KLT relations from the Einstein-Hilbert Lagrangian

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Abstract

The Kawai-Lewellen-Tye (KLT) relations derived from string theory tell us that perturbative gravity amplitudes are the "square" of the corresponding amplitudes in gauge theory. Starting from the light-cone Lagrangian for pure gravity we make these relations manifest off-shell, for three- and four-graviton vertices, at the level of the action.

1 Introduction

The Kawai-Lewellen-Tye (KLT) relations relate tree-level amplitudes in closed and open string theories [1]. In the field theory limit the KLT relations, for three- and four-point amplitudes, reduce to[1]

$$\begin{aligned} M_3^{\text{tree}}(1,2,3) &= A_3^{\text{tree}}(1,2,3)\, A_3^{\text{tree}}(1,2,3)\ , \\ M_4^{\text{tree}}(1,2,3,4) &= -i\, s_{12}\, A_4^{\text{tree}}(1,2,3,4)\, A_4^{\text{tree}}(1,2,4,3)\ , \end{aligned} \tag{1.1}$$

where the M_n represent gravity amplitudes and the A_n are color-ordered [3,4] amplitudes in pure Yang-Mills theory ($s_{ij} \equiv -(p_i + p_j)^2$). Although the KLT relations apply only at the tree-level they have been used, with great success, in conjunction with unitarity based methods to derive loop amplitudes in gravity [2,5]. In particular, these relations have proven invaluable in studying the ultra-violet properties of $\mathcal{N} = 8$ supergravity [6]. The question of whether the KLT relations are valid only for on-shell amplitudes or, more generally, at the level of the Lagrangian remains open [7]. This is the issue we focus on in this letter.

The tree-level amplitudes take a very compact form in a helicity basis. Thus when attempting to derive the KLT relations starting from the gravity Lagrangian it seems natural to work in light-cone gauge where only the helicity states propagate. Tree-level amplitudes in which precisely two external legs carry negative helicity are called maximally helicity violating (MHV) amplitudes. A very simple expression for all the MHV amplitudes in Yang-Mills theory was given in [8]. An MHV-Lagrangian (also referred to as the CSW Lagrangian) where the fundamental vertices are off-shell versions of the MHV amplitudes was proposed in [9]. In [10] and [11] it was shown how this MHV-Lagrangian can be derived from the usual light-cone Yang-Mills Lagrangian by a suitable field redefinition.

In this letter we perform a field redefinition, similar to that in [10,11], on the light-cone gravity Lagrangian. Although the shifted Lagrangian is not simply the sum of MHV-vertices, the off-shell KLT relations, to the order examined in this letter, are manifest.

2 Yang-Mills

We start by sketching schematically, the proposal of [10,11] for Yang-Mills. The light-cone Yang-Mills Lagrangian is of the form

$$L \sim L_{+-} + L_{++-} + L_{+--} + L_{++--}\ , \tag{2.1}$$

where the indices, in no particular order, refer to helicity. The field redefinition maps the first two terms (the kinetic and one cubic term) into a purely kinetic term. This transformation also generates an infinite series of higher order terms producing exactly the MHV-Lagrangian

$$L_{YM} \sim L_{+-} + L_{+--} + L_{++--} + L_{+++--} + L_{++++--} + \ldots + L_{(+)^n--} + \ldots\ . \tag{2.2}$$

Again, this is merely a formal way of writing the Lagrangian. For example, L_{++--} receives contributions from the two inequivalent orderings $\mathrm{tr}(A\bar{A}A\bar{A})$ and $\mathrm{tr}(AA\bar{A}\bar{A})$ where A and

[1] For higher-point generalizations see [2].

$\bar{A}$ are gluons of helicity[2] $+1$ and -1 respectively. Each trace is multiplied by an off-shell continuation (cf. appendix A) of the appropriate Parke-Taylor amplitude [4, 8]

$$\frac{\langle k\, l\rangle^4}{\prod_{i=1}^{n}\,\langle i\,(i+1)\rangle}\ , \qquad n+1\equiv 1\ . \tag{2.3}$$

We will not go into details regarding the derivation of these results which can be found in [10–13]. The analysis in the gravity case is completely analogous and is presented in detail in section 3. The hope is that a similar field redefinition in pure gravity will generate interaction terms which make KLT factorization manifest. The purpose of this letter is to examine this issue.

