, the object A is always a rigid object of $\mathcal{C}$. In $\mathbf{Vect}_k$ this means that A is finite-dimensional.

The unit object $\mathbb{1} \in |\mathcal{C}|$ forms an algebra object $(\mathbb{1}, \lambda_{\mathbb{1}}, \mathrm{id}_{\mathbb{1}})$ in $\mathcal{C}$ with multiplication $\lambda_{\mathbb{1}}\colon \mathbb{1} \otimes \mathbb{1} \to \mathbb{1}$ and unit $\mathrm{id}_{\mathbb{1}}\colon \mathbb{1} \to \mathbb{1}$ as well as a coalgebra object $(\mathbb{1}, \lambda_{\mathbb{1}}^{-1}, \mathrm{id}_{\mathbb{1}})$ defining a commutative Frobenius

algebra object in $\mathcal{C}$. Given two algebra objects (A, μ_A, η_A) and (B, μ_B, η_B) in $\mathcal{C}$, the tensor product $(A \otimes B, \mu_{A \otimes B}, \eta_{A \otimes B})$ forms an algebra object in $\mathcal{C}$ with,

$$
\begin{aligned}
\mu_{A \otimes B} \;&=\; (\mu_A \otimes \mu_B) \circ \alpha^{-1}_{A,A,B \otimes B} \circ (\mathrm{id}_A \otimes \alpha_{A,B,B}) \circ (\mathrm{id}_A \otimes (\tau_{B,A} \otimes \mathrm{id}_B)) \\
&\quad \circ (\mathrm{id}_A \otimes \alpha^{-1}_{B,A,B}) \circ \alpha_{A,B,A \otimes B}, \tag{2.10} \\
\eta_{A \otimes B} \;&=\; (\eta_A \otimes \eta_B) \circ \lambda^{-1}_{\mathbb{1}}. \tag{2.11}
\end{aligned}
$$

A similar result holds for coalgebra objects and for Frobenius algebra objects in $\mathcal{C}$. Given two homomorphisms of algebra objects $f \colon (A, \mu_A, \eta_A) \to (A', \mu_{A'}, \eta_{A'})$ and $g \colon (B, \mu_B, \eta_B) \to (B', \mu_{B'}, \eta_{B'})$, their tensor product $f \otimes g \colon (A \otimes B, \mu_{A \otimes B}, \eta_{A \otimes B}) \to (A' \otimes B', \mu_{A' \otimes B'}, \eta_{A' \otimes B'})$ forms a homomorphism of algebra objects. A similar result holds for homomorphisms of coalgebra and of Frobenius algebra objects.

The following definition plays a central role for the structure of open-closed TQFTs.

Definition 2.3. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \tau)$ be a symmetric monoidal category. A *knowledgeable Frobenius algebra* $\mathbb{A} = (A, C, \imath, \imath^*)$ in $\mathcal{C}$ consists of,

- a symmetric Frobenius algebra $A = (A, \mu_A, \eta_A, \Delta_A, \varepsilon_A)$,

- a commutative Frobenius algebra $C = (C, \mu_C, \eta_C, \Delta_C, \varepsilon_C)$,

- morphisms $\imath \colon C \to A$ and $\imath^* \colon A \to C$ of $\mathcal{C}$,

such that $\imath \colon C \to A$ is a homomorphism of algebra objects in $\mathcal{C}$ and,

$$
\begin{aligned}
\mu_A \circ (\imath \otimes \mathrm{id}_A) &= \mu_A \circ \tau_{A,A} \circ (\imath \otimes \mathrm{id}_A) && \text{(knowledge)}, \tag{2.12} \\
\varepsilon_C \circ \mu_C \circ (\mathrm{id}_C \otimes \imath^*) &= \varepsilon_A \circ \mu_A \circ (\imath \otimes \mathrm{id}_A) && \text{(duality)}, \tag{2.13} \\
\mu_A \circ \tau_{A,A} \circ \Delta_A &= \imath \circ \imath^* && \text{(Cardy condition)}. \tag{2.14}
\end{aligned}
$$

Condition (2.13) says that $\imath^*$ is the morphism dual to $\imath$. Together with the fact that $\imath$ is an algebra homomorphism, this implies that $\imath^* \colon A \to C$ is a homomorphism of coalgebras in $\mathcal{C}$.

If $\mathcal{C} = \mathbf{Vect}_k$, the condition (2.12) states that the image of C under $\imath$ is contained in the centre of A, $\imath(C) \subseteq Z(A)$. The name *knowledgeable Frobenius algebra* is meant to indicate that the symmetric Frobenius algebra A knows something about its centre. This is specified precisely by C, $\imath$ and $\imath^*$. Notice that the centre $Z(A)$ itself cannot be characterized[4] by requiring the commutativity of diagrams labeled by objects and morphisms of $\mathcal{C}$.

It is not difficult to see that every semisimple algebra A can be equipped with the structure of a Frobenius algebra such that $(A, Z(A), \imath, \imath^*)$ forms a knowledgeable Frobenius algebra with the inclusion $\imath \colon Z(A) \to A$ and an appropriately chosen Frobenius algebra structure on $Z(A)$.

Definition 2.4. A *homomorphism of knowledgeable Frobenius algebras*

$$
\varphi \colon (A, C, \imath, \imath^*) \to (A', C', \imath', \imath'^*) \tag{2.15}
$$

in the symmetric monoidal category $\mathcal{C}$ is a pair $\varphi = (\varphi_1, \varphi_2)$ of Frobenius algebra homomorphisms $\varphi_1 \colon A \to A'$ and $\varphi_2 \colon C \to C'$ such that

$$
\begin{array}{ccc}
C \xrightarrow{\ \varphi_2\ } C' & & A \xrightarrow{\ \varphi_1\ } A' \\
\ \ \downarrow{\scriptstyle \imath} \qquad \downarrow{\scriptstyle \imath'} & \text{and} & \ \ \downarrow{\scriptstyle \imath^*} \qquad \downarrow{\scriptstyle \imath'^*} \\
A \xrightarrow{\ \varphi_1\ } A' & & C \xrightarrow{\ \varphi_2\ } C'
\end{array} \tag{2.16}
$$

commute.

---

[4] We thank James Dolan and John Baez for pointing this out.

Definition 2.5. Let $\mathcal{C}$ be a symmetric monoidal category. By **K-Frob**$(\mathcal{C})$ we denote the category of knowledgeable Frobenius algebras in $\mathcal{C}$ and their homomorphisms.

Proposition 2.6. Let $\mathcal{C}$ be a symmetric monoidal category. The category **K-Frob**$(\mathcal{C})$ forms a symmetric monoidal category as follows. The tensor product of two knowledgeable Frobenius algebra objects $\mathbb{A} = (A, C, \imath, \imath^*)$ and $\mathbb{A}' = (A', C', \imath', \imath'^*)$ is defined as $\mathbb{A} \otimes \mathbb{A}' := (A \otimes A', C \otimes C', \imath \otimes \imath', \imath^* \otimes \imath'^*)$. The unit object is given by $\underline{\mathbb{1}} := (\mathbb{1}, \mathbb{1}, \mathrm{id}_{\mathbb{1}}, \mathrm{id}_{\mathbb{1}})$, and the associativity and unit constraints and the symmetric braiding are induced by those of $\mathcal{C}$. Given two homomorphisms $\varphi = (\varphi_1, \varphi_2)$ and $\psi = (\psi_1, \psi_2)$ of knowledgeable Frobenius algebras, their tensor product is defined as $\varphi \otimes \psi := (\varphi_1 \otimes \psi_1, \varphi_2 \otimes \psi_2)$.

2.2 The category Th(K-Frob)

In this section, we define the category **Th(K-Frob)**, called the *theory of knowledgeable Frobenius algebras*. The description that follows is designed to make **Th(K-Frob)** the symmetric monoidal category freely generated by a knowledgeable Frobenius algebra, and the terminology 'theory of ...' indicates that knowledgeable Frobenius algebras in any symmetric monoidal category $\mathcal{C}$ arise precisely as the symmetric monoidal functors **Th(K-Frob)** $\to \mathcal{C}$. This is in analogy to the theory of algebraic theories in which one uses 'with finite products' rather than 'symmetric monoidal'. Readers who are interested in the topology of open-closed cobordisms rather than in the abstract description of knowledgeable Frobenius algebras may wish to briefly look at Proposition 2.8 and then directly proceed to Section 3.

The subsequent definition follows the construction of the 'free category with group structure' given by Laplaza [18]. It forms an example of a structure slightly more general than an operad or a PROP.

Definition 2.7. The category **Th(K-Frob)** is defined as follows. Its objects are the elements of the free $\{\mathbb{1}, \otimes\}$-algebra over the two element set $\{A, C\}$. These are words of a formal language that are defined by the following requirements,

- The symbols $\mathbb{1}$, A and C are objects of **Th(K-Frob)**.

- If X and Y are objects of **Th(K-Frob)**, then $(X \otimes Y)$ is an object of **Th(K-Frob)**.

We now describe the edges of a graph $\mathcal{G}$ whose vertices are the objects of **Th(K-Frob)**. There are edges,

$$
\begin{aligned}
\mu_A &: A \otimes A \to A, & \eta_A &: \mathbb{1} \to A, & \Delta_A &: A \to A \otimes A, & \varepsilon_A &: \mathbb{1} \to A, \\
\mu_C &: C \otimes C \to C, & \eta_C &: \mathbb{1} \to C, & \Delta_C &: C \to C \otimes C, & \varepsilon_C &: \mathbb{1} \to C, \\
& & \imath &: C \to A, & \imath^* &: A \to C,
\end{aligned} \tag{2.17}
$$

and for all objects X,Y,Z there are to be edges

$$
\begin{aligned}
\alpha_{X,Y,Z} &: (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z), & \tau_{X,Y} &: X \otimes Y \to Y \otimes X, \\
\lambda_X &: \mathbb{1} \otimes X \to X, & \rho_X &: X \otimes \mathbb{1} \to X, \\
\end{aligned} \tag{2.18}
$$

$$
\begin{aligned}
\bar{\alpha}_{X,Y,Z} &: X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z, & \bar{\tau}_{X,Y} &: Y \otimes X \to X \otimes Y, \\
\bar{\lambda}_X &: X \to \mathbb{1} \otimes X, & \bar{\rho}_X &: X \to X \otimes \mathbb{1}. \\
\end{aligned} \tag{2.19}
$$

For every edge $f \colon X \to Y$ and for every object Z, there are to be edges $Z \otimes f \colon Z \otimes X \to Z \otimes Y$, $f \otimes Z \colon X \otimes Z \to Y \otimes Z$. These edges are to be interpreted as words in a formal language and are considered distinct if they have distinct names.

Let $\mathcal{H}$ be the category freely generated by the graph $\mathcal{G}$. We now describe a congruence on the category $\mathcal{H}$. We define a relation $\sim$ as follows. We require the relations making $(A, \mu_A, \eta_A, \Delta_A, \varepsilon_A)$ a symmetric Frobenius algebra object, those making $(C, \mu_C, \eta_C, \Delta_C, \varepsilon_C)$ a commutative Frobenius algebra object, those making $\imath \colon C \to A$ an algebra homomorphism as well as (2.12), (2.13),

and (2.14). The relations making $\alpha_{X,Y,Z}$, λ_X, and ρ_X satisfy the pentagon and triangle axioms of a monoidal category as well as those making $\tau_{X,Y}$ a symmetric braiding, are required for all objects X,Y,Z. We also require the following relations for all objects X,Y and morphisms p,q,t,s of $\mathcal{H}$,

$$(X \otimes p)(X \otimes q) \sim X \otimes (pq), \qquad (p \otimes X)(q \otimes X) \sim (pq) \otimes X,$$
$$(t \otimes Y)(X \otimes s) \sim (X \otimes s)(t \otimes Y), \quad \mathrm{id}_{X \otimes Y} \sim X \otimes \mathrm{id}_Y \sim \mathrm{id}_X \otimes Y, \tag{2.20}$$

that make $\otimes$ a functor. Then we require the relations that assert the naturality of $\alpha, \lambda, \rho, \tau, \bar{\alpha}, \bar{\lambda}, \bar{\rho}, \bar{\tau}$ and that each pair e and $\bar{e}$ of edges of the graph form the inverses of each other. Finally, we have all expansions by $\otimes$, *i.e.* for each relation $a \sim b$, we include the relations $a \otimes X \sim b \otimes X$ and $X \otimes a \sim X \otimes b$ for all objects X, and all those relations obtained from these by a finite number of applications of this process. The category $\mathbf{Th}(\mathbf{K\text{-}Frob})$ is the category $\mathcal{H}$ modulo the category congruence generated by $\sim$.

It is clear from the description above that $\mathbf{Th}(\mathbf{K\text{-}Frob})$ contains a knowledgeable Frobenius algebra object $(A, C, \imath, \imath^*)$ which we call the knowledgeable Frobenius algebra object *generating* $\mathbf{Th}(\mathbf{K\text{-}Frob})$. Indeed, $\mathbf{Th}(\mathbf{K\text{-}Frob})$ is the symmetric monoidal category freely generated by a knowledgeable Frobenius algebra. Its basic property is that for any knowledgeable Frobenius algebra $\mathbb{A}' = (A', C', \imath', \imath'^*)$ in $\mathcal{C}$, there is exactly one *strict* symmetric monoidal functor $F_{\mathbb{A}'} \colon \mathbf{Th}(\mathbf{K\text{-}Frob}) \to \mathcal{C}$ which maps $(A, C, \imath, \imath^*)$ to $(A', C', \imath', \imath'^*)$.

An interesting question to ask is whether or not homomorphisms of knowledgeable Frobenius algebras are induced in some way by $\mathbf{Th}(\mathbf{K\text{-}Frob})$. This question is answered by the following proposition.

Proposition 2.8. Let $\mathcal{C}$ be a symmetric monoidal category. The category

$$\mathbf{Symm\text{-}Mon}(\mathbf{Th}(\mathbf{K\text{-}Frob}), \mathcal{C}) \tag{2.21}$$

of symmetric monoidal functors $\mathbf{Th}(\mathbf{K\text{-}Frob}) \to \mathcal{C}$ and their monoidal natural transformations is equivalent as a symmetric monoidal category to the category $\mathbf{K\text{-}Frob}(\mathcal{C})$.

This proposition is the reason for calling $\mathbf{Th}(\mathbf{K\text{-}Frob})$ the *theory of knowledgeable Frobenius algebras*. For easier reference, we collect the definitions of symmetric monoidal functors, monoidal natural transformations and of $\mathbf{Symm\text{-}Mon}$ in Appendix A.

Proof. Let $(A, C, \imath, \imath^*)$ be the knowledgeable Frobenius algebra generating $\mathbf{Th}(\mathbf{K\text{-}Frob})$, and let $\psi \colon \mathbf{Th}(\mathbf{K\text{-}Frob}) \to \mathcal{C}$ be a symmetric monoidal functor. It is clear that the image of $(A, C, \imath, \imath^*)$ under ψ, together with the coherence isomorphisms ψ_0 and ψ_2 of $\psi = (\psi, \psi_2, \psi_0)$, defines a knowledgeable Frobenius algebra $(\psi(A), \psi(C), \psi(\imath), \psi(\imath^*))$ in $\mathcal{C}$. The symmetric Frobenius algebra structure on $\psi(A)$ is given by

$$\psi(A) = \left(\psi(A), \psi(\mu_A) \circ \psi_2, \psi(\eta_A) \circ \psi_0, \psi_2^{-1} \circ \psi(\Delta_A), \psi_0^{-1} \circ \psi(\varepsilon_A) \right). \tag{2.22}$$

The commutative Frobenius algebra structure on $\psi(C)$ is given by

$$\psi(C) = \left(\psi(C), \psi(\mu_C) \circ \psi_2, \psi(\eta_C) \circ \psi_0, \psi_2^{-1} \circ \psi(\Delta_C), \psi_0^{-1} \circ \psi(\varepsilon_C) \right). \tag{2.23}$$

This defines a mapping on objects

$$\Gamma \colon \mathbf{Symm\text{-}Mon}(\mathbf{Th}(\mathbf{K\text{-}Frob}), \mathcal{C}) \quad \to \quad \mathbf{K\text{-}Frob}(\mathcal{C}) \tag{2.24}$$
$$\psi \quad \mapsto \quad (\psi(A), \psi(C), \psi(\imath), \psi(\imath^*)).$$

We now extend Γ to a functor by defining it on morphisms.

If $\varphi \colon \psi \Rightarrow \psi'$ is a monoidal natural transformation, then φ assigns to each object X in $\mathbf{Th}(\mathbf{K\text{-}Frob})$ a map $\varphi_X \colon \psi(X) \to \psi'(X)$ in $\mathcal{C}$. However, since every object in $\mathbf{Th}(\mathbf{K\text{-}Frob})$ is the tensor product of A's and C's and $\mathbb{1}$'s, the fact that φ is a *monoidal* natural transformation means that the φ_X are completely determined by two maps $\varphi_1 \colon \psi(A) \to \psi'(A)$ and $\varphi_2 \colon \psi(C) \to \psi'(C)$.

The *naturality* of φ means that the φ_i are compatible with the images of all the morphisms in **Th(K-Frob)**. Since all of the morphisms in **Th(K-Frob)** are built up from the generators:

$$\mu_A \colon A \otimes A \to A, \quad \eta_A \colon \mathbb{1} \to A, \quad \Delta_A \colon A \to A \otimes A, \quad \varepsilon_A \colon \mathbb{1} \to A,$$
$$\mu_C \colon C \otimes C \to C, \quad \eta_C \colon \mathbb{1} \to C, \quad \Delta_C \colon C \to C \otimes C, \quad \varepsilon_C \colon \mathbb{1} \to C, \tag{2.25}$$
$$\imath \colon C \to A, \qquad \imath^* \colon A \to C,$$

(and the structure maps α, ρ, λ, τ), naturality can be expressed by the commutativity of 10 diagrams involving the 10 generating morphisms of **Th(K-Frob)**. For example, corresponding to $\mu_A \colon A \otimes A \to A$ and $\eta_A \colon \mathbb{1} \to A$, we have the two diagrams:

$$
\begin{array}{ccc}
\psi(A \otimes A) \xrightarrow{\psi_2^{-1}} \psi(A) \otimes \psi(A) \xrightarrow{\varphi_1 \otimes \varphi_1} \psi'(A) \otimes \psi'(A) \xrightarrow{\psi_2'^{-1}} \psi'(A \otimes A) & \\[4pt]
\psi(\mu_A) \downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \psi'(\mu_A) & (2.26) \\[4pt]
\psi(A) \xrightarrow{\qquad\qquad\qquad\qquad \varphi_1 \qquad\qquad\qquad\qquad} \psi'(A) &
\end{array}
$$

$$
\begin{array}{ccc}
\psi(\mathbb{1}) \xrightarrow{\psi_0^{-1}} \mathbb{1} \xrightarrow{\psi_0'} \psi'(\mathbb{1}) & \\[4pt]
\psi(\eta_A) \downarrow \qquad\qquad\qquad \downarrow \psi'(\eta_A) & (2.27) \\[4pt]
\psi(A) \xrightarrow{\quad \varphi_1 \quad} \psi'(A) &
\end{array}
$$

which amounts to saying that φ_1 is an algebra homomorphism $\psi(A) \to \psi'(A)$. Together with the conditions for the generators $\Delta_A \colon A \to A \otimes A$ and $\varepsilon_A \colon A \to \mathbb{1}$, we have that φ_1 is a Frobenius algebra homomorphism from $\psi(A)$ to $\psi'(A)$. Similarly, the diagrams corresponding to the generators with a C subscript imply that φ_2 is a Frobenius algebra homomorphism $\psi(C) \to \psi'(C)$. The conditions on the images of the generators $\imath \colon C \to A$ and $\imath^* \colon A \to C$ produce the requirement that the two diagrams:

$$
\begin{array}{ccccc}
\psi(C) \xrightarrow{\varphi_2} \psi'(C) & & & \psi(A) \xrightarrow{\varphi_1} \psi'(A) & \\[4pt]
\psi(\imath) \downarrow \qquad \downarrow \psi'(\imath) & & \text{and} & \psi(\imath^*) \downarrow \qquad \downarrow \psi'(\imath^*) & (2.28) \\[4pt]
\psi(A) \xrightarrow{\varphi_1} \psi'(A) & & & \psi(C) \xrightarrow{\varphi_2} \psi'(C) &
\end{array}
$$

commute. Hence, the monoidal natural transformation φ defines a morphism of knowledgeable Frobenius algebras in $\mathcal{C}$. This assignment clearly preserves the monoidal structure and symmetry up to isomorphism. Thus, it is clear that one can define a symmetric monoidal functor $\Gamma = (\Gamma, \Gamma_2, \Gamma_0) \colon \mathbf{Symm\text{-}Mon}(\mathbf{Th}(\mathbf{K\text{-}Frob})) \to \mathbf{K\text{-}Frob}(\mathcal{C})$.

