

Holography at string field theory level: Conformal three point functions of BMN operators

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Abstract

A general framework for applying the pp-wave approximation to holographic calculations in the AdS/CFT correspondence is proposed. By assuming the existence and some properties of string field theory (SFT) on $AdS_5 \times S^5$ background, we extend the holographic ansatz proposed by Gubser, Klebanov, Polyakov and Witten to SFT level. We extract relevant information of assumed SFT on $AdS_5 \times S^5$ from its approximation, pp-wave SFT. As an explicit example, we perform string theoretic calculations of the conformal three point functions of the BMN operators. The results agree with the previous calculations in gauge theory. We identify a broad class of field redefinitions, including known ambiguities of the interaction Hamiltonian, which does not affect the results.

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The AdS/CFT correspondence conjecture [1], which states that string theory on $AdS_5 \times S^5$ is equivalent to $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, is one of the most explicit proposals of equivalence between large N gauge theory and string theory. A characteristic feature of the correspondence is the holography[2][3]: observables of gauge theory seem to be related to the behaviour of string theory at the boundary of AdS space.

An important problem is to understand the fundamental mechanism of the correspondence, in other words, to understand how degrees of freedom of closed strings arise from those of gauge theory. Solving this problem will considerably improve our understanding of string theory. As a first step to understand the mechanism, it will be useful to calculate corresponding observables in both string theory and gauge theory independently and check the agreement.

However, as is well known, there are difficulties in general to carry out such comparisons. At the string theory side, quantum string theory on $AdS_5 \times S^5$ is not defined so that we cannot treat non-zero modes of closed strings. At the gauge theory side, we have no general methods of computation since perturbative methods do not work, the expansion parameter $g_{\text{YM}}^2 N$ being large.

A breakthrough has been made in [4]. At the string theory side, the pp-wave approximation is found to be very useful for states which have large orbital angular momentum J on S^5 . Quantum theory is well defined under the approximation, so that in particular one can treat non-zero modes of closed strings. At the gauge theory side, corresponding operators (BMN operators) are proposed. The expansion parameter becomes $\frac{g_{\text{YM}}^2 N}{J^2}$ for them. Hence, perturbative methods are effective in the regime $\frac{g_{\text{YM}}^2 N}{J^2} \ll 1$.

By these methods, many tests are performed based on the postulated equivalence between the energy in string theory and the dilatation operator in gauge theory. However, holographic aspects of the AdS/CFT correspondence via the pp-wave approximation remain much unexplored. A step was made towards this direction by the work by Dobashi, the author, Yoneya[5]. The main observation is that, in order to directly apply the pp-wave approximation to the holography, we should interpret the closed strings as in a state of tunnelling under a barrier of the gravitational potential in AdS background.

In this letter we propose, under the tunnelling picture, a general framework to apply the pp-wave approximation to holographic evaluation of conformal n -point correlation functions of BMN operators. We in particular perform explicit calculations for the three point functions of scalar BMN operators to the leading order. We find agreement with the previous calculation in gauge theory up to an overall factor. Our methods do not suffer from known ambiguities in the pp-wave SFT Hamiltonian. We identify a broad class of field redefinitions including them which does not affect the physical observable.

The basic strategy is as follows. We begin by assuming the existence of string field theory

(SFT) on the $AdS_5 \times S^5$ background. Then we show that there is a very straightforward extension, from supergravity level to SFT level, of the holographic ansatz given in [2][3]. Full construction of SFT on $AdS_5 \times S^5$ would be a hard task. However, we can extract relevant information of SFT on $AdS_5 \times S^5$ from pp-wave SFT which can be considered as an approximation of the near BPS sector of it. We introduce coordinates and basis (see (5),(13), (14) below) which facilitate the extraction of information. We obtain a novel representation (21) of the observable in gauge theory by an infinite series of matrix elements of SFT on $AdS_5 \times S^5$, which reduces to (23) under the pp-wave approximation.

Let us start by making minimum assumptions on the nature of SFT on $AdS_5 \times S^5$. Firstly, we assume that the string field consists of infinitely many fields on AdS_5 . Secondly, we assume that the free part of the SFT action is given by the usual Klein-Gordon operators on AdS_5 , at least for scalar fields (on AdS_5) in the string field.

The two assumptions enable us to propose the following extension of the holographic ansatz. Firstly, we consider a correspondence between fields ϕ_L in SFT on $AdS_5 \times S^5$ and non-BPS composite operators \mathcal{O}_L (with definite conformal dimensions) in gauge theory, extending the correspondence between fields in supergravity theory and BPS operators. The extend ansatz is then given by ¹

$$\langle e^{-\int J_L(x) \cdot \mathcal{O}_L(x) dx} \rangle = e^{-S[\phi_{cl}]}, \quad (1)$$

where ϕ_{cl} is the classical solution of the equation of motion of SFT such that the asymptotic behaviour near the boundary (i.e. $z \approx 0$) is given by $\phi_L \approx z^{4-\Delta_L} J_L(x)$ and S is the action of SFT. The asymptotic behaviour follows from the Klein-Gordon equation, so it is the same as the supergravity case. We work on the Euclideanized AdS_5 and use the Poincaré coordinates, $ds^2 = \frac{R^2(dz^2 + (dx^\mu)^2)}{z^2}$. R is the radius of AdS_5 , which is related to gauge theory by $R^4/\alpha'^2 \sim g_{\text{YM}}^2 N$. We choose the length scale so that R is equal to unity. Indices μ, ν, \dots run from 0 to 3. Δ_L is the conformal dimension of \mathcal{O}_L which is related to the mass m_L of ϕ_L by $\Delta_L = 2 + \sqrt{4 + m_L^2}$. The important advantage of the original ansatz is that the n -point functions given by it automatically satisfy the correct conformal transformation law. This property is preserved upon our extension.

