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Large N Expansion of q-Deformed Two-Dimensional Yang-Mills Theory and Hecke Algebras

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Abstract: We derive the q-deformation of the chiral Gross-Taylor holomorphic string large N expansion of two dimensional $SU(N)$ Yang-Mills theory. Delta functions on symmetric group algebras are replaced by the corresponding objects (canonical trace functions) for Hecke algebras. The role of the Schur-Weyl duality between unitary groups and symmetric groups is now played by q-deformed Schur-Weyl duality of quantum groups. The appearance of Euler characters of configuration spaces of Riemann surfaces in the expansion persists. We discuss the geometrical meaning of these formulae.

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1. Introduction and Summary of the Results

Two-dimensional Yang-Mills theory, on a Riemann surface of genus G and of area A, can be solved exactly. The partition function is

$$Z_{\mathrm{YM}}(G, A) = \sum_R (\dim(R))^{2-2G} e^{-g_{\mathrm{YM}}^2 A\, C_2(R)} . \tag{1.1}$$

This result was first obtained using the lattice formulation, followed by a continuum limit [1]. The sum is over all irreducible representations of the gauge group, the cases $U(N)$ or $SU(N)$ will be of interest here.

Gross and Taylor [2–4] studied the large N expansion of two-dimensional Yang-Mills theory with gauge group $U(N)$ and $SU(N)$ and showed that it is equivalent to a string theory. They showed that the large N expansion is given by a non-chiral expansion, which is a sum involving chiral and anti-chiral factors. The chiral expansion of (1.1)[1] is given by

$$Z^+_{\mathrm{YM}}(G) = \sum_{n=0}^{\infty} \frac{1}{N^{(2G-2)n}} \sum_{s_i, t_i \in S_n} \frac{1}{n!}\, \delta\left(\Omega_n^{2-2G} \prod_{i=1}^{G} s_i t_i s_i^{-1} t_i^{-1} \right) . \tag{1.2}$$

It is a sum consisting of delta functions over symmetric groups, which count homomorphisms from the fundamental group of punctured Riemann surfaces to the symmetric groups. These homomorphisms are known to count branched covers of Σ_G. It was shown in [5, 6] that the chiral sum actually computes an Euler character of moduli spaces of holomorphic maps with fixed target space. This was done by expanding the Ω factors, and recognising that the coefficients in the expansion are Euler characters of configuration spaces of (branch) points on Σ_G. Topological string theory constructions were then used to derive a path integral which localizes to an integral of the Euler class on the moduli space of holomorphic maps. For simplicity we are discussing only the chiral part of the partition function here, but there is an analogous expansion for the full partition function. A different string action involving harmonic maps was proposed in [7].

Two-dimensional Yang-Mills has recently found a surprising new application in connection with topological strings on a non-compact Calabi-Yau and black hole entropy [8]. The q-deformation of two-dimensional Yang-Mills has also found an application in this context [9, 10]. The partition function of q-deformed Yang-Mills has been obtained by replacing the scalar dual field of the Yang-Mills field strength by a compact scalar.

[1] In this paper we work at zero area. The computations can be generalized to the case of finite area along the lines of [3, 4].

Such a compact scalar is natural from the point of view of the worldvolume of D4-branes wrapping a 4-cycle of the non-compact Calabi-Yau. New connections with Turaev invariants have also been suggested [11]. The q-deformation of two-dimensional Yang-Mills theory was studied earlier [12, 13] (see also [14]).

The q-deformation of the zero area partition function of two-dimensional Yang-Mills is

$$Z_{q\mathrm{YM}}(G) = \sum_R (\dim_q R)^{2-2G} . \tag{1.3}$$

In the context of [10] this is the limit where the degree p of one of the line bundles is zero. In the q-deformed Yang-Mills, the universal enveloping algebra of $U(N)$ is replaced by $U_q(u(N))$. The exact partition function for a closed Riemann surface, which is expressed in terms of dimensions of irreducible representations of $U(N)$, is now expressed in terms of q-dimensions of $U_q(u(N))$ representations. The same remarks apply to $U_q(su(N))$.

The underlying algebraic relation which leads to the relation between the sum over $U(N)$ representations in (1.1) and the delta functions over symmetric groups in (1.2) is Schur-Weyl duality, which we describe further in Sect. 2. The q-deformation of the Schur-Weyl duality between $U(N)$ and S_n is known [15]. In this q-deformation, the role of the group algebra of S_n (denoted by $\mathbb{C}S_n$) is played by the Hecke algebra $H_n(q)$.

In this paper, we show that the large N, chiral Gross-Taylor expansion, in terms of symmetric group data can be q-deformed to give an expansion in terms of Hecke algebra data. In this case we find the following result:

$$\begin{aligned} Z_{q\mathrm{YM}}(G) = \sum_{n=0}^{\infty} \sum_{s_i t_i \in S_n} \frac{1}{g} [N]^{(2-2G)n} \, \delta \\ \times \left(D \, \Omega_n^{2-2G} \prod_{i=1}^{G} q^{-l(s_i)-l(t_i)} h(s_i) h(t_i) h(s_i^{-1}) h(t_i^{-1}) \right). \end{aligned} \tag{1.4}$$

Here, $h(s) \in H_n$ is the Hecke algebra element associated to $s \in S_n$. That such an expansion is possible at all in the quantum case is highly non-trivial and very much suggestive of a geometric interpretation in terms of deformations of maps, on which we comment in Sect. 6. The possibility of the expansion (1.4) depends crucially on the existence of suitable central elements of the Hecke algebra (like D and Ω_n, to be defined later). These central elements play an important role in that they also determine the data on manifolds with closed boundary:

$$\begin{aligned} Z(\Sigma_G; C_1, \ldots, C_B) = \sum_R [N]^{(2-2G-B)n} \sum_{s_i t_i} \frac{1}{g} \\ \times \delta \left(D^{1-B} \, \Omega_n^{2-2G-B} \prod_{i=1}^{G} q^{-l(s_i)-l(t_i)} h(s_i) h(t_i) h(s_i^{-1}) h(t_i^{-1}) \prod_{j=1}^{B} C_j \right). \end{aligned} \tag{1.5}$$

In this formula, the central elements of the Hecke algebra take over the role of the holonomies of the gauge field around the B boundaries of Σ_G.

We also work out the case of non-intersecting Wilson loops. We develop an analog of the Verlinde formula for the tensor product multiplicity coefficients of $SU(N)$ in terms of characters of the Hecke algebra. To our knowledge, this formula has not appeared in the literature. Expectation values of Wilson loops can now again be written as Hecke delta functions which are natural deformations of the symmetric group delta functions.

In four appendices we give some of the facts and proofs about Hecke algebras that we use in the main text. To our knowledge, some of the formulas proven in these appendices are not available in the mathematical literature before.

2. Hecke Algebras and the Chiral Expansion of q-Deformed 2dYM

2.1. Review of the Gross-Taylor expansion. Before we do the q-deformed case, we will review the main tools used in the derivation of the partition function of 2d Yang-Mills as a topological theory counting branched covers of the Riemann surface. For full details we refer to [5]. For simplicity, we discuss the case of zero-area and no Wilson loops in this section. We start writing out the partition function as a sum over Young tableaux:

$$Z_{\mathrm{2dYM}}(\Sigma_G;A)=\sum_R(\dim(R))^{2-2G}=\sum_{n=0}^{\infty}\sum_{Y\in\mathcal{Y}_n^N}(\dim(R(Y)))^{2-2G}\ , \tag{2.1}$$

where we sum over the set $\mathcal{Y}_n^N$ of $SU(N)$ Young diagrams with n boxes and number of rows less than N. Of course, we also sum over diagrams with arbitrary number of boxes. The chiral expansion is derived by dropping the constraint on the number of rows. Next we use Schur-Weyl duality to derive the following fomula:

$$\dim(R)=\frac{N^n}{n!}\,\chi_R(\Omega_n)\ . \tag{2.2}$$

We are using a notation where $R=R(Y)$ denotes both the $SU(N)$ and the S_n representation corresponding to a Young tableau with n boxes, Y. χ_R is a character of the symmetric group, and Ω_n is a particular central element in $\mathbb{C}S_n$ given in [3, 4]. The chiral Gross-Taylor expansion is obtained as

$$\begin{aligned}Z_{\mathrm{2dYM}}(\Sigma_G;A)&=\sum_{n=0}^{\infty}\sum_R N^{(2-2G)n}\left(\frac{d_R}{n!}\right)^{2-2G}\frac{1}{d_R}\,\chi_R(\Omega_n^{2-2G})\\&=\sum_{n=0}^{\infty}N^{(2-2G)n}\frac{1}{n!}\sum_{s_i,t_i\in S_n}\delta\left(\Omega_n^{2-2G}\prod_{i=1}^{G}s_it_is_i^{-1}t_i^{-1}\right)\ .\end{aligned} \tag{2.3}$$

The fact that Ω_n is a central element in the group algebra $\mathbb{C}S_n$ is important. This is explained in more detail and generalized to the q-deformed case in Sect. (2.3). Another important identity which enters (2.3) is

$$\sum_{s,t\in S_n}\frac{1}{d_R}\,\chi_R(sts^{-1}t^{-1})=\left(\frac{n!}{d_R}\right)^2\ , \tag{2.4}$$

where it is easy to see that $\sum_{s,t} sts^{-1}t^{-1}$ is a central element of $\mathbb{C}S_n$. We find (2.50), which gives the q-deformation of this equation, and we prove related centrality properties for $H_n(q)$ in Appendix A.

2.2. Hecke algebras and Schur-Weyl duality. There is a natural generalization of the previous formulas using Hecke algebras. In this subsection we review basic facts about Hecke algebras and derive some formulas that we will use in what follows.

The symmetric group S_n can be defined in terms of generators s_i $(i = 1, \dots, n-1)$, which obey relations

$$\begin{aligned} s_i^2 &= 1 && \text{for } i = 1, \dots, n-1, \\ s_i\, s_{i+1}\, s_i &= s_{i+1}\, s_i\, s_{i+1} && \text{for } i = 1, \dots, n-2, \\ s_i\, s_j &= s_j\, s_i && \text{for } |i-j| \geq 2. \end{aligned} \tag{2.5}$$

The minimal length of a word in the s_i which is equal to a permutation σ is called the length of the permutation and is denoted as $l(\sigma)$.

The Hecke algebra $H_n(q)$ is defined in terms of generators g_i which obey [16]

$$\begin{aligned} g_i^2 &= (q-1)\, g_i + q && \text{for } i = 1, \dots, n-1, \\ g_i\, g_{i+1}\, g_i &= g_{i+1}\, g_i\, g_{i+1} && \text{for } i = 1, \dots, n-2, \\ g_i\, g_j &= g_j\, g_i && \text{for } |i-j| \geq 2. \end{aligned} \tag{2.6}$$

The Hecke algebra has, as a vector space, a basis $h(\sigma)$ labelled by the elements σ of S_n. This is often called the "standard basis" in the literature. These elements $h(\sigma)$ are obtained by expressing the σ as a minimal length word in the s_i and then replacing the s_i by g_i. These Hecke algebras arise as the algebra of operators on $V^{\otimes n}$, the n-fold tensor product of the fundamental representation of $U(N)$ or $SU(N)$, which commute with the action of $U_q(u(N))$ or $U_q(su(N))$, the q-deformation of the universal enveloping algebra of $u(N)$ or $su(N)$, respectively. The action of the q-deformed enveloping algebras on $V \otimes V$ is given by the co-product Δ. This obeys the following relations with respect to the R-matrix:

$$\begin{aligned} \Delta R &= \Delta' R, \\ (PR)\Delta &= \Delta(PR)\, . \end{aligned} \tag{2.7}$$

For $h \in U_q$, if we write $\Delta(h) = h_1 \otimes h_2$, then $\Delta'(h) = h_2 \otimes h_1$. P is the permutation operator. PR is also commonly denoted by $\check{R}$. For the R-matrix we will use the conventions of [17]. To make that explicit, we write R_{FRT}. The Hecke algebra is related to the algebra of the $\check{R}_{\mathrm{FRT}}$ as:

$$g = \sqrt{q}\, \check{R}_{\mathrm{FRT}}\Big(q_{\mathrm{FRT}} = \sqrt{q}\Big)\, . \tag{2.8}$$

g_i corresponds to $\check{R}_{\mathrm{FRT}}$ acting in the tensor product $V_i \otimes V_{i+1}$ and is sometimes called a braid operator.

Since the centralizer of U_q is the Hecke algebra, we can construct the projectors for irreducible representations of U_q in terms of words in the g_i. g_1 acts on the product space $V_1 \otimes V_2$, therefore there are two possible projectors that we can construct [17]:

$$\begin{aligned} P_{\square\square} &= \frac{q^{-1}}{q+q^{-1}}(1 + q\check{R}) = \frac{1}{1+q}(1+g), \\ P_{\substack{\square \\ \square}} &= \frac{q}{q+q^{-1}}(1 - q^{-1}\check{R}) = \frac{q}{1+q}(1 - q^{-1} g)\, , \end{aligned} \tag{2.9}$$

which project onto the totally symmetric and antisymmetric tensor products of the fundamental representation, respectively. Using (2.6), one easily checks that they satisfy

$$P_R^2 = P_R \; . \tag{2.10}$$

The symmetric projector is illustrated in Appendix D in terms of properties of Clebsch-Gordan coefficients of $U_q(su(2))$.

Projectors are useful to compute characters in a particular representation in terms of lower-dimensional representations. For example, taking the trace of the above,

$$\begin{aligned} \mathrm{Tr}_{\square\square} U &= \frac{q^{-1}}{q+q^{-1}} \left((\mathrm{tr} U)^2 + q \, \mathrm{tr} \otimes \mathrm{tr} \left(\check{R} \, (U \otimes 1)(1 \otimes U) \right) \right), \\ \mathrm{Tr}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} U &= \frac{q}{q+q^{-1}} \left((\mathrm{tr} U)^2 - q^{-1} \, \mathrm{tr} \otimes \mathrm{tr} \left(\check{R} \, (U \otimes 1)(1 \otimes U) \right) \right) \; , \end{aligned} \tag{2.11}$$

where the traces on the right-hand side are taken in the fundamental representation, $\mathrm{Tr}_\square = \mathrm{tr}_V = \mathrm{tr}$. From now on we will indicate such traces by $\mathrm{tr}_n = \mathrm{tr}_{V^{\otimes n}} = \mathrm{tr} \otimes \ldots \otimes \mathrm{tr}$. The U's in (2.11), which are matrix elements of representations of U_q , generate the dual algebra to U_q denoted by $\mathrm{Fun}_q(SU(N))$ or $\mathrm{Fun}_q(U(N))$ (see for example [18, 19, 13]).

