Where are the black hole entropy degrees of freedom ?

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Understanding the area-proportionality of black hole entropy (the 'Area Law') from an underlying fundamental theory has been one of the goals of all models of quantum gravity. A key question that one asks is: *where are the degrees of freedom giving rise to black hole entropy located?* Taking the point of view that entanglement between field degrees of freedom inside and outside the horizon can be a source of this entropy, we show that when the field is in its ground state, the degrees of freedom near the horizon contribute most to the entropy, and the area law is obeyed. However, when it is in an excited state, degrees of freedom far from the horizon contribute more significantly, and deviations from the area law are observed. In other words, we demonstrate that horizon degrees of freedom are responsible for the area law.

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The area proportionality of black hole entropy (the '*Area Law*' (AL)),

$$S_{\rm BH} = \frac{A_H}{4\ell_P^2}, \qquad \left(\ell_P \equiv \sqrt{\hbar G/c^3} = \text{Planck length}\right) \quad (1)$$

which differs from the volume proportionality of familiar thermodynamic systems, has been conjectured to be more fundamental in some sense (the *Holographic Hypothesis*). Black holes are also regarded as theoretical laboratories for quantum gravity. Thus, candidate models of quantum gravity, such as string theory and loop quantum gravity, have attempted to derive the 'macroscopic' AL (1) by counting 'microscopic' degrees of freedom (DOF), using the Von-Neumann/Boltzmann formula [1]:

$$S = -Tr[\rho \ln(\rho)] \; = \ln\Omega, \qquad k_B = 1 \quad (2)$$

where ρ and Ω correspond to the density matrix and number of accessible micro-states, respectively. Depending on the approach, one either counts certain DOF on the horizon, or abstract DOF related to the black hole, and there does not appear to be a consensus about which DOF are relevant or about their precise location [2].

In this article, we attempt to answer these questions in a more general setting, which in fact, may be relevant in any approach. We adopt the point of view that the entanglement between quantum field DOF lying inside and outside of the horizon leads to black-hole entropy. It was shown in Refs. [3, 4] (see also, Refs. [5, 6]) that the entanglement entropy of a massless scalar field propagating in flat space-time (by tracing over a spherical region of radius R) is proportional to the area of the sphere

$$S_{\rm ent} = 0.3 \left(\frac{R}{a}\right)^2 \qquad a \text{ is the UV cutoff}. \quad (3)$$

This suggests that the area law is a generic feature of entanglement, and acquires a physical meaning in the case of black-holes, the latter's horizon providing a natural 'boundary' of the region to trace over. Note that Eq.(3) will continue to hold if the region outside the sphere is traced over instead.

The relevance of $S_{\rm ent}$ to $S_{\rm BH}$ can also be understood from the fact that both entropies are associated with the existence of the *horizons* [7]. Consider a scalar field on a background of a collapsing star. At early times, there is no horizon, and both the entropies are zero. However, once the horizon forms, $S_{\rm BH}$ is non-zero, and if the scalar field DOF inside the horizon are traced over, $S_{\rm ent}$ obtained from the reduced density matrix is non-zero as well.

In Refs. [3, 4], along with the fact that calculations were done in flat space-time, a crucial assumption was made that the scalar field is in the *Ground State* (GS). In Refs. [8, 9], the current authors studied the robustness of the AL by relaxing the second assumption, and showed that for *Generic Coherent States* (GCS) and a class of *Squeezed States* (SS), the law continues to hold, whereas for the *Excited States* (ES), one obtains:

$$S_{\rm ent} = \kappa \left(\frac{R}{a}\right)^{2\alpha}, \qquad \kappa = \mathcal{O}(1) \quad (4)$$

where $\alpha < 1$, and higher the excitation, the smaller its value. In this article, we attempt to provide a physical understanding of this deviation from the AL, by showing that for ES, DOF far from the horizon contribute more significantly than for the GS. We also extend our flat space-time analyses for any $(3+1)-$D spherically symmetric non-degenerate black-hole space-times [10].