3 Gravity in light-cone gauge

We follow closely, in this section, the light-cone formulation of gravity in [14]. Here, we only review the key features of this formulation and refer the reader to appendix C in [14] for a detailed derivation of the results presented below.
The Einstein-Hilbert action reads

$$S_{EH} = \int d^4x\,\mathcal{L}\; = \;\frac{1}{2\,\kappa^2}\int d^4x\,\sqrt{-g}\,R\ , \tag{3.1}$$

where $g=\det g_{\mu\nu}$ and R is the curvature scalar. Light-cone gauge is chosen by setting

$$g_{--}\; = \;g_{-i}\; = \;0\ , \qquad i=1,2\ . \tag{3.2}$$

Our conventions and notation are explained in appendix A. The metric is parameterized as follows

$$g_{+-}\; = \;-\,e^{\frac{\psi}{2}}\ , \quad g_{ij}\; = \;e^{\psi}\,\gamma_{ij}\ . \tag{3.3}$$

The field ψ is real while γ_{ij} is a 2×2 real, symmetric, unimodular matrix. The $R_{-i}=0$ constraint allows us to eliminate g^{-i}. From the $R_{--}=0$ constraint we find

$$\psi = \frac{1}{4}\,\frac{1}{\partial_-^2}\left(\partial_-\gamma^{ij}\partial_-\gamma_{ij}\right)\ . \tag{3.4}$$

The Lagrangian density now reads

$$\mathcal{L} = \frac{1}{2\kappa^2}\,\sqrt{-g}\,\left(2g^{+-}R_{+-}+g^{ij}R_{ij}\right)\ . \tag{3.5}$$

We expand this to find [15]

$$\begin{aligned}\mathcal{L}\;\; = \;\; & \frac{1}{2\kappa^2}\Big\{\,\mathrm{e}^{\psi}\left(\frac{3}{2}\partial_+\partial_-\psi-\frac{1}{2}\partial_+\gamma^{ij}\partial_-\gamma_{ij}\right)\\ & -\mathrm{e}^{\frac{\psi}{2}}\gamma^{ij}\left(\frac{1}{2}\partial_i\partial_j\psi-\frac{3}{8}\partial_i\psi\partial_j\psi-\frac{1}{4}\partial_i\gamma^{kl}\partial_j\gamma_{kl}+\frac{1}{2}\partial_i\gamma^{kl}\partial_k\gamma_{jl}\right)\\ & -\frac{1}{2}\mathrm{e}^{-\frac{3}{2}\psi}\gamma^{ij}\frac{1}{\partial_-}R_i\frac{1}{\partial_-}R_j\,\Big\}\ ,\end{aligned} \tag{3.6}$$

[2]The helicity label assumes that the particle is outgoing.

where

$$R_i = \mathrm{e}^{\psi}\bigg(-\frac{1}{2}\partial_-\gamma^{jk}\partial_i\gamma_{jk} + \frac{3}{2}\partial_-\partial_i\psi - \frac{1}{2}\partial_i\psi\partial_-\psi\bigg) - \partial_k\Big(\mathrm{e}^{\psi}\gamma^{jk}\partial_-\gamma_{ij}\Big) \ . \tag{3.7}$$

This is the closed form of the Lagrangian.