Using known facts about Hecke algebras and the q-deformation of the Schur-Weyl duality between $U(N)$ and S_n, we will now derive the generalization for arbitrary irreducible representations:

$$P_R = \frac{d_R(q)}{g} \sum_\sigma q^{-l(\sigma)} \chi_R(h(\sigma^{-1})) \, h(\sigma) \; , \tag{2.12}$$

where $l(\sigma)$ is the length of the permutation, i.e. the number of elements in the minimal presentation of the permutation as a product of simple transpositions. The character is taken in the Hecke algebra H_n. Without danger of confusion, we will denote H_n and $\mathrm{Fun}_q(SU(N))$ characters with the same symbol. The characters for low values of n can be read off from the tables in [16, 20]. $d_R(q)$ is the q-deformation of the dimension of a representation of the symmetric group, and g reduces to $n!$ in the classical limit:

$$\begin{aligned} d_R(q) &= \frac{\prod_{i \langle j} (q^{l_i} - q^{l_j})}{\prod_{i=1}^m (q-1)(q^2-1)\ldots(q^{l_i}-1)} \, \frac{(q-1)(q^2-1)\ldots(q^n-1)}{q^{\frac{m(m-1)(m-2)}{6}}} , \\ g &= \frac{(1-q)(1-q^2)\ldots(1-q^n)}{(1-q)^n} \; , \end{aligned} \tag{2.13}$$

where $l_i = \lambda_i + m - i$ and $\lambda_1 \geq \lambda_2 \geq ..\lambda_m \geq 0$ are the row lengths of the Young diagram, and m is the number of non-zero λ's.

In order to derive (2.12), recall the familiar relation in the $q = 1$ case:

$$\chi_R(U) = \frac{1}{n!} \sum_\sigma \chi_R(\sigma) \, \mathrm{tr}_n(\sigma U) \; . \tag{2.14}$$

Here R is both the $U(N)$ reprsentation corresponding to a Young diagram and the S_n rep corresponding to the same diagram. The trace on the right-hand side is taken in $V^{\otimes n}$, that is U acts as $U \otimes U \otimes ... \otimes U$ and σ acts by permuting the vectors of the tensor product.

The above is obtained from the fact that, if V is the fundamental representation of $U(N)$ or the universal enveloping algebra $U(u(N))$, then $V^{\otimes n}$ can be decomposed on the terms of the product group $U(N) \times S_n$ as

$$V^{\otimes n} = \oplus_R V_R^{U(N)} \otimes V_R^{S_n} \ . \tag{2.15}$$

The sum is over Young diagrams of S_n, $V_R^{S_n}$ is the irrep of S_n corresponding to the Young diagram R, while $V_R^{U(N)}$ is the irrep of $U(N)$ corresponding to the same Young diagram. Similar relations hold when $U(N)$ is replaced by $SU(N)$. An immediate consequence of the above expansion is

$$\mathrm{tr}\,(\sigma U) = \sum_R \chi_R(\sigma)\, \chi_R(U). \tag{2.16}$$

Then we can use orthogonality of characters of S_n,

$$\sum_\sigma \chi_R(\sigma)\, \chi_S(\sigma^{-1}) = n!\, \delta_{RS}, \tag{2.17}$$

to obtain (2.14). From (2.15) it also follows that

$$d_R\, \chi_R(U) = \mathrm{tr}_n(P_R U)\ , \tag{2.18}$$

hence we can read off

$$P_R = \frac{d_R}{n!} \sum_\sigma \chi_R(\sigma^{-1})\, \sigma\ . \tag{2.19}$$

The decomposition analogous to (2.15) holds for $U_q(u(N))$, when $\mathbb{C}S_n$ is replaced by the Hecke algebra $H_n(q)$ [15]:

$$V^{\otimes n} = \oplus_R V_R^{U_q} \otimes V_R^{H_n}. \tag{2.20}$$

Here $V_R^{U_q}$ is the irrep of $U_q(u(N))$ corresponding to the Young diagram R and $V_R^{H_n}$ is the representation of H_n corresponding to the same Young diagram. It follows from (2.20) that

$$\mathrm{tr}_n\ (h(\sigma)\, U) = \sum_R \chi_R(h(\sigma))\, \chi_R(U). \tag{2.21}$$

U lives in the deformed algebra of functions on $U(N)$ denoted as $\mathrm{Fun}_q(U(N))$. This can be defined as the dual to $U_q(U(N))$. For further discussion on the duality see for example [18, 19, 13]. In (2.21) U acts as

$$(U\otimes 1\otimes 1\otimes \cdots)(1\otimes U\otimes 1 \otimes \cdots)(1\otimes 1\otimes U \otimes 1\otimes \cdots)\cdots(1 \otimes 1 \otimes \cdots \otimes 1 \otimes U). \tag{2.22}$$

This product of n U's is dual to the co-product which defines the action of U_q on $V^{\otimes n}$.

As will be explained in Sect. 5 (see also Appendix D), quantum traces contain the u-element associated to the Hopf algebra $U_q(su(N))$. We get the quantum trace if we take a trace of the action of $u\,U$ on the left-hand side of (2.20) to get

$$\mathrm{tr}_n\ (h(\sigma)\, \rho_n(u\, U)) = \sum_R \chi_R(h(\sigma))\, \chi_R(u\, U). \tag{2.23}$$

Here $\rho_n(u) = u^{\otimes n}$ and U acts as above. For the case of diagonal U, the formula (2.21) is used in [20].

Multiplying the left- and right-hand side of (2.21) with $q^{-l(\sigma)}\chi_S(h(\sigma^{-1}))$, and using the orthogonality relation [21] for Hecke characters

$$\sum_\sigma q^{-l(\sigma)}\chi_R(h(\sigma))\,\chi_S(h(\sigma^{-1})) = g\,\frac{d_R(1)}{d_R(q)}\,\delta_{RS}\,, \tag{2.24}$$

we get

$$\sum_\sigma q^{-l(\sigma)}\chi_R(h(\sigma^{-1}))\,\mathrm{tr}_n(h(\sigma)U) = g\,\frac{d_R(1)}{d_R(q)}\,\chi_R(U)\,. \tag{2.25}$$

This means that the character can be expressed as

$$\chi_R(U) = \frac{1}{g}\frac{d_R(q)}{d_R(1)}\sum_\sigma q^{-l(\sigma)}\chi_R(h(\sigma^{-1}))\,\mathrm{tr}_n(h(\sigma)U)\,. \tag{2.26}$$

This equation can be interpreted as giving us the projection on a fixed Young diagram from the sum in (2.15). Indeed, note that (2.20) implies, by projecting on a fixed Young diagram:

$$d_R(1)\,\chi_R(U) = \mathrm{tr}_n(P_R U)\,. \tag{2.27}$$

Comparing with (2.26) we see that the projector is

$$P_R = \frac{1}{g}\,d_R(q)\sum_\sigma q^{-l(\sigma)}\;\chi_R(h(\sigma^{-1}))\,h(\sigma)\,, \tag{2.28}$$

as claimed above. In the appendix we check that it satisfies (2.10).

If we use orthogonality starting from (2.23) rather than (2.21), then we get

$$\chi_R^{(q)}(U) \equiv \chi_R(u\,U) = \frac{1}{g}\frac{d_R(q)}{d_R(1)}\sum_\sigma q^{-l(\sigma)}\;\chi_R(h(\sigma^{-1}))\,\mathrm{tr}_n(h(\sigma)(u\,U))\,. \tag{2.29}$$

Note that $u^{\otimes n}$ commutes with $h(\sigma)$. We will specialize to $U = 1$ in order to get a new formula for the q-dimension in Sect. (2.3).

2.3. A Hecke formula for the q-dimension . Recall that in the case $q = 1$ there is a very useful formula for the dimension of $SU(N)$ reps which follows from Schur-Weyl duality [5]. This formula can be obtained by specializing (2.14) to $U = 1$. To that end we need to compute the trace of a permutation acting on $V^{\otimes n}$. If $\sigma = 1$, we just get N^n. If $\sigma = (12)(3)(4)..(n)$, we get N^{n-1}. In general we get one factor of N for each cycle in the permutation. If the permutation has cycles of length i occuring with multiplicity k_i the power of N is $N^{\sum k_i}$. In the 2d Yang-Mills literature this is also denoted as N^{K_σ}. So the useful formula for the dimension in 2d Yang-Mills [3, 4] is

$$\begin{aligned}\dim(R) &= \frac{1}{n!}\sum_\sigma \chi_R(\sigma)\,N^{K_\sigma}\\ &= \frac{N^n}{n!}\sum_\sigma \chi_R(\sigma)\,N^{-n+\sum_i k_i(\sigma)}\,. && (2.30)\\ &= \frac{N^n}{n!}\chi_R(\Omega_n). && (2.31)\end{aligned}$$

The last line defines the element Ω_n. It is convenient to write this as a sum over conjugacy classes. Let T be a conjugacy class, which is given by specification of the cycle decomposition of the permutations involved. We will write $C_T = \sum_{\sigma \in T} \sigma$. Note this is a central element of the group algebra $\mathbb{C}S_n$, i.e. it commutes with all the elements of S_n. So the above can be rewritten as

$$\dim(R) = \frac{1}{n!} \sum_T \chi_R(C_T)\, N^{\sum_i k_i(T)} . \tag{2.32}$$

We can now find the q-generalization of this formula by setting $U = 1$ in (2.29), to obtain

$$\dim_q(R) = \frac{1}{g} \frac{d_R(q)}{d_R(1)} \sum_{\sigma \in S_n} q^{-l(\sigma)}\, \chi_R\left(h(\sigma^{-1})\right) \mathrm{tr}_n(h(\sigma)u) . \tag{2.33}$$

We can manipulate the above sum, using cyclicity of the trace and the Hecke relations, to reduce it to a sum over conjugacy classes T in S_n, with the only terms appearing inside tr_n being the $\mathrm{tr}_n(u\, h(m_T))$. m_T are permutations in the conjugacy class T which have minimal length when expressed in terms of generators. They are the minimal words in [16]. For $n = 3$, m_T are $1, g_1, g_1 g_2$ for the 3 conjugacy classes. We prove in Appendix B (B.7),

$$\mathrm{tr}_n\, (u\, h(m_T)) = q^{\frac{N+1}{2}\, l(T)}\, [N]^{\sum_i k_i}, \tag{2.34}$$

where the q-number is

$$[N] = \frac{q^{N/2} - q^{-N/2}}{q^{1/2} - q^{-1/2}} , \tag{2.35}$$

and $l(T)$ is the length of the permutation m_T.

We will explain below that the Hecke algebra elements C_T appearing as the coefficients of $q^{-l(T)} \mathrm{tr}(u\, h(m_T))$ are central. Hence the formula for the q-dimension becomes

$$\dim_q(R) = \frac{1}{g} \frac{d_R(q)}{d_R(1)} \sum_T \chi_R(C_T(q))\, [N]^{\sum_i k_i(T)}\, q^{\frac{N-1}{2}\, l(T)} . \tag{2.36}$$

Examples of this formula are described in Appendix B, along with checks against the standard formula in terms of a product of q-numbers over the cells of the Young diagram.

We now explain the centrality property of C_T. Starting from the formula for the projector (2.12) we can express it in a reduced form using cyclicity and Hecke relations, where we only have the characters of the minimal words in each conjugacy class:

$$P_R = \frac{1}{g}\, d_R(q) \sum_T \chi_R(h(m_T))\, C_T . \tag{2.37}$$

Here T runs over conjugacy classes, and m_T are the minimal words. For the formulas up to $n = 4$, see Appendix C. We can get the projector to the form (2.37) because, by using cyclicity of χ_R and the Hecke relations, the Hecke characters can be expressed in terms of these basic characters [16]. Now for every R, P_R is a central element of the Hecke algebra since it is a projector for the irreducible representation R. There are as many conjugacy classes T as irreducible representations R. Hence C_T must be central elements. When we calculate the q-dimension we get (2.33). When we manipulate the expression to express it in terms of $q^{-l(T)}\,\mathrm{tr}_n(u\,h(m_T))$, we are using the same Hecke relations and cyclicity (of tr_n this time):

$$\dim_q(R) = \frac{1}{g}\frac{d_R(q)}{d_R(1)}\sum_T q^{-l(T)}\,\chi_R(C_T)\,\mathrm{tr}_n(h(m_T)\,u)\;. \tag{2.38}$$

This immediately leads to (2.36). Incidentally, (2.37) seems to give a relatively efficient way of calculating the central class elements compared to the ones we are aware of in the mathematical literature. Some interesting papers with explicit formulae for Hecke central elements, which we found useful, are [22, 23].

2.4. Hecke q-generalization of sums over symmetric groups of 2d Yang Mills. The string theory interpretation of 2d Yang Mills at $q = 1$ is centred on formulae derived from Schur-Weyl duality. The character relations following from Schur-Weyl give rise to a formula for dimensions of $SU(N)$ reps in terms of S_n reps. Then some group theory manipulations lead to an expression of the chiral partition function in terms of delta functions over the symmetric group.

The delta function is defined over the symmetric group or, more generally, over the group algebra of the symmetric group:

$$\begin{aligned}\delta(\sigma) &= 1 \quad \text{if } \sigma = 1,\\ \delta(\sigma) &= 0 \quad \text{otherwise}\;.\end{aligned} \tag{2.39}$$

A useful property of this delta function is that it can be expressed in terms of characters,

$$n!\,\delta(\sigma) = \sum_R d_R\,\chi_R(\sigma)\;. \tag{2.40}$$

The expressions arising in the 2d Yang-Mills string take the form

$$\delta\left(\sigma_1\sigma_2\cdots\sigma_k\right)\;, \tag{2.41}$$

and the weights depend on the genus G and on N in precisely such a way that the chiral partition function can be expressed in terms of a sum of Euler characters of moduli spaces of holomorphic maps (see Sect. 7 of [5]).

Now we will describe a q-generalization of this story, where the Hecke algebra will replace the group algebra of the symmetric group. A q-analog of the delta function on the symmetric group is known in the theory of Hecke algebras [21]. It is defined as:

$$\begin{aligned}\delta(h(\sigma)) &= 1 \quad \text{if } \sigma = 1,\\ \delta(h(\sigma)) &= 0 \quad \text{otherwise}\;.\end{aligned} \tag{2.42}$$

Our $\delta(h(\sigma))$ is $\frac{1}{g}\,\mathrm{tr}(h(\sigma))$ in the notation of [21] for the canonical trace function $\mathrm{tr}(h(\sigma))$. This q-deformed delta function reduces exactly to the delta function on the symmetric group defined above when $q \to 1$. It can be expressed as

$$g\,\delta(h(\sigma)) = \sum_R d_R(q)\,\chi_R(h(\sigma))\;, \tag{2.43}$$

where R runs over partitions of n or Young diagrams with n boxes, and g is given in (2.13).