Conversely, given any knowledgeable Frobenius algebra $\mathbb{A}' = (A', C', \imath', \imath'^*)$ in $\mathcal{C}$, then by the remarks preceding this proposition, there is an assignment

$$\overline{\Gamma} \colon \mathbf{K\text{-}Frob}(\mathcal{C}) \quad \to \quad \mathbf{Symm\text{-}Mon}(\mathbf{Th}(\mathbf{K\text{-}Frob}), \mathcal{C}) \tag{2.29}$$
$$(A', C', \imath', \imath'^*) \quad \mapsto \quad F_{\mathbb{A}'}. \tag{2.30}$$

where $F_{\mathbb{A}'}$ is the strict symmetric monoidal functor sending the knowledgeable Frobenius algebra $(A, C, \imath, \imath^*)$ generating **Th(K-Frob)** to the knowledgeable Frobenius algebra $(A', C', \imath', \imath'^*)$ in $\mathcal{C}$. Furthermore, it is clear from the discussion above that a homomorphism of knowledgeable Frobenius algebras $\varphi \colon \mathbb{A}_1 \to \mathbb{A}_2$ defines a monoidal natural transformation $\varphi \colon F_{\mathbb{A}_1} \to F_{\mathbb{A}_2}$. Thus, it is clear that $\overline{\Gamma}$ extends to a symmetric monoidal functor $\overline{\Gamma} = (\overline{\Gamma}, \overline{\Gamma}_2, \overline{\Gamma}_0) \colon \mathbf{K\text{-}Frob}(\mathcal{C}) \to \mathbf{Symm\text{-}Mon}(\mathbf{Th}(\mathbf{K\text{-}Frob}), \mathcal{C})$.

To see that Γ and $\overline{\Gamma}$ define an equivalence of categories, let $\mathbb{A}' = (A', C', \imath', \imath'^*)$ be a knowledgeable Frobenius algebra in $\mathcal{C}$. The composite $\Gamma\overline{\Gamma}(\mathbb{A}') = \Gamma(F_{\mathbb{A}'}) = \mathbb{A}'$ since $F_{\mathbb{A}'}$ is a strict symmetric

monoidal functor. Hence, $\Gamma\overline{\Gamma} = 1_{\mathbf{Th}(\mathbf{K}\text{-}\mathbf{Frob})}$. Now let $\psi\colon \mathbf{Th}(\mathbf{K}\text{-}\mathbf{Frob}) \to \mathcal{C}$ be a symmetric monoidal functor and consider the composite $\overline{\Gamma}\Gamma(\psi)$. Let $\widetilde{\mathbb{A}} = \big(\psi(A), \psi(\mu_A)\circ\psi_2, \psi(\eta_A)\circ\psi_0, \psi_2^{-1}\circ \psi(\Delta_A), \psi_0^{-1}\circ\psi(\varepsilon_A)\big)$ so that $\overline{\Gamma}\Gamma(\psi) = F_{\widetilde{\mathbb{A}}}$. We define a monoidal natural isomorphism $\vartheta\colon \psi \Rightarrow F_{\widetilde{\mathbb{A}}}$ on the generators as follows:

$$
\begin{aligned}
\vartheta_{\mathbb{1}} &:= \psi_1^{-1}, \\
\vartheta_A &:= 1_A, \\
\vartheta_C &:= 1_C.
\end{aligned}
\tag{2.31}
$$

The condition that ϑ be monoidal implies that $\vartheta_{A\otimes A} = \vartheta_{A\otimes C} = \vartheta_{C\otimes A} = \vartheta_{C\otimes C} = \psi_2^{-1}$. Since $\mathbf{Th}(\mathbf{K}\text{-}\mathbf{Frob})$ is generated by $\mathbb{1}$, A, and C, this assignment uniquely defines a monoidal natural isomorphism. Hence, $\overline{\Gamma}\Gamma(\psi) \cong \psi$ so that $\overline{\Gamma}$ and Γ define a monoidal equivalence of categories. $\qquad\square$

3 The category of open-closed cobordisms

In this section, we define and study the category $\mathbf{2Cob}^{\text{ext}}$ of open-closed cobordisms. Open-closed cobordisms form a special sort of compact smooth 2-manifolds with corners that have a particular global structure. If one decomposes their boundary minus the corners into connection components, these components are either *black* or *coloured* with elements of some given set S. Every corner is required to separate a black boundary component from a coloured one[5].

These 2-manifolds with corners are viewed as cobordisms between their black boundaries, and they can be composed by gluing them along their black boundaries subject to a matching condition for the colours of the other boundary components. In the conformal field theory literature, the coloured boundary components are referred to as *free boundaries* and the colours as *boundary conditions*.

2-manifolds with corners with this sort of global structure form a special case of $\langle 2\rangle$-manifolds according to Jänich [19]. For an overview and a very convenient notation, we refer to the introduction of the article [20] by Laures.

In the following two subsections, we present all definitions for a generic set of colours S. Starting in Subsection 3.3, the generators and relations description of $\mathbf{2Cob}^{\text{ext}}$ is developed only for the case of a single colour, $S = \{*\}$. We finally return to the case of a generic set of colours S in Section 5.

3.1 $\langle 2\rangle$-manifolds

3.1.1 Manifolds with corners

An *n-manifold with corners* M is a topological manifold with boundary that is equipped with a smooth structure with corners. A smooth structure with corners is defined as follows. A *smooth atlas with corners* is a family $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ of coordinate systems such that the $U_\alpha \subseteq M$ are open subsets which cover M, and the

$$
\varphi_\alpha\colon U_\alpha \to \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}_+^n
\tag{3.1}
$$

are homeomorphisms onto open subsets of $\mathbb{R}_+^n := [0,\infty)^n$. The transition functions

$$
\varphi_\beta\circ\varphi_\alpha^{-1}\colon \varphi_\alpha(U_{\alpha\beta}) \to \varphi_\beta(U_{\alpha\beta})
\tag{3.2}
$$

for $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$ are required to be the restrictions to $\mathbb{R}_+^n$ of diffeomorphisms between open subsets of $\mathbb{R}^n$. Two such atlases are considered equivalent if their union is a smooth atlas with corners, and a *smooth structure with corners* is an equivalence class of such atlases.

A *smooth map* $f\colon M \to N$ between manifolds with corners M and N is a continuous map for which the following condition holds. Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ and $\{(V_\beta, \psi_\beta)\}_{\beta\in J}$ be atlases that

---

[5]In this terminology, black is not considered a colour.

represent the smooth structures with corners of M and N, respectively. For every $p \in M$ and for every $\alpha \in I$, $\beta \in J$ with $p \in U_\alpha$ and $f(p) \in V_\beta$, we require that the map

$$\psi_\beta \circ f \circ \varphi_\alpha^{-1}|_{\varphi_\alpha(U_\alpha \cap f^{-1}(V_\beta))} \colon \varphi_\alpha(U_\alpha \cap f^{-1}(V_\beta)) \to \psi_\beta(f(U_\alpha) \cap V_\beta) \tag{3.3}$$

is the restriction to $\mathbb{R}_+^m$ of a smooth map between open subsets of $\mathbb{R}^m$ and $\mathbb{R}^p$ for some m and p.

3.1.2 Manifolds with faces

For each $p \in M$, we define $c(p) \in \mathbb{N}_0$ to be the number of zero coefficients of $\varphi_\alpha(p) \in \mathbb{R}^n$ for some $\alpha \in I$ for which $p \in U_\alpha$. A *connected face* of M is the closure of a component of $\{\, p \in M \colon c(p) = 1 \,\}$. A *face* is a free union of connected faces. This includes the possibility that a face can be empty.

A *manifold with faces* M is a manifold with corners such that each $p \in M$ is contained in $c(p)$ different connected faces. Notice that every face of M is itself a manifold with faces.

3.1.3 $\langle n \rangle$-manifolds

An $\langle n \rangle$-manifold M is a manifold with faces with a specified tuple $(\partial_0 M, \dots, \partial_{n-1} M)$ of faces of M such that the following two conditions hold.

1. $\partial_0 M \cup \cdots \cup \partial_{n-1} M = \partial M$. Here ∂M denotes the boundary of M as a topological manifold.

2. $\partial_j M \cap \partial_k M$ is a face of both $\partial_j M$ and $\partial_k M$ for all $j \neq k$.

Notice that a $\langle 0 \rangle$-manifold is just a manifold without boundary while a $\langle 1 \rangle$-manifold is a manifold with boundary. A diffeomorphism $f \colon M \to N$ between two $\langle n \rangle$-manifolds is a diffeomorphism of the underlying manifolds with corners such that $f(\partial_j M) = \partial_j N$ for all j.

The following notation is taken from Laures [20]. Let $\underline{2}$ denote the category associated with the partially ordered set $\{0, 1\}$, $0 \leq 1$, *i.e.* the category freely generated by the graph $0 \xrightarrow{*} 1$. Denote by $\underline{2}^n$ the n-fold Cartesian product of $\underline{2}$ and equip its set of objects $\{0, 1\}^n$ with the corresponding partial order. An $\langle n \rangle$-*diagram* is a functor $\underline{2}^n \to \mathbf{Top}$. We use the term $\langle n \rangle$-diagram *of inclusions* for an $\langle n \rangle$-diagram which sends each morphism of $\underline{2}^n$ to an inclusion, and so on.

Every $\langle n \rangle$-manifold M gives rise to an $\langle n \rangle$-diagram $M \colon \underline{2}^n \to \mathbf{Top}$ of inclusions as follows. For the objects $a = (a_0, \dots, a_{n-1}) \in |\underline{2}^n|$, write $a' := (1 - a_0, \dots, 1 - a_{n-1})$, and denote the standard basis of $\mathbb{R}^n$ by $(e_0, \dots, e_{n-1})$. The functor $M \colon \underline{2}^n \to \mathbf{Top}$ is defined on the objects by,

$$M(a) := \bigcap_{i \in \{\, i \colon\, a \leq e_i' \,\}} \partial_i M, \tag{3.4}$$

if $a \neq (1, \dots, 1)$, and by $M((1, \dots, 1)) := M$. The functor sends the morphisms of $\underline{2}^n$ to the obvious inclusions.

For all $a \in |\underline{2}^n|$, the face $M(a)$ of M forms a $\langle k \rangle$-manifold itself for which $k = \sum_{i=1}^{n-1} a_i$. An orientation of M induces orientations on the $M(a)$ as usual. The product of an $\langle n \rangle$-manifold M with a $\langle p \rangle$-manifold N forms an $\langle n + p \rangle$-manifold, denoted by $M \times N$. The structure of its faces can be read off from the functor

$$M \times N \colon \underline{2}^{n+p} \simeq \underline{2}^n \times \underline{2}^p \xrightarrow{M \times N} \mathbf{Top} \times \mathbf{Top} \xrightarrow{\times} \mathbf{Top}. \tag{3.5}$$

The half-line $\mathbb{R}_+ := [0, \infty)$ is a manifold with boundary, *i.e.* a $\langle 1 \rangle$-manifold. The product of (3.5) then equips $\mathbb{R}_+^n$ with the structure of an $\langle n \rangle$-manifold.

For $n = 2$, for example, a $\langle 2 \rangle$-manifold M is a manifold with faces with a pair of specified faces $(\partial_0 M, \partial_1 M)$ such that $\partial_0 M \cup \partial_1 M = \partial M$ and $\partial_0 M \cap \partial_1 M$ is a face of both $\partial_0 M$ and $\partial_1 M$. The following diagram shows the faces of one of the typical $\langle 2 \rangle$-manifolds M that are used below.

$$\tag{3.6}$$
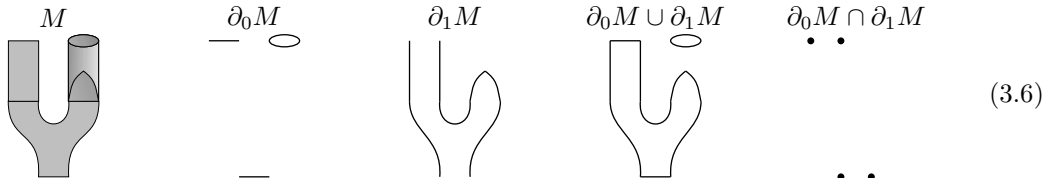
The $\langle 2 \rangle$-diagram $M \colon \underline{2}^2 \to \mathbf{Top}$ of inclusions is the following commutative square:

$$
\begin{array}{ccc}
\partial_0 M \cap \partial_1 M & \xrightarrow{\ M(\mathrm{id}_0 \times *)\ } & \partial_0 M \\
\Big\downarrow{\scriptstyle M(*\times \mathrm{id}_0)} & & \Big\downarrow{\scriptstyle M(*\times \mathrm{id}_1)} \\
\partial_1 M & \xrightarrow[\ M(\mathrm{id}_1 \times *)\ ]{} & M
\end{array}
\tag{3.7}
$$

Another example of a manifold with corners M which is embedded in $\mathbb{R}^3$ is depicted in (1.1). It has the structure of a $\langle 2 \rangle$-manifold when one chooses $\partial_0 M$ to be the union of the top and bottom boundaries of the picture, similarly to (3.6).

3.1.4   Collars and neat embeddings

In order to glue $\langle 2 \rangle$-manifolds along specified faces, we need the following technical results.

Lemma 3.1 (Lemma 2.1.6 of [20]). Each $\langle n \rangle$-manifold M admits an $\langle n \rangle$-diagram C of embeddings

$$
C(a \to b) \colon \mathbb{R}^n_+(a') \times M(a) \hookrightarrow \mathbb{R}^n_+(b') \times M(b)
\tag{3.8}
$$

such that the restriction $C(a \to b)|_{\mathbb{R}^n_+(b') \times M(a)} = \mathrm{id}_{\mathbb{R}^n_+(b')} \times M(a \to b)$ is the inclusion map.

In particular, for every $\langle 2 \rangle$-manifold M, there is a commutative square $C \colon \underline{2}^2 \to \mathbf{Top}$ of embeddings,

$$
\begin{array}{ccc}
\mathbb{R}^2_+ \times (\partial_0 M \cap \partial_1 M) & \xrightarrow{\ C(\mathrm{id}_0 \times *)\ } & \partial_0 \mathbb{R}^2_+ \times \partial_1 M \\
\Big\downarrow{\scriptstyle C(*\times \mathrm{id}_0)} & & \Big\downarrow{\scriptstyle C(*\times \mathrm{id}_1)} \\
\partial_1 \mathbb{R}^2_+ \times \partial_0 M & \xrightarrow[\ C(\mathrm{id}_1 \times *)\ ]{} & \{(0,0)\} \times M
\end{array}
\tag{3.9}
$$

such that the following restrictions are inclusions,

$$
\begin{aligned}
C(\mathrm{id}_0 \times *)|_{\partial_0 \mathbb{R}^2_+ \times \partial_1 M} &= \mathrm{id}_{\partial_0 \mathbb{R}^2_+} \times M(\mathrm{id}_0 \times *), & (3.10) \\
C(* \times \mathrm{id}_0)|_{\partial_1 \mathbb{R}^2_+ \times \partial_0 M} &= \mathrm{id}_{\partial_1 \mathbb{R}^2_+} \times M(* \times \mathrm{id}_0), & (3.11) \\
C(* \times \mathrm{id}_1)|_{\{(0,0)\} \times \partial_1 M} &= \mathrm{id}_{\{(0,0)\}} \times M(* \times \mathrm{id}_1), & (3.12) \\
C(\mathrm{id}_1 \times *)|_{\{(0,0)\} \times \partial_0 M} &= \mathrm{id}_{\{(0,0)\}} \times M(\mathrm{id}_1 \times *). & (3.13)
\end{aligned}
$$

The embedding $C(\mathrm{id}_1 \times *) \colon \partial_1 \mathbb{R}^2_+ \times \partial_0 M \to \{(0,0)\} \times M$ provides us with a diffeomorphism from $([0, \varepsilon] \times \{0\}) \times \partial_0 M \subseteq ([0, \infty) \times \{0\}) \times \partial_0 M = \partial_1 \mathbb{R}^2_+ \times \partial_0 M$ onto a submanifold of $\{(0,0)\} \times M$ for some $\varepsilon > 0$. It restricts to an inclusion on $\{(0,0)\} \times \partial_0 M$ and thereby yields a (smooth) collar neighbourhood for $\partial_0 M$.

Definition 3.2 (Definition 2.1.4 of [20]). A *neat embedding* of an $\langle n \rangle$-manifold M is a natural transformation $\imath \colon M \Rightarrow \mathbb{R}^n_+ \times \mathbb{R}^m$ for some m such that the following conditions hold:

1. $\imath(a)$ is an inclusion of a submanifold for all $a \in |\underline{2}^n|$.

2. The intersection $M(a) \cap (\mathbb{R}^n_+(b) \times \mathbb{R}^m) = M(b)$ are perpendicular for all morphisms $b \to a$ of $\underline{2}^n$.

Proposition 3.3 (Proposition 2.1.7 of [20]). Any compact $\langle n \rangle$-manifold admits a neat embedding in $\mathbb{R}^n_+ \times \mathbb{R}^m$ for some m.

3.2 Open-closed cobordisms

For a topological space M, we denote by $\Pi_0(M)$ the set of its connection components, and for $p \in M$, we note by $[p] \in \Pi_0(M)$ its component.