3.1 The perturbative expansion

In order to obtain a perturbative expansion of the metric we choose

$$\gamma_{ij} = \left(\mathrm{e}^{\kappa H}\right)_{ij} \ , \qquad H = \frac{1}{\sqrt{2}}\begin{pmatrix} h+\bar{h} & -i(h-\bar{h}) \\ -i(h-\bar{h}) & -h-\bar{h}\end{pmatrix} \ , \tag{3.8}$$

where h and $\bar{h}$ represent gravitons of helicity $+2$ and -2 respectively. The light-cone Lagrangian density for pure gravity, to order κ^2 [14], reads [3]

$$\begin{aligned}
\mathcal{L} \;=\;& \bar{h}\,\Box\, h \\
& +2\kappa\,\bar{h}\,\partial_-^2\left(\frac{\bar{\partial}}{\partial_-}h\frac{\bar{\partial}}{\partial_-}h - h\frac{\bar{\partial}^2}{\partial_-^2}h\right) + 2\kappa\, h\,\partial_-^2\left(\frac{\partial}{\partial_-}\bar{h}\frac{\partial}{\partial_-}\bar{h} - \bar{h}\frac{\partial^2}{\partial_-^2}\bar{h}\right) \\
& +2\kappa^2\bigg\{ \frac{1}{\partial_-^2}\left(\partial_-h\partial_-\bar{h}\right)\frac{\partial\bar{\partial}}{\partial_-^2}\left(\partial_-h\partial_-\bar{h}\right) + \frac{1}{\partial_-^3}\left(\partial_-h\partial_-\bar{h}\right)\left(\partial\bar{\partial}h\,\partial_-\bar{h} + \partial_-h\partial\bar{\partial}\bar{h}\right) \\
& -\frac{1}{\partial_-^2}\left(\partial_-h\partial_-\bar{h}\right)\left(2\,\partial\bar{\partial}h\,\bar{h} + 2\,h\partial\bar{\partial}\bar{h} + 9\,\bar{\partial}h\partial\bar{h} + \partial h\bar{\partial}\bar{h} - \frac{\partial\bar{\partial}}{\partial_-}h\,\partial_-\bar{h} - \partial_-h\frac{\partial\bar{\partial}}{\partial_-}\bar{h}\right) \\
& -2\frac{1}{\partial_-}\left(2\bar{\partial}h\,\partial_-\bar{h} + h\partial_-\bar{\partial}\bar{h} - \partial_-\bar{\partial}h\bar{h}\right)h\,\partial\bar{h} - 2\frac{1}{\partial_-}\left(2\partial_-h\,\partial\bar{h} + \partial_-\partial h\,\bar{h} - h\partial_-\partial\bar{h}\right)\bar{\partial}h\,\bar{h} \\
& -\frac{1}{\partial_-}\left(2\bar{\partial}h\,\partial_-\bar{h} + h\partial_-\bar{\partial}\bar{h} - \partial_-\bar{\partial}h\bar{h}\right)\frac{1}{\partial_-}\left(2\partial_-h\,\partial\bar{h} + \partial_-\partial h\,\bar{h} - h\partial_-\partial\bar{h}\right) \\
& -h\,\bar{h}\left(\partial\bar{\partial}h\,\bar{h} + h\partial\bar{\partial}\bar{h} + 2\,\bar{\partial}h\partial\bar{h} + 3\frac{\partial\bar{\partial}}{\partial_-}h\,\partial_-\bar{h} + 3\partial_-h\frac{\partial\bar{\partial}}{\partial_-}\bar{h}\right)\bigg\} \ .
\end{aligned} \tag{3.9}$$

As in (2.1) the three-vertex terms are of the form $(-,+,+)$ and $(+,-,-)$. In analogy to Yang-Mills, a solution to the self-duality condition

$$R_{\mu\nu\rho\sigma} = \frac{i}{2}\,\epsilon_{\mu\nu}{}^{\alpha\beta}R_{\alpha\beta\rho\sigma} \ , \tag{3.10}$$

is

$$\bar{h} = 0 \ , \quad \Box\, h + 2\kappa\,\partial_-^2\left(\frac{\bar{\partial}}{\partial_-}h\frac{\bar{\partial}}{\partial_-}h - h\frac{\bar{\partial}^2}{\partial_-^2}h\right) \;=\; 0 \ , \tag{3.11}$$

where the second relation is the $\bar{h}$ equation of motion (at $\bar{h}=0$). Thus, as in Yang-Mills, we will map the first two terms in (3.9) to a free theory. Further discussions regarding this point may be found in [12].