An important fact we will use in what follows is that for C a central element of the Hecke algebra, and for arbitrary $\sigma \in S_n$,

$$\chi_R(C)\,\chi_R\,(h(\sigma)) = d_R(1)\,\chi_R\,(C\,h(\sigma))\ , \tag{2.44}$$

which follows simply from Schur's lemma applied to the Hecke algebra.

We now have all the elements we need in order to rewrite the quantum dimensions in terms of central elements of the Hecke algebra. Using (2.36), we can write

$$\dim_q(R) = \frac{[N]^n}{g}\frac{d_R(q)}{d_R(1)}\,\chi_R(\Omega_n)\ . \tag{2.45}$$

In the quantum case the Ω's are expressed as

$$\begin{aligned}\Omega_n &= \sum_T [N]^{K_T-n}\,q^{\frac{N-1}{2}\,l(T)}\,C_T\\ &= 1 + \sum_T{}'\,[N]^{K_T-n}\,q^{\frac{N-1}{2}\,l(T)}\,C_T\\ &\equiv 1+\Omega'_n\ ,\end{aligned} \tag{2.46}$$

where the unprimed sum runs over the central elements of H_n. The restricted sum (denoted by the prime) runs over all central elements associated with conjugacy classes of S_n which are not the identity. The last line is a definition of Ω'_n. Making repeated use of (2.44), we find that for a central element we have:

$$\left(\frac{\chi_R(C)}{d_R(1)}\right)^m = \frac{\chi_R(C^m)}{d_R(1)}\ . \tag{2.47}$$

It now follows from (2.45) that

$$\begin{aligned}(\dim_q(R))^m &= \\ &= \left(\frac{[N]^n\,d_R(q)}{g}\right)^m \frac{\chi_R(\Omega^m)}{d_R(1)}\\ &= \left(\frac{[N]^n\,d_R(q)}{g}\right)^m \sum_{\ell=0}^{\infty}\frac{d(m,\ell)}{d_R(1)}\chi_R\\ &\quad\times\left(\prod_{i=1}^{\ell}\sum_{T_i}{}'C_{T_i}\right)[N]^{\sum_i K_{T_i}-n}q^{\frac{N-1}{2}\sum_i l(T_i)},\end{aligned} \tag{2.48}$$

where $d(m,\ell) = \frac{\Gamma(m+1)}{\Gamma(\ell+1)\Gamma(m-\ell+1)}$, and we wrote out the definition of Ω'.

Let us develop the q-deformed chiral Gross-Taylor expansion

$$\begin{aligned}Z &= \sum_{n=0}^{\infty}\sum_{R\in Y_n}\left(\dim_q(R)\right)^{2-2G}\\ &= \sum_{n=0}^{\infty}\sum_{R\in Y_n}[N]^{(2-2G)n}\left(\frac{d_R(q)}{g}\right)^{2-2G}\frac{1}{d_R(1)}\,\chi_R\left(\Omega^{2-2G}\right)\ .\end{aligned} \tag{2.49}$$

Now we can show (see Appendix A) that

$$\left(\frac{g}{d_R(q)}\right)^2 = \sum_{s,t\in S_n} q^{-l(s)-l(t)}\,\frac{1}{d_R(1)}\,\chi_R\left(h(s)h(t)h(s^{-1})h(t^{-1})\right)\,. \tag{2.50}$$

We also show in the appendix that the element

$$\sum_{s,t\in S_n} q^{-l(s)-l(t)}\,h(s)h(t)h(s^{-1})h(t^{-1}) \tag{2.51}$$

is central in H_n. Hence we have

$$\left(\frac{g}{d_R(q)}\right)^{2G} = \sum_{s_1,t_1\cdots s_G,t_G} q^{-\sum_i(l(s_i)+l(t_i))}\,\frac{1}{d_R(1)}\,\chi_R\left(\prod_{i=1}^{G} h(s_i)h(t_i)h(s_i^{-1})h(t_i^{-1})\right)\,. \tag{2.52}$$

Now we employ this equation in (2.49) to get

$$\begin{aligned} Z &= \sum_{n=0}^{\infty}\sum_{R\in Y_n}\sum_{s_i t_i} q^{-\sum_i(l(s_i)+l(t_i))}\,[N]^{(2-2G)n}\left(\frac{d_R(q)}{g\,d_R(1)}\right)^2 \\ &\quad\times\chi_R\left(\prod_{i=1}^{G} h(s_i)h(t_i)h(s_i^{-1})h(t_i^{-1})\right)\chi_R(\Omega^{2-2G}) \\ &= \sum_{n=0}^{\infty}\sum_{R\in Y_n}\sum_{s_i t_i}[N]^{(2-2G)n}\left(\frac{d_R(q)}{g}\right)^2\frac{q^{-\sum_i(l(s_i)+l(t_i))}}{d_R(1)} \\ &\quad\times\chi_R\left(\Omega^{2-2G}\prod_{i=1}^{G} h(s_i)h(t_i)h(s_i^{-1})h(t_i^{-1})\right), \end{aligned} \tag{2.53}$$

where we sum over S_n permutations $s_1, t_1, \ldots, s_G, t_G$. At this point the manipulations performed in the classical case do not generalize straightforwardly to the quantum case because of the different powers of $d_R(q)$ and $d_R(1)$. We need to introduce an element D of the Hecke algebra with the property

$$\chi_R(D) = d_R(q)\,. \tag{2.54}$$

The existence of this element is proven in Appendix A, where an explicit expression is given for it in terms of an infinite sum. Let us find it explicitly for low values of n. For $n = 2, 3$, we can solve the above equation explicitly. We find for $n = 2$,

$$D = \frac{1+q^2}{1+q} + \frac{1-q}{1+q}\,g_1,$$

and for $n = 3$,

$$\begin{aligned} D &= \frac{1+q^2+2q^3+q^4+q^6}{(1+q)(1+q+q^2)} + \frac{(1-q)(2+2q+q^2+2q^3+2q^4)}{(1+q)(1+q+q^2)}\,g_1 \\ &\quad+\frac{(1+q)(1-q)^2}{1+q+q^2}\,g_1g_2\,. \end{aligned} \tag{2.55}$$

We note that $D \to 1$ in the classical limit. Using the form of D in the appendix we can write $\frac{d_R(q)^2}{d_R(1)} = d_R(q)\,\frac{\chi_R(D)}{d_R(1)}$, which allows us to rewrite (2.53),

$$Z = \sum_{n=0}^{\infty} \sum_{s_i t_i} \frac{1}{g} [N]^{(2-2G)n}\, \delta\left(D\, \Omega^{2-2G} \prod_{i=1}^{G} q^{-l(s_i)-l(t_i)} h(s_i) h(t_i) h(s_i^{-1}) h(t_i^{-1}) \right). \tag{2.56}$$

In the last step we used (2.43). This is the q-analog of the Gross-Taylor expansion. We can expand the Ω-factors as follows:

$$Z = \sum_{n=0}^{\infty} \sum_{s_i t_i} q^{-\sum_i (l(s_i)+l(t_i))} \sum_{\ell=0}^{\infty} \sum_{T_1 \dots T_\ell}{}' \frac{1}{g} [N]^{(2-2G)n + \sum_i (K(T_i)-n)}\, q^{\frac{N-1}{2} \sum_{i=1}^{\ell} l(T_i)}$$
$$\times d\,(2-2G, \ell)\; \delta\left(D\, C_{T_1} \dots C_{T_\ell} \prod_{i=1}^{G} h(s_i) h(t_i) h(s_i^{-1}) h(t_i^{-1}) \right). \tag{2.57}$$

As explained in [5], the factor of $d(2-2G, \ell)$ is the Euler character of the configuration space of ℓ points on Σ_G, denoted as $\chi(\Sigma_{G,\ell})$. Hence we can write

$$Z = \sum_{n=0}^{\infty} \sum_{s_i t_i} q^{-\sum_i (l(s_i)+l(t_i))} \sum_{\ell=0}^{\infty} \sum_{T_1 \dots T_\ell}{}' \frac{1}{g} [N]^{(2-2G)n + \sum_{j=1}^{\ell} (K(T_j)-n)}\, q^{\frac{N-1}{2} \sum_{j=1}^{\ell} l(T_j)}$$
$$\times \chi(\Sigma_{G,\ell})\;\; \delta\left(D\, C_{T_1} \dots C_{T_\ell} \prod_{i=1}^{G} q^{-(l(s_i)+l(t_i))} h(s_i) h(t_i) h(s_i^{-1}) h(t_i^{-1}) \right). \tag{2.58}$$

3. Manifolds with Boundary

We now describe the chiral large $[N]$ expansion of q-deformed 2d Yang-Mills theory on manifolds with boundary, in terms of Hecke algebras. We recall the classical case first. For a Riemann surface of genus G with B boundaries and boundary holonomies $U_1, \dots, U_B$ in $SU(N)$, the parition function is

$$Z_{\mathrm{YM}}(G, B; U_1, \dots, U_B) = \sum_R (\dim R)^{2-2G-B} \chi_R(U_1)\, \chi_R(U_2) \dots \chi_R(U_B)\,. \tag{3.1}$$

It is useful in that case to multiply by $(\frac{1}{n!})^B\, \mathrm{tr}_n(T_1\, U_1^\dagger)\, \mathrm{tr}_n(T_2\, U_2^\dagger) \dots \mathrm{tr}(T_B\, U_B^\dagger)$ and integrate over the holonomies, where $T_1, \dots, T_n$ are sums of permutations in fixed conjugacy classes in S_n. Then the chiral Gross-Taylor expansion becomes

$$Z_{\mathrm{YM}}(G, B; T_1, \dots, T_B) = \sum_{s_i, t_i} \frac{1}{n!} N^{n(2-2G-B)} \delta\left(T_1 \dots T_B \Omega_n^{2-2G-B} \prod_{i=1}^{G} s_i t_i s_i^{-1} t_i^{-1} \right). \tag{3.2}$$

This is basically a Fourier transformation, and the derivation is explained in [24].

For q-deformed 2d Yang-Mills, the holonomies along the boundaries are specified by the quantum characters [13, 12] of $U_q(SU(N))$:

$$Z_{q\mathrm{YM}}(G, B; U_1, \ldots, U_B) = \sum_R (\dim_q R)^{2-2G-B} \chi_R(U_1)\, \chi_R(U_2) \ldots \chi_R(U_B). \quad (3.3)$$

Now we can insert $(\frac{1}{g})^B \operatorname{tr}_n(C_{T_1} U_1^\dagger) \operatorname{tr}_n(C_{T_2} U_2^\dagger) \ldots \operatorname{tr}(C_{T_B} U_B^\dagger)$. In this case, $C_{T_1}, \ldots, C_{T_B}$ are central elements in $H_n(q)$ which approach the class sums $T_1, T_2, \ldots, T_B$ in the limit $q \to 1$. They have appeared in the formulae for the q-dimension earlier. We use the expansion

$$\operatorname{tr}(C_T U^\dagger) = \sum_S \chi_S(C_T)\, \chi_S(U^\dagger) , \quad (3.4)$$

where $\chi_S(C_T)$ is the Hecke algebra character in the representation S. Then we integrate the quantum group elements $U_1, \ldots, U_B$, and use the orthogonality [13, 12]

$$\int \mathrm{d}U\, \chi_R(U)\, \chi_S(U^\dagger) = \delta_{RS} . \quad (3.5)$$

The result is

$$\begin{aligned}
&Z_{q\mathrm{YM}}(G, B; C_{T_1}, \ldots, C_{T_B}) \\
&= \sum_{R \in Y_n} \left(\dim_q R\right)^{2-2G-B} \prod_{j=1}^{B} \left(\frac{\chi_R(C_{T_j})}{g}\right) \\
&= \sum_R [N]^{(2-2G-B)n} \left(\frac{d_R(q)}{g}\, \frac{\chi_R(\Omega)}{d_R(1)}\right)^{2-2G-B} \prod_{j=1}^{B} \left(\frac{\chi_R(C_{T_j})}{g}\right) \\
&= \frac{1}{g} [N]^{(2-2G-B)n} \sum_{s_i t_i} \delta \\
&\quad \times \left(\left(\frac{E}{g}\right)^{B-1} \Omega^{2-2G-B} \prod_{i=1}^{G} q^{-l(s_i)-l(t_i)}\, h(s_i) h(t_i) h(s_i^{-1}) h(t_i^{-1}) \prod_{j=1}^{B} C_{T_j} \right).
\end{aligned} \quad (3.6)$$

In the second line we used (2.48), and in the last line we employed (2.52). The element E is defined in (A.14). As in manipulations of the partition function we repeatedly used (2.44) to combine products of characters. Finally to obtain the delta function from the Hecke characters, we used (2.43).

In the $q = 1$ limit (3.6) reduces to a delta function over the group algebra of S_n, counting maps with specified conjugacy classes of permutations at the boundaries. There is now some deformation of this geometry, involving central elements of the Hecke algebra $H_n(q)$ associated with the boundaries. It is very intersting that for $B = 1$ we do not have the $\frac{E}{g}$ factors. Recall also that $\frac{E}{g} = 1$ in the $q = 1$ limit.

In the q-deformed theory there is a notion of a delta-function over the quantum group -valued holonomies [13]. It is the partition function on the disk, therefore the case $G = 0$,

$B = 1$ of the above. We compute directly:

$$\begin{aligned} \delta(U, 1) &= \sum_{n;\sigma\in S_n} \frac{1}{g} [N]^n \, q^{-l(\sigma)} \delta(D\Omega\, h(\sigma^{-1}))\, \mathrm{tr}_n(h(\sigma)\, u\, U) \\ &= \sum_{n;\sigma\in S_n} \frac{1}{g} [N]^n \, Q^\sigma \, \mathrm{tr}_n(h(\sigma)\, u\, U) \, , \end{aligned} \tag{3.7}$$

where we defined $D\Omega = \sum_\sigma Q^\sigma h(\sigma)$. Using (3.5), we can integrate this expression against any test function to obtain a form that depends purely on the Hecke algebra. In particular, the above gives another expression for the quantum dimensions. Thus, in the q-deformed theory the partition function on a disk of zero area continues to be associated to a flat connection, in the quantum group sense [13].