3.2.1 Cobordisms

Definition 3.4. Let S be some set. An *S-coloured open-closed cobordism* (M, γ) is a compact oriented $\langle 2 \rangle$-manifold M whose distinguished faces we denote by $(\partial_0 M, \partial_1 M)$, together with a map $\gamma \colon \Pi_0(\partial_1 M) \to S$. The face $\partial_0 M$ is called the *black boundary*, $\partial_1 M$ the *coloured boundary*, and γ the *colouring*. An *open-closed cobordism* is an S-coloured open-closed cobordism for which S is a one-element set.

Two S-coloured open-closed cobordisms (M, γ_M) and (N, γ_N) are considered *equivalent* if there is an orientation preserving diffeomorphism of $\langle 2 \rangle$-manifolds $f \colon M \to N$ that restricts to the identity on $\partial_0 M$ and that preserves the colouring, *i.e.* $\gamma_N \circ f = \gamma_M$.

The face $\partial_0 M$ is a compact 1-manifold with boundary and therefore diffeomorphic to a free union of circles S^1 and unit intervals $[0, 1]$. For each of the unit intervals, there is thus an orientation preserving diffeomorphism $\varphi \colon [0, 1] \to \varphi([0, 1]) \subseteq \partial_0 M$ onto a component of $\partial_0 M$ such that the boundary points are mapped to the corners, *i.e.* $\varphi(\{0, 1\}) \subseteq \partial_0 M \cap \partial_1 M$. We say that the cobordism (M, γ) *equips* the unit interval $[0, 1]$ *with the colours* $(\gamma_+, \gamma_-) \in S \times S$ if $\gamma_+ := \gamma([\varphi(1)])$ and $\gamma_- := \gamma([\varphi(0)])$.

3.2.2 Gluing

Let (M, γ_M) and (N, γ_N) be S-coloured open-closed cobordisms and $f \colon S^{1*} \to M$ and $g \colon S^1 \to N$ be orientation preserving diffeomorphisms onto components of $\partial_0 M$ and $\partial_0 N$, respectively. Here we have equipped the circle S^1 with a fixed orientation, and S^{1*} denotes the one with opposite orientation. Then we obtain an S-coloured open-closed cobordism $M \,_f \coprod_g N$ by *gluing M and N along S^1* as follows. As a topological manifold, it is the pushout. As mentioned in Section 3.1, $\partial_0 M$ and thereby all its components have smooth collar neighbourhoods, and so the standard techniques are available to equip $M \,_f \coprod_g N$ with the structure of a manifold with corners whose smooth structure is unique up to a diffeomorphism that restricts to the identity on $\partial_0 M \cup \partial_0 N$. It is obvious that $M \,_f \coprod_g N$ also has the structure of a $\langle 2 \rangle$-manifold with $\partial_1(M \,_f \coprod_g N) = \partial_1 M \cup \partial_1 N$ and, furthermore, that of an S-coloured open-closed cobordism.

Similarly, let $f \colon [0, 1]^* \to M$ and $g \colon [0, 1] \to N$ be orientation preserving diffeomorphisms onto components of $\partial_0 M$ and $\partial_0 N$, respectively, such that (M, γ_M) equips the interval $f([0, 1]^*)$ with the colours $(\gamma_+, \gamma_-) \in S$ and (N, γ_N) equips the interval $g([0, 1])$ precisely with the colours (γ_-, γ_+). Then we obtain the gluing of M and N along $[0, 1]$ again as the pushout $M \,_f \coprod_g N$ equipped with the smooth structure that is unique up to a diffeomorphism which restricts to the identity on $\partial_0 M \cup \partial_0 N$. It is easy to see that $M \,_f \coprod_g N$ also has the structure of a $\langle 2 \rangle$-manifold with $\partial_1(M \,_f \coprod_g N) = \partial_1 M \cup \partial_1 N$ and, moreover, due to the matching of colours, that of an S-coloured open-closed cobordism.

3.2.3 The category $\mathbf{2Cob}^{\text{ext}}(S)$

The following definition of the category of open-closed cobordisms is inspired by that of Baas, Cohen and Ramírez [1]. Note, however, that although our objects are embedded manifolds, our cobordisms are not, and we consider two cobordisms equivalent once they are related by an orientation preserving diffeomorphism that restricts to the identity on their black boundaries.

Definition 3.5. Let S be a set. The category $\mathbf{2Cob}^{\text{ext}}(S)$ is defined as follows. Its objects are triples $(\vec{n}, \gamma_+, \gamma_-)$ consisting of a finite sequence $\vec{n} := (n_1, \ldots, n_k)$, $k \in \mathbb{N}_0$, with $n_j \in \{0, 1\}$, $1 \leq j \leq k$, and maps $\gamma_\pm \colon \{1, \ldots, k\} \to S \cup \{\emptyset\}$ for which $\gamma_\pm(j) \neq \emptyset$ if $n_j = 1$ and $\gamma_\pm(j) = \emptyset$ if

$n_j = 0$[6]. We denote the *length* of such a sequence by $|\vec{n}| := k$.

Each sequence $\vec{n} = (n_1, \ldots, n_k)$ represents the diffeomorphism type of a compact oriented 1-dimensional submanifold of $\mathbb{R}^2$,

$$C_{\vec{n}} := \bigcup_{j=1}^{k} I(j, n_j), \tag{3.14}$$

where $I(j, 0)$ is the circle of radius $1/4$ centred at $(j, 0) \in \mathbb{R}^2$ and $I(j, 1) = [j - 1/4, j + 1/4] \times \{0\}$, both equipped with the induced orientation. Taking the disjoint union of two such manifolds $C_{\vec{n}}$ and $C_{\vec{m}}$ is done as follows,

$$C_{\vec{n}} \coprod C_{\vec{m}} := C_{\vec{n}} \cup T_{(|\vec{n}|, 0)}(C_{\vec{m}}), \tag{3.15}$$

where $T(x, y) \colon \mathbb{R}^2 \to \mathbb{R}^2$ denotes the translation by (x, y) in $\mathbb{R}^2$.

A morphism $f \colon (\vec{n}, \gamma_+, \gamma_-) \to (\vec{n}', \gamma_+', \gamma_-')$ is a pair $f = ([f], \Phi)$ consisting of an equivalence class $[f]$ of S-coloured open-closed cobordisms and a specified orientation preserving diffeomorphism

$$\Phi \colon C_{\vec{n}}^* \coprod C_{\vec{n}'} \to \partial_0 f, \tag{3.16}$$

such that the following conditions hold:

1. For each $j \in \{1, \ldots, |\vec{n}|\}$ for which $n_j = 1$, the S-coloured open-closed cobordism (f, γ) representing $[f]$ equips the corresponding unit interval $\Phi(I(j, n_j))$ with the colours $(\gamma_+(j), \gamma_-(j))$.

2. For each $j \in \{1, \ldots, |\vec{n}'|\}$ for which $n_j' = 1$, (f, γ) equips the corresponding unit interval $\Phi(I(j, n_j'))$ with the colours $(\gamma_+'(j), \gamma_-'(j))$.

The composition $g \circ f$ of two morphisms $f = ([f], \Phi_f) \colon (\vec{n}, \gamma_+, \gamma_-) \to (\vec{n}', \gamma_+', \gamma_-')$ and $g = ([g], \Phi_g) \colon (\vec{n}', \gamma_+', \gamma_-') \to (\vec{n}'', \gamma_+'', \gamma_-'')$ is defined as $g \circ f := ([g \coprod_{C_{\vec{n}'}} f], \Phi_{g \circ f})$. Here $[g \coprod_{C_{\vec{n}'}} f]$ is the equivalence class of the S-coloured open-closed cobordism $g \coprod_{C_{\vec{n}'}} f$ which is obtained by successively gluing f and g along all the components of $C_{\vec{n}'}$. $\Phi_{g \circ f} \colon C_{\vec{n}}^* \coprod C_{\vec{n}''} \to \partial_0(g \coprod_{C_{\vec{n}'}} f)$ is the obvious orientation preserving diffeomorphism obtained from restricting $\Phi_f \colon C_{\vec{n}}^* \coprod C_{\vec{n}'} \to \partial_0 f$ and $\Phi_g \colon C_{\vec{n}'}^* \coprod C_{\vec{n}''} \to \partial_0 g$.

For any object $(\vec{n}, \gamma_+, \gamma_-)$, the cylinder $\mathrm{id}_{(\vec{n}, \gamma_+, \gamma_-)} := [0, 1] \times C_{\vec{n}}$ forms an S-coloured open-closed cobordism such that $\partial_0 \mathrm{id}_{(\vec{n}, \gamma_+, \gamma_-)} = C_{\vec{n}}^* \coprod C_{\vec{n}}$. It plays the role of the identity morphism.

The category $\mathbf{2Cob}^{\mathrm{ext}}$ is defined as the category $\mathbf{2Cob}^{\mathrm{ext}}(S)$ for the singleton set $S = \{*\}$. When we describe the objects of $\mathbf{2Cob}^{\mathrm{ext}}$, we can suppress the γ_+ and γ_- and simply write the sequences $\vec{n} = (n_1, \ldots, n_{|\vec{n}|})$.

Examples of morphisms of $\mathbf{2Cob}^{\mathrm{ext}}$ are depicted here,

 $: (1, 1) \to (1)$, $: (0) \to (0, 0)$, $: (0) \to (1)$. $\tag{3.17}$

In these pictures, the source of the cobordism is drawn at the top and the target at the bottom. The morphism depicted in (1.1) goes from $(1, 0, 1, 1, 1)$ to $(0, 1, 1, 0, 0)$.

The concatenation $\vec{n} \coprod \vec{m} := (n_1, \ldots, n_{|\vec{n}|}, m_1, \ldots, m_{|\vec{m}|})$ of sequences together with the free union of S-coloured open-closed cobordisms, also denoted by $\coprod$, provides the category $\mathbf{2Cob}^{\mathrm{ext}}(S)$ with the structure of a strict symmetric monoidal category.

Let $k \in \mathbb{N}$. The symmetric group $\mathcal{S}_k$ acts on the subset of objects $(\vec{n}, \gamma_+, \gamma_-) \in |\mathbf{2Cob}^{\mathrm{ext}}(S)|$ for which $|\vec{n}| = k$. This action is defined by,

$$\sigma \triangleright (\vec{n}, \gamma_+, \gamma_-) := ((n_{\sigma^{-1}(1)}, \ldots, n_{\sigma^{-1}(|\vec{n}|)}), \gamma_+ \circ \sigma^{-1}, \gamma_- \circ \sigma^{-1}). \tag{3.18}$$

For each object $(\vec{n}, \gamma_+, \gamma_-) \in |\mathbf{2Cob}^{\mathrm{ext}}(S)|$ and any permutation $\sigma \in \mathcal{S}_{|\vec{n}|}$, we define a morphism,

$$\sigma^{(\vec{n}, \gamma_+, \gamma_-)} \colon (\vec{n}, \gamma_+, \gamma_-) \to \sigma \triangleright (\vec{n}, \gamma_+, \gamma_-), \tag{3.19}$$

---

[6]This is done simply because the values $\gamma_{\pm}(j)$ are never used if $n_j = 0$, but we nevertheless want to keep the indices j in line with those of the sequence $\vec{n}$.

by taking the underlying S-coloured open-closed cobordism of the cylinder $\mathrm{id}_{(\vec{n},\gamma_+,\gamma_-)}$, and replacing the orientation preserving diffeomorphism

$$\Phi\colon C_{\vec{n}}^* \coprod C_{\vec{n}} \to \partial_0 \mathrm{id}_{(\vec{n},\gamma_+,\gamma_-)} \tag{3.20}$$

by one that has the components of the $C_{\vec{n}}$ for the target permuted accordingly. For example, for $S = \{*\}$, $\vec{n} = (1,0,0,1)$ and $\sigma = (234) \in \mathcal{S}_4$ in cycle notation, we obtain the morphism $\sigma^{(\vec{n})}$ that is depicted in (3.91) below. As morphisms of $\mathbf{2Cob}^{\mathrm{ext}}(S)$, *i.e.* up to the appropriate diffeomorphism, these cobordisms satisfy,

$$\tau^{\sigma \triangleright (\vec{n},\gamma_+,\gamma_-)} \circ \sigma^{(\vec{n},\gamma_+,\gamma_-)} = (\tau \circ \sigma)^{(\vec{n},\gamma_+,\gamma_-)}. \tag{3.21}$$

If the source of $\sigma^{(\vec{n},\gamma_+,\gamma_-)}$ is obvious from the context, we just write σ.

3.2.4 Invariants for open-closed cobordisms

In order to characterize the S-coloured open-closed cobordisms of $\mathbf{2Cob}^{\mathrm{ext}}(S)$ topologically, we need the following invariants. The terminology is taken from Baas, Cohen, and Ramírez [1].

Definition 3.6. Let S be a set and $[f] \in \mathbf{2Cob}^{\mathrm{ext}}(S)[(\vec{n},\gamma_+,\gamma_-),(\vec{n}',\gamma_+',\gamma_-')]$ be a morphism of $\mathbf{2Cob}^{\mathrm{ext}}(S)$ from $(\vec{n},\gamma_+,\gamma_-)$ to $(\vec{n}',\gamma_+',\gamma_-')$.

1. The *genus* $g(f)$ is defined to be the genus of the topological 2-manifold underlying f.

2. The *window number* of f is a map $\omega(f)\colon S \to \mathbb{N}_0$ such that $\omega(f)(s)$ is the number of components of the face $\partial_1 f$ that are diffeomorphic to S^1 and that are equipped by $\gamma\colon \Pi_0(\partial_1 f) \to S$ with the colour $s \in S$. In the case $S = \{*\}$, we write $\omega(f) \in \mathbb{N}_0$ instead of $\omega(f)(*)$.

3. Let k be the number of coefficients of $\vec{n} \coprod \vec{n}'$ that are 1, *i.e.* the number of components of the face $\partial_0 f$ that are diffeomorphic to the unit interval. Number these components by $1,\ldots,k$. The *open boundary permutation* $(\sigma(f),\gamma_\partial(f))$ of f consists of a permutation $\sigma(f) \in \mathcal{S}_k$ and a map $\gamma_\partial(f)\colon \{1,\ldots,k\} \to S$. We define $\sigma(f)$ as a product of disjoint cycles as follows. Consider every component X of the boundary ∂f of f viewed as a topological manifold, for which $X \cap \partial_0 f \cap \partial_1 f \neq \emptyset$. These are precisely the components that contain a corner of f. The orientation of f induces an orientation of X and thereby defines a cycle $(i_1,\ldots,i_\ell)$ where the $i_j \in \{1,\ldots,k\}$ are the numbers of the intervals of $\partial_0 f$ that are contained in X. The permutation $\sigma(f)$ is the product of these cycles for all such components X. The map $\gamma_\partial(f)\colon \{1,\ldots,k\} \to S$ is defined such that the S-coloured open-closed cobordism (f,γ) representing $[f]$ equips the interval with the number $j \in \{1,\ldots,k\}$ with the colours $(\gamma_\partial(j),\gamma_\partial(\sigma^{-1}(j)))$.

For example, the morphism depicted in (1.1) has 6 components of its black boundary diffeomorphic to the unit interval. Its open boundary permutation is $\sigma(f) = (256)(34) \in \mathcal{S}_6$ if one numbers the intervals in the source (top of the diagram) from left to right by $1,2,3,4$ and those in the target (bottom of the diagram) from left to right by $5,6$.

3.3 Generators

Beginning with this subsection, we restrict ourselves to the case of $\mathbf{2Cob}^{\mathrm{ext}}$ in which there is only one boundary colour, *i.e.* $S = \{*\}$. We use a generalization of Morse theory to manifolds with corners in order to decompose each open-closed cobordism into a composition of open-closed cobordisms each of which contains precisely one critical point. The components of these form the *generators* for the morphisms of the category $\mathbf{2Cob}^{\mathrm{ext}}$.

For the generalization of Morse theory to manifolds with corners, we follow Braess [21]. We first summarize the key definitions and results.

We need a notion of tangent space for a point $p \in \partial M$ if M is a manifold with corners. Every $p \in M$ has a neighbourhood $U \subseteq M$ which forms a submanifold of M and for which

there is a diffeomorphism $\varphi\colon U \to \varphi(U)$ onto an open subset $\varphi(U) \subseteq \mathbb{R}^n_+$. Using the fact that $\varphi(p) \in \mathbb{R}^n_+ \subseteq \mathbb{R}^n$, we define the tangent space of p in M as $T_pM := d\varphi_p^{-1}(T_{\varphi(p)}\mathbb{R}^n)$, *i.e.* just identifying it with that of $\varphi(p)$ in $\mathbb{R}^n$ via the isomorphism $d\varphi_p^{-1}$.

Definition 3.7. Let M be a manifold with corners.

1. For each $p \in M$, we define the *inwards pointing tangential cone* $C_pM \subseteq T_pM$ as the set of all tangent vectors $v \in T_pM$ for which there exists a smooth path $\gamma\colon [0, \varepsilon] \to M$ for some $\varepsilon > 0$ such that $\gamma(0) = p$ and the one-sided derivative is:

$$\lim_{t \to 0+} (\gamma(t) - \gamma(0))/t = v. \tag{3.22}$$

2. Let $f\colon M \to \mathbb{R}$ be smooth. A point $p \in M$ is called a *critical point* and $f(p) \in \mathbb{R}$ its *critical value* if the restriction of the derivative $df_p\colon T_pM \to \mathbb{R}$ to the inwards pointing tangential cone is not surjective, *i.e.* if

$$df_p(C_pM) \neq \mathbb{R}. \tag{3.23}$$

   The point $p \in M$ is called $(+)$-*critical* if $df_p(C_pM) \subseteq \mathbb{R}_+$ and it is called $(-)$-*critical* if $df_p(C_pM) \subseteq \mathbb{R}_- := -\mathbb{R}_+$.

Note that $df_p\colon T_pM \to \mathbb{R}$ is linear and therefore maps cones to cones, and so $df_p(C_pM)$ is either $\{0\}$, $\mathbb{R}_+$, $\mathbb{R}_-$, or $\mathbb{R}$. If $p \in M$ is a critical point, then $df_p(C_pM)$ is either $\{0\}$, $\mathbb{R}_+$, or $\mathbb{R}_-$. If $p \in M\backslash\partial M$, then $C_pM = T_pM$, and so p is critical if and only if $df_p = 0$. If $p \in \partial M$, $c(p) = 1$, and p is critical, then the restriction of f to ∂M has vanishing derivative, *i.e.* $d(f|_{\partial M})_p = 0$.

Definition 3.8. Let M be a manifold with corners and $f\colon M \to \mathbb{R}$ be a smooth function.

1. A critical point $p \in M$ of f is called *non-degenerate* if the Hessian of f at p, restricted to the kernel of df_p, has full rank, *i.e.* if

$$\det \mathrm{Hess}_p(f)|_{\ker df_p \otimes \ker df_p} \neq 0. \tag{3.24}$$

2. The function f is called a *Morse function* if all its critical point are non-degenerate.

Note that if $p \in M\backslash\partial M$, then the notion of non-degeneracy is as usual. If $p \in \partial M$, $c(p) = 1$, and p is a non-degenerate critical point, then p is a non-degenerate critical point of the restriction $f|_{\partial M}\colon \partial M \to \mathbb{R}$ in the usual sense. All non-degenerate critical points are isolated [21].