[3]As seen in appendix C of [14], a field redefinition which removes occurences of ∂_+ from the interaction terms has been performed.

3.2 The field redefinition

We seek a transformation $(h, \bar{h}) \to (C, \bar{C})$ such that[4]

$$K = -\bar{h}\partial_+\partial_- h + \bar{h}V(h) = -\bar{C}\partial_+\partial_- C + \bar{C}\partial\bar{\partial}C \ , \tag{3.12}$$

where

$$V(h) = \partial\bar{\partial}h + \kappa\,\partial_-^2\left(\frac{\bar{\partial}}{\partial_-}h\,\frac{\bar{\partial}}{\partial_-}h - h\,\frac{\bar{\partial}^2}{\partial_-^2}h\right) . \tag{3.13}$$

The remaining three- and four-point vertices in (3.9) all involve exactly two negative helicity gravitons. Since MHV amplitudes also involve exactly two negative helicity legs, we aim to preserve this structure[5]. In analogy with Yang-Mills, we choose h to be a function of C alone while $\bar{h}$ is chosen to be a function of both C and $\bar{C}$. This field redefinition is not unique and we will comment on this below.

To find the explicit transformation, which is in fact a canonical transformation on the phase space with coordinates (C, π_C), we start with a generating function of the form $G(C, \pi_h) = \int g(C)\,\pi_h$. Then

$$\pi_C \equiv \frac{\partial L}{\partial(\partial_+ C)} = \partial_-\bar{C} = \frac{\delta G}{\delta C} = \int \frac{\delta g}{\delta C}\pi_h \ , \qquad h = \frac{\delta G}{\delta \pi_h} = g(C) \ . \tag{3.14}$$

Since $\pi_h = \partial_-\bar{h}$ we have

$$\partial_-\bar{C}(y) = \int d^3x\,\partial_-\bar{h}(x)\,\frac{\delta h(x)}{\delta C(y)} \ , \tag{3.15}$$

where the integral is performed on a surface of constant x^+. The Lagrangian density then reads (here and below we drop surface terms)

$$\mathcal{L} = -\bar{C}\partial_+\partial_- C + \bar{C}\partial\bar{\partial}C \ = \partial_-\bar{C}\,\partial_+ C - \partial_-\bar{C}\,\frac{\partial\bar{\partial}}{\partial_-}C \ . \tag{3.16}$$

Using (3.15) the Lagrangian becomes

$$L = \int d^3x\,\partial_-\bar{h}(x)\partial_+ h(x) - \int d^3x \int d^3y\,\partial_-\bar{h}(y)\frac{\partial\bar{\partial}}{\partial_-}C(x)\,\frac{\delta h(y)}{\delta C(x)} \ . \tag{3.17}$$

We want this to be equal to

$$L = \int d^3x \left(\partial_-\bar{h}(x)\partial_+ h(x) - \partial_-\bar{h}(x)\frac{1}{\partial_-}V(h(x))\right) , \tag{3.18}$$

implying that

$$\frac{\partial\bar{\partial}}{\partial_-}h(x) + \kappa\,\partial_-\,\left(\frac{\bar{\partial}}{\partial_-}h\,\frac{\bar{\partial}}{\partial_-}h \ - \ h\,\frac{\bar{\partial}^2}{\partial_-^2}h\right)(x) = \int d^3y\,\frac{\partial\bar{\partial}}{\partial_-}C(y)\frac{\delta h(x)}{\delta C(y)} \ . \tag{3.19}$$

[4]Note that the d'Alembertian is $\Box = 2(\partial\bar{\partial} - \partial_+\partial_-)$. See appendix A for further details.

[5]We point out that higher order terms in (3.6) do not possess this structure.

