

$\mathcal{N} = 2$ supergravity in three dimensions and its Gödel supersymmetric background

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Abstract

The four dimensional Gödel spacetime is known to have the structure $M_3 \times \mathfrak{R}$. It is also known that the three-dimensional factor M_3 is an exact solution of three-dimensional gravity coupled to a Maxwell-Chern-Simons theory. We build in this paper a $\mathcal{N} = 2$ supergravity extension for this action and prove that the Gödel background preserves half of all supersymmetries.

Gödel-type solutions to general relativity have recently been under scrutiny due to the discovery [1] of their supersymmetric properties. Black holes on these backgrounds have also been found [2]. Since these black holes have unusual asymptotics, issues like first law of thermodynamics and the proper definition of charges are subtle and require detailed analysis (see [3] and references therein for a detailed discussion). Gödel spacetimes in string theory have been considered in [4] and [5].

Most of the discussion that followed the work [1] concerned a class of Gödel solutions existing in five dimensions. As it is well-known this theory contains in its bosonic sector the graviphoton, that is the metric coupled to a Maxwell theory with the addition of a Chern-Simons term $AdAdA$.

On the other hand, the original four dimensional Gödel spacetime, discovered in 1949, has a direct product structure $M_4 = M_3 \times \mathfrak{R}$ where the three dimensional factor M_3 encodes most of its interesting properties. Motivated by [1] it was shown in [6] that indeed the M_3 factor is a solution to three-dimensional gravity coupled to a Maxwell theory including the 3d Chern-Simons term AdA . Particles and black holes on this background were also discussed in [6]. The next step, which we take in this paper, is to study the supersymmetric properties of this solution.

Consider the bosonic action,

$$I[g_{\mu\nu}, A_\mu] = \frac{1}{16\pi G} \int d^3x \left[\sqrt{-g} \left(R + \frac{2}{l^2} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) - \alpha \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \right]. \quad (1)$$

The field,

$$ds^2 = -dt^2 - 4\alpha r dt d\varphi + \left[2r - (\alpha^2 l^2 - 1) \frac{2r^2}{l^2} \right] d\varphi^2 + \left(2r + (\alpha^2 l^2 + 1) \frac{2r^2}{l^2} \right)^{-1} dr^2 \quad (2)$$

$$A = \sqrt{\alpha^2 l^2 - 1} \frac{2r}{l} d\varphi. \quad (3)$$

is an exact solution of the equations deriving from (1) [6]. The metric represents the 3d factor M_3 of the original Gödel solution (actually, its generalization containing two parameters, l and α [7]). Given the high symmetry of this solution – it has four Killing vectors – it is a natural question to ask whether it preserves some supersymmetries.

Note the strong similarities between the 3d bosonic action (1) and the corresponding 5d supergravity action. Nonetheless, the supergravity theory corresponding to (1) has some subtleties. In particular we would like to have the cosmological radius l and Chern-Simons coupling α as arbitrary parameters.

The minimal $\mathcal{N} = 1$ supergravity extension to (1) consists of two super-multiplets:¹ the gravity multiplet $\{g_{\mu\nu}, \psi_\mu\}$ and a vector multiplet $\{A_\mu, \lambda\}$. ψ_μ is the spin 3/2 Rarita-Schwinger

¹We work on-shell, i.e. no auxiliary fields appear.

field, while λ is a spin 1/2 fermion. Both are Majorana fermions. However, it is easy to see (assuming an action with no higher derivatives) that the Gödel background cannot be a supersymmetric solution to this system. In fact, the transformation for the Majorana spinor field λ has the form $\delta\lambda = F^{\mu\nu}\gamma_{\mu\nu}\epsilon$. For the background (3) $F^{\mu\nu}$ is non-zero and one easily verifies that the equation $\delta\lambda = 0$ implies $\gamma_0\epsilon = 0$ and hence $\epsilon = 0$.

We thus explore extended supergravity, or, more precisely, the three-dimensional $\mathcal{N} = 2$ vector supermultiplet coupled to $\mathcal{N} = 2$ supergravity. In three dimensions the former consists of a vector, a real scalar and complex Dirac fermion, $\{A_\mu, \phi, \lambda\}$. The gravity multiplet contains [12, 13] the metric $g_{\mu\nu}$, a complex Rarita-Schwinger field ψ_μ , and an Abelian $U(1)$ gauge field B_μ . The field B_μ is independent of A_μ . One might think that the gravity multiplet suffices for our purposes. However, at the level of at most two derivatives, the gauge field only enters through the Chern-Simons term and it does not allow for the free parameter α .