4. Chiral Large N Expansion for Wilson Loops

After having computed the partition function on closed Riemann surfaces and Riemann surfaces with boundaries, we should now discuss the chiral expansion of Wilson loops. For simplicity, we will consider non-intersecting Wilson loops in this section. The basic object we need to take into account are the $SU(N)$ tensor multiplicity coefficients [13, 12]. Indeed, consider a surface of genus $G = G_1 + G_2$ with a Wilson loop in representation S, where G_1 and G_2 are the genera of the inner and outer faces of the Wilson loop. The expectation value of this Wilson loop is

$$W_S(G) = \sum_{R_1 R_2} \int \mathrm{d}U \, \left(\dim_q R_1\right)^{1-2G_1} \left(\dim_q R_2\right)^{1-2G_2} \, \chi_{R_1}(U)\, \chi_S(U)\, \chi_{R_2}(U^\dagger) \, , \tag{4.1}$$

where R_1 and R_2 are the representations of the inner and outer faces, respectively. Since we are discussing the case of q non-root of unity, the result of the above quantum integral is the usual $SU(N)$ tensor multiplicity coefficients (Littlewood-Richardson coefficients). Thus we are set to compute

$$W_S(G) = \sum_{R_1 R_2} (\dim_q R_1)^{1-2G_1} (\dim_q R_2)^{1-2G_2} \, N_{R_1 S}^{R_2} \, . \tag{4.2}$$

Our next task is to look for an expression for the Littlewood-Richardson coefficients that we can interpret as a deformation of the Riemann surface. Thus, we want to write them as delta functions on the Hecke algebra. We start from the definition:

$$N_{R_1 R_2}^{R_3} = \int \mathrm{d}U \, \chi_{R_1}(U)\, \chi_{R_2}(U)\, \chi_{R_3}(U^\dagger), \tag{4.3}$$

and observe that the above is a trace of the following operator acting in $R_1 \otimes R_2$:

$$\int \mathrm{d}U \, \chi_{R_3}(U^\dagger)\, \rho_{R_1\otimes R_2}(U) \, . \tag{4.4}$$

Now R_1 can be realized in $V^{\otimes n_1}$ with multiplicity $d_{R_1}(1)$ when we project on the given Young diagram, and likewise for R_2. It is also useful to note that the above operator is proportional to a projector for the representation R_3,

$$\int \mathrm{d}U\, \chi_{R_3}(U^\dagger)\, \rho_{R_1\otimes R_2}(U) = \frac{1}{\dim_q R_3}\, \rho_{R_1\otimes R_2}(P_{R_3})\, . \tag{4.5}$$

Using the expression for the projectors for R_1 and R_2 in terms of the Hecke algebra, we obtain

$$\begin{aligned} N^{R_3}_{R_1 R_2} = {} & \frac{1}{g_1 g_2 g_3} \frac{d_{R_1}(q)}{d_{R_1}(1)} \frac{d_{R_2}(q)}{d_{R_2}(1)} \sum_{\sigma_1\sigma_2} q^{-l(\sigma_1)-l(\sigma_2)}\, \chi_{R_1}(h(\sigma_1^{-1}))\, \chi_{R_2}(h(\sigma_2^{-1})) \\ & \times \frac{1}{\dim_q R_3} \mathrm{tr}_{V^{\otimes n_1}\otimes V^{\otimes n_2}} \Big((h(\sigma_1)\cdot h(\sigma_2))\; P_{R_3} \Big)\, . \end{aligned} \tag{4.6}$$

Here and in what follows we take $\sigma_i \in S_{n_i}$ for $i = 1, 2, 3$. Writing out the projector (2.28), we get

$$\begin{aligned} N^{R_3}_{R_1 R_2} = {} & \frac{1}{g_1 g_2 g_3} \frac{d_{R_1}(q)}{d_{R_1}(1)} \frac{d_{R_2}(q)}{d_{R_2}(1)} \\ & \times \sum_{\sigma_1\sigma_2\sigma_3} q^{-l(\sigma_1)-l(\sigma_2)-l(\sigma_3)}\, \chi_{R_1}(h(\sigma_1^{-1}))\, \chi_{R_2}(h(\sigma_2^{-1}))\, \chi_{R_3}(h(\sigma_3^{-1})) \\ & \times \frac{d_{R_3}(q)}{\dim_q R_3} \mathrm{tr}_{V^{\otimes n_1}\otimes V^{\otimes n_2}} \Big((h(\sigma_1)\cdot h(\sigma_2))\; h(\sigma_3) \Big)\, , \end{aligned} \tag{4.7}$$

and expanding the trace in a basis of Young tableaux with $n_1 + n_2$ boxes, we get

$$\begin{aligned} N^{R_3}_{R_1 R_2} = {} & \frac{1}{g_1 g_2 g_3} \frac{d_{R_1}(q)}{d_{R_1}(1)} \frac{d_{R_2}(q)}{d_{R_2}(1)} \\ & \times \sum_{\sigma_1\sigma_2\sigma_3} q^{-l(\sigma_1)-l(\sigma_2)-l(\sigma_3)}\, \chi_{R_1}(h(\sigma_1^{-1}))\, \chi_{R_2}(h(\sigma_2^{-1}))\, \chi_{R_3}(h(\sigma_3^{-1})) \\ & \times \frac{d_{R_3}(q)}{\dim_q R_3} \sum_{S\in Y_{n_1+n_2}} \chi_S \Big((h(\sigma_1)\cdot h(\sigma_2))\; h(\sigma_3) \Big) \dim_q S\, . \end{aligned} \tag{4.8}$$

If we now use the projector property

$$\chi_S(P_{R_3} h(\sigma)) = \delta_{R_3 S}\, \chi_{R_3}(h(\sigma)) \tag{4.9}$$

and the explicit form of the projector in (2.12) then we have the useful orthogonality relation

$$\begin{aligned} & \sum_{\sigma_3} q^{-l(\sigma_3)} \frac{d_{R_3}(q)}{g_3} \chi_{R_3}(h(\sigma_3^{-1}))\, \chi_S \Big(h(\sigma_3)(h(\sigma_1)\cdot h(\sigma_2)) \Big) \\ & = \chi_{R_3}(h(\sigma_1)\cdot h(\sigma_2))\, \delta_{R_3 S}. \end{aligned} \tag{4.10}$$

This can be used to simplify the expression (4.8) further to

$$\begin{aligned} N^{R_3}_{R_1, R_2} = {} & \frac{1}{g_1 g_2} \frac{d_{R_1}(q)}{d_{R_1}(1)} \frac{d_{R_2}(q)}{d_{R_2}(1)} \sum_{\sigma_1\sigma_2} q^{-l(\sigma_1)-l(\sigma_2)} \\ & \times \chi_{R_1}(h(\sigma_1^{-1}))\, \chi_{R_2}(h(\sigma_2^{-1}))\, \chi_{R_3}(h(\sigma_1)\cdot h(\sigma_2)). \end{aligned} \tag{4.11}$$

This formula is reminiscent of the Verlinde formula for the fusion coefficients of orbifold conformal field theories [25], or alternatively of Chern-Simons theory with finite groups [26, 27]. It would be interesting to understand the connection.

If we go from the character basis to the basis in terms of central elements of the Hecke algebra, and using the above, we get

$$
\begin{aligned}
&\frac{1}{g_1 g_2 g_3} \sum_{R_1 R_2 R_3} N_{R_1 R_2}^{R_3}\, \chi_{R_1}(C_1)\, \chi_{R_2}(C_2)\, \chi_{R_3}(C_3) \\
&\quad = \frac{1}{g_1 g_2} \sum_{\sigma_1 \sigma_2} \delta(h(\sigma_1^{-1})C_1)\, \delta(h(\sigma_2^{-1})C_2)\, q^{-l(\sigma_1)-l(\sigma_2)} \\
&\qquad \times \sum_{R_3} \frac{1}{g_3}\, d_{R_3}(1)\, \chi_{R_3}(h(\sigma_1)\cdot h(\sigma_2) C_3) \\
&\quad = \frac{1}{g_1 g_2} \sum_{\sigma_1 \sigma_2} \delta(h(\sigma_1^{-1})C_1)\, \delta(h(\sigma_2^{-1})C_2)\, q^{-l(\sigma_1)-l(\sigma_2)}\, \frac{1}{g_3}\, \delta\left(E\, C_3\, (h(\sigma_1)\cdot h(\sigma_2))\right) \\
&\quad = \frac{1}{g_1 g_2 g_3} \sum_{\sigma_1} \sum_{\sigma_2} C_1^{\sigma_1}\, C_2^{\sigma_2}\, \delta\left(\, E\, C_3\, (h(\sigma_1)\cdot h(\sigma_2))\right). \qquad (4.12)
\end{aligned}
$$

E is the element defined in (A.14) of Appendix A. We have denoted by C^{σ} the coefficients which appear in the expansion of the central element C,

$$
C = \sum_{\sigma} C^{\sigma}\, h(\sigma)\,, \qquad (4.13)
$$

and we used the following property of the trace [21]:

$$
\begin{aligned}
&\delta(h(\sigma)h(\sigma')) = q^{l(\sigma)} \quad \text{if} \quad \sigma\sigma' = 1, \\
&\delta(h(\sigma)h(\sigma')) = 0 \qquad \text{otherwise.} \qquad (4.14)
\end{aligned}
$$

Consider now the computation of a simple Wilson loop, in the representation S, separating a region with G_1 handles from another region with G_2 handles,

$$
\begin{aligned}
W_S &= \sum_{n_1, n_2} \sum_{R_1 R_2} (\dim_q R_1)^{1-2G_1} (\dim_q R_2)^{1-2G_2} N_{R_1 S}^{R_2} \\
&= \sum_{R_1} [N]^{n_1(1-2G_1)} \left(\frac{d_{R_1}(q)}{g_1 d_{R_1}(1)}\right)^{1-2G_1} (\chi_{R_1}(\Omega))^{1-2G_1} \\
&\quad \times \sum_{R_2} [N]^{n_2(1-2G_2)} \left(\frac{d_{R_2}(q)}{g_2 d_{R_2}(1)}\right)^{1-2G_2} (\chi_{R_2}(\Omega))^{1-2G_2}\;\; N_{R_1 S}^{R_2}. \qquad (4.15)
\end{aligned}
$$

We now use (4.11) with the fusion coefficient, multiply by the character of some central element C in $H_{n_S}(q)$ and sum over S

$$
W(C, G_1, G_2) = \sum_{S} \frac{\chi_S(C)}{g_S}\, W_S. \qquad (4.16)
$$

Collecting all S dependences we have

$$\sum_S \frac{1}{g_S} \frac{d_S(q)}{d_S(1)} \chi_S(C)\, \chi_S(h(\sigma_2^{-1})) = \delta(C\, h(\sigma_2^{-1})). \tag{4.17}$$

Hence we obtain

$$\begin{aligned} W(C; G_1, G_2) = \sum_{n_1, n_2} \frac{1}{g_1 g_2}\, \delta_{n_1+n_S, n_2}\, [N]^{n_1(1-2G_1)+n_2(1-2G_2)} \sum_{\sigma_1 \sigma_2} q^{-l(\sigma_1)-l(\sigma_2)} \\ \times\, \delta(C\, h(\sigma_2^{-1}))\, \delta\left(D\, \Pi_1^{G_1}\, \Omega^{1-2G_1}\, h(\sigma_1^{-1})\right) \\ \times\, \delta\left(\Pi_1^{G_2} \Omega^{1-2G_2} (h(\sigma_1) \cdot h(\sigma_2))\right). \end{aligned} \tag{4.18}$$

The factors of $[N]$ are as above. We have defined

$$\begin{aligned} \Pi_1^{G_1} &= \sum_{s_1, t_1 .. s_{G_1}, t_{G_1}} q^{-\sum_i l(s_i) - l(t_i)} \prod_{i=1}^{G_1} h(s_i) h(t_i) h(s_i^{-1}) h(t_i^{-1}), \\ \Pi_1^{G_2} &= \sum_{s_1, t_1 .. s_{G_2}, t_{G_2}} q^{-\sum_i l(s_i) - l(t_i)} \prod_{i=1}^{G_2} h(s_i) h(t_i) h(s_i^{-1}) h(t_i^{-1}). \end{aligned} \tag{4.19}$$

Expanding

$$\begin{aligned} C &= \sum_\sigma C^\sigma\, h(\sigma) \\ P &= D\, \Omega^{1-2G_1} \Pi_1^{G_1} = \sum_\sigma P^\sigma\, h(\sigma)\, , \end{aligned} \tag{4.20}$$

we finally get

$$W(C; G_1, G_2) = \sum_{n_1=0}^{\infty} \frac{1}{g_1 g_2} [N]^\gamma \sum_{\sigma \sigma'} P^\sigma C^{\sigma'} \delta\left(\Omega^{1-2G_2} \Pi_1^{G_2} (h(\sigma) \cdot h(\sigma'))\right). \tag{4.21}$$

We defined $\gamma = n_1 + n_2 - 2(n_1 G_1 + n_2 G_2) = (2 - 2G) n_1 + n_S (1 - 2G_2)$, where we used $n_2 = n_1 + n_S$.

5. On the Role of Quantum Characters in q-Deformed 2d YM

In this paper we have used quantum $U_q(SU(N)$, characters rather than classical $SU(N)$ characters. For the computations in [10] it seemed enough to consider classical $SU(N)$ characters. So one can ask: does one need to compute with quantum characters, or do the classical ones suffice? In this section we argue that quantum characters are needed in the generic situation; in fact, they are extremely natural and they provide the simplest solution to the problem of crossings and gluing along open lines. Our arguments are consistent with [10], where the dimensions appearing in the partition function (1.3) were quantum dimensions but the characters associated with boundaries and Wilson loops were classical $SU(N)$ characters. In particular, this paper did not consider crossings on the surface, and gluing constructions involved closed curves only. In the absence of crossing points, both the classical and the quantum characters lead to a topological invariant theory. It is a well-known fact from Chern-Simons theory that one can do without R-matrices or other quantum group structure as long as one considers simple Wilson loops – for example, toric ones, whose expectation value follows from surgery. In the 2d Yang-Mills case, the basic gluing formula along circles is (3.5), which is valid both for classical and quantum characters, and ensures topological invariance of the gluing construction along circles. More precisely, the need for quantum characters in qYM can be seen:

1) in the presence of Wilson loops with non-trivial crossings;
2) when gluing along open lines.

The original definition of qYM is well-known [13, 12] and it involves quantum characters. In the following subsections we collect several arguments that show the need for quantum characters.

5.1. Consistency of Wilson loops. One of the basic consistency conditions to be imposed on a Wilson loop is that, if the charge of the particle is zero, the expectation value of the Wilson loop should be that of the unit operator; in other words, it should give back the partition function of the theory. In our case, if $W_R(G; C)$ is the Wilson loop operator in representation R around the curve C on the Riemann surface of genus G, consistency requires

$$W_{R=\rho}(G; C) = \langle 1 \rangle = Z_{q\mathrm{YM}}(\Sigma_G), \tag{5.1}$$

where ρ is the Weyl vector labeling the trivial representation. Thus, we should reproduce:

$$W_\rho(G; C) = \sum_S (\dim_q S)^{2-2g} q^{-\frac{1}{2} A\, C_2(S)} . \tag{5.2}$$

We will check whether quantum dimensions and classical characters are consistent with this for a Wilson loop with crossings.