In momentum space, this becomes

$$\frac{p\bar{p}}{p_-}h(p_-) \;-\; \int d^3m\, \frac{m\bar{m}}{m_-}C(m)\,\frac{\delta h(p)}{\delta C(m)} \;=$$
$$-\,\kappa \int d^3k\, d^3l\, \delta^{(3)}(p-k-l)\ (k_- + l_-)\left(\frac{\bar{k}\bar{l}}{k_-l_-} - \frac{\bar{l}^2}{l_-^2}\right) h(k)\, h(l)\ . \tag{3.20}$$

For h, we choose the ansatz

$$h(p) = \sum_{n=1}^{\infty} \int \prod_{i=1}^{n} d^3k_i\ Z^{(n)}(p_1, k_1, \ldots, k_n)\, C(k_1) \ldots C(k_n)\ , \tag{3.21}$$

so (3.20) implies

$$\int d^3k\, d^3l \left(\frac{p\bar{p}}{p_-} \;-\; \frac{k\bar{k}}{k_-} \;-\; \frac{l\bar{l}}{l_-}\right) Z^{(2)}(p,k,l)\, C(k)\, C(l) \;=$$
$$-\kappa \int d^3k\, d^3l\, (k_- + l_-) \left(\frac{\bar{k}\bar{l}}{k_-l_-} \;-\; \frac{\bar{l}^2}{l_-^2}\right) C(k)\, C(l)\, \delta^{(3)}(p-k-l)\ . \tag{3.22}$$

Thus

$$\begin{aligned} Z^{(1)}(p,k) \;&=\; \delta^{(3)}(p-k)\ , \\ Z^{(2)}(p,k,l) \;&=\; \frac{\kappa}{2}\ (k_- + l_-)\, \frac{\frac{\bar{l}^2}{l_-^2} \;+\; \frac{\bar{k}^2}{k_-^2} \;-\; 2\frac{\bar{k}\bar{l}}{k_-l_-}}{\frac{p\bar{p}}{p_-} \;-\; \frac{k\bar{k}}{k_-} \;-\; \frac{l\bar{l}}{l_-}}\, \delta^{(3)}(p-k-l) \\ &=\; -\frac{\kappa}{2}\, \frac{p_-^2}{k_-l_-}\, \frac{[k\,l]}{\langle k\,l\rangle}\, \delta^{(3)}(p-k-l)\ . \end{aligned} \tag{3.23}$$

From (3.15) we also find

$$p_-\bar{h}(p) = p_-\bar{C}(p) - \int d^3k\, d^3l\, k_-\big(\,Z^{(2)}(-k,-p,l) + Z^{(2)}(-k,l,-p)\big)\bar{C}(k)C(l) + \ldots \tag{3.24}$$

which can be rewritten as

$$\bar{h}(p) \;=\; \bar{C}(p) + \kappa \int d^3k\, d^3l\, \frac{k_-^3}{p_-^2 l_-}\, \frac{[k\,l]}{\langle k\,l\rangle}\, \bar{C}(k)C(l) \;+\; \ldots \tag{3.25}$$

It is straightforward to work out a recursion relation for the coefficients $Z^{(n)}$ which can then be solved to any desired order. We will not present the details here.

3.3 The shifted gravity action

After performing the field redefinition described in the previous section we find that the gravity action, to order κ^2, is

$$\int d^4p\, \bar{C}(-p)\, p^2\, C(p) + \kappa \int d^4p\ d^4k\ d^4l\ \frac{\langle k\,l\rangle^6}{\langle l\,p\rangle^2\, \langle p\,k\rangle^2}\, C(p)\bar{C}(k)\bar{C}(l)\ \delta^{(4)}(p{+}k{+}l) \tag{3.26}$$
$$+\ \kappa^2 \int d^4p\ d^4q\ d^4k\ d^4l\ \frac{\langle k\,l\rangle^8\, [k\,l]}{\langle k\,l\rangle\, \langle k\,p\rangle\, \langle k\,q\rangle\, \langle l\,p\rangle\, \langle l\,q\rangle\, \langle p\,q\rangle^2}\, C(p)C(q)\bar{C}(k)\bar{C}(l)\ \delta^{(4)}(p{+}q{+}k{+}l)$$
$$+\ \kappa^2 \int d^4p\, d^4q\, d^4k\, d^4l\, \big(J(p,q,k,l)\, p^2 + K(p,q,k,l)\, k^2\big)\, C(p)C(q)\bar{C}(k)\bar{C}(l)\delta^{(4)}(p{+}q{+}k{+}l)\ .$$