For our open-closed cobordisms, we need a special sort of Morse functions that are compatible with the global structure of the cobordisms.

Definition 3.9. Let $M \in \mathbf{2Cob}^{\mathrm{ext}}[\vec{n}, \vec{m}]$ be an open-closed cobordism with source $C_{\vec{n}}$ and target $C_{\vec{m}}$. Here we have suppressed the diffeomorphisms from $C_{\vec{n}}$ onto a component of ∂_0M, *etc.*, and we write M for any representative of its equivalence class. A *special Morse function* for M is a Morse function $f\colon M \to \mathbb{R}$ such that the following conditions hold:

1. $f(M) \subseteq [0, 1]$.

2. $f(p) = 0$ if and only if $p \in C_{\vec{n}}$, and $f(p) = 1$ if and only if $p \in C_{\vec{m}}$.

3. Neither $C_{\vec{n}}$ nor $C_{\vec{m}}$ contain any critical points.

4. The critical points of f have pairwise distinct critical values.

Using the standard techniques, one shows that every open-closed cobordism $M \in \mathbf{2Cob}^{\mathrm{ext}}[\vec{n}, \vec{m}]$ admits a special Morse function $f\colon M \to \mathbb{R}$. Since M is compact and since all non-degenerate critical points are isolated, the set of critical points of f is a finite set. If neither $a \in \mathbb{R}$ nor $b \in \mathbb{R}$ are critical values of f, the pre-image $N := f^{-1}([a, b])$ forms an open-closed cobordism with $\partial_0N = f^{-1}(\{a, b\})$ and $\partial_1N = \partial_1M \cap N$. If $[a, b]$ does not contain any critical value of f, then $f^{-1}([a, b])$ is diffeomorphic to the cylinder $f^{-1}(\{a\}) \times [0, 1]$.

The following proposition classifies in terms of Morse data the non-degenerate critical points that can occur on open-closed cobordisms.

Proposition 3.10. Let $M \in \mathbf{2Cob}^{\mathrm{ext}}[\vec{n}, \vec{m}]$ be a connected open-closed cobordism and $f \colon M \to \mathbb{R}$ a special Morse function such that f has precisely one critical point. Then M is diffeomorphic to one of the following open-closed cobordisms:

$$\mu_A \qquad\qquad \Delta_A \qquad \eta_A \quad \varepsilon_A \qquad \mu_C \qquad\qquad \Delta_C \qquad \eta_C \quad \varepsilon_C \quad \imath \qquad \imath^* \tag{3.25}$$
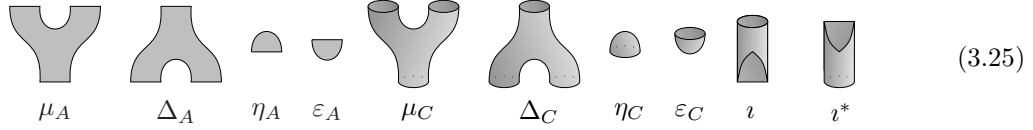
These diagrams show open-closed cobordisms embedded in $\mathbb{R}^3$ and are drawn in such a way that the vertical axis of the drawing plane is $-f$. The source is at the top, and the target at the bottom of the diagram.

Proof. We analyze the properties of the non-degenerate critical point $p \in M$ case by case.

1. If $p \in M \backslash \partial M$, then the critical point is characterized by its index $i(p)$ (the number of negative eigenvalues of $\mathrm{Hess}_p(f)$) as usual; see, for example [22]. There exists a neighbourhood $U \subseteq M$ of p and a coordinate system $x \colon U \to \mathbb{R}^2$ in which the Morse function has the normal form,

$$f(p) = -\sum_{j=1}^{i(p)} x_j^2(p) + \sum_{j=i(p)+1}^{2} x_j^2(p) \tag{3.26}$$

   for all $p \in U$.

   (a) If the index is $i(p) = 2$, then the Morse function has a maximum at p, and so the neighbourhood (and thereby the entire open-closed cobordism) is diffeomorphic to ε_C of (3.25). Recall that the vertical coordinate of our diagrams is $-f$ rather than $+f$.

   (b) If the index is $i(p) = 1$, then f has a saddle point, and the usual argument shows that the possible cases are either μ_C or Δ_C of (3.25).

   (c) If $i(p) = 0$, then f has a minimum, and the cobordism is diffeomorphic to η_C of (3.25).

2. Otherwise, $p \in \partial_1 M \backslash \partial_0 M$, *i.e.* the critical point is on the coloured boundary, but does not coincide with a corner of M. Consider the restriction $f|_{\partial_1 M} \colon \partial_1 M \to \mathbb{R}$ which then has a non-degenerate critical point at p with index $i'(p) \in \{0, 1\}$.

   (a) If $i'(p) = 1$, then $f|_{\partial M}$ has a maximum at p.

      i. If p is a $(-)$-critical point of f, the cobordism is diffeomorphic to ε_A of (3.25).

      ii. If p is a $(+)$-critical point of f, the neighbourhood of p looks as follows,

$$\tag{3.27}$$

      Consider the component of the boundary ∂M of M as a topological manifold. The set of corners $\partial_0 M \cap \partial_1 M$ contains at least two elements. If it contains precisely two elements, then the cobordism is diffeomorphic to $\imath^*$ of (3.25). Otherwise, it contains four elements, and the cobordism is diffeomorphic to μ_A of (3.25).

      iii. Otherwise p is neither $(+)$-critical nor $(-)$-critical, and so $df_p = 0$. Non-degeneracy now means that $\mathrm{Hess}_p(f)$ is non-degenerate. Let $i''(p) \in \{0, 1, 2\}$ be the number of negative eigenvalues of $\mathrm{Hess}_p(f)$. The case $i''(p) = 0$ is ruled out by the assumption that $i'(p) = 1$.

      A. If $i''(p) = 2$, then we are in the same situation as in case 2(a)i.

      B. Otherwise $i''(p) = 1$, and we are in the same situation as in case 2(a)ii.

   (b) If $i'(p) = 0$, then $f|_{\partial M}$ has a minimum at p.

      i. If p is a $(+)$-critical point of f, the cobordism is diffeomorphic to η_A of (3.25).

ii. If p is a $(-)$-critical point of f, the neighbourhood of p looks as follows,

$$\tag{3.28}$$

Similarly to case 2(a)ii above, the cobordism is either diffeomorphic to Δ_A or to $\imath$ of (3.25).

iii. Otherwise $df_p = 0$, and by considering $\mathrm{Hess}_p(f)$ similarly to case 2(a)iii above, we find that we are either in case 2(b)i or 2(b)ii.

<div align="right">□</div>

The structure of arbitrary open-closed cobordisms can then be established by using a special Morse function and decomposing the cobordism into a composition of pieces that have precisely one critical point each. This result generalizes the conventional handle decomposition to the case of our sort of 2-manifolds with corners.

Proposition 3.11. Let $f \in \mathbf{2Cob}^{\mathrm{ext}}[\vec{n}, \vec{n}']$ be any morphism. Then f is the composition $f = f_\ell \circ \cdots \circ f_1$ of a finite number of morphisms f_j each of which is of the form $f_j = \mathrm{id}_{\vec{m}_j} \coprod g_j \coprod \mathrm{id}_{\vec{p}_j}$ where g_j is one of the morphisms depicted in (3.25) and $\mathrm{id}_{\vec{m}_j}$ and $\mathrm{id}_{\vec{p}_j}$ are identities, *i.e.* cylinders over their source.

Our pictures, for example (1.1), indicate how the morphisms are composed from the generators. In order to keep the height of the diagram small, we have already used relations such as $(g \coprod \mathrm{id}_{\vec{n}'}) \circ (\mathrm{id}_{\vec{m}} \coprod f) = g \coprod f$ for $f \colon \vec{n} \to \vec{n}'$ and $g \colon \vec{m} \to \vec{m}'$ which obviously hold in $\mathbf{2Cob}^{\mathrm{ext}}$.

Notice that the flat strip, twisted by 2π when we draw it as embedded in $\mathbb{R}^3$, is nevertheless equivalent to the flat strip:
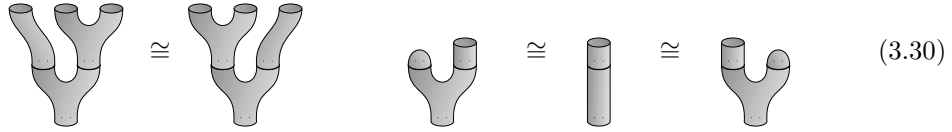
$$\cong \tag{3.29}$$

3.4   Relations
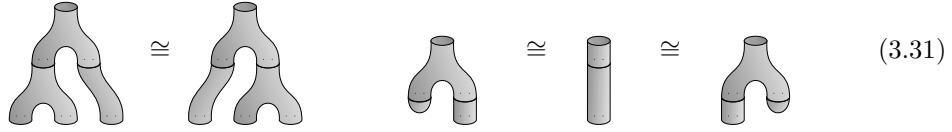
Below we provide a list of relations that the generators of $\mathbf{2Cob}^{\mathrm{ext}}$ satisfy. In Section 3.5, we summarize some consequences of these relations. As already mentioned in the introduction, these relations have emerged from work on boundary conformal field theory. See, for example [7–11]. In Section 3.6, we define a normal form for open-closed cobordisms with a specified genus, window number and open boundary permutation. In Section 3.7, we then provide an inductive proof which constructs a finite sequence of diffeomorphisms that puts an arbitrary open-closed cobordism into the normal form using only the relations given below. Hence, we provide a constructive proof that the relations are sufficient to completely describe the category $\mathbf{2Cob}^{\mathrm{ext}}$.

Proposition 3.12. The following relations hold in the symmetric monoidal category $\mathbf{2Cob}^{\mathrm{ext}}$:
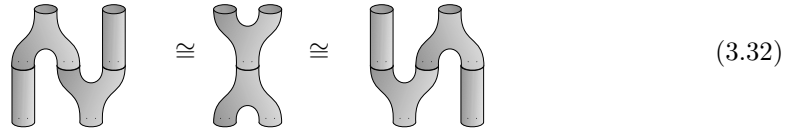
1. The object $\vec{n} = (0)$, *i.e.* the circle $C_{\vec{n}} \cong S^1$, forms a commutative Frobenius algebra object.

$$\cong \qquad\qquad\qquad \cong \qquad \cong \qquad\qquad (3.30)$$

$$\cong \qquad\qquad\qquad \cong \qquad \cong \qquad\qquad (3.31)$$
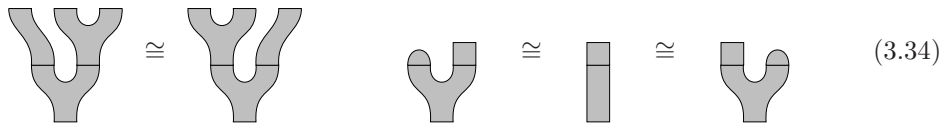
$$\cong \qquad \cong \qquad\qquad\qquad (3.32)$$
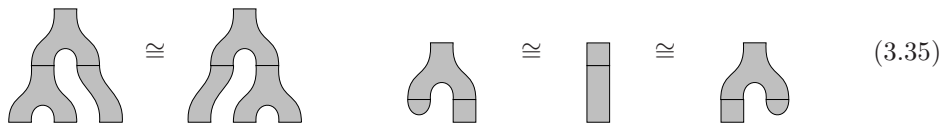
$$\cong \qquad\qquad\qquad (3.33)$$
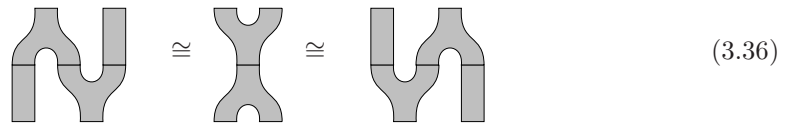
2. The object $\vec{n} = (1)$, *i.e.* the interval $C_{\vec{n}} \cong I$, forms a symmetric Frobenius algebra object.

$$\cong \qquad\qquad\qquad \cong \qquad \cong \qquad\qquad (3.34)$$

$$\cong \qquad\qquad\qquad \cong \qquad \cong \qquad\qquad (3.35)$$
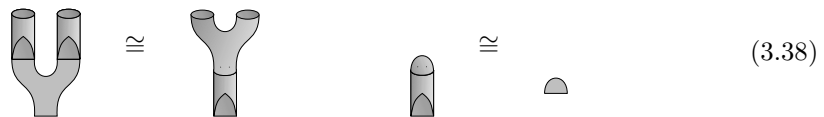
$$\cong \qquad \cong \qquad\qquad\qquad (3.36)$$
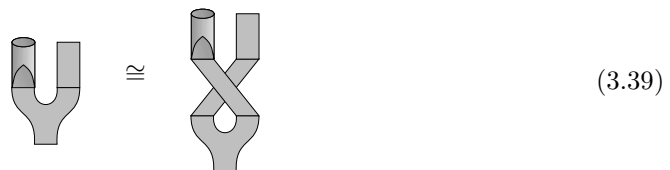
$$\cong \qquad\qquad\qquad (3.37)$$

3. The 'zipper' forms an algebra homomorphism:

$$\cong \qquad\qquad\qquad\qquad \cong \qquad\qquad (3.38)$$

4. This relation describes the 'knowledge' about the centre, *c.f.* (2.12):

$$\cong \qquad\qquad\qquad (3.39)$$

5. The 'cozipper' is dual to the zipper:

$$\text{(3.40)}$$

6. The Cardy condition:

$$\text{(3.41)}$$

Proof. It can be show in a direct computation that the depicted open-closed cobordisms are equivalent. Writing out this proof would be tremendously laborious, but of rather little insight. For the Cardy condition (3.41), the right hand side is most naturally depicted as:

$$\text{(3.42)}$$

$\square$
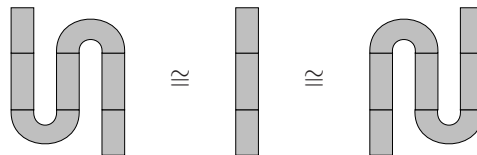
3.5   Consequences of relations

In this section, we collect some additional diffeomorphisms that can be constructed from the diffeomorphisms in Proposition 3.12. To simplify the diagrams, we define:
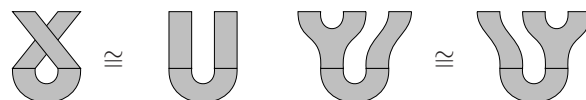
$$\text{(3.43)}$$

These open-closed cobordisms which we sometimes call the *open pairing* and *open copairing*, respectively, satisfy the zig-zag identities:

$$\text{(3.44)}$$

This follows directly from the Frobenius relations, the left and right unit laws, and the left and right counit laws. From Equations (3.37) and (3.34), the pairing can be shown to be symmetric and invariant,

$$\text{(3.45)}$$

and the same holds for the copairing. Similarly, we define the *closed pairing* and the *closed copairing*:

$$\text{(3.46)}$$

These also satisfy the zig-zag identities,

$$\cong \quad \cong \tag{3.47}$$

and the closed pairing is symmetric and invariant,

$$\cong \quad \cong \tag{3.48}$$

A similar result holds for the closed copairing.

Proposition 3.13. The following open-closed cobordisms are diffeomorphic by compositions of the diffeomorphisms of Proposition 3.12:

$$\cong \tag{3.49}$$

$$\cong \tag{3.50}$$

$$\cong \tag{3.51}$$

$$\cong \tag{3.52}$$

Proof. Equation (3.49) is just a restatement of the second axiom in Equation (3.39). The proof of Equation (3.50) is as follows:

$$\underset{(3.33)}{\cong} \quad \underset{\text{Nat}}{\cong} \quad \underset{(3.39)}{\cong} \quad \underset{\text{Nat}}{\cong} \quad \underset{(3.37)}{\cong}$$

$$\tag{3.53}$$

By 'Nat' we have denoted the obvious diffeomorphisms which, algebraically speaking, expresses the naturality of the symmetric braiding. The proof of Equation (3.51) is as follows:

$$\underset{(3.47)}{\cong} \quad \underset{(3.50)}{\cong} \quad \underset{(3.44)}{\cong} \tag{3.54}$$

We leave the proof of Equation (3.52) as an exercise for the reader. □

Proposition 3.14. The following open-closed cobordisms are diffeomorphic:

$$\cong \quad \cong \tag{3.55}$$

$$\cong \qquad\qquad\qquad\qquad (3.56)$$

$$\cong \qquad\cong \qquad\qquad (3.57)$$

$$\cong \qquad\qquad\qquad\qquad (3.58)$$

Proof. The first diffeomorphism in Equation (3.55) is constructed from the following sequence of diffeomorphisms:

$$\underset{(3.30)}{\cong} \qquad \underset{(3.32)}{\cong} \qquad \underset{(3.46)}{\cong} \qquad\qquad (3.59)$$

The second diffeomorphism in Equation (3.55) is constructed similarly. The diffeomorphism in Equation (3.56) is constructed as follows:

$$\underset{(3.55)}{\cong} \qquad \underset{(3.47)}{\cong} \qquad \underset{(3.48)}{\cong} \qquad\qquad (3.60)$$

The proofs of Equations (3.57) and (3.58) are identical as those above with the closed cobordisms replaced by their open counterparts. □

Proposition 3.15. The cozipper ⬚ is a homomorphism of coalgebras.