We begin by considering a scalar field $\phi(x)$ propagating in a Schwarzschild space-time [17]:

$$ds^2 = -f(r)dt^2 + \frac{dr}{f(r)} + r^2 d\Omega_2^2 \quad , \quad f(r) = 1 - \frac{r_0}{r}\,. \quad (5)$$

where r_0 is the horizon. Transforming to Lemaître coordinates (τ, R) [12]:

$$\tau \;=\; t + r_0\left[\ln\left(\frac{1-\sqrt{r/r_0}}{1+\sqrt{r/r_0}}\right) + 2\sqrt{\frac{r}{r_0}}\right] \quad (6)$$

$$R \;=\; \tau + \frac{2\, r^{\frac{3}{2}}}{3\sqrt{r_0}} \quad \Longrightarrow \quad \frac{r}{r_0} = \left[\frac{3}{2}\frac{(R-\tau)}{r_0}\right]^{2/3},$$

the line-element (5) becomes:

$$ds^2 = -d\tau^2 + \left[\frac{3}{2}\frac{(R-\tau)}{r_0}\right]^{-\frac{2}{3}} dR^2 + \left[\frac{3}{2}\frac{(R-\tau)}{r_0}\right]^{\frac{4}{3}} r_0^{\frac{2}{3}} d\Omega^2 \,. \quad (7)$$

Note that $R(\tau)$ is everywhere space-like (time-like), the metric is non-singular at $r = r_0$ (corresponding to $3(R-\tau)/2r_0 = 1$) and the metric is explicitly time-dependent. The Hamiltonian of a massless, minimally coupled scalar field in the background (7) is given by

$$H(\tau) = \sum_{lm} \frac{1}{2} \int_\tau^\infty dR \left[\frac{2P_{lm}^2(\tau,R)}{3(R-\tau)} + \frac{3r}{2}(R-\tau) \times \left[\partial_R \phi_{lm}(\tau,R)\right]^2 + \sqrt[3]{\frac{r_0}{r}}\, \ell(\ell+1)\phi_{lm}^2(\tau,R)\right] , \quad (8)$$

where $P_{lm}(\tau,R)$ is the canonical conjugate momenta of the spherically reduced scalar field $\phi_{lm}(\tau,R)$, such that $[\hat{\phi}_{lm}(R,\tau_0), \hat{P}_{lm}(R',\tau_0)] = i\delta(R-R')$ and ℓ denotes angular momenta. Note that, in these coordinates, the scalar field and its Hamiltonian explicitly depend on time.

Next, choosing a *fixed* Lemaître time (say, $\tau_0 = 0$) and performing the following canonical transformation [16]

$$P_{lm}(r) = \sqrt{r}\,\pi_{lm}(r) \quad , \quad \phi_{lm}(r) = \frac{\varphi_{lm}(r)}{r} \,, \quad (9)$$

the Hamiltonian (8) transforms to:

$$H(0) = \sum_{lm} \int_0^\infty \frac{dr}{2} \left[\pi_{lm}^2(r) + r^2 \left[\partial_r \frac{\varphi_{lm}}{r}\right]^2 + \frac{\ell(\ell+1)}{r^2}\varphi_{lm}^2(r)\right] \quad (10)$$

which is simply the Hamiltonian of a free scalar field in flat space-time! Eq.(10) holds for *any* fixed τ, for which the results of Refs. [4, 9], namely relations (3) and (4), go through, provided one traces over either the region $R \in [0, \frac{2}{3}r_0)$ or the region $R \in [\frac{2}{3}r_0, \infty)$ [10]. Extension to any non-degenerate spherically symmetric space-times is straightforward [18].

Remaining steps in the computation of $S_{\rm ent}$ are:
1. *Discretize the Hamiltonian (10)*, i. e.,

$$H = \sum_{lm} \frac{1}{2a} \sum_{j=1}^N \left[\pi_{lm,j}^2 + \left(j+\frac{1}{2}\right)^2 \left(\frac{\varphi_{lm,j}}{j} - \frac{\varphi_{lm,j+1}}{j+1}\right)^2 + \frac{l(l+1)}{j^2}\, \varphi_{lm,j}^2\right] \quad , \quad (11)$$

where $\pi_{lm,j}$ denotes the conjugate momenta of $\varphi_{lm,j}$, $L = (N+1)a$ is the box size and a is the radial lattice size. This is of the form of the Hamiltonian of N coupled HOs

$$H = \frac{1}{2}\sum_{i=1}^N p_i^2 + \frac{1}{2}\sum_{i,j=1}^N x_i K_{ij} x_j \qquad i,j = 1,\ldots,N \quad (12)$$

with the interaction matrix elements K_{ij} given by:

$$K_{ij} = \frac{1}{i^2}\left[l(l+1)\,\delta_{ij} + \frac{9}{4}\,\delta_{i1}\delta_{j1} + \left(N-\frac{1}{2}\right)^2 \delta_{iN}\delta_{jN} + \left(\left(i+\frac{1}{2}\right)^2 + \left(i-\frac{1}{2}\right)^2\right)\delta_{i,j(i\neq 1,N)}\right] - \left[\frac{(j+\frac{1}{2})^2}{j(j+1)}\right]\delta_{i,j+1} - \left[\frac{(i+\frac{1}{2})^2}{i(i+1)}\right]\delta_{i,j-1} \,, \quad (13)$$

where the last two terms are the nearest-neighbor interactions.
2. *Choose the state of the quantum field:* For GS ($r = 1, \alpha_i = 0$), GCS ($r = 1, \alpha_i =$ arbitrary), or a class of SS ($\alpha = 0, r =$ arbitrary), the N-particle wave function $\psi(x_1 \ldots x_N)$ is given by [9]

$$\psi(x_1 \cdots x_N) = \left|\frac{\Omega}{\pi^N}\right| \exp\left[-\frac{1}{2}\sum_i r\kappa_{Di}^{1/2}\left(\underline{x}_i - \alpha_i\right)^2\right] . \quad (14)$$

For ES (or 1-Particle state), the N-particle wave function $\psi(x_1 \ldots x_N)$ is given by

$$\psi(x_1 \cdots x_N) = \left|\frac{2\Omega}{\pi^N}\right|^{\frac{1}{4}} \sum_{i=1}^N a_i k_{Di}^{\frac{1}{4}}\, \underline{x}_i \exp\left[-\frac{1}{2}\sum_j k_{Dj}^{\frac{1}{2}}\, \underline{x}_j^2\right] \quad (15)$$

where $a^T = (a_1, \ldots, a_N)$ are the expansion coefficients (normalization requires $a^T a = 1$). For our computations, we choose $a^T = 1/\sqrt{o}(0, \cdots 0, 1 \cdots 1)$ with the last o columns being non-zero. For details, see Ref. [9].
3. *Obtain the density matrix:* For an arbitrary wavefunction $\psi(x_1, \ldots, x_N)$, the density matrix — tracing over first n of the N field points — is given by:

$$\rho\,(t;t') = \int \prod_{i=1}^n dx_i\, \psi(x_1, \ldots, x_n; t_1, \ldots, t_{N-n}) \times \psi^\star(x_1, \ldots, x_n; t'_1, \ldots, t'_{N-n}) \quad (16)$$

where: $t_j \equiv x_{n+j}, t \equiv t_1, \ldots, t_{N-n},\ j = 1..(N-n), x^T = (x_1, \ldots, x_n; t_1, \ldots, t_{N-1}) = (x_1, \ldots, x_n; t)$. This step, in general, cannot be evaluated analytically and requires numerical techniques. For GS/CS/SS, substituting (14) in the above expression and using the relation (2) gives Eq. (3). For ES, on the other hand, this leads to the relation (4) [9].

Now, to locate those DOF which are responsible for entropy, we take a closer look at the interaction matrix (13). As mentioned earlier, the last two terms are the nearest-neighbor interactions between the oscillators and constitute entanglement. As expected, if these terms are set to zero (by hand), $S_{\rm ent}$ vanishes. Instead, let us do the following:
(i) Set the interactions to zero (by hand) everywhere except in the 'window' ($q-s \le i \le q+s$), with $s \le q \le n-s$, i.e., restrict the thickness of the interaction region to $2s+1$ radial lattice points, while moving it rigidly across

from the origin to horizon. In Fig. (1), we have plotted the percentage contribution to the total entropy as a function of q for the GS/GCS/SS ($o = 0$, solid thin curve) and the ES ($o = 30(50)$, bold (light) thick curve). [We choose $N = 300$ and $n = 100, 150, 200$. The numerical error in the evaluation of the entropy is less than 0.1%.] Fig. (1) shows that:

- When the interaction region does not include the horizon, the entanglement entropy is zero and it is maximum if the horizon lies exactly in its middle. The first observation confirms that entanglement between DOF inside and outside the horizon contributes to entropy, while the second suggests that DOF close to the horizon contribute more to the entropy compared to those far from it.

- When the number of excited states is increased (i. e. $o = 30, 50$), the percentage contribution to the total entropy close to the horizon is less compared to that of GS and the (bold/light thick) curves are flatter. These indicate that, for ES, there is a significant contribution from the DOF far-away from the horizon.