The full $\mathcal{N} = 2$ Lagrangian (to quadratic order in the fermions) is ²

$$L = \frac{1}{\kappa^2}L_{-2} + L_0 + \kappa L_1 + \kappa^2 L_2 \quad (4)$$

where

$$\begin{aligned} L_{-2} &= \frac{1}{2}e \left(R + \frac{2}{l_0^2} \right) \\ L_0 &= -i\epsilon^{\mu\nu\rho}\bar{\psi}_\mu D_\nu\psi_\rho + \epsilon^{\mu\nu\rho}B_\mu\partial_\nu B_\rho - \frac{1}{4}eF^2 - \frac{e}{2}(\partial\phi)^2 - \alpha_1\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho \\ &\quad + 2\alpha_1\alpha_2e\phi^2 - ie\bar{\lambda}\gamma^\mu D_\mu\lambda - 2i\alpha_2e\bar{\lambda}\lambda \\ L_1 &= e\bar{\lambda} \left[-\frac{i}{4}\gamma^\mu\gamma^{\nu\rho}F_{\nu\rho} - \frac{1}{2}\gamma^\mu\gamma^\nu\partial_\nu\phi - \alpha_1\gamma^\mu\phi \right] \psi_\mu + e\bar{\psi}_\mu \left[-\frac{i}{4}\gamma^{\nu\rho}\gamma^\mu F_{\nu\rho} + \frac{1}{2}\gamma^\nu\gamma^\mu\partial_\nu\phi - \alpha_1\gamma^\mu\phi \right] \lambda \\ L_2 &= -\frac{e}{2}\bar{\psi}_\mu\psi_\nu F^{\mu\nu}\phi - \frac{e}{4}\bar{\lambda}\gamma^{\mu\nu}\lambda F_{\mu\nu}\phi - \frac{ie\alpha_1}{2}\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu\phi^2 - \frac{ie\alpha_1}{2}\bar{\lambda}\lambda\phi^2 + \alpha_1^2 e\phi^4. \end{aligned}$$

The covariant derivative D_μ is

$$D_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab} - \frac{1}{2l_0}\gamma_\mu - iB_\mu;$$

²We have used the following conventions: Our metric $g_{\mu\nu}$ has signature $(-, +, +)$. The Dirac matrices satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and $\gamma^\mu\gamma^\nu = g^{\mu\nu} + \epsilon^{\mu\nu\rho}\gamma_\rho$. $\epsilon^{\mu\nu\rho} = \pm 1$, $\epsilon_{\mu\nu\rho} = \mp 1$. $\bar{\lambda} = \lambda^\dagger\gamma_0$. $(\gamma^\mu)^\dagger = \gamma_0\gamma^\mu\gamma_0$. In the 1.5 formalism the variation of the spin connection is obtained from its algebraic equation of motion, and not needed explicitly.

$F_{\mu\nu}$ is the field strength of A_μ . The Lagrangian is invariant, up to a total derivative, under the following linearized supersymmetry transformations,

$$\begin{aligned}
\delta e_\mu^a &= \frac{i\kappa}{2} (\bar{\epsilon}\gamma^a\psi_\mu - \bar{\psi}_\mu\gamma^a\epsilon) \\
\delta\psi_\mu &= \frac{1}{\kappa} D_\mu\epsilon - \kappa \left(\frac{ie}{4}\epsilon_{\mu\nu\rho}F^{\nu\rho}\phi + \frac{\alpha_1}{2}\phi^2\gamma_\mu \right) \epsilon \\
\delta B_\mu &= -\frac{1}{2\kappa} (\bar{\epsilon}\psi_\mu - \bar{\psi}_\mu\epsilon) \\
\delta A_\mu &= \frac{i}{2} (\bar{\epsilon}\gamma_\mu\lambda - \bar{\lambda}\gamma_\mu\epsilon) - \frac{\kappa}{2} (\bar{\epsilon}\psi_\mu - \bar{\psi}_\mu\epsilon)\phi \\
\delta\phi &= \frac{1}{2} (\bar{\epsilon}\lambda - \bar{\lambda}\epsilon) \\
\delta\lambda &= -\frac{1}{4} F_{\mu\nu}\gamma^{\mu\nu}\epsilon + \frac{i}{2}\partial_\mu\phi\gamma^\mu\epsilon + i\alpha_1\phi\epsilon
\end{aligned} \tag{5}$$