Proof.

$$\underset{(3.47)}{\cong} \qquad \underset{(3.46)}{\cong} \qquad \underset{(3.39)}{\cong} \qquad\qquad (3.61)$$

$$\underset{(3.46)}{\cong} \qquad \underset{(3.31)}{\cong} \qquad \underset{(3.38)}{\cong} \qquad \underset{(3.39)}{\cong}$$

$$\cong_{(3.56)} \qquad \cong_{(3.39)} \qquad (3.62)$$

$$\cong_{(3.38)} \qquad \cong_{(3.58)}$$

$\Box$

Proposition 3.16. Open-closed cobordisms of the form which we sometimes call *closed windows*, can be moved around freely in any closed diagram. By this we mean that the following open-closed cobordisms are diffeomorphic,

$$\cong \qquad \cong \qquad (3.63)$$

$$\cong \qquad \cong \qquad (3.64)$$

$$\cong \qquad (3.65)$$

Proof. Equation (3.63) holds because,

$$\cong_{(3.61)} \qquad \cong_{(3.57)} \qquad \cong_{(3.51)}$$

$$\cong_{(3.38)} \qquad \cong_{(3.55)}$$

Equation (3.64) can be proven similarly. Equation (3.65) follows readily from the prior two:

$$\cong_{(3.70)} \qquad \cong_{(3.71)} \qquad (3.66)$$

□

Proposition 3.17. Open-closed cobordisms of the form ⬡ which we sometimes call *open windows*, can be moved around freely in any open diagram. More precisely, the following open-closed cobordisms are diffeomorphic,

$$\cong \qquad \cong \qquad (3.67)$$

$$\cong \qquad \cong \qquad (3.68)$$

Proof. We show only the proof of one of these relations,

$$\cong_{(3.36)} \qquad \cong_{(3.35)} \qquad (3.69)$$

□

Proposition 3.18. Open-closed cobordisms of the form ⬡, also called *genus-one operators*, can be moved around freely in any closed diagram. More precisely,

$$\cong \qquad \cong \qquad (3.70)$$

$$\cong \qquad \cong \qquad (3.71)$$

Proof. The proof is similar to that of Proposition 3.17.   □

Proposition 3.19. The following open-closed cobordisms are diffeomorphic by compositions of the diffeomorphisms of Proposition 3.12:

$$\cong \tag{3.72}$$

$$\cong \qquad \cong \tag{3.73}$$

$$\cong \qquad \cong \tag{3.74}$$

Proof. The proof of Equation (3.72) is as follows:

$$\underset{(3.44)}{\cong} \qquad \underset{(3.37)}{\cong} \qquad \underset{\text{Nat}}{\cong} \qquad \underset{(3.44)}{\cong} \tag{3.75}$$

The proof of Equation (3.73) is as follows:

$$\underset{(3.57)}{\cong} \qquad \underset{(3.39)}{\cong} \qquad \underset{\text{Nat}}{\cong} \tag{3.76}$$

$$\underset{(3.72)}{\cong} \qquad \underset{\text{Nat}}{\cong} \qquad \underset{(3.57)}{\cong} \tag{3.77}$$

The second part of Equation (3.72) is proven similarly. Finally, Equation (3.74) follows from the Cardy condition as shown below:

$$\underset{(3.73)}{\cong} \qquad \underset{(3.41)}{\cong} \tag{3.78}$$

$\square$

3.6 The normal form of an open-closed cobordism

In this section, we describe the normal form of an arbitrary connected open-closed cobordism. This normal form is characterized by its genus, window number, and open boundary permutation (*c.f.* Definition 3.6). For non-connected open-closed cobordisms, the normal form has to be taken for each component independently.

3.6.1 The case of open source and closed target

We begin by describing the normal form of a connected open-closed cobordism whose source consists only of intervals and whose target consists only of circles. More precisely, we consider those open-closed cobordisms $f \in \mathbf{2Cob}^{\text{ext}}[\vec{n}, \vec{m}]$ for which $\vec{n} = (1, 1, \ldots, 1)$ and $\vec{m} = (0, 0, \ldots, 0)$ and denote the set of all such cobordisms by $\mathbf{2Cob}^{\text{ext}}_{\text{O}\to\text{C}}[\vec{n}, \vec{m}]$. Some examples are shown below:

$$(3.79)$$
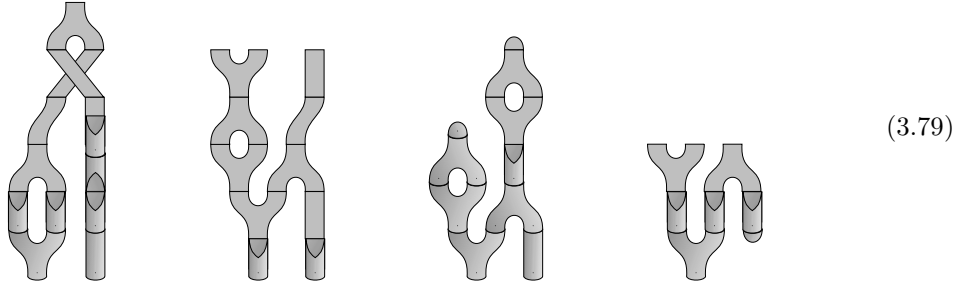
Once we have defined the normal form for this class of cobordisms, we describe in Section 3.6.2 the normal form of an arbitrary connected open-closed cobordism by exploiting the duality on the interval and circle, *c.f.* (3.44) and (3.47). To provide the reader with some intuition about the normal form, the cobordisms of (3.79) are shown in normal form below:

$$(3.80)$$
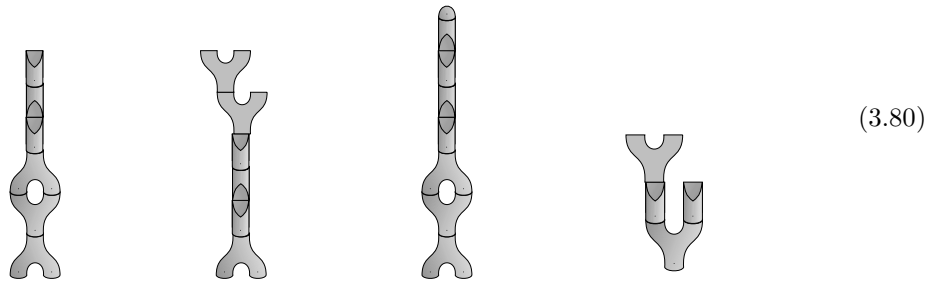
Definition 3.20. Let $f \in \mathbf{2Cob}^{\text{ext}}_{\text{O}\to\text{C}}[\vec{n}, \vec{m}]$ be connected with open boundary permutation $\sigma(f)$, window number $\omega(f)$, and genus $g(f)$. Write the open boundary permutation as a product $\sigma(f) = \sigma_1 \cdots \sigma_r$, $r \in \mathbb{N}_0$, of disjoint cycles $\sigma_j = (i_1^{(j)}, \ldots, i_{q_j}^{(j)})$ of length $q_j \in \mathbb{N}$, $1 \leq j \leq r$. The normal form is the composite,

$$\text{NF}_{\text{O}\to\text{C}}(f) := E_{|\vec{m}|} \circ D_{g(f)} \circ C_{\omega(f)} \circ B_r \circ \left(\coprod_{j=1}^{r} A(q_j) \right) \circ \overline{\sigma(f)}, \qquad (3.81)$$

of the following open-closed cobordisms.

- For each cycle σ_j, the open-closed cobordism $A(q_j)$ consists of $q_j - 1$ flat multiplications and
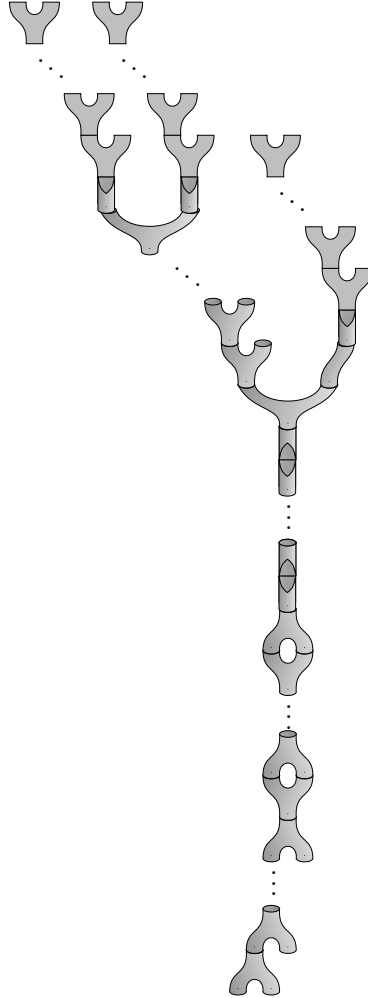
then a cozipper,

$$A(q_j) := \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(3.82)}$$

The normal form (3.81) contains the free union of such a cobordism for each cycle σ_j, $1 \leq j \leq r$. Cycles of length one are represented by a single cozipper. If $|\vec{n}| = 0$, then we have $r = 0$, and the free union is to be replaced by the empty set.

- If $r \geq 1$, then the open-closed cobordism B_r consists of $r - 1$ closed multiplications,

$$B_r := \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(3.83)}$$

  and otherwise $B_0 :=$.

- The open-closed cobordism $C_{\omega(f)}$ is defined as,

$$C_{\omega(f)} := \underbrace{C' \circ C' \circ \cdots \circ C'}_{\omega(f)}, \qquad C' := \qquad\qquad \text{(3.84)}$$

  if $\omega(f) \geq 1$ and empty otherwise.

- Similarly,

$$D_{g(f)} := \underbrace{D' \circ D' \circ \cdots \circ D'}_{g(f)}, \qquad D' := \qquad\qquad \text{(3.85)}$$

  if $g(f) \geq 1$ and empty otherwise.

- $E_{|\vec{m}|}$ consists of $|\vec{m}| - 1$ closed comultiplications,

$$E_{|\vec{m}|} := \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(3.86)}$$

  if $|\vec{m}| \geq 1$ and a closed cup $E_0 :=$ otherwise.

- Finally, $\overline{\sigma(f)}$ denotes the open-closed cobordism that represents the permutation $\overline{\sigma(f)}$ (as defined in (3.19)) given in the following. Let $\tau(f)$ be the open boundary permutation of the open-closed cobordism

$$E_{|\vec{m}|} \circ D_{g(f)} \circ C_{\omega(f)} \circ B_r \circ \Big(\coprod_{j=1}^{r} A(q_j)\Big). \qquad\qquad \text{(3.87)}$$

Figure 1:   This figure depicts the normal form of an open-closed cobordism in $\mathbf{2Cob}^{\text{ext}}_{\text{O}\to\text{C}}[\vec{n}, \vec{m}]$ without precomposition with a permutation, *i.e.* it shows the open-closed cobordism (3.87)
.

Since by construction both $\tau(f)$ and $\sigma(f)$ have the same cycle structure, characterized by the partition $|\vec{n}| = \sum_{j=1}^{r} q_j$, there exists a permutation $\overline{\sigma(f)}$ such that,

$$\sigma(f) = \left(\overline{\sigma(f)}\right)^{-1} \cdot \tau(f) \cdot \overline{\sigma(f)}. \tag{3.88}$$

Note that multiplying $\overline{\sigma(f)}$ by an element in the centralizer of $\sigma(f)$ yields the same open-closed cobordism $\text{NF}_{\text{O}\to\text{C}}(f)$ up to equivalence because of the relations (3.34) and (3.73), and so $\text{NF}_{\text{O}\to\text{C}}(f)$ is well defined.

When we prove the sufficiency of the relations in Section 3.7 below, we provide an algorithm which automatically produces the required $\overline{\sigma(f)}$. Figure 1 depicts the structure of the normal form up to the $\overline{\sigma(f)}$, *i.e.* it shows a cobordism of the form (3.87).

Any cobordism in normal form is invariant (up to equivalence) under composition with certain permutation morphisms as follows.

Proposition 3.21. Let $[f] \in \mathbf{2Cob}^{\text{ext}}_{\text{O}\to\text{C}}[\vec{n}, \vec{m}]$. Then

$$[\sigma^{(\vec{m})} \circ \text{NF}_{\text{O}\to\text{C}}(f)] = [\text{NF}_{\text{O}\to\text{C}}(f)] \tag{3.89}$$

for any $\sigma \in \mathcal{S}_{|\vec{m}|}$, and

$$[\mathrm{NF}_{\mathrm{O}\to\mathrm{C}}(f) \circ \sigma_j^{(\vec{n})}] = [\mathrm{NF}_{\mathrm{O}\to\mathrm{C}}(f)] \tag{3.90}$$

for all cycles $\sigma_j \in \mathcal{S}_{|\vec{n}|}$ from the decomposition of the open boundary permutation $\sigma(f) = \sigma_1 \cdots \sigma_r$ into disjoint cycles.

Proof. Equation (3.89) follows from (3.33), (3.47), and (3.48) while (3.90) follows from (3.34) and (3.45). □

3.6.2   The case of generic source and target

We now extend the definition of the normal form of connected cobordisms from $\mathbf{2Cob}^{\mathrm{ext}}_{\mathrm{O}\to\mathrm{C}}[\vec{n}, \vec{m}]$ to $\mathbf{2Cob}^{\mathrm{ext}}[\vec{n}, \vec{m}]$. Let $[f]$ be a representative of the diffeomorphism class of an open closed cobordism in $\mathbf{2Cob}^{\mathrm{ext}}[\vec{n}, \vec{m}]$. Let $\vec{n}_0 = (0, 0, \ldots, 0)$ and $\vec{n}_1 = (1, 1, \ldots, 1)$ such that $\vec{n}_0 \coprod \vec{n}_1$ is a permutation of $\vec{n}$, and similarly for $\vec{m}$.

We define a map $\Lambda \colon \mathbf{2Cob}^{\mathrm{ext}}[\vec{n}, \vec{m}] \to \mathbf{2Cob}^{\mathrm{ext}}_{\mathrm{O}\to\mathrm{C}}[\vec{m}_1 \coprod \vec{n}_1, \vec{m}_0 \coprod \vec{n}_0]$ as follows: Let $\sigma_1 \in \mathcal{S}_{|\vec{n}|}$ denote the permutation that sends $\vec{n}$ to $\vec{n}_1 \coprod \vec{n}_0$. Let $\sigma_2 \in \mathcal{S}_{|\vec{m}|}$ denote the permutation that sends $\vec{m}$ to $\vec{m}_1 \coprod \vec{m}_0$. For example, if $\vec{n} = (1, 0, 0, 1)$, then σ_1 is represented as an open-closed cobordism by:

$$\sigma_1 \quad = \quad \text{} \tag{3.91}$$

For $[f] \in \mathbf{2Cob}^{\mathrm{ext}}[\vec{n}, \vec{m}]$ we define $\Lambda([f])$ to be the open closed cobordism obtained from $[f]$ by precomposing with σ_1^{-1}, postcomposing with σ_2, gluing closed copairings on each circle in $\vec{n}_0$, and gluing open pairings on each interval in $\vec{m}_1$. For example, let $[f]$ be an arbitrary open closed cobordism from $(1, 0, 0, 1)$ to $(0, 1, 1, 0)$, then $\Lambda([f])$ is illustrated below:

$$\Lambda \colon \quad \text{} \quad \longmapsto \quad \text{} \tag{3.92}$$

Up to equivalence, this assignment does not depend on the choice of representative in the class $[f]$; if f' is a different representative then there exists a black boundary preserving diffeomorphism from f to f'. Applying this diffeomorphism in the interior of $\Lambda([f])$ shows that $\Lambda([f'])$ is equivalent to $\Lambda([f])$,

$$[\Lambda([f])] = [\Lambda([f'])]. \tag{3.93}$$

$\Lambda([f])$ is connected if and only if f is.

Given some extra structure, an inverse mapping can be defined. Let $[g]$ be a representative of a diffeomorphism class in $\mathbf{2Cob}^{\mathrm{ext}}_{\mathrm{O}\to\mathrm{C}}[\vec{n}', \vec{m}']$ equipped with:

- a decomposition of its source into a free union $\vec{n}' = \vec{n}'_t \coprod \vec{n}'_s$,

- a decomposition of its target into a free union $\vec{m}' = \vec{m}'_t \coprod \vec{m}'_s$,

- an element of the symmetric group $\sigma'_1 \in \mathcal{S}_{|\vec{n}'_s| + |\vec{m}'_s|}$, and

- an element of the symmetric group $\sigma'_2 \in \mathcal{S}_{|\vec{n}'_t| + |\vec{m}'_t|}$.

Note that the image of an $[f] \in \mathbf{2Cob}^{\text{ext}}[\vec{n}, \vec{m}]$ under the mapping Λ is equipped with such structure. The decompositions are given by distinguishing which intervals and circles came from the source and the target. The permutations can be taken to be $\sigma'_1 = \sigma_1$ and $\sigma'_2 = \sigma_2^{-1}$.

We define $\Lambda^{-1}([g])$ to be the open-closed cobordism in $\mathbf{2Cob}^{\text{ext}}[\sigma'_2(\vec{n}'_t \coprod \vec{m}'_t), \sigma'_1(\vec{n}'_s \coprod \vec{m}'_s)]$ given by gluing open copairings to the intervals in $\vec{n}'_t$ and closed pairings to the circles in $\vec{m}'_s$. The result of this gluing is then precomposed with a cobordism representing σ'_1 and postcomposed with a cobordism representing σ'_2.

$$\tag{3.94}$$

Again, this assignment does not depend on the choice of representative of the class $[g]$. One can readily verify that this defines a bijection between the equivalence classes of open-closed cobordisms in $\mathbf{2Cob}^{\text{ext}}[\vec{n}, \vec{m}]$ and those of open-closed cobordisms in $\mathbf{2Cob}^{\text{ext}}_{\text{O}\to\text{C}}[\vec{m}_1 \coprod \vec{n}_1, \vec{m}_0 \coprod \vec{n}_0]$ equipped with the extra structure described above. One direction of this verification,

$$[\Lambda^{-1}([\Lambda([f])])] = [f], \tag{3.95}$$

is depicted schematically below:

$$\tag{3.96}$$

In Theorem 3.24 below, we show that for any connected $[g] \in \mathbf{2Cob}^{\text{ext}}_{\text{O}\to\text{C}}[\vec{n}, \vec{m}]$, g is equivalent to its normal form, *i.e.*

$$[g] = [\mathrm{NF}_{\text{O}\to\text{C}}(g)]. \tag{3.97}$$

Applying this result to $[g] := \Lambda([f])$ is the motivation for the definition of the normal form for generic connected $[f] \in \mathbf{2Cob}^{\text{ext}}[\vec{n}, \vec{m}]$.

Definition 3.22. Let $[f] \in \mathbf{2Cob}^{\text{ext}}[\vec{n}, \vec{m}]$ be connected. Then we define its normal form by,

$$[\mathrm{NF}(f)] := \Lambda^{-1}([\mathrm{NF}_{\text{O}\to\text{C}}(f)]), \tag{3.98}$$

which can be depicted as follows

$$(3.99)$$

3.7 Proof of sufficiency of relations

In this section, we show that any connected open-closed cobordism $[f] \in \mathbf{2Cob}^{\mathrm{ext}}_{\mathrm{O} \to \mathrm{C}}[\vec{n}, \vec{m}]$ can be related to its normal form $\mathrm{NF}_{\mathrm{O} \to \mathrm{C}}(f)$ by applying the relations of Proposition 3.12 a finite number of times. We know that f is equivalent to an open-closed cobordism of the form stated in Proposition 3.11.

For convenience, we designate the following composites,

$$(3.100)$$

as being distinct generators. This simplifies the proof of the normal form below. We continue to use the shorthand (3.43) and (3.46). However, we do *not* consider these as distinct generators.

In our diagrams, we denote an arbitrary open-closed cobordism X, whose source is a general object $\vec{n}_X$ that contains at least one 1, as follows:

$$(3.101)$$

Similarly, to denote an arbitrary open-closed cobordism Y, whose source is a general object $\vec{n}_Y$ containing at least one 0, we use the notation:

$$(3.102)$$

Finally, for an arbitrary open-closed cobordism Z, whose target $\vec{m}$ is not glued to any other cobordism in the decomposition of Σ, we use the following notation:

$$(3.103)$$

and similarly if the source is not glued to anything.