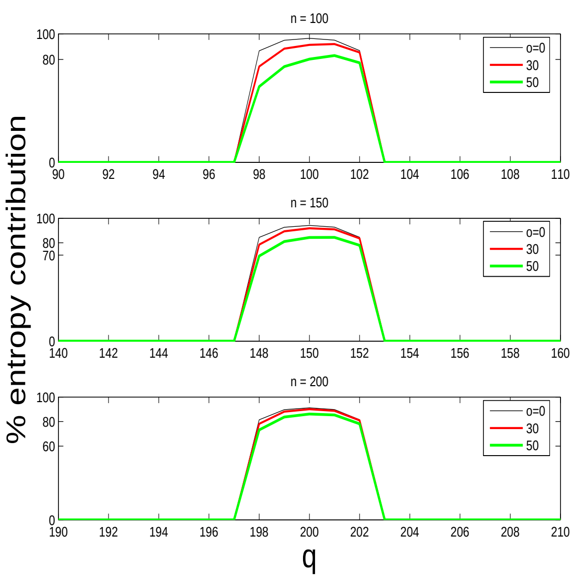

FIG. 1: Plot of the percentage contribution to the entropy for the GS and ES as we move the window $q-2 \leq i < q+2$ from $q = 2$ to $q = n$, for $N = 300$ and $n = 100, 150$ and 200. The solid thin curve is for GS $o = 0$ and bold (light) thick curve for $o = 30(50)$.

(ii) To further investigate, we now set the interactions to zero (by hand) everywhere except in the window $p \leq i \leq n$, with the horizon as its outer boundary, and vary the width of the window $t \equiv n - p$ from 0 to n. For $t = n$, the total entropy is recovered, while for $t = 0$, i.e. zero interaction region, entropy vanishes. In Fig. 2, we have plotted the normalized GS/GCS/SS and ES entropies $[S_{ent}(t)]$ vs t for $n = 100, 150$ and 200. [Here again, the solid thin curve is for GS and bold (light) thick curve for $o = 30(50)$.] We infer the following:

- For the GS/GCS/SS, the entropy reaches the GS entropy for as little as $t = 3$. In other words, the interaction region encompassing DOF close to the horizon contribute to most of the entropy for the GS/GCS/SS.

- In the case of ES, the entropy reaches the ES entropy only for $t = 15(20)$ [for $o = 30(50)$]. This indicates that: (a) The DOF far-away from the horizon have a greater contribution than that of GS and (b) As the number of excited states increases, contribution far-away from the horizon also increases.

  This leads us to *one of the main conclusions* of this article: the greater the deviation from the AL, the larger is the contribution of the DOF far-away from the horizon. It can be shown that the density matrix for the ES is *more spread out* than for the GS.

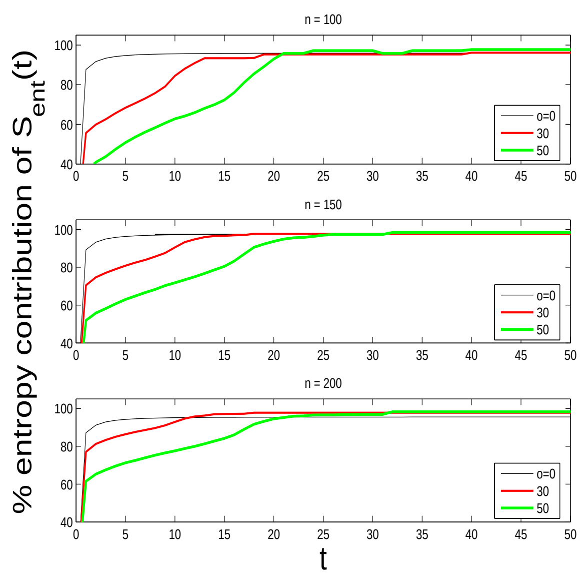

FIG. 2: Plot of the percentage contribution of $S_{ent}(t)$ for the GS and ES. Here again, $N = 300$, $n = 100, 150$ and 200, and the solid thin curve is for GS and bold (light) thick curve for $o = 30(50)$.