provided that the parameters α_1, α_2 and l_0 satisfy the condition

$$\alpha_1 + \alpha_2 = \frac{1}{l_0}. \tag{6}$$

This action thus has two arbitrary parameters, the cosmological radius l_0 and the Chern-Simons coupling α_1 .

For the construction of the Lagrangian and of the transformation rules we followed the standard Noether method. L_{-2} is the gravitational part. L_0 contains the kinetic terms and masses of all matter fields. L_1 are the Noether currents (of global supersymmetry) coupled to the complex Rarita-Schwinger field. Finally L_2 ensures linearized supersymmetry. One may also check that the commutator of two supersymmetry transformations is a combination of a diffeomorphism and a gauge transformation.

Setting all fermions and the bosons B_μ and ϕ to zero, the Lagrangian (4) reduces to the bosonic system (1) with $l = l_0$ and $\alpha = \alpha_1$. Thus the metric and the $U(1)$ gauge field (2) and (3) also solve the equations of motion of the supersymmetric theory. However, with this background we meet the same problems as with the $\mathcal{N} = 1$ theory. With $\phi = 0$ we find again $\delta\lambda \sim F_{\mu\nu}\gamma^{\mu\nu}\epsilon \sim \gamma_0\epsilon$, thus, $\delta\lambda = 0$ implies $\epsilon = 0$.

We observe, however, that the real scalar field in the $\mathcal{N} = 2$ supersymmetric theory has a potential

$$V(\phi) = -2\alpha_1\alpha_2\phi^2 - \kappa^2\alpha_1^2\phi^4. \tag{7}$$

This means that it could develop a non-zero vacuum expectation value

$$\phi_0 = \pm \sqrt{\frac{-\alpha_2}{\kappa^2 \alpha_1}}. \quad (8)$$

provided that the ratio α_2/α_1 is negative (ϕ is a real scalar field). Let's assume this condition holds such that the vev exists. In this case ϕ_0 are the two maxima of a potential which is unbounded from below. $\phi = 0$ is a local minimum.

The vev ϕ_0 has two effects. First, it contributes non-trivially to the supersymmetry transformations (5), in particular of the fermionic fields λ and ψ_μ . Second, the value of the potential on ϕ_0 is not zero. Setting all fermions to zero and $B_\mu = 0$ and $\phi = \phi_0$ we recover the action (1) with $\alpha = \alpha_1$ and a shifted value for the cosmological constant:

$$\begin{aligned} \frac{1}{l^2} &= \frac{1}{l_0^2} - \kappa^2 V(\phi_0) \\ &= \frac{1}{l_0^2} - \alpha_2^2 \end{aligned} \quad (9)$$

Note that the effective cosmological constant $1/l^2$ can be positive, negative, or zero. We will see below that in all three cases half of the supersymmetries are preserved. Of course, there arises the question of stability of this background. We will not try to answer it here but one should keep in mind that experience with the AdS vacuum tells us that the stability properties of fields in non-trivial backgrounds should be analyzed with care [11]. de Sitter supergravity theories in three dimensions have been studied in [8].

We will now analyze the question whether the background specified by eqs.(2,3,8), with all other fields set to zero, preserves some supersymmetry. For convenience we will set

$$\kappa^2 = \frac{1}{2} \quad (10)$$

from now on.

The bosonic fields are of course invariant because all fermions are zero on the background. The variations of λ and ψ_μ give rise to the equations

$$\delta\lambda = 0 \quad \Rightarrow \quad -\frac{1}{4}F_{\mu\nu}\gamma^{\mu\nu}\epsilon + \frac{i}{2}\partial_\mu\phi\gamma^\mu\epsilon + i\alpha_1\phi\epsilon = 0 \quad (11)$$

$$\delta\psi_\mu = 0 \quad \Rightarrow \quad D_\mu\epsilon - \frac{1}{2}\left(\frac{ie}{4}\epsilon_{\mu\nu\rho}F^{\nu\rho}\phi + \frac{\alpha_1}{2}\phi^2\gamma_\mu\right)\epsilon = 0 \quad (12)$$

which must be evaluated on the background defined by eqs.(2), (3), and (8) and, for supersymmetry to be preserved, must have a nontrivial solution for the supersymmetry parameter ϵ .