Definition 3.23. Let $[f] \in \mathbf{2Cob}^{\mathrm{ext}}[\vec{n}, \vec{m}]$ be written in the form of Proposition 3.11. The *height* of a generator in the decomposition of f is the following number defined inductively, ignoring all identity morphisms in the decomposition:

- $h(\boxed{X}) := 0$

- $h(\smile) = h(\bullet) := 0$

- $h(Y) = h(Y) = h(Y) = h(Y) := 1 + h(Y)$

- $h(\,\includegraphics{Y}\,) = h(\,\includegraphics{Y}\,) = h(\,\includegraphics{Y}\,) = h(\,\includegraphics{Y}\,) = h(\,\includegraphics{Y}\,) := 1 + h(Y)$

- $h(\,\includegraphics{YZ}\,) = h(\,\includegraphics{YZ}\,) := h(Y) + h(Z) + 1.$

Theorem 3.24. Let $f \in \mathbf{2Cob}^{\mathrm{ext}}_{\mathrm{O}\to\mathrm{C}}[\vec{n}, \vec{m}]$ be a connected open-closed cobordism. Then f is equivalent to its normal form, *i.e.*

$$[f] = [\mathrm{NF}_{\mathrm{O}\to\mathrm{C}}(f)]. \tag{3.104}$$

Proof. We construct a diffeomorphism from f to $\mathrm{NF}_{\mathrm{O}\to\mathrm{C}}(f)$ by applying the composition of a finite number of diffeomorphisms from Proposition 3.12.

We say *decomposition* for a presentation of f as a composition of the generators as in Proposition 3.11, *i.e.* for a generalized handle decomposition. We use the term *move* for the application of a diffeomorphism from Proposition 3.12, we just say diffeomorphism here meaning diffeomorphism relative to the black boundary, and we use the term *configuration* of a generator in a decomposition to refer to the generators immediately pre- and postcomposed to it.

The proof proceeds step by step as follows.

I) The decomposition of f is diffeomorphic to one without any *open cups* $\includegraphics{cup}$ or *open caps* $\includegraphics{cap}$. This is achieved by applying the following moves.

   a) $\includegraphics{cup} \overset{\longmapsto}{\underset{(3.61)}{}} \includegraphics{}$

   b) $\includegraphics{cap} \overset{\longmapsto}{\underset{(3.38)}{}} \includegraphics{}$

   to every instance of the open cup and cap.

II) The resulting decomposition of f is diffeomorphic to one in which every *open comultiplication* $\includegraphics{}$ appears in one of the following three configurations:

$$\tag{3.105}$$

   where the '?' may be any open-closed cobordism which may or may not be attached to the multiplication at the bottom. We prove this step-by-step by considering every possible configuration and providing the moves to reduce the decomposition into one of the above mentioned configurations.

   a) The cases $\includegraphics{}$ and $\includegraphics{}$ are excluded by step (Ia).

   b) Wherever possible apply the move: $\includegraphics{} \overset{\longmapsto}{\underset{(3.34)}{}} \includegraphics{}$.

   c) We consider all of the remaining possible configurations and provide a list of moves which either remove the open comultiplication or reduce its height. Since there are no longer any open cups after (Ia) and since the target of f is of the form $\vec{m} = (0, \ldots, 0)$, *i.e.* a free union of circles, the open comultiplication is either removed from the diagram or takes the form claimed in (II) before its height is reduced to zero.

   Apply the following moves wherever possible:

1) $\underset{\text{Def}}{\longmapsto}$

2) $\underset{(3.41)}{\longmapsto}$

3) $\underset{(3.55)}{\longmapsto}$ $\underset{(3.52)}{\longmapsto}$ and $\underset{(3.55)}{\longmapsto}$ $\underset{(3.51)}{\longmapsto}$

4) $\underset{(3.36)}{\longmapsto}$ $\underset{(3.36)}{\longleftarrow}$

5) $\underset{(3.67)}{\longmapsto}$ $\underset{(3.67)}{\longleftarrow}$
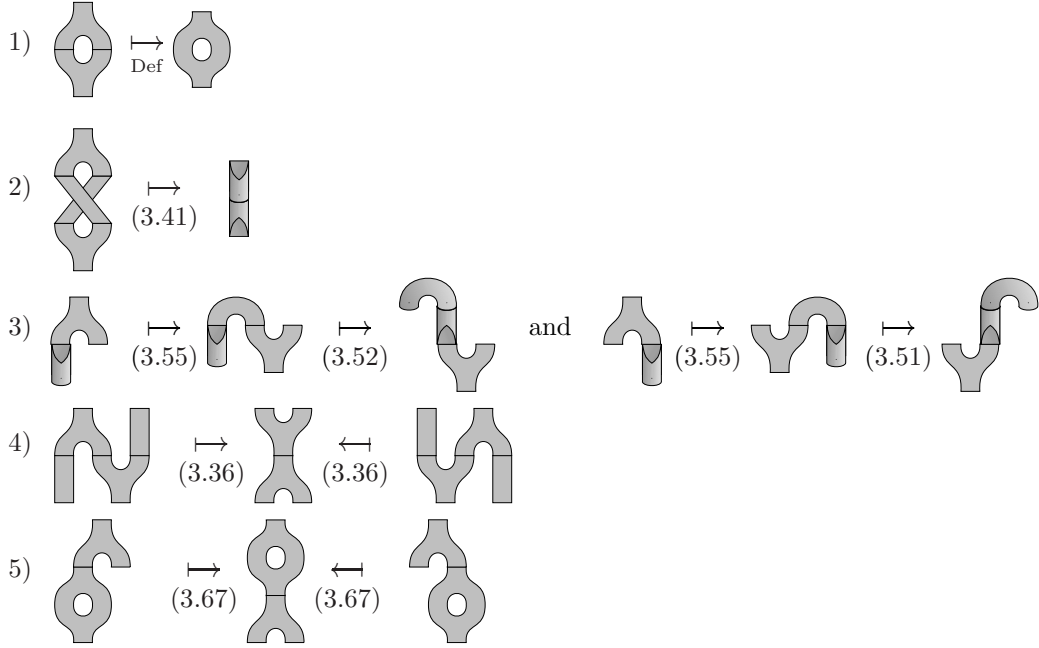
d) Iterate steps (IIb) and (IIc). Since each iteration either removes the open comultiplication or reduces its height, this process is guaranteed to terminate with every comultiplication in one of the three configurations of (3.105).

III) Now we apply a sequence of moves to the decomposition of f which reduces the number of possible configurations that need to be considered.

a) To begin, we provide a sequence of moves to put every *open multiplication* in the decomposition of f into one of the following configurations:

(3.106)
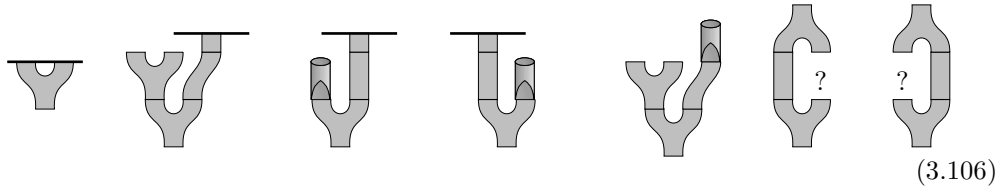
Again, we prove this claim by considering all possible configurations *of the* open multiplication.

Apply the following moves which either removes the open multiplication or increases its height or attains the desired configuration.

1) $\underset{(3.34)}{\longmapsto}$

2) $\underset{(3.68)}{\longmapsto}$ $\underset{(3.68)}{\longleftarrow}$

3) $\underset{(3.38)}{\longmapsto}$
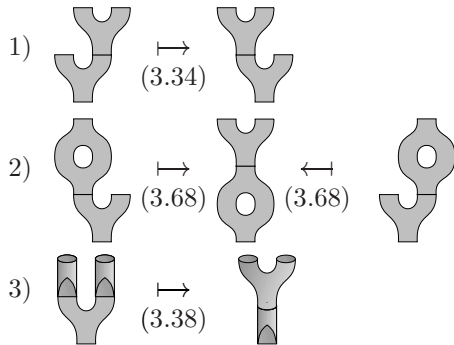
4) The moves of steps II(c)1, II(c)2, II(c)4.

All other configurations are excluded by step I. Since none of these steps increases the number of generators in the decomposition of f, iterating this process either removes all open multiplications or puts them into the configurations in (3.106) as claimed above.

b) Now we show that the source of every cozipper can be put into either of the following configurations:

$$(3.107)$$

We establish the above claim by applying the following sequence of moves wherever they are possible.

1) $\longmapsto$ (3.74)

2) $\longmapsto$ Def
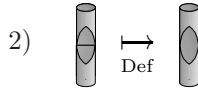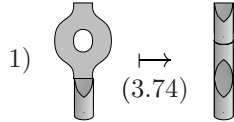
3) The moves of step II(c)3.

All other configurations are excluded by step I.

c) In this step we show that every instance of the open window can be removed.

Iterate the following sequence of moves wherever possible.

1) $\longmapsto$ (3.67)
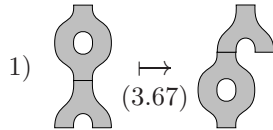
2) The moves of steps III(a)2 and III(b)1.

All other configurations are excluded by step I. Iterating these moves is guaranteed to remove all instances of the open window since each iteration either removes the window or reduces its height. The height of the open window cannot be zero.

d) From the sequence of moves applied thus far, it follows that the target of every open multiplication is in one of the following configurations:

$$(3.108)$$
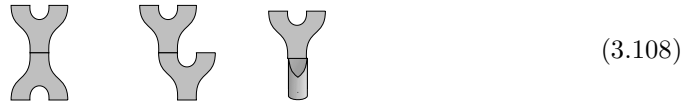
All other possibilities are excluded by steps II(c)1, I and IIIc.

IV) In this step, we apply a sequence of moves that removes all open comultiplications. After step II, we need to consider only three cases. Step III has not changes this situation. From the set of open comultiplications in the decomposition of f, choose one of minimal height.

a) The case has been excluded by the assumption that the open comultiplication is of minimal height. Hence the only remaining configurations to consider are

$$(3.109)$$
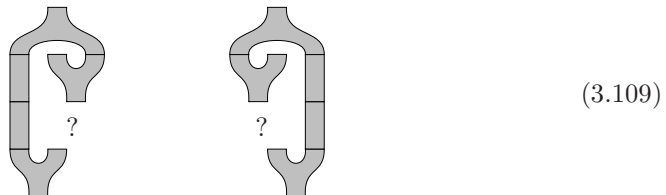
where no other open comultiplication occurs in '?' above.

b) By symmetry it suffices to consider one of the remaining configurations, say . We
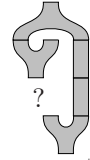
proceed by considering all possible configurations of '?' above. The first generator in the decomposition of '?' is determined by step IIId and the assumption that the open comultiplication under consideration is of minimal height. Hence, only two situations are possible:

(3.110)
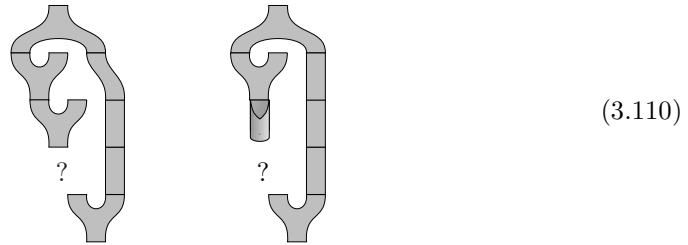
1) In the first case, iteratively apply the move so that the only
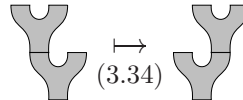
possible configurations are
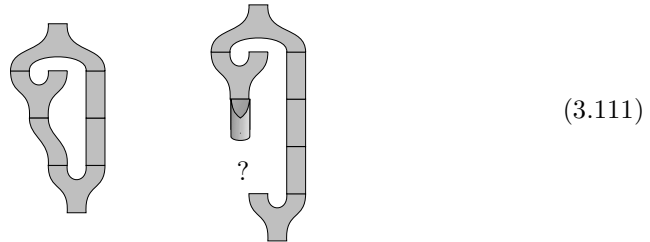
(3.111)

In the following two steps we remove the open comultiplication from the above two situations.

2) Consider the first case in (3.111) above. The comultiplication is removed by the following sequence of moves:

(3.112)

(3.113)
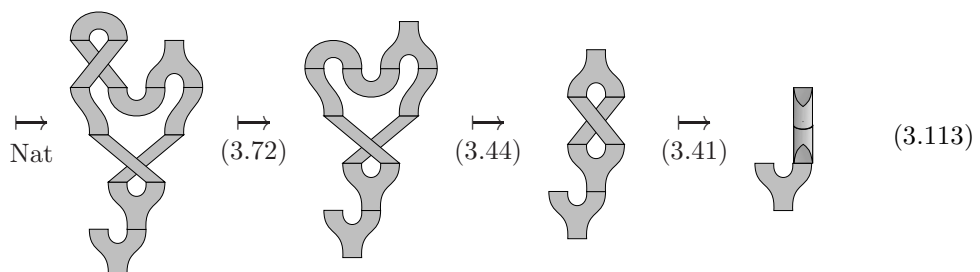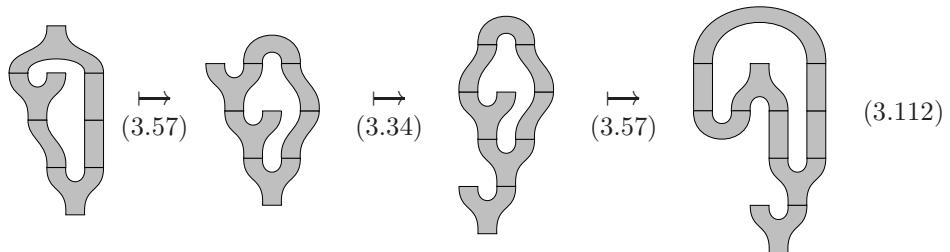
3) Consider now the second case in (3.111). In this case the comultiplication is removed by the following sequence of moves:

$$(3.114)$$
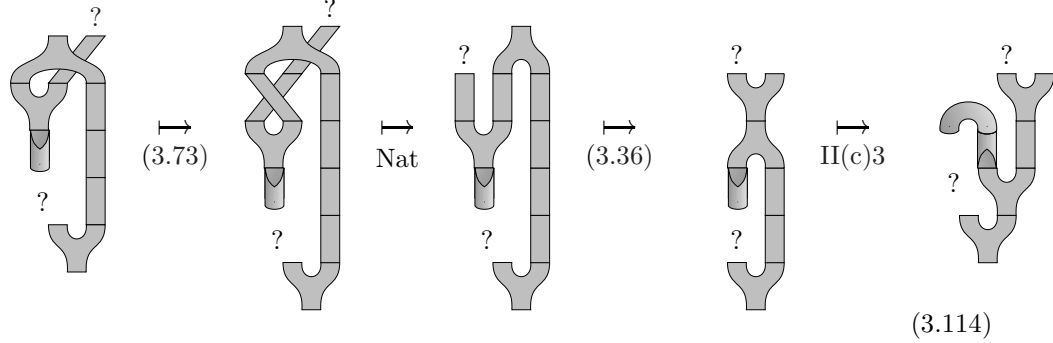
c) Step IVb has changed the cobordism so much that the claims made in steps II and III need not hold any longer. We therefore reapply the steps II and III.

d) Then we iterate the sequence of steps IVb and IVc until all open comultiplications have disappeared. This iteration terminates because neither step II nor step III (which are invoked in IVc) increase the number of open comultiplications, but step IVb always decreases this number by one.

e) When the last open comultiplication has disappeared in step IVd, step IVc ensures that the claims made in steps II and III are satisfied again.

V) At this stage of the proof, all open caps, open cups and open comultiplications have been removed from the decomposition of f. The decomposition has the following further properties.

a) After the step IIIa, it is clear that any open multiplication has its source in one of the following configurations:

$$(3.115)$$
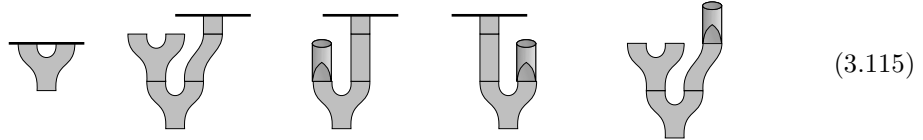
b) Every instance of the cozipper is in the configuration claimed in step IIIb.

c) All instances of have been removed by step IIIc.

d) From step IIId and step IV, the only possible configuration for the target of an open multiplication are

$$(3.116)$$

VI) Now we remove every instance of the zipper in the decomposition of f. We consider all remaining possible configurations involving the zipper and provide the moves to get rid of it.

a) The following configurations:

$$(3.117)$$
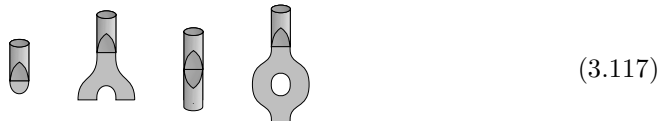
are excluded by steps I, IV, IIIb, and IIIc, respectively.

b) The remaining possibilities are

$$(3.118)$$

Using step Vd together with possibly repeated applications of the following moves:

1) $\xmapsto{}$ (3.39)

2) $\xmapsto{}$ (3.34)

3) $\xmapsto{}$ (3.57) $\xmapsto{}$ (3.13) $\xmapsto{}$ (3.38) $\xmapsto{}$ (3.32)
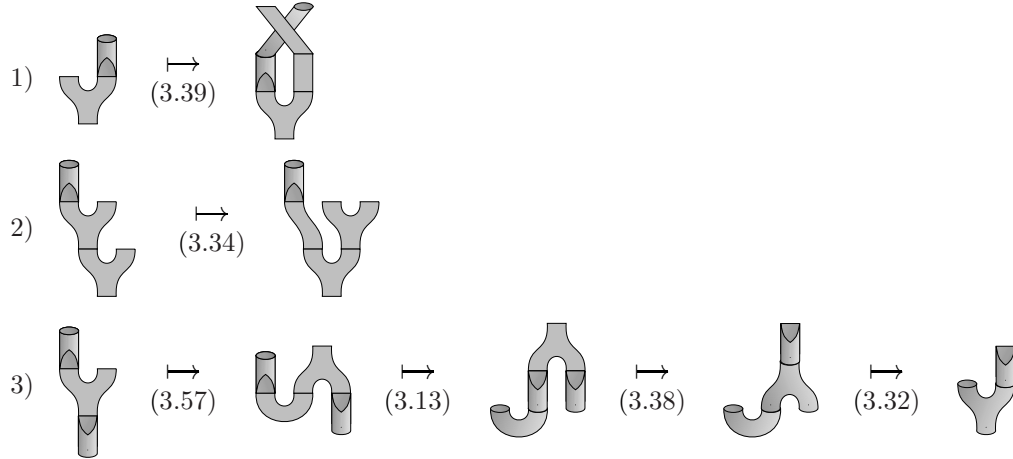
we make sure that all instances of have disappeared.