Consider the expectation value of the Wilson loop $W_R(G; C)$ in Fig. 1. In this case we have $A = A_1 + A_2 + A_3$, where A_1 is the area of the outer face, which has genus G. We get:

$$\begin{aligned} &W_R(G; C) \\ &\quad = \sum_{R_1 R_2 R_3} (\dim_q(R_1))^{1-2g} \dim_q(R_2) \dim_q(R_3)\, q^{-\frac{1}{2}(A_1 C_2(R_1) + A_2 C_2(R_2) + A_3 C_2(R_3))} \\ &\qquad \times \int \mathrm{d}U \mathrm{d}V\, \chi_{R_1}(U^{-1}V^{-1})\, \chi_{R_2}(U)\, \chi_{R_3}(V) \chi_R(UV^{-1}), \end{aligned} \tag{5.3}$$

where, since we are dealing with classical characters, dU is the Haar measure. Let us compute this in the trivial case: $R = \rho$. We can compute the integrals using the character formula

$$\int \mathrm{d}U\, \chi_{R_2}(U)\chi_{R_3}(U^{-1}V) = \delta_{R_2 R_3} \frac{\chi_{R_2}(V)}{\dim(R_2)}. \tag{5.4}$$

We get

$$\sum_S \frac{(\dim_q(S))^{3-2g}}{\dim(S)} q^{-\frac{1}{2} A C_2(S)}, \tag{5.5}$$

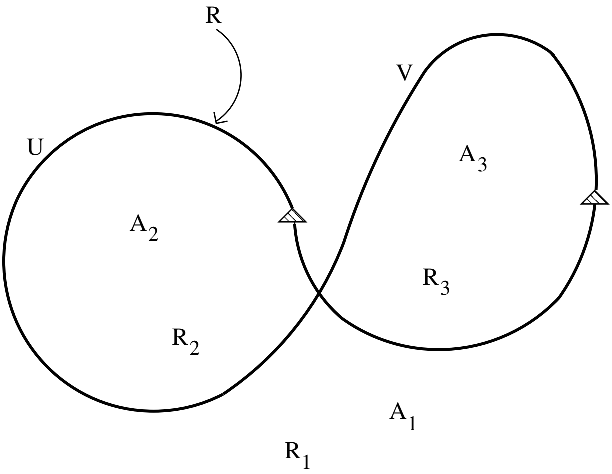

Fig. 1. A Wilson loop with a crossing.

which disagrees with (5.2). The reason that the dimensions do not come out right is that we were forced to use formula (5.4). We conclude that this procedure is not consistent. On the other hand, the same computation can be carried out with quantum characters, and in that case we do get the quantum dimension in (5.4).

5.2. Gauge invariance of Wilson loops. There is a short proof of gauge invariance for the Wilson loops and boundary elements we have discussed in previous sections. Let $U \in \mathrm{Fun}_q(SU(N))$ (for more details on this see Appendices B and D), and consider the ad-action of $\mathrm{Fun}_q(SU(N))$ on itself:

$$\mathrm{ad} : U \mapsto h\, U S(h)\ , \tag{5.6}$$

where we are considering $\mathrm{Fun}_q(SU(N))$ as a Hopf algebra with antipode S [17]. It is easy to see that the quantum trace

$$\mathrm{Tr}\,(u\, U) \tag{5.7}$$

is left invariant under this action (for the definition of the u-element, see Appendix B). We get:

$$\begin{aligned}\mathrm{Tr}\,(u\, hU S(h)) &= \mathrm{Tr}\,(S^2(h)uU S(h)) = (S^2(h))_{ij}(uU)_{jk}(S(h))_{ki}\\ &= (S^2(h))_{ij}(S(h))_{ki}(uU)_{jk} = S(h_{ki}(S(h))_{ij})(uU)_{jk}\\ &= \mathrm{Tr}\,(u\, U)\ ,\end{aligned} \tag{5.8}$$

where we used $\epsilon(h) = 1$, and the fact that u satisfies

$$u\, x = S^2(x)\, u \tag{5.9}$$

for any $x \in \mathrm{Fun}_q(SU(N))$. Thus, gauge invariance in $\mathrm{Fun}_q(SU(N))$ is ensured provided we include the u-element.

We have proven that the triple

$$\Big(\text{Migdal gluing , quantum dimensions , classical characters}\Big) \tag{5.10}$$

is inconsistent in the generic case. To get a consistent theory, we need to modify one of the above. If instead of quantum dimensions we use classical dimensions, we of course get back the usual 2d Yang-Mills. If we want the dimensions to be quantum, we either need quantum characters, or a modification of the gluing rules. The possibility to have quantum characters has been discussed at length in this paper, and it has been shown to be consistent in [13]. In particular, the theory is gauge invariant and independent of the triangulation. We do not exclude that there might be a complicated modification of the gluing rules that would allow to keep quantum dimensions and classical characters even in the presence of crossings.

Additional features of the quantum characters are the following. The natural expansion of the quantum dimensions is in terms of quantum characters, which are most easily expressed in terms of a Hecke algebra, as we have shown. This gives a natural deformation of the symmetric group description of covering maps of the Riemann surface. Also in the case with boundaries, the use of quantum characters was essential for this. Finally, q-deformed 2d Yang-Mills computes invariants of knots in Seifert manifolds [28, 11]. This is also expected from open-closed string duality in the A-model with branes. This relation will however only work if on the qYM side we deform the gauge symmetry as well so as to get quantum characters, since only that will give the quantum 6j-symbols that appear in the Reshetikhin-Turaev invariant relevant for knots in Chern-Simons [11].

6. Discussion and Outlook

We have shown that the chiral large N expansion (note that the q-number $[N]$ appears as the natural expansion parameter) for q-deformed Yang-Mills can be described by Hecke algebras. The full large N expansion is expected to be given by a coupled product of chiral and anti-chiral contributions. We expect that techniques of this paper can be extended to give a precise description of this non-chiral expansion in terms of Hecke algebras.

The string interpretation of q-deformed 2d Yang-Mills on Σ_G has been developed in [9, 10]. The leading order terms in the expansion, obtained by setting the Ω factors to 1, were shown to compute Gromov-Witten invariants of a Calabi-Yau space X which is a direct sum of line bundles $L_p \oplus L_{2g-2-p}$ fibered over Σ_G. The sub-leading terms, due to the Ω factors were intepreted in terms of D-brane insertions at $2G-2$ points. This picture develops the Gross-Taylor interpretation (at $q=1$) of the Ω factors in terms of fixed points on the Riemann surface [3, 4]. An alternative interpretation of the Ω factors underlies the topological string theory developed in [5, 6] for $q=1$. The latter topological string is different from the standard one. It has been labelled a balanced topological string and has been observed to be an example of a general class of balanced topological field theories naturally related to Euler characters of moduli spaces [29]. It integrates over the moduli space of holomorphic maps the Euler class of the tangent bundle to that moduli space.

The concrete connection between Euler characters and the large N expansion of two dimensional Yang-Mills is manifest when one expands the Ω factors and recognizes the binomial coefficients as Euler characters of configuration spaces of points on the Riemann surface Σ_G [5]. Our treatment of the Ω factors in the q-deformed case, which has expressed it in terms of central elements of the Hecke algebra, naturally lends itself to this interpretation. Euler characters of configuration spaces continue to appear in the expansion for the same reasons as at $q = 1$. This suggests that a closed topological string interpretation exists for the large N expansion of q-deformed two-dimensional Yang-Mills in terms of a balanced topological string. The simplest proposal along these lines is that the balanced topological string with target space X would give a closed string interpretation for the all orders expansion of q-deformed two-dimensional Yang-Mills. The relation of such a picture to the D-brane insertions of [10] would involve an interesting incarnation of open-string/closed-string duality. Developing these relations requires a clearer understanding of the coupling between holomorphic and anti-holomorphic sectors in the context of the balanced topological string. The connection between the Gross-Taylor expansion and the Gromov-Witten invariants appearing in [9, 10] has also been discussed in [30, 31].

Given the rather simple Hecke q-deformation we have uncovered, of the sums over symmetric group delta functions related to the classical Hurwitz counting of branched covers, it is also natural to speculate that there is an intrinsically two-dimensional picture which would account for the Hecke delta functions, without appealing to the Calabi-Yau X. One possiblity is that we have q-deformed Riemann surfaces and maps between such Riemann surfaces. In fact q-deformed planes, known as Manin planes, have been studied and holomorphy has been discussed (see for example [32]). One could construct Riemann surfaces which, in some sense, locally look like Manin planes, and consider holomorphic maps between them. As far as we are aware, such a theory of Hurwitz spaces for q-deformed Riemann surfaces has not yet been developed.

While Hecke algebras are more familiar to mathematical physicists as centralizers of quantum groups acting in tensor spaces, they have another pure mathematical origin (see for example [33]). $H_n(q)$ is an algebra of double cosets $B_n(F_q) \setminus GL_n(F_q)/B_n(F_q)$. Here F_q is the finite field with q elements, where q is a power of a prime p. (If $q = p$ then F_q is just the field of residue classes modulo p.) $GL_n(F_q)$ is the group of $n \times n$ matrices with entries in F_q. $B_n(F_q)$ is the subgroup of the upper triangular matrices. This generalises the fact that S_n appears from double cosets $B_n(\mathbb{C}) \setminus GL_n(\mathbb{C})/B_n(\mathbb{C})$. Hence the deformation of $\mathbb{C}S_n$ to the Hecke algebra $H_n(q)$ corresponds to going from $\mathbb{C}$ to F_q. This suggests that, at least for q equal to a power of a prime, our Hecke-q-deformed Hurwitz counting problem might be related to Riemann surfaces over F_q. It is interesting that, in this context, fundamental groups can be defined and they still take the form

$$a_1\, b_1\, a_1^{-1}\, b_1^{-1}\, a_2\, b_2\, a_2^{-1}\, b_2^{-1}\, \cdots\, a_G\, b_G\, a_G^{-1}\, b_G^{-1}\, u_1 \cdots u_B = 1\,. \tag{6.1}$$

There are also results on the moduli spaces of branched covers in this set-up, generalizing properties of classical Hurwitz space [34]. An interesting direction for the future is to determine if there is a relation between Hecke algebras $H_n(q)$ and these moduli spaces, and if such a relation provides the geometrical meaning for the q-deformed Hecke counting problems in (2.56), (2.57).

Classical and q-deformed 2d Yang-Mills are closely connected to Chern-Simons theory on Seifert manifolds [35, 11, 10, 36, 37]. On the other hand, some of the formulas in this paper, such as (4.11) , are suggestive of some connection of the chiral large N expansion of q-deformed 2d Yang-Mills and orbifold conformal field theories [25] or Chern-Simons theory for finite gauge groups [26, 27]. It is known that the Chung-Fukuma-Shapere three-dimensional topological field theory [38] is the absolute value squared of the partition function of the Dijkgraaf-Witten theory. It seems very likely that the chiral expansion in terms of Hecke characters worked out in this paper can be formulated in the two-dimensional topological field theory framework of [39, 38] with additional insertions coming from the branch points. It would be interesting to see in detail to what extent the chiral q-deformed 2d Yang-Mills theory is related to the Dijkgraaf-Witten theory. In view of the connection to Chern-Simons theory, it will be interesting to explore the q-deformed chiral as well when q approaches roots of unity. q-Schur Weyl duality at roots of unity has been discussed in [40, 41].

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A. Central Elements

A.1. Centrality of q-deformed conjugation sum . We want to show that

$$\sum_s q^{-l(s)} h(s)h(t)h(s^{-1}) \tag{A.1}$$

is central in $H_q(n)$. Since $H_q(n)$ is generated by $g_1, ...g_{n-1}$, it suffices to show that the above element commutes with these generators. We will first show it for g_1, and it will be clear the same proof can be repeated for g_2, etc.

First recall how this works in the case $q = 1$. We write

$$\begin{aligned}\sum_s s_1 s t s^{-1} &= \sum_s (\tilde{s}) t \tilde{s}^{-1} s_1 \\ &= \sum_{\tilde{s}} \tilde{s} t \tilde{s}^{-1} s_1 \ ,\end{aligned}$$

where we defined $\tilde{s} = s_1 s$. The cancellation only uses a pair of terms at a time. For a fixed s,

$$\begin{aligned}s_1 s t s^{-1} &= \tilde{s} t \tilde{s}^{-1} s_1, \\ s_1 \tilde{s} t \tilde{s}^{-1} &= s t s^{-1} s_1 \ ,\end{aligned}$$

which means that

$$[s_1, sts^{-1}] + [s_1, \tilde{s} t \tilde{s}^{-1}] = 0 \ . \tag{A.2}$$

It turns out that *the same pairwise cancellation works for $q \neq 1$.* It is instructive to check it explicitly for $n = 3, 4$. Below we give the general argument.

Suppose s is of the form $s_1 u$, where u is a word in the generators. Now recall that before applying the map h to s we must express it in reduced form. This means that if $s = s_1 u$, the leftmost term in u is not s_1. The following can be derived easily

$$\begin{aligned}h(s) &= g_1 h(u), \\ l(s) &= l(u) + 1, \\ h(s^{-1}) &= h(u^{-1}) g_1 \ .\end{aligned}$$

Then $\tilde{s} = s_1 s = u$. Now we write the pair of elements from (A.1) for the fixed s, $\tilde{s}$.

$$q^{-l(s)} h(s)h(t)h(s^{-1}) = q^{-l(u)-1} g_1 h(u)h(t)h(u^{-1})g_1,$$
$$q^{-l(\tilde{s})} h(\tilde{s})h(t)h(\tilde{s}^{-1}) = q^{-l(u)} h(u)h(t)h(u^{-1}) \, . \tag{A.3}$$

The commutator with the first term is

$$\begin{aligned}&[g_1, q^{-l(s)} h(s)h(t)h(s^{-1})]\\ &= q^{-l(u)} h(u)h(t)h(u^{-1})g_1 + q^{-l(u)-1}(q-1)g_1 h(u)h(t)h(u^{-1})g_1\\ &\quad - q^{-l(u)} g_1 h(u)h(t)h(u^{-1}) - q^{-l(u)-1}(q-1)g_1 h(u)h(t)h(u^{-1})g_1 \, .\end{aligned} \tag{A.4}$$

The commutator with the second term in (A.3) is

$$[g_1, q^{-l(u)} h(u)h(t)h(u^{-1})] = q^{-l(u)} g_1 h(u)h(t)h(u^{-1}) - q^{-l(u)} h(u)h(t)h(u^{-1})g_1 \, . \tag{A.5}$$

Combining the terms in (A.4) and (A.5) we see that the terms proportional to a power of q cancel between the two equations (as they must for this to work at $q = 1$). The terms containing a factor $q - 1$ cancel within (A.4).