We start by evaluating (11), which is purely algebraic. A straightforward calculation gives the condition

$$(i\alpha_1\phi l - \sqrt{\alpha_1^2 l^2 - 1}\gamma_0)\epsilon = 0. \quad (13)$$

where γ_0 is the Dirac matrix with flat (i.e.tangent space) index. Since $\gamma_0^2 = -1$, it has eigenvalues $\pm i$ and we find that eq.(13) requires

$$\phi = \pm \frac{1}{\alpha_1 l} \sqrt{\alpha_1^2 l^2 - 1} \quad (14)$$

which agrees with the extrema of the potential, eq.(8).

Next, we analyze the Killing spinor equations (12). Using that ϵ is an eigenvector of γ_0 with eigenvalue $\pm i$ we obtain the three equations

$$\partial_t \epsilon = \mp \frac{i}{2\alpha_1 l^2} (\alpha_1^2 l^2 + 1) \epsilon \quad (15)$$

$$\partial_r \epsilon = 0 \quad (16)$$

$$\partial_\varphi \epsilon = \pm \frac{i}{2} \epsilon \quad (17)$$

which can be easily solved

$$\epsilon = \epsilon_0^\pm e^{\frac{\pm i}{2} \frac{1 + \alpha_1^2 l^2}{\alpha_1 l^2} t \mp \frac{i}{2} \phi}. \quad (18)$$

where ϵ_0^\pm are constant eigenspinors of the flat γ_0 with eigenvalues $\pm i$.

The third equation indicates that $\epsilon(t, \varphi)$ is periodic in φ with period 4π , as it must be for a regular spinor [14].

The Gödel background defined by eqs.(2,3,8) is thus a supersymmetric solution. For a given choice of the vev (8), i.e. for a given sign, there exists one Killing spinor. In that sense, this solution preserves half of the supersymmetries.

Let us now comment on the supersymmetric properties of the other solutions to the action (1), constructed from (2) via identifications [6]. As pointed out in that reference, the theory

described by the action (1) has two sectors $\alpha^2 l^2 \geq 1$ and $\alpha^2 l^2 < 1$. For $\alpha^2 l^2 \geq 1$, the identifications produce ‘particles’ which have conical singularities at the origin. The Killing spinors on this backgrounds do not have the right periodicity, neither periodic nor antiperiodic, and thus are singular in the quotient manifold. In other words there are no globally defined supercharges. More details on this point can be found in [9, 10].

Identifications on the sector $\alpha^2 l^2 < 1$ produce black holes [6]. However, it turns out that the vev (8) for the scalar field is real only in the sector $\alpha^2 l^2 > 1$. In fact, from (9) and (6) we can express α_2 and l_0 as functions of α_1 and l :

$$\frac{1}{l_0} = \frac{1}{2\alpha_1} \left(\frac{1}{l^2} + \alpha_1^2 \right) \quad (19)$$

$$\alpha_2 = \frac{1}{2\alpha_1} \left(\frac{1}{l^2} - \alpha_1^2 \right) \quad (20)$$

From (20) we conclude that $-2\alpha_2/\alpha_1 = (\alpha_1^2 l^2 - 1)/\alpha_1^2 l^2$. Thus, the vev (8) is real only in the sector $\alpha_1^2 l^2 > 1$.

It would be interesting to find a supergravity theory yielding supersymmetric backgrounds for $\alpha^2 l^2 < 1$. In that sector black holes are present and one could then ask whether extreme ones are supersymmetric or not.

We conclude with some comments. The action we have constructed is supersymmetric at the linear order. In principle, the higher fermionic terms in the action and the transformation rules can be constructed via the Noether procedure. But this is very tedious. A more promising approach is to use superfields and we leave this for the future. Another immediate question is how the three dimensional Gödel background which we have studied here can be obtained from the five-dimensional solution of [1] via compactification. Finally, as we have already mentioned, the stability issue might be worth studying.

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