VII) The resulting decomposition of f is diffeomorphic to one in which each *closed multiplication* has its source in one of these configurations:

$$(3.119)$$

a) The cases , and are excluded by the assumption that the source of f is of the form $\vec{n} = (1, \ldots, 1)$, *i.e.* is a free union of intervals I.

b) $\xmapsto{}$ (3.30) $\xmapsto{}$ (3.30) $\xmapsfrom{}$ (3.30)

c) $\xmapsto{}$ Def $\xmapsfrom{}$ (3.33)

d) $\xmapsto{}$ (3.32) $\xmapsfrom{}$ (3.32)

e) $\xmapsto{}$ (3.63) $\xmapsfrom{}$ (3.63)

f) $\xmapsto{}$ (3.71) $\xmapsfrom{}$ (3.71)
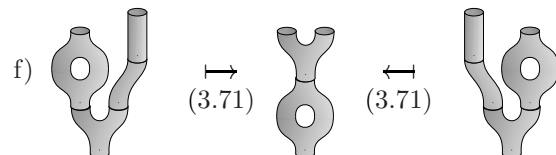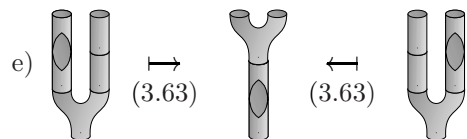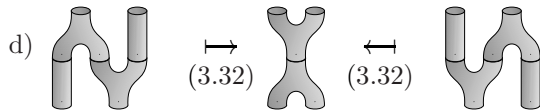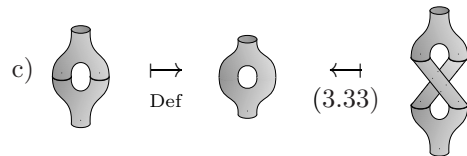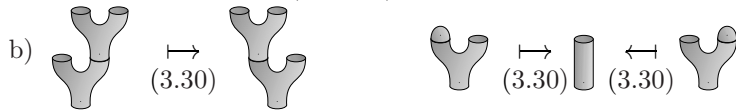
Since each of the above moves either removes the closed multiplication or increases its height while not increasing the number of generators, iterating the above diffeomorphisms is guaranteed to terminate with the closed multiplication in one of the specified configurations.

VIII) The decomposition of f is diffeomorphic to one in which each closed comultiplication is in one of the following two configurations:

$$\tag{3.120}$$
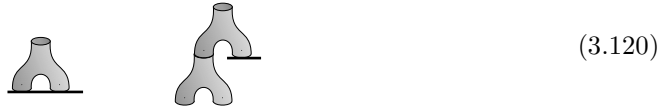
We consider all possible configurations of closed comultiplications.

a) The cases:

$$\tag{3.121}$$
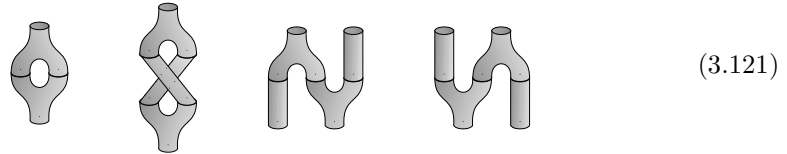
are excluded by step VII.

b) The cases:

$$\tag{3.122}$$
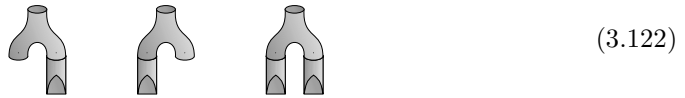
are excluded by step VI.

c) To prove the claim, we iterate the following sequences of moves wherever possible:

1) $\overset{\longmapsto}{(3.31)}$

2) $\overset{\longmapsto}{(3.31)}$ $\overset{\longleftarrow}{(3.31)}$

3) $\overset{\longmapsto}{(3.63)}$ $\overset{\longleftarrow}{(3.63)}$

4) $\overset{\longmapsto}{(3.70)}$ $\overset{\longleftarrow}{(3.70)}$
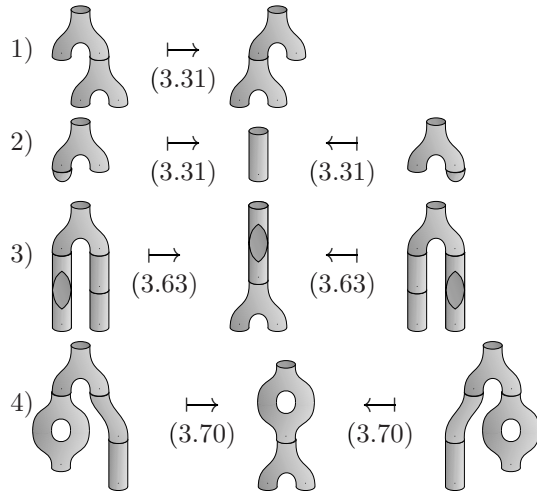
This iteration is guaranteed to terminate since each move either decreases the height of the closed comultiplication or removes it.

IX) In the resulting decomposition, each instance of the closed window has above it one of the following: , , , or . There are only two remaining cases to consider.

a) The cases and are excluded by step VIII(c)3.

b) The claim follows by iterating the moves $\longmapsto$.

$$\text{(3.65)}$$
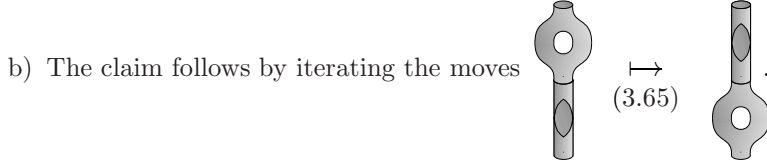
At this point, the decomposition of f is in the normal form desired. In order to see this, we need the claims made in the steps VIII, Va, VI, Vd, VII and the following two results.

X) If a *closed cap* occurs anywhere in the resulting decomposition of f, then the source of f is the object $\vec{n} = \emptyset$, and the has its target in one of the following configurations:

$$\text{(3.123)}$$
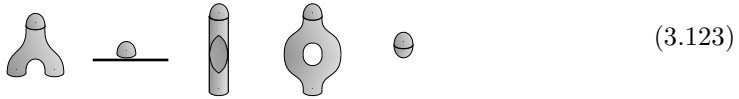
This follows since all other possible configurations are excluded by steps VIIb and VI.

XI) If a *closed cup* occurs anywhere in the resulting decomposition of f then the target of f is the object $\vec{m} = \emptyset$, and the source of the is in one of the following configurations:

$$\text{(3.124)}$$
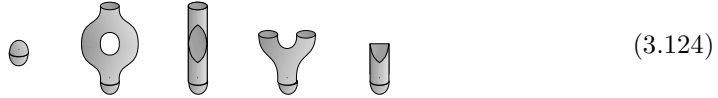
The remaining cases are excluded by step VIIIc.

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

The main result for arbitrary connected open-closed cobordisms then follows.

Corollary 3.25. Let $[f] \in \mathbf{2Cob}^{\mathrm{ext}}[\vec{n}, \vec{m}]$ be connected. Then $[f] = [\mathrm{NF}(f)]$.

Proof. Using Definition 3.22, then applying Theorem 3.24 to $\Lambda(f)$, and then applying (3.95), we find,

$$[\mathrm{NF}(f)] = [\Lambda^{-1}([\mathrm{NF}_{\mathrm{O}\to\mathrm{C}}(\Lambda([f]))])] = [\Lambda^{-1}([\Lambda([f])])] = [f]. \qquad\qquad \text{(3.125)}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Since the normal form is already characterized by the invariants of Definition 3.6, we also obtain the following result.

Corollary 3.26. Let $[f], [f'] \in \mathbf{2Cob}^{\mathrm{ext}}[\vec{n}, \vec{m}]$ be connected such that their genus, window number, and open boundary permutation agree, then $[f] = [f']$.

4   Open-closed TQFTs

In this section, we define the notion of open-closed TQFTs. We show that the categories $\mathbf{2Cob}^{\mathrm{ext}}$ and $\mathbf{Th}(\mathbf{K\text{-}Frob})$ are equivalent as symmetric monoidal categories which implies that the category of open-closed TQFTs is equivalent to the category of knowledgeable Frobenius algebras.

Definition 4.1. Let $\mathcal{C}$ be a symmetric monoidal category. An *open-closed Topological Quantum Field Theory (TQFT)* in $\mathcal{C}$ is a symmetric monoidal functor $\mathbf{2Cob}^{\mathrm{ext}} \to \mathcal{C}$. A *homomorphism* of open-closed TQFTs is a monoidal natural transformation of such functors. By $\mathbf{OC\text{-}TQFT}(\mathcal{C}) := \mathbf{Symm\text{-}Mon}(\mathbf{2Cob}^{\mathrm{ext}}, \mathcal{C})$, we denote the category of open-closed TQFTs.

Theorem 4.2. The category $\mathbf{2Cob}^{\mathrm{ext}}$ is equivalent as a symmetric monoidal category to the category $\mathbf{Th}(\mathbf{K\text{-}Frob})$.

This theorem states the precise correspondence between topology (Section 3) and algebra (Section 2). The second main result of the present article follows from this theorem and from Proposition 2.8.

Corollary 4.3. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \tau)$ be a symmetric monoidal category. The category **K-Frob**$(\mathcal{C})$ of knowledgeable Frobenius algebras in $\mathcal{C}$ is equivalent as a symmetric monoidal category to the category **OC-TQFT**$(\mathcal{C})$.

These results also guarantee that one can use the diagrams of Section 3.3 and the relations of Section 3.4 in order to perform computations in knowledgeable Frobenius algebras. Recall that **2Cob**$^{\text{ext}}$ is a strict monoidal category whereas **Th**(**K-Frob**) is weak. When one translates from diagrams to algebra, one chooses parentheses for all tensor products and then inserts the structure isomorphisms α, λ, ρ as appropriate. The coherence theorem of MacLane guarantees that all ways of inserting these isomorphisms yield the same morphisms, and so the morphisms on the algebraic side are well defined by their diagrams.

In particular, we could have presented the second half of Section 3, starting with Subsection 3.5, entirely in the algebraic rather than in the topological language.

Proof of Theorem 4.2. Define a mapping Ξ from the objects of **2Cob**$^{\text{ext}}$ to the objects of the category **Th**(**K-Frob**) by mapping the generators as follows:

$$\Xi \colon \emptyset \;\mapsto\; \mathbb{1} \tag{4.1}$$

$$\Xi \colon \bigcirc \;\mapsto\; C \tag{4.2}$$

$$\Xi \colon \text{—} \;\mapsto\; A \tag{4.3}$$

and extending to the general object $\vec{n} \in$ **2Cob**$^{\text{ext}}$ by mapping $\vec{n}$ to the tensor product in **Th**(**K-Frob**) of copies of A and C with all parenthesis to the left. More precisely, if $\vec{n} = (n_1, n_2, n_3, \cdots, n_k)$ with each $n_i \in \{0, 1\}$, then $\Xi(\vec{n}) = (((\Xi(n_1) \otimes \Xi(n_2)) \otimes \Xi(n_3)) \cdots \Xi(n_k))$ with each $\Xi(0) := C$ and $\Xi(1) := A$. On the generating morphisms in **2Cob**$^{\text{ext}}$, Ξ is defined as follows:

$$\mapsto \quad 1_C \colon C \to C \tag{4.4}$$

$$\mapsto \quad 1_A \colon A \to A \tag{4.5}$$

$$\mapsto \quad \tau_{C,C} \colon C \otimes C \to C \otimes C \tag{4.6}$$

$$\mapsto \quad \tau_{A,A} \colon A \otimes A \to A \otimes A \tag{4.7}$$

$$\mapsto \quad \tau_{A,C} \colon A \otimes C \to C \otimes A \tag{4.8}$$

$$\mapsto \quad \tau_{C,A} \colon C \otimes A \to A \otimes C \tag{4.9}$$

$$\mapsto \quad \mu_A \colon A \otimes A \to A \tag{4.10}$$

$$\mapsto \quad \eta_A \colon \mathbb{1} \to A \tag{4.11}$$

$$\mapsto \quad \Delta_A \colon A \to A \otimes A \tag{4.12}$$

$$\mapsto \quad \varepsilon_A \colon A \to \mathbb{1} \tag{4.13}$$

$$\mapsto \quad \mu_C \colon C \otimes C \to C \tag{4.14}$$

$$\mapsto \quad \eta_C \colon \mathbb{1} \to C \tag{4.15}$$

$$\mapsto \quad \Delta_C \colon C \to C \otimes C \tag{4.16}$$

$$\mapsto \quad \varepsilon_C \colon C \to \mathbb{1} \tag{4.17}$$

$$\mapsto \quad \imath \colon C \to A \tag{4.18}$$

$$\mapsto \quad \imath^* \colon A \to C \tag{4.19}$$

Without loss of generality we can assume that every general morphism f in $\mathbf{2Cob}^{\mathrm{ext}}$ is decomposed into elementary generators in such a way that each critical point in the decomposition of f has a unique critical value. We can then extend Ξ to a map on all the morphisms of $\mathbf{2Cob}^{\mathrm{ext}}$ inductively using the following assignments:

$$\mapsto \quad \mu_A \otimes 1_A \colon (A \otimes A) \otimes A \to A \otimes A \tag{4.20}$$

$$\mapsto \quad 1_A \otimes \mu_A \circ \alpha_{A,A,A} \colon (A \otimes A) \otimes A \to A \otimes A \tag{4.21}$$

$$\mapsto \quad \eta_A \otimes 1_A \circ \lambda_A^{-1} \colon A \to A \otimes A \tag{4.22}$$

$$\mapsto \quad 1_A \otimes \eta_A \circ \rho_A^{-1} \colon A \to A \otimes A \tag{4.23}$$

$$\mapsto \quad \Delta_A \otimes 1_A \colon A \otimes A \to (A \otimes A) \otimes A \tag{4.24}$$

$$\mapsto \quad \alpha_{A,A,A}^{-1} \circ 1_A \otimes \Delta_A \colon A \otimes A \to (A \otimes A) \otimes A \tag{4.25}$$

$$\mapsto \quad \lambda_A \circ \varepsilon_A \otimes 1_A \colon A \otimes A \to A \tag{4.26}$$

$$\mapsto \quad \rho_A \circ 1_A \otimes \varepsilon_A \colon A \otimes A \to A \tag{4.27}$$

$$\mapsto \quad \mu_C \otimes 1_C \colon (C \otimes C) \otimes C \to C \otimes C \tag{4.28}$$

$$\mapsto \quad 1_C \otimes \mu_C \circ \alpha_{C,C,C} \colon (C \otimes C) \otimes C \to C \otimes C \tag{4.29}$$

$$\mapsto \quad \eta_C \otimes 1_C \circ \lambda_C^{-1} \colon C \to C \otimes C \tag{4.30}$$

$$\mapsto \quad 1_C \otimes \eta_C \circ \rho_C^{-1} \colon C \to C \otimes C \tag{4.31}$$

$$\mapsto \quad \Delta_C \otimes 1_C \colon C \otimes C \to (C \otimes C) \otimes C \tag{4.32}$$

$$\mapsto \quad \alpha_{C,C,C}^{-1} \circ 1_C \otimes \Delta_C \colon C \otimes C \to (C \otimes C) \otimes C \tag{4.33}$$

$$\mapsto \quad \lambda_C \circ \varepsilon_C \otimes 1_C \colon C \otimes C \to C \tag{4.34}$$

$$\mapsto \quad \rho_C \circ 1_C \otimes \varepsilon_C \colon C \otimes C \to C \tag{4.35}$$

This assignment is well defined and extends to all the general morphisms in $\mathbf{2Cob}^{\mathrm{ext}}$ by the coherence theorem for symmetric monoidal categories, which ensures that there is a unique morphism from one object to another composed of associativity constraints and unit constraints. The relations in Proposition 3.12 and the proof that these are all the required relations in $\mathbf{2Cob}^{\mathrm{ext}}$ imply that the image of Ξ is in fact a knowledgeable Frobenius algebra in $\mathbf{Th}(\mathbf{K\text{-}Frob})$. Hence, Ξ defines a functor $\mathbf{2Cob}^{\mathrm{ext}} \to \mathbf{Th}(\mathbf{K\text{-}Frob})$.

Define a natural isomorphism $\Xi_2 \colon \Xi(\vec{n}) \otimes \Xi(\vec{m}) \to \Xi(\vec{n} \coprod \vec{m})$ for $X, Y \in \mathbf{2Cob}^{\mathrm{ext}}$ as follows: Let $\vec{n} = (n_1, n_2, n_3, \ldots, n_k)$ and $\vec{m} = (m_1, m_2, m_3, \cdots, m_\ell)$ so that

$$\Xi(\vec{n}) \quad = \quad (((\Xi(n_1) \otimes \Xi(n_2)) \otimes \Xi(n_3)) \cdots \Xi(n_k)), \tag{4.36}$$

$$\Xi(\vec{m}) \quad = \quad (((\Xi(m_1) \otimes \Xi(m_2)) \otimes \Xi(m_3)) \cdots \Xi(m_\ell)), \tag{4.37}$$

$$\Xi(\vec{n} \coprod \vec{m}) \quad = \quad (((((\Xi(n_1) \otimes \Xi(n_2)) \otimes \Xi(n_3)) \cdots \Xi(n_k)) \otimes \Xi(m_1)) \cdots \otimes \Xi(m_\ell)). \tag{4.38}$$

Hence the map $\Xi_2 \colon \Xi(\vec{n}) \otimes \Xi(\vec{m}) \to \Xi(\vec{n} \coprod \vec{m})$ is composed entirely of composites of the natural isomorphism α. By the coherence theorem for monoidal categories, any choice of composites from the source to the target is unique. One can easily verify that if $\Xi_0 := 1_{\mathbb{1}}$, then collection (Ξ, Ξ_2, Ξ_0) defines a monoidal natural transformation. Furthermore, our choice for the assignment of Ξ of the open-closed cobordisms generating $\mathbf{2Cob}^{\mathrm{ext}}$'s symmetry ensures that (Ξ, Ξ_2, Ξ_0) is a symmetric monoidal functor.

Using the assignments from equations (4.1)-(4.19) we see that the generating open-closed cobordisms in $\mathbf{2Cob}^{\mathrm{ext}}$ define a knowledgeable Frobenius algebra structure on the interval and circle. Hence, by the remarks preceding Proposition 2.8 we get a strict symmetric monoidal functor $\bar{\Xi} \colon \mathbf{Th}(\mathbf{K\text{-}Frob}) \to \mathbf{2Cob}^{\mathrm{ext}}$. In this case, if X is related to Y in $\mathbf{Th}(\mathbf{K\text{-}Frob})$ by a sequence of

associators and unit constraints then X and Y are mapped to the same object in $\mathbf{2Cob}^{\mathrm{ext}}$. We now show that Ξ and $\bar{\Xi}$ define an equivalence of categories.

Let $\vec{n}$ be a general object in $\mathbf{2Cob}^{\mathrm{ext}}$. From the discussion above we have that $\bar{\Xi}\Xi(\vec{n}) = \vec{n}$, so that $\bar{\Xi}\Xi(\vec{n}) = 1_{\mathbf{2Cob}^{\mathrm{ext}}}$. If X is an object of $\mathbf{Th}(\mathbf{K\text{-}Frob})$ then X is a parenthesized word consisting of the symbols $\mathbb{1}, A, C, \otimes$. Let $\bar{\Xi}(X) = (n_1, n_2, \cdots, n_n)$ were the ordered sequence $(n_1, n_2, \cdots, n_n)$ corresponds to the ordered sequence of A's and C's in X. Hence, $\Xi\bar{\Xi}(X)$ is the word obtained from X by removing all the symbols $\mathbb{1}$ and putting all parenthesis to the left. Thus, $\Xi\bar{\Xi}(X)$ is isomorphic to X by a sequence of associators and unit constraints. We have therefore established the desired monoidal equivalence of symmetric monoidal categories. $\qquad\square$

The following special cases are covered by Corollary 4.3.