This proves that the sum (A.1) commutes with g_1. It has been done by decomposing the sum over S_n into a sum over left coset elements by the subgroup S_2 generated by s_1, and a sum over representatives in each coset. The vanishing of the commutator with g_1 works within the sum over representatives in each coset. To prove that it commutes with $g_2 \cdots g_{n-1}$ we similarly decompose with respect to left cosets of $s_2, \cdots s_{n-1}$. Hence (A.1) is central in $H_q(n)$. It follows that its matrix representation in any irreducible representation must be diagonal. Using the matrices given in [16], we have checked this explicitly up to $n = 4$.

A special case of (A.1) is given by the choice $t = 1$. Based on evidence described below, we conjecture that its character in an irreducible representation is

$$\sum_s q^{-l(s)} \frac{\chi_R}{d_R(1)}(h(s^{-1})h(s)) = \frac{g \, d_R(1)}{d_R(q)} \, , \tag{A.6}$$

with $d_R(q)$ and g as given in (2.13). Since the Hecke element in the character is central (after summation over s), it suffices to calculate it on one state in the irrep. We have checked this for general completely symmetric reps and completely antisymmetric reps, as well as for all representations up to $n = 4$, using the explicit matrices given in [16]. Another check of this formula is to multiply by $d_R(q)d_R(1)$ and sum over young diagrams R with n boxes. Using (2.43), the LHS becomes

$$g \, \delta\left(\sum_s q^{-l(s)} \, h(s^{-1})h(s)\right). \tag{A.7}$$

But from [21] $\delta(h(s^{-1})h(s)) = q^{l(s)}$. Hence the LHS is equal to $(g \, n!)$. On the RHS we have $g \sum_R (d_R(1))^2 = (g \, n!)$. This gives a consistency check of (A.6) for any n.

Using (A.6) and (2.44),

$$\sum_s q^{-l(s)} \frac{\chi_R}{d_R(1)}(h(s)h(t)h(s^{-1})) = \sum_s q^{-l(s)} \frac{\chi_R}{d_R(1)}(h(s)h(s^{-1})h(t)) = \frac{g\, d_R(1)}{d_R(q)} \frac{\chi_R}{d_R(1)}(h(t)) = \frac{g}{d_R(q)} \chi_R(h(t)) . \tag{A.8}$$

Hence

$$\sum_{s,t} q^{-l(t)-l(s)} \frac{\chi_R}{d_R(1)}(h(s)h(t)h(s^{-1})h(t^{-1})) = \sum_{s,t} q^{-l(t)} q^{-l(s)} \frac{\chi_R}{d_R(1)}(h(s)h(t)h(s^{-1})) \frac{\chi_R}{d_R(1)}(h(t^{-1})) = \frac{g}{d_R(q)} \sum_t q^{-l(t)} \chi_R(h(t)) \frac{\chi_R}{d_R(1)}(h(t^{-1})) = \frac{g}{d_R(q)d_R(1)} \sum_t q^{-l(t)} \chi_R(h(t))\chi_R(h(t^{-1})) = \frac{g}{d_R(q)d_R(1)} \frac{g d_R(1)}{d_R(q)} = \left(\frac{g}{d_R(q)}\right)^2 . \tag{A.9}$$

The last sum over characters was done by using orthogonality (2.24). This shows the desired identity (2.50)

A.2. Centrality of q-deformed commutator sum . We prove that the element

$$C \equiv \sum_{s,t} q^{-l(s)-l(t)} h(s)h(t)h(s^{-1})h(t^{-1}) \tag{A.10}$$

of the Hecke algebra $H_n(q)$ is central. In the $q = 1$ limit, this is $\sum_{s,t} sts^{-1}t^{-1}$, a sum of commutators of all group elements. Hence C is a q-deformed sum of commutators. Since $H_n(q)$ is generated by $g_1 \ldots g_{n-1}$ it suffices to prove that $g_i C = C g_i$ for any g_i. We will start with g_1 and it will be clear how to generalize to the other generators.

Given the centrality of the q-deformed conjugation sum (A.1) we can write

$$\Delta_1 \equiv g_1 C - C g_1 = \sum_{s,t} q^{-l(s)-l(t)} h(s)h(t)h(s^{-1}) g_1 h(t^{-1}) - \sum_{s,t} q^{-l(s)-l(t)} h(s) g_1 h(t)h(s^{-1})h(t^{-1}) . \tag{A.11}$$

We want to prove $\Delta_1 = 0$. For $q = 1$ this can be proved as follows. If we define $t = \hat{t}s_1$, $s = \hat{s}s_1$, we can write

$$\sum_{s,t} s s_1 t s^{-1} t^{-1} = \sum_{\hat{s},\hat{t}} \hat{s}\hat{t}\hat{s}^{-1} s_1 \hat{t}^{-1}. \tag{A.12}$$

This shows that it is useful to think about the sums over S_n in terms of the cosets S_n/S_2, where the S_2 is generated by s_1. Let us choose expressions for the elements of S_n in terms of words of minimal length in $s_1..s_n$. Let S_+ be the set of words not ending with s_1 on the right, and S_- the set of elements of the form $\hat{s}s_1$. Clearly $\hat{s}$ does not end with s_1 : if it did s would not be in reduced form. Hence $\hat{s} \in S_+$. For such $s = \hat{s}s_1$, it is easy to see that

$$\begin{aligned}
h(s) &= h(\hat{s})g_1, \\
l(s) &= l(\hat{s}) + 1, \\
h(s^{-1}) &= g_1 h(\hat{s}^{-1}).
\end{aligned} \tag{A.13}$$

We can write Δ_1 as

$$\begin{aligned}
\Delta_1 = &\Big(\sum_{s\in S_+} + \sum_{s=\hat{s}s_1\in S_-\ ;\ \hat{s}\in S_+} \Big)\Big(\sum_{t\in S_+} + \sum_{t=\hat{t}s_1\in S_-\ ;\ \hat{t}\in S_+} \Big)\, h(s)h(t)h(s^{-1})g_1h(t^{-1})q^{-l(s)-l(t)} \\
&-\Big(\sum_{s\in S_+} + \sum_{s=\hat{s}s_1\in S_-\ ;\ \hat{s}\in S_+} \Big)\Big(\sum_{t\in S_+} + \sum_{t=\hat{t}s_1\in S_-\ ;\ \hat{t}\in S_+} \Big)\, h(s)g_1h(t)h(s^{-1})h(t^{-1})q^{-l(s)-l(t)} \\
= &\sum_{s,t\in S_+} q^{-l(s)-l(t)}h(s)h(t)h(s^{-1})g_1h(t^{-1}) + \sum_{s,\hat{t}\in S_+} q^{-l(s)-l(\hat{t})-1}h(s)h(\hat{t})g_1h(s^{-1})g_1^2h(\hat{t}^{-1}) \\
&+\sum_{\hat{s},t\in S_+} q^{-l(\hat{s})-l(t)-1}h(\hat{s})g_1h(t)g_1h(\hat{s}^{-1})g_1h(t^{-1}) + \sum_{\hat{s},\hat{t}\in S_+} q^{-l(\hat{s})-l(\hat{t})-2}h(\hat{s})g_1h(\hat{t})g_1^2h(\hat{s}^{-1})g_1^2h(\hat{t}^{-1}) \\
&-\sum_{s,t\in S_+} q^{-l(s)-l(t)}h(s)g_1h(t)h(s^{-1})h(t^{-1}) - \sum_{s,\hat{t}\in S_+} q^{-l(s)-l(\hat{t})-1}h(s)g_1h(t)g_1h(s^{-1})g_1h(t^{-1}) \\
&-\sum_{\hat{s},t\in S_+} q^{-l(\hat{s})-l(t)-1}h(\hat{s})g_1^2h(t)g_1h(\hat{s}^{-1})h(t^{-1}) - \sum_{\hat{s},\hat{t}\in S_+} q^{-l(\hat{s})-l(\hat{t})-2}h(\hat{s})g_1^2h(\hat{t})g_1^2h(\hat{s}^{-1})g_1h(t^{-1})\,.
\end{aligned}$$

This can be simplified by using $g_1^2 = (q-1)g_1+q$. We get terms with powers $q^{-l(s)-l(t)}$ in the summand but without powers of $(q-1)$, terms proportional to $(q-1)$ and terms proportional to $(q-1)^2$. The terms without powers of $(q-1)$ cancel pairwise among the 8 terms. The other terms can be written out explicitly, and seen to cancel. This proves that $[g_1, C] = 0$. When checking for commutation with g_i, we organise the sums over S_n according to cosets of the S_2 subgroup generated by s_i. Then the same argument as above applies to show that any of the generating g_i commute with C. Hence C is central.

A.3. The elements D and E of $H_n(q)$. Equation (A.6) also allows us to give an expression for D defined in (2.54). Let us write

$$\begin{aligned}
E &= \sum_s q^{-l(s)}h(s^{-1})h(s) \\
&= 1 + {\sum_s}' q^{-l(s)}h(s^{-1})h(s) \\
&\equiv 1 + E'.
\end{aligned}$$

The primed sum extends over elements in S_n excluding the identity. Then we can write

$$\frac{\chi_R(E)}{d_R(1)} = \frac{g d_R(1)}{d_R(q)}. \tag{A.14}$$

Using that E is central

$$\left(\frac{\chi_R(E)}{d_R(1)}\right)^m = \frac{\chi_R(E^m)}{d_R(1)} = \left(\frac{g\, d_R(1)}{d_R(q)}\right)^m . \tag{A.15}$$

Now let $m = -1$ to get

$$\chi_R(E^{-1}) = \frac{d_R(q)}{g}. \tag{A.16}$$

Hence

$$\begin{aligned} D &= gE^{-1} \\ &= g\sum_{k=0}^{\infty}(-1)^k (E')^k \\ &= g\sum_{k=0}^{\infty}(-1)^k \sum_{u_1,u_2\ldots u_k}{}' q^{-l(u_1)-l(u_2)-\ldots-l(u_k)} h(u_1^{-1})h(u_1)\cdots h(u_k^{-1})h(u_k) \end{aligned} \tag{A.17}$$

B. Quantum Dimensions

The irreducible representations R of $U_q(U(N))$ can be realized as subspaces of $V^{\otimes n}$, where V is the fundamental representation. The matrix elements of the fundamental representation are denoted by U, with entries U_i^j (see Appendix D for explicit expressions), and the algebra generated by the U's is dual to $U_q(U(N))$ and is denoted by $\mathrm{Fun}_q(U(N))$. The commutation relations of the U_i^j are given in terms of the R-matrix in the reference we denote as FRT [17] (we are using U for T of this reference).

We first derive formula (2.34), used in Sect. 2 to obtain a Hecke formula for the quantum dimensions. Thus we need to compute the trace $\mathrm{tr}_n(u\, h(m_T))$ that comes from the quantum character expression. The element u is:

$$u = q^{2\sum_{i=1}^{N}\left(\frac{N+1}{2}-i\right)E_{ii}} . \tag{B.1}$$

The E_{ij} act on the fundamental representation in the usual way

$$E_{ij}\, v_k = \delta_{jk}\, v_i . \tag{B.2}$$

Now we can use the FRT formula for the R-matrix to show that

$$(\mathrm{tr}\otimes 1)(u\otimes 1)\, PR = q^N\, \mathbf{1} , \tag{B.3}$$

and $\mathrm{tr}(u) = \frac{q^N-q^{-N}}{q-q^{-1}}$. This means that

$$(\mathrm{tr}\otimes \mathrm{tr})(u\otimes u)\, PR = q^N\, \frac{q^N-q^{-N}}{q-q^{-1}} . \tag{B.4}$$

Going back to the Hecke algebra conventions using (2.8) ($q \to \sqrt{q}$), we get

$$\begin{aligned} (\mathrm{tr}\otimes 1)(u\otimes 1)\, g_1 &= q^{\frac{N+1}{2}}\, \mathbf{1}, \\ (\mathrm{tr}\otimes \mathrm{tr})(u\otimes u)\, g_1 &= q^{\frac{N+1}{2}}\, [N] . \end{aligned} \tag{B.5}$$

More generally, tensor products of traces act on $uh(m_T)$ as

$$(\mathrm{tr}\otimes\mathrm{tr}\otimes\ldots\otimes\mathrm{tr}_i)(u\otimes u\otimes\cdots\otimes u)(g_1g_2\cdots g_{i-1}) = q^{(i-1)\frac{N+1}{2}}\,[N]\,. \tag{B.6}$$

We now need to find out how to built $h(m_T)$ out of the g_i's. Consider a conjugacy class in S_n, denoted by T, made of permutations which have K_i cycles of length i. When expressed in terms of the generators s_i, the minimal length permutation in this conjugacy class, denoted by m_T, has length $\sum_i(i-1)K_i$. The minimal permutations are given in terms of words of the form $g_ig_{i+1}\ldots g_{i+j}$, such as the one appearing in (B.6). For such minimal words, we can use (B.6) to obtain

$$\mathrm{tr}_n(u\,h(m_T)) \equiv \mathrm{tr}^{\otimes n}\left(u^{\otimes n}h(m_T)\right) = q^{\frac{N+1}{2}\sum_i(i-1)K_i}\,[N]^{\sum_i K_i} = q^{\frac{N+1}{2}\,l(m_T)}\,[N]^{\sum_i K_i}\,. \tag{B.7}$$

This is the formula (2.34) used in the derivation of the q-dimension formula in Sect. 2.

We now show explicitly how formula (2.29) works in some examples, and that it leads to a q-dimension formula in terms of central elements (2.36). For q-traces in $V^{\otimes 3}$, i.e traces with $u^{\otimes 3}$ inserted, we have

$$\begin{aligned}\mathrm{tr}_q\,(1) &= [N]^3,\\ \mathrm{tr}_q\,(g_1) &= [N]^2\,q^{\frac{N+1}{2}},\\ \mathrm{tr}_q\,(g_1g_2) &= [N]\,q^{N+1}\,,\end{aligned} \tag{B.8}$$

therefore

$$\begin{aligned}\mathrm{tr}_q(g_2g_1g_2) &= (q-1)\,\mathrm{tr}_q(g_2g_1) + q\,\mathrm{tr}_q(g_1)\\ &= (q-1)\,q^{N+1}\,[N] + q\,q^{\frac{N+1}{2}}\,[N]^2\,.\end{aligned} \tag{B.9}$$

Now (2.29) gives for the q-dimension

$$\begin{aligned}\dim_q(R) = \frac{1}{g}\frac{d_R(q)}{d_R(1)}\Bigg(&[N]^3\chi_R(1) + 2q^{-1}\,q^{\frac{N+1}{2}}\,[N]^2\,\chi_R(g_1)\\ &+q^{-3}\,\chi_R(g_1g_2g_1)\,\mathrm{tr}_q(g_2g_1g_2)\\ &+q^{-2}\,\chi_R(g_1g_2)\,q^{N+1}\,[N] + q^{-2}\,\chi_R(g_2g_1)\,q^{N+1}[N]\Bigg).\end{aligned} \tag{B.10}$$

Filling in the above, we finally find

$$\begin{aligned}\dim_q(R) &= \frac{1}{g}\frac{d_R(q)}{d_R(1)}\Big([N]^3\,\chi_R(1) + q^{\frac{N-1}{2}}\,[N]^2\chi_R(g_1+g_2+q^{-1}g_1g_2g_1)\\ &\quad + q^{N-1}\,[N]\,\chi_R(g_1g_2+g_2g_1+q^{-1}(q-1)g_1g_2g_1)\Big)\\ &= \frac{1}{g}\,\frac{d_R(q)}{d_R(1)}\,\Big([N]^3\,\chi_R(1) + q^{\frac{N-1}{2}}\,[N]^2\,\chi_R(C_{T(2,1)}) + q^{N-1}\,[N]\,\chi_R(C_{T(3)})\Big)\,.\end{aligned} \tag{B.11}$$

The final expression contains central elements C_T associated to conjugacy classes of S_n. There is the trivial conjugacy class containing the identity element, for which $C_T(q) = 1$. There is $C_{(2,1)}(q) = g_1 + g_2 + q^{-1}g_1g_2g_1$, for the conjugacy class corresponding to a single transposition. Finally there is $C_{(3)}(q) = g_1g_2 + g_2g_1 + \frac{(q-1)}{q} g_1g_2g_1$. It is easy to check that these elements commute with g_1, g_2. The above central elements and their generalizations are described in [43, 44]. They approach the correct classical limit of a sum of permutations in the appropriate conjugacy class.