(iii) To understand this further, let us define

$$\Delta pc(t) = pc(t) - pc(t-1) \quad \text{where} \quad pc(t) = \frac{S_{\rm ent}(t)}{S_{\rm ent}} \times 100\,,$$

where $pc(t)$ is the percentage contribution to the total entropy by an interaction region of thickness t and $\Delta pc(t)$ is the percentage increase in entropy when the interaction region is incremented by one radial lattice point. In other words, $\Delta pc(t)$ is the slope of the curves in Fig. (2). In Fig. (3), we have plotted $\Delta pc(t)$ vs $(n - t)$ for

GS and ES. For the GS/GCS/SS, it is seen that when the first lattice point just inside the horizon is included in the interaction region, the entropy increases from 0 to 85% of the GS entropy. Inclusion of the next three points add another $9\%, 3\%, 1\%$ respectively. The contributions to the entropy decrease rapidly and by the time the $(n/3)^{th}$ point is included, entropy barely increases by a hundredth of a percent! For ES however, inclusion of one lattice point adds 70(50)%, for $o = 30(50)$, to the entropy, while the next few points add $9\%, 4\%, 3\% \cdots$ respectively. The corresponding slopes are smaller.

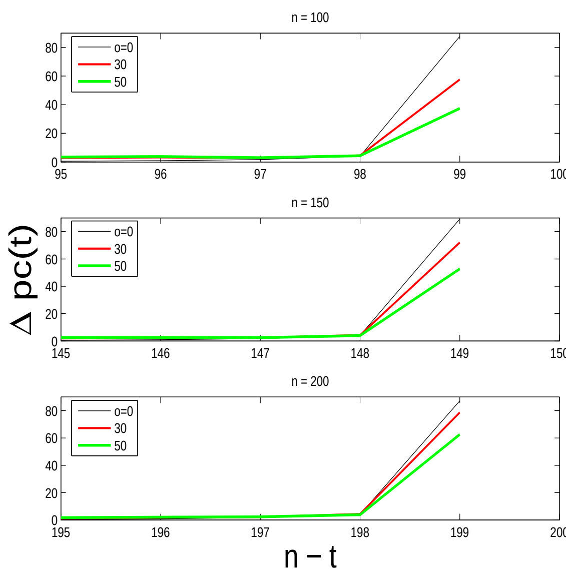

FIG. 3: Plot of $\Delta pc(t)$ vs $n - t$ for GS and ES. Here again, $N = 300$, $n = 100, 150$ and 200, and the solid thin curve is for GS and bold (light) thick curve for $o = 30(50)$.

Figures (2) and (3) suggest the next key result of this article - most of the entropy comes from the DOF close to the horizon. However, the DOF deep inside must also be taken into account for the AL (3) to emerge for the GS/GCS/SS, and the law (4) to emerge for the ES. These DOF affect the horizon DOF via the nearest-neighbor interactions in (13).

Our work clearly demonstrates the close relationship between the AL and the horizon DOF, and that when the latter become less important, the entropy scales as a power of area less than unity. This can be understood from the following heuristic picture: taking the point of view that S_{BH} is proportional to $\mathcal{N}$, the number of DOF *on the horizon*, and that for the GS/CS/SS there is one DOF per Planck-area, such that $\mathcal{N} \propto A_H$, the AL follows (This is known as the *it-from-bit picture* [14]). For ES however, since this number is seen to get diminished, it will be given by another function $\mathcal{N} = f(A) < A$. Now, since current results are expected to be valid when $A \gg 1$ (in Planck units) (such that backreaction effects on the background can be neglected), it is quite plausible that $f(A) \sim A^\alpha$, $\alpha < 1$, just as we obtain. Note that the above reasoning continues to hold when the outside of the horizon is traced over. Another way of understanding our result is as follows: all interactions being of the nearest-neighbor type [cf. Eq.(13)], the degrees of freedom deep inside the horizon influence the ones near the horizon (and hence contribute to the entropy) only indirectly, i.e. via the intermediate degrees of freedom. Evidently, their influence diminishes with their increasing distance from the horizon. When they are excited however, they have more energy to spare, resulting in an increased overall effect. Further investigations with superpositions of GS and ES are expected to shed more light on this [15].

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[17] The perturbations of a $(3+1)$-dimensional static spherically symmetric black holes result in two kinds – axial

and polar – of gravitational perturbations. The equation of motion of the axial perturbations is same as that of a test scalar field propagating in the black-hole background [11, 15]. Hence, by computing $S_{\rm ent}$ of the test scalar fields, we obtain entropy of black-hole metric perturbations.

[18] Ideally, one would like to fix the vacuum state at some Lemaitre time $\tau = 0$ and study the evolution of the modes at late time leading to Hawking effect. Such an analysis is nontrivial for the $(3+1)-D$ black-holes. Even in the case of $(1+1)-D$ black-holes, this issue has been addressed with limited success [13]. In the rest of the article, we will not consider these effects.