Definition 4.4. Let $\mathbf{2Cob}^{\mathrm{open}}$, $\mathbf{2Cob}^{\mathrm{closed}} = \mathbf{2Cob}$, and $\mathbf{2Cob}^{\mathrm{planar}}$ denote the subcategories of $\mathbf{2Cob}^{\mathrm{ext}}$ consisting only of purely open cobordisms, purely closed cobordisms, and purely open cobordisms that can be embedded into the plane. An open (respectively closed, planar open) TQFT is a functor from $\mathbf{2Cob}^{\mathrm{open}}$ (respectively $\mathbf{2Cob}^{\mathrm{closed}}$, $\mathbf{2Cob}^{\mathrm{planar}}$) into a symmetric monoidal category $\mathcal{C}$ ($\mathcal{C}$ need not be symmetric in the planar open context).

Corollary 4.5. Let $\mathcal{C}$ be a symmetric monoidal category. The category of open TQFTs in $\mathcal{C}$ is equivalent as a symmetric monoidal category to the category of symmetric Frobenius algebras in $\mathcal{C}$.

The following well-known result on 2-dimensional closed TQFTs [4, 6] follows from Corollary 4.3, as does the 2-dimensional case of [17].

Corollary 4.6. Let $\mathcal{C}$ be a symmetric monoidal category. The category of closed TQFTs in $\mathcal{C}$ is equivalent as a symmetric monoidal category to the category of commutative Frobenius algebras in $\mathcal{C}$.

Corollary 4.7. Let $\mathcal{C}$ be a monoidal category. The category of planar open topological quantum field theories in $\mathcal{C}$ is equivalent to the category of Frobenius algebras in $\mathcal{C}$.

5   Boundary labels

In this section, we generalize the results on knowledgeable Frobenius algebras and on open-closed cobordisms to free boundaries labeled with elements of some set S. The proofs of these results are very similar to the unlabeled case, and so we state only the results.

Definition 5.1. Let S be a set. An *S-coloured knowledgeable Frobenius algebra*

$$(\{A_{ab}\}_{a,b \in S}, \{\mu_{abc}\}_{a,b,c \in S}, \{\eta_a\}_{a \in S}, \{\Delta_{abc}\}_{a,b,c \in S}, \{\varepsilon_a\}_{a \in S}, C, \{\imath_a\}_{a \in S}, \{\imath_a^*\}_{a \in S}) \tag{5.1}$$

in some symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \tau)$ consists of,

- a commutative Frobenius algebra object $(C, \mu, \eta, \Delta, \varepsilon)$ in $\mathcal{C}$,

- a family of objects A_{ab} of $\mathcal{C}$, $a, b \in S$,

- families of morphisms $\mu_{abc} \colon A_{ab} \otimes A_{bc} \to A_{ac}$, $\eta_a \colon \mathbb{1} \to A_{aa}$, $\Delta_{abc} \colon A_{ac} \to A_{ab} \otimes A_{bc}$, $\varepsilon_a \colon A_{aa} \to \mathbb{1}$, $\imath_a \colon C \to A_{aa}$ and $\imath^* \colon A_{aa} \to C$ of $\mathcal{C}$ for all $a, b, c \in S$ such that the following

conditions are satisfied for all $a, b, c, d \in S$:

$$\mu_{abd} \circ (\mathrm{id}_{A_{ab}} \otimes \mu_{bcd}) \circ \alpha_{A_{ab}, A_{bc}, A_{cd}} \; = \; \mu_{acd} \circ (\mu_{abc} \otimes \mathrm{id}_{A_{cd}}), \tag{5.2}$$

$$\mu_{aab} \circ (\eta_{aa} \otimes \mathrm{id}_{A_{ab}}) \; = \; \lambda_{A_{ab}}, \tag{5.3}$$

$$\mu_{abb} \circ (\mathrm{id}_{A_{ab}} \otimes \eta_{bb}) \; = \; \rho_{A_{ab}}, \tag{5.4}$$

$$\alpha_{A_{ab}, A_{bc}, A_{cd}} \circ (\Delta_{abc} \otimes \mathrm{id}_{A_{cd}}) \circ \Delta_{acd} \; = \; (\mathrm{id}_{A_{ab}} \otimes \Delta_{bcd}) \circ \Delta_{abd}, \tag{5.5}$$

$$(\varepsilon_{aa} \otimes \mathrm{id}_{A_{ab}}) \circ \Delta_{aab} \; = \; \lambda_{A_{ab}}^{-1}, \tag{5.6}$$

$$(\mathrm{id}_{A_{ab}} \otimes \varepsilon_{bb}) \circ \Delta_{abb} \; = \; \rho_{A_{ab}}^{-1}, \tag{5.7}$$

$$\Delta_{abd} \circ \mu_{acd} \; = \; (\mathrm{id}_{A_{ab}} \otimes \mu_{bcd}) \circ \alpha_{A_{ab}, A_{bc}, A_{cd}} \circ (\Delta_{abc} \otimes \mathrm{id}_{A_{cd}})$$
$$= \; (\mu_{abc} \otimes \mathrm{id}_{A_{cd}}) \circ \alpha_{A_{ab}, A_{bc}, A_{cd}}^{-1}$$
$$\circ (\mathrm{id}_{A_{ab}} \otimes \Delta_{bcd}), \tag{5.8}$$

$$\varepsilon_{aa} \circ \mu_{aba} \; = \; \varepsilon_{bb} \circ \mu_{bab} \circ \tau_{A_{ab}, A_{ba}}, \tag{5.9}$$

$$\mu_{aaa} \circ (\imath_a \otimes \imath_a) \; = \; \imath_a \circ \mu, \tag{5.10}$$

$$\eta_{aa} \; = \; \imath_a \circ \eta, \tag{5.11}$$

$$\mu_{aab} \circ (\imath_a \otimes \mathrm{id}_{A_{ab}}) \; = \; \mu_{abb} \circ \tau_{A_{bb}, A_{aa}} \circ (\imath_b \otimes \mathrm{id}_{A_{aa}}), \tag{5.12}$$

$$\varepsilon \circ \mu \circ (\mathrm{id}_C \otimes \imath_a^*) \; = \; \varepsilon_{aa} \circ \mu_{aaa} \circ (\imath_a \otimes \mathrm{id}_{A_{aa}}), \tag{5.13}$$

$$\imath_a \circ \imath_b^* \; = \; \mu_{aba} \circ \tau_{A_{ba}, A_{ab}} \circ \Delta_{bab}. \tag{5.14}$$

It is easy to see that the notion of an S-coloured knowledgeable Frobenius algebra precisely models the topological relations of Proposition 3.12 for all possible ways of labeling the free boundaries with elements of the set S. The following consequences of this definition are not difficult to see from the diagrams of Proposition 3.12.

Corollary 5.2. Let $(\{A_{ab}\}, \{\mu_{abc}\}, \{\eta_a\}, \{\Delta_{abc}\}, \{\varepsilon_a\}, C, \{\imath_a\}, \{\imath_a^*\})$ be an S-coloured knowledgeable Frobenius algebra in some symmetric monoidal category $\mathcal{C}$.

1. Each A_{ab}, $a, b \in S$, is a rigid object of $\mathcal{C}$ whose left- and right-dual is given by A_{ba}.

2. Each A_{aa}, $a \in S$, forms a symmetric Frobenius algebra object in $\mathcal{C}$.

3. Each $\imath_a \colon C \to A_{aa}$, $a \in S$, forms a homomorphism of algebras in $\mathcal{C}$.

4. Each $\imath_a^* \colon A_{aa} \to C$, $a \in S$, forms a homomorphism of coalgebras in $\mathcal{C}$.

5. Each A_{ab} forms an A_{aa}-left-A_{bb}-right-bimodule in $\mathcal{C}$.

6. Each A_{ab} forms an A_{aa}-left-A_{bb}-right-bicomodule in $\mathcal{C}$.

Definition 5.3. A *homomorphism*

$$\begin{aligned} f \quad : \quad & (C, \{A_{ab}\}, \{\mu_{abc}\}, \{\eta_a\}, \{\Delta_{abc}\}, \{\varepsilon_a\}, \{\imath_a\}, \{\imath_a^*\}) \\ \to \quad & (C', \{A'_{ab}\}, \{\mu'_{abc}\}, \{\eta'_a\}, \{\Delta'_{abc}\}, \{\varepsilon'_a\}, \{\imath'_a\}, \{\imath'^*_a\}) \end{aligned} \tag{5.15}$$

of S-coloured knowledgeable Frobenius algebras is a pair $f = (\{f_{ab}\}_{a,b \in S}, f_C)$ consisting of a homomorphism of Frobenius algebras $f_C \colon C \to C'$ and a family of morphisms $f_{ab} \colon A_{ab} \to A'_{ab}$, $a, b \in S$ that satisfy the following conditions for all $a, b, c \in S$:

$$\mu'_{abc} \circ (f_{ab} \otimes f_{bc}) \; = \; f_{ac} \circ \mu_{abc}, \tag{5.16}$$

$$\eta'_a \; = \; f_{aa} \circ \eta_a, \tag{5.17}$$

$$\Delta'_{abc} \circ f_{ac} \; = \; (f_{ab} \otimes f_{bc}) \circ \Delta_{abc}, \tag{5.18}$$

$$\varepsilon'_a \circ f_{aa} \; = \; \varepsilon_a, \tag{5.19}$$

$$\imath'_a \circ f_C \; = \; f_{aa} \circ \imath_a, \tag{5.20}$$

$$\imath'^*_a \circ f_{aa} \; = \; f_C \circ \imath_a^*. \tag{5.21}$$

Definition 5.4. By **K-Frob**$^{(S)}(\mathcal{C})$ we denote the category of S-coloured knowledgeable Frobenius algebras in some symmetric monoidal category $\mathcal{C}$ and their homomorphisms.

Definition 5.5. The category of *open-closed TQFTs* in some symmetric monoidal category $\mathcal{C}$ *with free boundary labels* in some set S is the category

$$\mathbf{OC\text{-}TQFT}^{(S)}(\mathcal{C}) := \mathbf{Symm\text{-}Mon}(\mathbf{2Cob}^{\mathrm{ext}}(S), \mathcal{C}). \tag{5.22}$$

In the S-coloured case, the correspondence between the algebraic and the topological category of Corollary 4.3 generalizes to the following result.

Theorem 5.6. Let S be some set and $\mathcal{C}$ be a symmetric monoidal category. The categories **K-Frob**$^{(S)}(\mathcal{C})$ and **OC-TQFT**$^{(S)}(\mathcal{C})$ are equivalent as symmetric monoidal categories.

One can verify [23] that the groupoid algebra of a finite groupoid gives rise to an S-coloured knowledgeable Frobenius algebra for which S is the set of objects of the groupoid.

6   Conclusion

In this paper, we have extended the results of classical cobordism theory to the context of 2-dimensional open-closed cobordisms. Using manifolds with faces with a particular global structure, rather than the full generality of manifolds with corners, we have defined an appropriate category of open-closed cobordisms. Using a generalization of Morse theory to manifolds with corners, we have found a characterization of this category in terms of generators and relations. In order to prove the sufficiency of the relations, we have explicitly constructed the diffeomorphism between an arbitrary cobordism and a normal form which is characterized by topological invariants.

All of the technology outlined above is defined for manifolds with faces of arbitrary dimension. Thus, our work suggests a natural framework for studying extended topological quantum field theories in dimensions three and four. Using 3-manifolds or 4-manifolds with faces, one can imagine defining a category (most likely higher-category) of extended three or four dimensional cobordisms. In both cases, gluing will produce well defined composition operations using the existing technology for manifolds with faces. One could then extract a list of generating cobordisms, again using a suitable generalization of Morse theory.

The main difficulty in obtaining a complete generators and relations description of these higher-dimensional extended cobordism categories is the lack of general theory producing the relations. Specifically, the handlebody theory for manifolds with boundaries and corners is not as advanced as the standard Morse theory for closed manifolds. For the 2-dimensional case, we were able to use relations previously proposed in the literature and to show the sufficiency of these relations by finding the appropriate normal form for 2-dimensional open-closed cobordisms. Our induction proof shows that the proposed relations are in fact necessary and sufficient to reduce an arbitrary open-closed cobordism to the normal form. To extend these results to higher-dimensions, it is expected that a more sophisticated procedure will be required, most likely involving a handlebody theory for manifolds with faces.

We close by commenting on a different approach to TQFTs with corners. In the literature, for example [24], extended TQFTs are often defined for manifolds with corners in which the basic building blocks have the shape of bigons [24] with only one sort of boundary along which one can always glue. This is a special case of our definition which is obtained if every coloured boundary between two corners is shrunk until it disappears and there is a single corner left that now separates two black boundaries.

A   Symmetric monoidal categories

In this appendix, we collect some key definitions for easier reference.

Definition A.1. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ and $(\mathcal{C}', \otimes, \mathbb{1}', \alpha', \lambda', \rho')$ be monoidal categories. A *monoidal functor* $\psi \colon \mathcal{C} \to \mathcal{C}'$ is a triple $\psi = (\psi, \psi_2, \psi_0)$ consisting of,

- a functor $\psi \colon C \to C'$,

- a natural isomorphism $\psi_2 \colon \psi(X) \otimes \psi(Y) \to \psi(X \otimes Y)$, where for brevity we suppress the subscripts indicating the dependence of this isomorphism on X and Y, and

- an isomorphism $\psi_0 \colon \mathbb{1}' \to \psi(\mathbb{1})$,

such that the following diagrams commute for all objects $X, Y, Z \in \mathcal{C}$:

$$
\begin{array}{ccccc}
(\psi(X) \otimes \psi(Y)) \otimes \psi(Z) & \xrightarrow{\psi_2 \otimes \mathbb{1}} & \psi(X \otimes Y) \otimes \psi(Z) & \xrightarrow{\psi_2} & \psi((X \otimes Y) \otimes Z) \\
\Big\downarrow{\scriptstyle a_{\psi(X),\psi(Y),\psi(Z)}} & & & & \Big\downarrow{\scriptstyle \psi(\alpha_{X,Y,Z})} \\
\psi(X) \otimes (\psi(Y) \otimes \psi(Z)) & \xrightarrow{\mathbb{1} \otimes \psi_2} & \psi(X) \otimes \psi(Y \otimes Z) & \xrightarrow{\psi_2} & \psi(X \otimes (Y \otimes Z))
\end{array}
\tag{A.1}
$$

$$
\begin{array}{ccc}
\mathbb{1}' \otimes \psi(X) & \xrightarrow{\lambda'_{\psi(X)}} & \psi(X) \\
\Big\downarrow{\scriptstyle \psi_0 \otimes \mathbb{1}} & & \Big\uparrow{\scriptstyle \psi(\lambda_X)} \\
\psi(\mathbb{1}) \otimes \psi(X) & \xrightarrow{\psi_2} & \psi(\mathbb{1} \otimes X)
\end{array}
\tag{A.2}
$$

$$
\begin{array}{ccc}
\psi(X) \otimes \mathbb{1}' & \xrightarrow{\rho'_{\psi(X)}} & \psi(X) \\
\Big\downarrow{\scriptstyle \mathbb{1} \otimes \psi_0} & & \Big\uparrow{\scriptstyle \psi(\rho_X)} \\
\psi(X) \otimes \psi(\mathbb{1}) & \xrightarrow{\psi_2} & \psi(X \otimes \mathbb{1})
\end{array}
\tag{A.3}
$$

The monoidal functor is called *strict* if ψ_2 and ψ_0 are identities.

Definition A.2. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \tau)$ and $(\mathcal{C}', \otimes, \mathbb{1}', \alpha', \lambda', \rho', \tau')$ be symmetric monoidal categories. A *symmetric monoidal functor* $\psi \colon \mathcal{C} \to \mathcal{C}'$ is a monoidal functor for which the following additional diagram commutes for all $X, Y \in \mathcal{C}$:

$$
\begin{array}{ccc}
\psi(X) \otimes \psi(Y) & \xrightarrow{\tau'_{X,Y}} & \psi(Y) \otimes \psi(X) \\
\Big\downarrow{\scriptstyle \psi_2} & & \Big\downarrow{\scriptstyle \psi_2} \\
\psi(X \otimes Y) & \xrightarrow{\psi(\tau)} & \psi(Y \otimes X)
\end{array}
\tag{A.4}
$$

Definition A.3. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ and $(\mathcal{C}', \otimes, \mathbb{1}', \alpha', \lambda', \rho')$ be monoidal categories and $\psi \colon \mathcal{C} \to \mathcal{C}'$ and $\psi' \colon \mathcal{C} \to \mathcal{C}'$ be monoidal functors. A *monoidal natural transformation* $\varphi \colon \psi \Rightarrow \psi'$ is a natural transformation such that for all objects X, Y of $\mathcal{C}$, the following diagrams commute,

$$
\begin{array}{ccc}
\psi(X) \otimes \psi(Y) & \xrightarrow{\varphi_X \otimes \varphi_Y} & \psi'(X) \otimes \psi'(Y) \\
\Big\downarrow{\scriptstyle \psi_2} & & \Big\downarrow{\scriptstyle \psi'_2} \\
\psi(X \otimes Y) & \xrightarrow{\varphi_{X \otimes Y}} & \psi'(X \otimes Y)
\end{array}
\qquad \text{and} \qquad
\begin{array}{c}
\mathbb{1}' \\
{\scriptstyle \psi_0}\Big\downarrow \quad \searrow{\scriptstyle \psi'_0} \\
\psi(\mathbb{1}) \xrightarrow{\varphi_1} \psi'(\mathbb{1})
\end{array}
\tag{A.5}
$$

Definition A.4. Let $\mathcal{C}$ be a small symmetric monoidal category and let $\mathcal{C}'$ be an arbitrary symmetric monoidal category. We denote by **Symm-Mon**$(\mathcal{C}, \mathcal{C}')$ the category of symmetric monoidal functors $\mathcal{C} \to \mathcal{C}'$ and monoidal natural transformations between them. It is clear that the tensor product of symmetric monoidal functors and monoidal natural transformations defines a symmetric monoidal structure on the category **Symm-Mon**$(\mathcal{C}, \mathcal{C}')$.

Definition A.5. Let $\mathcal{C}$ and $\mathcal{C}'$ be monoidal categories. We say that $\mathcal{C}$ and $\mathcal{C}'$ are *equivalent as monoidal categories* if there is an equivalence of categories $\mathcal{C} \simeq \mathcal{C}'$ given by functors $F \colon \mathcal{C} \to \mathcal{C}'$ and $G \colon \mathcal{C}' \to \mathcal{C}$ and natural isomorphisms $\eta \colon 1_{\mathcal{C}} \Rightarrow GF$ and $\varepsilon \colon FG \Rightarrow 1_{\mathcal{C}'}$ such that both F and G are monoidal functors and η and ε are monoidal natural transformations.

If $\mathcal{C}$ and $\mathcal{C}'$ are symmetric monoidal categories, we say that they are *equivalent as symmetric monoidal categories* if in addition F and G are symmetric monoidal functors.

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