Using the Hecke characters given in [16, 20], we have checked that the above is consistent with the standard formula for the q-dimension as a product over the cells of the Young diagram:

$$\dim_q R = \prod_{1 \le i \langle j \le N} \frac{q^{(\lambda_i - \lambda_j + j - i)/2} - q^{-(\lambda_i - \lambda_j + j - i)/2}}{q^{(j-i)/2} - q^{-(j-i)/2}}, \tag{B.12}$$

where $\lambda_1, \ldots, \lambda_N$ are the lengths of the rows of the Young tableau, and, for $SU(N)$, $\lambda_N = 0$. For $n = 2$, an easy check gives for the symmetric representation $\square\!\square$:

$$\frac{[N][N+1]}{[2]}, \tag{B.13}$$

and for the antisymmetric representation $\begin{smallmatrix}\square\\\square\end{smallmatrix}$:

$$\frac{[N][N-1]}{[2]}. \tag{B.14}$$

We have also obtained by the above manipulations, explicit formulae for central elements for $n = 4$ which agree with those given in [22]. We have also checked that our formula for the quantum dimensions (2.36) agrees with the standard formula (B.12) for all representations up to $n = 4$.

C. Projectors

Below we give explicit checks that (2.28) indeed defines projectors. We do this for $n = 3$ and $n = 4$, that is for the Hecke algebras of H_3 and H_4, and outline the method of [21] for the general case.

C.1. H_3. We work out the projector for a general representation of H_3. It contains $3! = 6$ independent terms, corresponding to the six elements of H_3. Using (B.9), we get

$$\begin{aligned} P_R = \frac{1}{c_R} \Bigg(& \chi_R(1) + \frac{1}{q}\, \chi_R(g_1)\,(g_1 + g_2) + \left(\frac{q-1}{q^3}\, \chi_R(g_1g_2) + \frac{1}{q^2}\, \chi_R(g_1) \right) g_1g_2g_1 \\ & + \frac{1}{q^2}\, \chi_R(g_1g_2)\,(g_1g_2 + g_2g_1) \Bigg), \end{aligned} \tag{C.1}$$

where the term $g_1g_2g_1$ corresponds to the (13) permutation. Notice that in S_n, $s_1s_2s_1$ is in the same conjugacy class as s_1. Indeed, in the classical case where $q = 1$ the first term in (B.9) is absent and $\mathrm{tr}(g_1g_2g_1) = \mathrm{tr}(g_1)$. In the quantum case, $g_1g_2g_1$ has contributions from both $\chi(g_1g_2)$ and $\chi(g_1)$. This implies that it contributes to two different class elements.

Using the expressions for the characters in [16], we get for the three H_3 representations:

$$
\begin{aligned}
P_{\yng(3)} &= \frac{1}{c_{\yng(3)}}\,(1+g_1+g_2+g_1g_2+g_2g_1+g_1g_2g_1)\,,\\
P_{\yng(2,1)} &= \frac{1}{c_{\yng(2,1)}}\left(2+\frac{q-1}{q}\,(g_1+g_2)-\frac{1}{q}\,(g_1g_2+g_2g_1)\right),\\
P_{\yng(1,1,1)} &= \frac{1}{c_{\yng(1,1,1)}}\left(1-\frac{1}{q}\,(g_1+g_2)+\frac{1}{q^2}\,(g_1g_2+g_2g_1)-\frac{1}{q^3}\,g_1g_2g_1\right).
\end{aligned}
\tag{C.2}
$$

We have checked by explicit computation that they satisfy the projection equation (2.10) provided

$$
\begin{aligned}
c_{\yng(2,1)} &= \frac{q^2+q+1}{q},\\
c_{\yng(3)} &= (q+1)(q^2+q+1),\\
c_{\yng(1,1,1)} &= \frac{(q+1)(q^2+q+1)}{q^3}\,.
\end{aligned}
\tag{C.3}
$$

This agrees exactly with the values given in [21], Eq. (C.7) below.

C.2. H_4. For H_4, the projector contains $4! = 24$ independent terms. The projector is:

$$
\begin{aligned}
c_RP_R &= a+b(g_1+g_2+g_3)+c(g_1g_2+g_2g_3+g_2g_1+g_3g_2)+dg_1g_3\\
&\quad+\ f(g_1g_2g_3+g_1g_3g_2+g_2g_1g_3+g_3g_2g_1)+h(g_1g_2g_1+g_2g_3g_2)\\
&\quad+\ k(g_1g_2g_1g_3+g_1g_2g_3g_2+g_1g_3g_2g_1+g_2g_3g_2g_1)+lg_2g_1g_3g_2\\
&\quad+\ m(g_1g_2g_1g_3g_2+g_2g_1g_3g_2g_1)+ng_1g_2g_3g_2g_1+pg_2g_1g_3g_2g_1g_3\,.
\end{aligned}
\tag{C.4}
$$

The coefficients $a, b, c, d, f, h, k, l, m, n, p$ depend on the representation. They are characters of $q^{-l(\sigma)}\chi(h(\sigma^{-1}))$ which can be simplified, using cyclicity and the Hecke relations, to

$$
\begin{aligned}
a &= \chi(1)\,,\ b=q^{-1}\chi(g_1)\,,\ c=q^{-2}\chi(g_1g_2),\\
d &= q^{-2}\chi(g_1g_3)\,,\ f=q^{-3}\chi(g_1g_2g_3),\\
h &= q^{-3}[(q-1)\chi(g_1g_2)+q\chi(g_1)],\\
k &= q^{-4}[(q-1)\chi(g_1g_2g_3)+q\chi(g_1g_2)],\\
l &= q^{-4}[(q-1)\chi(g_1g_2g_3)+q\chi(g_1g_3)],\\
m &= q^{-5}[(q^2-q+1)\chi(g_1g_2g_3)+q(q-1)\chi(g_2g_3)],\\
n &= q^{-5}[(q-1)^2\chi(g_1g_2g_3)+2q(q-1)\chi(g_1g_2)+q^2\chi(g_1)],\\
p &= q^{-6}[(q-1)(q^2+1)\chi(g_1g_2g_3)+q(q-1)^2\chi(g_1g_2)+q^2\chi(g_1g_3)]\,.
\end{aligned}
\tag{C.5}
$$

Again, the mixing between different terms comes from using formulas like (B.9) and is related to the contribution of a single term to different central elements. In the limit $q = 1$, each of the $a, d, \ldots, p$ depend on a single character, the one corresponding to the conjugacy class of the element that $a, d, \ldots, p$ multiply in the projector.

Using computer algebra, we have checked that the above are projectors for the five $n = 4$ representations, provided

$$
\begin{aligned}
c_{\yng(4)} &= (q+1)(q^2+q+1)(q^3+q^2+q+1), \\
c_{\yng(3,1)} &= \frac{(q+1)(q^3+q^2+q+1)}{q}, \\
c_{\yng(2,2)} &= \frac{(q+1)^2(q^2+q+1)}{q^2}, \\
c_{\yng(2,1,1)} &= \frac{(q+1)(q^3+q^2+q+1)}{q^3}, \\
c_{\yng(1,1,1,1)} &= \frac{(q+1)(q^2+q+1)(q^3+q^2+q+1)}{q^6} .
\end{aligned}
\tag{C.6}
$$

C.3. The construction for H_n. In the general case, the projector contains $n!$ elements. Gyoja has given a formula for the coefficients[2] c_R:

$$
c_R = \frac{\prod_{i=1}^{m}(q-1)(q^2-1)\dots(q^{\lambda_i+m-i}-1)}{\prod_{1\le i\langle j\le m}(q^{\lambda_i+m-i}-q^{\lambda_j+m-j})}\, q^{\frac{1}{6}m(m-1)(m-2)}(q-1)^{-n} . \tag{C.7}
$$

From (C.3) and (C.6) we easily see that the coefficients satisfy

$$
c_R(q^{-1}) = c_{R^T}(q) , \tag{C.8}
$$

where R^T is the representation with transposed Young tableau. This is indeed a general property of the projectors (C.7) [45].

It is easy to see that in the classical limit,

$$
c_R(q=1) = \frac{\prod_{i=1}^{m} l_i!}{\prod_{1\le i\langle j\le m}(l_i-l_j)}, \tag{C.9}
$$

where $l_i = \lambda_i + m - i$. This is the coefficient of the Young symmetrizer, and is given by the hook formula. Also the quantum coefficients (C.7) can be expressed in terms of a hook formula.

For high n, it is tedious to check idempotency of the projector. Also, it relies on having explicit formulas for the characters of the Hecke algebra. Gyoja [21] has given a construction to compute projectors in general without recourse to characters. In this construction, to associate a projector to a particular representation R, we first associate a projector to every state of the representation. Every state is represented by a standard tableau T. A standard tableau is a tableau where the entries (numbered with elements from $\{1,\dots,n\}$) are increasing across each row and down each column. The number of states in a given representation is $d_R(1)$. Thus, P_R will be a sum of $d_R(1)$ primitive projectors, which we call E_T, where T is the standard tableau they correspond to. The construction proceeds by defining two special tableaux, T_+ and T_-. These are the tableaux where the entries of the tableaux are numbered from 1 to n successively across the first row (column), then the second, third, etc. I_+ and I_- are the subgroups of S_n that

[2] We corrected a typo in the formula in [21].

preserve the rows (columns) of T_+ (T_-). We associate to them parabolic subgroups $W_\pm$ of S_n and define

$$
\begin{aligned}
e_+ &= \sum_{w \in W_+} h(w), \\
e_- &= \sum_{w \in W_-} (-q)^{-l(w)} h(w) \, .
\end{aligned}
\tag{C.10}
$$

The primitive projector (up to normalization) associated to T is then

$$
E(T) = h_- e_- h_-^{-1} h_+ e_+ h_+^{-1}, \tag{C.11}
$$

where $h_+ = h_+(T)$ and $h_- = h_-(T)$ are the elements of the Hecke algebra corresponding to the permutation that transforms T_+ (resp. T_-) to the standard tableau T. Gyoja showed that the E's are idempotents. The projector is then the sum of the orthogonal primitive idempotents:

$$
P_R = \frac{1}{c_R} \sum_{i=1}^{d_R(1)} E(T_i) \, , \tag{C.12}
$$

where c_R was given before[3]. We checked the previously constructed projectors for n up to 4 using this construction. The first non-trivial case for $n = 3$ is the representation $\yng(2,1)$. There are two standard tableaux: $T_+ = [\{1, 2\}\{3\}]$ and $T_- = [\{1, 3\}\{2\}]$. The permutation relating both is (23), which is $h((23)) = g_2$. In this case the parabolic subgroups are $W_+ = W_- = \{1, s_1\}$, and

$$
\begin{aligned}
e_+ &= 1 + g_1, \\
e_- &= 1 - \frac{1}{q} g_1 .
\end{aligned}
\tag{C.13}
$$

We further have $h_+(T_+) = 1$, $h_-(T_+) = g_2$, therefore

$$
E(T_+) = 1 + g_1 + \frac{q-1}{q} g_2 - \frac{1}{q} g_1 g_2 + \frac{q-1}{q} g_2 g_1 - \frac{1}{q} g_1 g_2 g_1 \, . \tag{C.14}
$$

For $E(T_-)$, $h_+ = g_2$ and $h_- = 1$, so

$$
E(T_-) = 1 - \frac{1}{q} g_1 - g_2 g_1 + \frac{1}{q} g_1 g_2 g_1 . \tag{C.15}
$$

The primitive idempotents are automatically orthogonal. We get

$$
\begin{aligned}
P_{\yng(2,1)} &= \frac{q}{q^2 + q + 1} \left(E(T_+) + E(T_-) \right) \\
&= \frac{q}{q^2 + q + 1} \left(2 + \frac{q-1}{q} (g_1 + g_2) - \frac{1}{q} (g_1 g_2 + g_2 g_1) \right) \, ,
\end{aligned}
\tag{C.16}
$$

[3] Gyoja showed that the primitive idempotents are orthogonal using a certain ordering. In order for (C.12) to be a projector, they must be orthogonal independently of the ordering. This can be done defining new primitive idempotents in terms of the old ones, see Theorem 4.5 in [21]. For n up to 4, however, we found that the primitive projectors are automatically orthogonal.

in agreement with the formula obtained earlier.

As another example, we do $n = 4$ for the representation $\yng(3,1)$. There are three standard tableaux: $T_+ = [\{1, 2, 3\}, \{4\}]$, $T_- = [\{1, 3, 4\}, \{2\}]$, and $T_3 = [\{1, 2, 4\}, \{3\}]$. We have $I_+ = \{1, s_1, s_2\}$ and $I_- = \{1, s_1\}$, so

$$\begin{aligned} e_+ &= 1 + g_1 + g_2 + g_1 g_2 + g_2 g_1 + g_1 g_2 g_1, \\ e_- &= 1 - \frac{1}{q}\, g_1 \,. \end{aligned} \tag{C.17}$$

In this case $h_+(T_+) = 1$, $h_-(T_+) = g_3 g_2$. Thus:

$$E(T_+) = g_3\, g_2\, e_-(T)\, g_2^{-1}\, g_3^{-1}\, e_+(T)\,, \tag{C.18}$$

which we worked out with the help of computer algebra. In the same way we have $h_-(T_-) = 1$, $h_+(T_-) = g_2 g_3$, so

$$E(T_-) = e_-(T)\, g_2\, g_3\, e_+(T)\, g_3^{-1}\, g_2^{-1}\,. \tag{C.19}$$

For T_3, $h_-(T_3) = g_2$, $h_+(T_3) = g_3$, hence

$$E(T_3) = g_2\, e_-(T)\, g_2^{-1}\, g_3\, e_+(T)\, g_3^{-1}\,. \tag{C.20}$$

The projector is the sum of the three, with the appropriate coefficient, and it agrees with the one computed directly. Notice that the primitive idempotents were automatically orthogonal in this case as well.