We stress that the coefficients in the action above are off-shell. Note that the four-graviton amplitude does not receive exchange contributions due to the structure of the action at the cubic level after the field redefinitions (3.21) and (3.25). The functions J and K turn out to be fairly complicated but are irrelevant for on-shell four-point scattering since the third line vanishes on-shell. In particular, when interaction vertices are proportional to the free equations of motion they can be eliminated by a suitable field redefinition [16]. The required field redefinitions are[6]

$$
\begin{aligned}
C(p) \to\ & C(p) - \kappa^2 \int d^4q\, d^4k\, d^4l\ \ K(k,q,-p,l)\ C(k)C(q)\bar{C}(l)\ \delta^{(4)}(-p+q+k+l)\ , \\
\bar{C}(p) \to\ & \bar{C}(p) - \kappa^2 \int d^4q\, d^4k\, d^4l\ \ J(-p,q,k,l)\ \bar{C}(k)\bar{C}(l)\, C(q)\ \delta^{(4)}(-p+q+k+l)\ ,
\end{aligned} \tag{3.27}
$$

and these eliminate the third line in (3.26). The light-cone action for gravity to order κ^2 thus reads

$$
\begin{aligned}
&\int d^4p\, \bar{C}(-p)\, p^2\, C(p) + \kappa \int d^4p\ d^4k\ d^4l\ \frac{\langle k\, l\rangle^6}{\langle l\, p\rangle^2 \langle p\, k\rangle^2}\, C(p)\bar{C}(k)\bar{C}(l)\ \delta^{(4)}(p+k+l) \qquad (3.28) \\
&+\ \kappa^2 \int d^4p\ d^4q\ d^4k\ d^4l\ \frac{\langle k\, l\rangle^8\, [k\, l]}{\langle k\, l\rangle \langle k\, p\rangle \langle k\, q\rangle \langle l\, p\rangle \langle l\, q\rangle \langle p\, q\rangle^2}\, C(p)C(q)\bar{C}(k)\bar{C}(l)\, \delta^{(4)}(p+q+k+l)\ .
\end{aligned}
$$

These off-shell vertices clearly factorize into products of off-shell MHV vertices in Yang-Mills. In particular this confirms, *off-shell*, the relations (1.1) for three- and four-point vertices. It will be interesting to see if this KLT factorization extends to higher orders in the action where non-MHV vertices appear.

* * *

In contrast to the Yang-Mills case, the MHV vertices in gravity appear only after a further field redefinition (3.27) that removes interaction vertices proportional to the free equations of motion. This was to be expected given that the gravity Lagrangian, unlike Yang-Mills, does not stop at quartic order and that the MHV gravity amplitudes are non-holomorphic [17]. Furthermore MHV vertices in the gravity Lagrangian are not sufficient to compute all the non-MHV diagrams, at least for our choice of field variables. For example the 5-point amplitude $M^{\text{tree}}(+,+,-,-,-)$ has contributions from the MHV vertices but also from a direct contact vertex present in the original Lagrangian[7]. The five-point MHV amplitude $M^{\text{tree}}(+,+,+,-,-)$ is special in that it has three contributions: one term from the original Lagrangian and two from the field redefinition acting on the three- and four-point vertices. Otherwise, as in Yang-Mills, all n-point ($n > 5$) MHV amplitudes are generated by the field redefinitions alone.

The discussion in the main body of this letter dealt with light-cone gravity at tree-level. At the loop level, field redefinitions have to be considered with much greater care.

[6]This field redefinition changes the structure $h = h(C)$ to $h = h(C, \bar{C})$ but affects only four- and higher-point vertices. This demonstrates the non-uniqueness of the field redefinition in section 3.2.

[7]In [18], the five-point non-MHV graph is simply a sum of MHV-exchange diagrams. In our case there is also a direct contribution: this is not surprising since, in our Lagrangian, we have eliminated the three-vertex $M(+,+,-)$ and so do not have a contribution equivalent to D_2 in equation (3.14) of that reference.

If the Jacobian of the field redefinition is not unity it will lead to additional interaction terms [16]. Even if the Jacobian is classically one there may be anomalies which lead to additional interaction terms as proposed in the context of the MHV Lagrangian for Yang-Mills in [10]; see also the discussion in [12, 13].

An interesting question is whether the Lagrangians of $\mathcal{N}=8$ supergravity and $\mathcal{N}=4$ superYang-Mills share a similar relationship. Since there exist superfield formulations, in light-cone gauge, for both these theories [19] a similar analysis is certainly worth performing.

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A Conventions and notation

We work with the metric $(-,+,+,+)$ and define

$$x^{\pm} = \frac{1}{\sqrt{2}}\,(x^0 \pm x^3)\ , \quad \partial_{\pm} = \frac{1}{\sqrt{2}}\,(\partial_0 \pm \partial_3)\ . \tag{A.1}$$

x^+ plays the role of light-cone time and ∂_+ the light-cone Hamiltonian. ∂_- is now a spatial derivative and its inverse, $\frac{1}{\partial_-}$, is defined using the prescription in [20]. We define

$$\begin{aligned} x &= \frac{1}{\sqrt{2}}\,(x^1 + i\,x^2)\ , \quad \bar{\partial} \equiv \frac{\partial}{\partial x} = \frac{1}{\sqrt{2}}\,(\partial_1 - i\,\partial_2)\ , \\ \bar{x} &= \frac{1}{\sqrt{2}}\,(x^1 - i\,x^2)\ , \quad \partial \equiv \frac{\partial}{\partial \bar{x}} = \frac{1}{\sqrt{2}}\,(\partial_1 + i\,\partial_2)\ . \end{aligned} \tag{A.2}$$

A four-vector p_μ may be expressed as a bispinor $p_{a\dot{a}}$ using the $\sigma^\mu = (-\mathbf{1}, \sigma)$ matrices

$$p_{a\dot{a}} \equiv p_\mu\,(\sigma^\mu)_{a\dot{a}} = \begin{pmatrix} -p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_0 - p_3 \end{pmatrix} = \sqrt{2}\begin{pmatrix} -p_- & \overline{p} \\ p & -p_+ \end{pmatrix}\ . \tag{A.3}$$

The determinant of this matrix is

$$\det\,(\,p_{a\dot{a}}\,) = -2\,(\,p\overline{p} - p_+p_-\,) = -\,p^\mu p_\mu\ . \tag{A.4}$$

When the vector p_μ is light-like we have $p_+ = \frac{p\overline{p}}{p_-}$ which is the on-shell condition. We then define holomorphic and anti-holomorphic spinors [8]

$$\lambda_a = \frac{2^{\frac{1}{4}}}{\sqrt{p_-}}\begin{pmatrix} p_- \\ -p \end{pmatrix}\ , \qquad \tilde{\lambda}_{\dot{a}} = -(\lambda_a)^* = -\,\frac{2^{\frac{1}{4}}}{\sqrt{p_-}}\begin{pmatrix} p_- \\ -\overline{p} \end{pmatrix}\ , \tag{A.5}$$

such that $\lambda_a\tilde{\lambda}_{\dot{a}}$ agrees with (A.3) on-shell. We define the off-shell holomorphic and anti-holomorphic spinor products [13]

$$\langle i\,j\rangle = \sqrt{2}\,\frac{p^i\,p^j_- - p^j\,p^i_-}{\sqrt{p^i_-\,p^j_-}}\ , \qquad [i\,j] = \sqrt{2}\,\frac{\overline{p}^i\,p^j_- - \overline{p}^j\,p^i_-}{\sqrt{p^i_-\,p^j_-}}\ . \tag{A.6}$$

Their product is

$$\langle i\,j\rangle\,[j\,i] = s_{ij} \equiv -(p_i + p_j)^2\ . \tag{A.7}$$

[8] When working with a Lorentzian signature, choosing $\tilde{\lambda}_{\dot{a}} = \pm(\lambda_a)^*$ ensures that $p_{a\dot{a}}$ is real.

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