D. q-Schur-Weyl Duality and q-Characters

In this appendix, we explain concretely the relation between quantum characters of the q-deformed $SU(N)$ and the symmetric group, in the special case of $SU(2)$. We will use the quantum group conventions of [46] and [47].

We will use the formulae for matrix elements of spin-one representations from [46] in terms of spin-half representations and show that they are consistent with expressing the characters in spin-one in terms of the characters of spin half, using the Hecke algebra generators, or R-matrices. For the R-matrix we will use the notation of [47].

D.1. $U_q(su(2))$ conventions. We first summarize some of the formulas of [46, 47] that we will use later. The $U_q(su(2))$ algebra and coproduct are [46]:

$$\begin{aligned} He - eH &= 2e, \\ Hf - fH &= -2f, \\ ef - fe &= \frac{q^{H/2} - q^{-H/2}}{q^{1/2} - q^{-1/2}}, \\ \Delta(e) &= e \otimes q^{H/4} + q^{-H/4} \otimes e\,. \end{aligned} \tag{D.1}$$

For later convenience, we note that the map to the notation of [47] is

$$\begin{aligned} q &\to q, \\ e &\to X_+, \\ f &\to X_-, \\ H &\to H. \end{aligned} \tag{D.2}$$

The universal R-matrix in this basis is [47]

$$R = q^{\frac{H\otimes H}{4}} \sum_{n=0}^{\infty} \frac{(1-q^{-1})^n}{[n]!} \, (q^{H/4}X_+)^n \otimes (q^{-H/4}X_-)^n, \tag{D.3}$$

where $[n]$ is as in (2.35). Together with the action of the generators on spin-half states,

$$\begin{aligned} e\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle &= |\tfrac{1}{2},\tfrac{1}{2}\rangle, \\ f\,|\tfrac{1}{2},\tfrac{1}{2}\rangle &= |\tfrac{1}{2},-\tfrac{1}{2}\rangle, \\ H\,|\tfrac{1}{2},\pm\tfrac{1}{2}\rangle &= \pm|\tfrac{1}{2},\tfrac{1}{2}\rangle\,, \end{aligned} \tag{D.4}$$

this determines the R-matrix as follows:

$$\begin{aligned} R^{\frac{1}{2},\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}} &= R^{-\frac{1}{2},-\frac{1}{2}}_{-\frac{1}{2},-\frac{1}{2}} = q^{1/4}, \\ R^{\frac{1}{2},-\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}} &= R^{-\frac{1}{2},\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}} = q^{-1/4}, \\ R^{\frac{1}{2},-\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}} &= q^{-1/4}\,(q^{1/2}-q^{-1/2})\,. \end{aligned} \tag{D.5}$$

D.2. Schur-Weyl duality in spin-one. As in the classical case, the q-characters in higher representations can be written in terms of q-characters of lower representations. Consider for concreteness the case of spin-one, which is contained in the tensor product of two spin-half representations V. There is a projector (2.9) acting on $V \otimes V$ that leads to the symmetric representation. In the classical case it is just $\frac{1}{2}(1+P)$, where P is the permutation of the two tensor factors. In the quantum case P does not commute with the coproduct, but $PR \equiv \check{R}$ does:

$$\Delta\check{R} = \check{R}\Delta. \tag{D.6}$$

When $\check{R}$ acts on the tensor product of two spin half irreps, it satisfies a relation of the form

$$\check{R}^2 = q^{-1/4}(q^{1/2}-q^{-1/2})\,\check{R} + q^{-1/2}. \tag{D.7}$$

A rescaling $g = q^{3/4}\check{R}$ can be done to map to the standard form of the Hecke algebra used in the main text. A matrix element of some element h of $U_q(su(2))$ in the spin 1 representation can be written in terms of a product of spin half reps by using the Clebsch-Gordan coefficients. Consider now the following matrix element in the spin-one representation:

$$\langle j=1,n|h|j=1,m\rangle = d^n_{j=1;m}(h) = \langle h, d^n_{j=1;m}\rangle. \tag{D.8}$$

$d^n_{j;m}$ is the representation matrix in representation j with indices n, m, and $d^m_{j;m}$ its trace. In the last equation we have expressed the fact that the matrix elements can be viewed as living in the dual space $U_q(su(2))$, denoted by $\mathrm{Fun}_q(SU(2))$. For more details on this duality see for example [18, 19, 13].

We now express this in terms of matrix elements of the fundamental representation. They generate $\mathrm{Fun}_q(SU(2))$, the deformed algebra of functions on $SU(2)$. Using the Clebsch-Gordan coefficients, we can rewrite the above as follows:

$$\begin{aligned}\langle j=1,n|h|j=1,m\rangle &= \sum_{m_1,m_2;n_1,n_2} C^n_{n_1n_2}\, C^{m_1m_2}_m \\ &\quad \times\langle j=\frac{1}{2},n_1|\otimes\langle j=\frac{1}{2},n_2|(h_1\otimes h_2)|j=\frac{1}{2},m_1\rangle\otimes|j=\frac{1}{2},m_2\rangle \\ &= \sum_{m_1,m_2;n_1,n_2} C^n_{n_1n_2}\, C^{m_1m_2}_m\, d^{n_1}_{j=\frac{1}{2};m_1}(h_1)\, d^{n_2}_{j=\frac{1}{2};m_2}(h_2) \\ &= \sum_{m_1,m_2;n_1,n_2} C^n_{n_1n_2}\, C^{m_1m_2}_m\, \langle h, d^{n_1}_{j=\frac{1}{2};m_1}\, d^{n_2}_{j=\frac{1}{2};m_2}\rangle\,. \end{aligned} \tag{D.9}$$

In the first equality, the co-product $\Delta(h)=h_1\otimes h_2$ gives the action of h on the tensor product $V\otimes V$. In the last equality, we used the fact that the dual pairing of a product of two elements in $\mathrm{Fun}_q(SU(2))$ is given by the co-product. Now we can sum over m and use the identity between Clebsch-Gordan coefficients and projectors (see for example [42])

$$\sum_m C^m_{n_1n_2}\, C^{m_1m_2}_m = P^{m_1m_2}_{n_1n_2}\left(\frac{1}{2},\frac{1}{2};1\right). \tag{D.10}$$

The projector is a linear combination of the identity and the $\check{R}$. For $j=1$, the projector is in the tensor product of two spin-half representations. It has to be a linear combination of 1 and $\check{R}$ since the Hecke algebra generates the centralizer of the quantum group action in the tensor product:

$$P\left(\frac{1}{2},\frac{1}{2};1\right) = a + b\,\check{R}, \tag{D.11}$$

and for the matrix elements we have

$$d^n_{j=1;m} = \sum_{m_1,m_2;n_1,n_2} d^{n_1}_{j=\frac{1}{2};m_1}\, d^{n_2}_{j=\frac{1}{2};m_2}\, C^{m_1m_2}_m\, C^n_{n_1n_2}\,. \tag{D.12}$$

To compute the character, we want the trace of this equation. Using (D.10), and expanding the projector in terms of the R-matrix as in (D.11), we get:

$$\mathrm{tr}_1 d = a\,(\mathrm{tr}_{\frac{1}{2}}\,d)\,(\mathrm{tr}_{\frac{1}{2}}\,d) + b\,\mathrm{tr}_1(\check{R}(d_{\frac{1}{2}}\otimes 1)(1\otimes d_{\frac{1}{2}}))\,, \tag{D.13}$$

which, written out in indices, reads:

$$\sum_m d^m_{j=1;m} = \sum_{m_1,m_2;n_1,n_2}\left(a\,\delta^{m_1}_{n_1}\delta^{m_2}_{n_2} + b\,R^{m_2m_1}_{n_1n_2}\right) d^{n_1}_{\frac{1}{2};m_1}\, d^{n_2}_{\frac{1}{2};m_2}\,. \tag{D.14}$$

We will show that the above equation can indeed be solved for constants a, b. The left-hand side can be calculated to give:

$$\begin{aligned}\sum_m d^m_{j=1;m} &= x^2 + (xy+\sqrt{q}uv) + y^2 \\ &= x^2 + y^2 + xy(1+q) - q, \end{aligned} \tag{D.15}$$

where we have used Eqs. (36-40) of [46] (recalling that x, y, u, v are the matrix entries of d in the fundamental representation). For the right-hand side of (D.14) we get

$$(a + bq^{1/4})(x^2 + y^2) + axy + 2buvq^{-1/4} + (a + bq^{-1/4}(\sqrt{q} - 1/\sqrt{q}))yx. \quad \text{(D.16)}$$

Using the relations

$$\begin{aligned} yx &= (1-q) + qxy, \\ uv &= q^{1/2}(xy - 1), \end{aligned} \quad \text{(D.17)}$$

we can rewrite (D.16) in terms of x^2, y^2, xy, 1. Comparing with (D.15) and considering the coefficient of $x^2 + y^2$ we immediately see that

$$a + b\,q^{1/4} = 1\ . \quad \text{(D.18)}$$

With this condition the coefficient of xy becomes $(q + 1)$ as desired. Comparing coefficients of the constant term then determines

$$a = \frac{1}{1+q}\ ,\ \ b = \frac{q^{3/4}}{1+q}. \quad \text{(D.19)}$$

Putting everything together, and going back to the notation used in the main text, we get:

$$\mathrm{tr}_1\, U = \frac{1}{1+q}\,\mathrm{tr}\, U\,\mathrm{tr}\, U + \frac{q^{3/4}}{1+q}\,\mathrm{tr}\otimes\mathrm{tr}\left(\check{R}\ (U\otimes 1)(1\otimes U)\right) \quad \text{(D.20)}$$

which is q-Schur-Weyl duality (2.20) for $n = 1$. By comparing (D.7) and with the first of (2.6) we can see that we can define $g = q^{3/4}\check{R}$. Then the projector can be read from above,

$$P_{\square\square} = \frac{1}{1+q}(1 + g), \quad \text{(D.21)}$$

and agrees with (2.9) and the general form (2.28).

D.3. Quantum characters in spin-one representation. The quantum characters can be obtained from the above by including the u-element (B.1) in the trace, which is basically q^{-H}. In fact, we will do a slightly more general computation of the trace with an insertion of q^{AH}. Thus, we consider the matrix element in the spin one representation of hq^{AH} where A is an arbitrary number and h is an arbitrary element o $U_q(su(2))$:

$$\langle j = 1, n|h\,q^{AH}|j = 1, m\rangle = q^{Am} d^n_{j=1;m}(h). \quad \text{(D.22)}$$

As before, we now rewrite this in spin-half matrix coefficients using the Clebsch-Gordan coefficients:

$$\begin{aligned} \langle j = 1, n|hq^{AH}|j = 1, m\rangle &= \sum_{m_1,m_2;n_1,n_2} C^n_{n_1 n_2}\, C^{m_1 m_2}_m \\ &\quad \times \langle j{=}\tfrac{1}{2}\,, n_1|\otimes\langle j{=}\tfrac{1}{2}\,, n_2|(h_1\otimes h_2)(q^{AH}\otimes q^{AH})|j{=}\tfrac{1}{2}\,, m_1\rangle\otimes|j = \tfrac{1}{2}\,, m_2\rangle \\ &= \sum_{m_1,m_2;n_1,n_2} C^n_{n_1 n_2}\, C^{m_1 m_2}_m\, d^{n_1}_{j=1/2;m_1}(h_1)\, d^{n_2}_{j=\frac{1}{2}\,;m_2}(h_2)\, q^{Am_1+Am_2} \\ &= \sum_{m_1,m_2;n_1,n_2} C^n_{n_1 n_2}\, C^{m_1 m_2}_m\, q^{Am_1+Am_2}\, \langle h, d^{n_1}_{j=\frac{1}{2}\,;m_1}\ \ d^{n_2}_{j=\frac{1}{2}\,;m_2}\rangle\ . \end{aligned}$$

Again, we can sum over $m = n$ to take the trace (notice the presence of q^{AH} so this gives the quantum trace) and use (D.10) to get

$$\sum_m q^{Am}\, d^m_{j=1;m} = \sum_{m_1,m_2;n_1,n_2} q^{A(m_1+m_2)} \left(a\, \delta^{m_1}_{n_1}\, \delta^{m_2}_{n_2} + b\, R^{m_2 m_1}_{n_1 n_2}\right) d^{n_1}_{\frac{1}{2};m_1}\, d^{n_2}_{\frac{1}{2};m_2}\,. \tag{D.23}$$

For comparison to the classical formulae, it is useful to rewrite it as

$$\mathrm{tr}_1\,(q^{AH}U) = a\,\mathrm{tr}\,(q^{AH}U)\,\mathrm{tr}\,(q^{AH}U) + b\,\mathrm{tr}\otimes\mathrm{tr}\left(q^{AH}\,\check{R}\,(U\otimes 1)(1\otimes U)\right). \tag{D.24}$$

In the $q \to 1$ limit, $\check{R}$ goes to the permutation P and the second term becomes $\frac{1}{2}\mathrm{tr}(U^2)$.

We still need to compute the constants a, b in this case. Writing out the traces using [46], we get:

$$\begin{aligned}
\mathrm{tr}_1\,(q^{AH}\,U) &= q^A x^2 + q^{-A} y^2 + 1 + (q^{1/2} + q^{-1/2}) uv,\\
\mathrm{tr}\otimes\mathrm{tr}\left(q^{A\,H\otimes H}\,(U\otimes 1)(1\otimes U)\right) &= (\mathrm{tr}\,(q^{AH}\,U))^2 = q^A x^2 + q^{-A} y^2 + 2 + (q^{1/2}+q^{-1/2})uv,\\
\mathrm{tr}\otimes\mathrm{tr}\left(q^{A\,H\otimes H}\,\check{R}\,(U\otimes 1)(1\otimes U)\right) &= q^{1/4}[q^A x^2 + q^{-A} y^2 + 1 - q^{-1} + (q^{1/2}+q^{-1/2})uv],
\end{aligned} \tag{D.25}$$

where A denotes an arbitrary power. It is now easy to see that the values of a, b (D.19) are still the same, independently of the value of A.

From the explicit computation (D.25) we also get the special $N = 2$ relations,

$$\begin{aligned}
\mathrm{Tr}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} U &= 1,\\
\mathrm{Tr}_{\square\square} U &= (\mathrm{tr}\, U)^2 - 1\,.
\end{aligned} \tag{D.26}$$

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