For the near-BPS sector, we can determine Δ , appearing in the asymptotic behaviour near boundary, by the BMN formula

$$\Delta - J = \sum N_m \sqrt{1 + \frac{R^4 m^2}{J^2 \alpha'^2}}, \quad (2)$$

¹Actually, the left hand side of the equation should be considered as a tree level approximation (in string theory) of the path integral of e^{-S} .

which holds approximately when J is large. Thus we have been able to extract the masses of the fields, the only information of the free part of the SFT action, in a very simple way.

The interaction part, to which we now turn, requires much more work. One reason is that the interaction has strongly non-local features: we cannot assume the local interaction form as had been done, for example in [6], in the supergravity case. From now on, we shall concentrate on three point functions. Let us first identify what we should calculate according to the holographic ansatz. We have three scalar BMN operators, $\mathcal{O}_r (r = 1, 2, 3)$, and corresponding scalar fields on AdS_5 , ϕ_r . The holographic ansatz reads, for three point functions,

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle = S_{int}[K_1, K_2, K_3], \quad (3)$$

where S_{int} is the cubic interaction part of the SFT action and K_r 's are the boundary-bulk propagators for the fields ϕ_r given by ²

$$K_r = \frac{\Gamma(\Delta_r)}{\sqrt{\pi^4} \Gamma(\Delta_r - 2)} \left(\frac{z}{z^2 + (x - x_r)^2} \right)^{\Delta_r}. \quad (4)$$

Thus, what we want is the value of the interaction action evaluated on some special field configurations, namely boundary-bulk propagators.

On the other hands, what we have is the three string vertex, which embodies the joining (or splitting) amplitudes of strings. It gives roughly the value of coefficients before the $A^\dagger A A$ (or $A^\dagger A^\dagger A$) terms in the interaction Hamiltonian, where A, A^\dagger respectively are the annihilation, creation operators of the closed string (in the second-quantized sense). The interaction action is given by the time integral of the interaction Lagrangian, which in turn is the Legendre transformation of the interaction Hamiltonian. Hence, we should be able to calculate the interaction action from the three string vertex. However, we should clarify some points. Firstly, we have to understand what is meant by the time in our case. Moreover, we should decide the annihilation-creation operators, or the basis of the positive and negative energy solutions to the equations of motion of the free fields, to use. We shall specify the basis and then expand the boundary-bulk propagators by them below.

In flat space, it is most natural to take the basis of the form like e^{ikx} . However, we are working in AdS space so that there is an external potential. Since it has the harmonic oscillator form under pp-wave approximation, the natural choice made in pp-wave SFT is the wavefunctions for the harmonic oscillator: the wavefunction given by a gaussian (the ground state) and gaussian multiplied by hermite polynomials (excited states). We shall seek for the exact basis which reduce to these gaussian wavefunctions under the pp-wave

²The prefactor depends only on Δ_r . We neglect it in this letter. By taking care of overall factors, comparison to works [7] should be possible. We defer the study to further publication.

approximation. We will see that the discussion crucially depends on the use of the exact basis.

To this end, we introduce new coordinates \tilde{y}, \tilde{x}^μ on AdS_5 , which capture qualitative features of the pp-wave background,

$$\tilde{y} = \log \sqrt{z^2 + (x^\mu)^2}, \quad \tilde{x}^\mu = \frac{x^\mu}{z}. \quad (5)$$

The metric becomes

$$ds^2 = (1 + \tilde{x}^2)d\tilde{y}^2 + (d\tilde{x}^\mu)^2 - \frac{(\tilde{x}^\mu d\tilde{x}^\mu)^2}{1 + \tilde{x}^2}. \quad (6)$$

We emphasize that this is only a coordinate transformation so that there are no approximations involved. We are still quantitatively working in AdS_5 . The isometry $z' = \alpha z, x' = \alpha x$, which corresponds to the dilatation transformation in gauge theory, is realized by the translation of $\tilde{y}, \tilde{y}' = \tilde{y} + \log \alpha$. Also, the isometry of AdS_5 corresponding to inversion

$$z' = \frac{z}{z^2 + x^2}, \quad x'^\mu = \frac{x^\mu}{z^2 + x^2}, \quad (7)$$

is realized as the \tilde{y} -reversal, $\tilde{y}' = -\tilde{y}$.

We consider \tilde{y} as Euclidean time. This identification follows naturally from the tunnelling picture of [5], and enables us to directly identify the energy in string theory and the dilatation operator in gauge theory.

Let us consider the well-known solution to the Klein-Gordon equation z^Δ , in the new coordinates,

$$z^\Delta = \left(\frac{1}{\sqrt{1 + \tilde{x}^2}} \right)^\Delta e^{\Delta \tilde{y}}. \quad (8)$$

For large Δ , we can approximate the lorentzian in the above expression by a gaussian,

$$\left(\frac{1}{\sqrt{1 + \tilde{x}^2}} \right)^\Delta \approx e^{-\frac{\Delta}{2}\tilde{x}^2}. \quad (9)$$

Thus, we have found the exact solution which reduces to the gaussian wavefunction under the pp-wave approximation, large J implying large Δ .³ From the dependence on \tilde{y} , we see that it is the negative energy solution (in the sense of Euclidean field theory) of energy Δ . By inversion, we get the corresponding positive energy solution

$$(z')^\Delta = \left(\frac{z}{z^2 + x^2} \right)^\Delta = \left(\frac{1}{\sqrt{1 + \tilde{x}^2}} \right)^\Delta e^{-\Delta \tilde{y}}. \quad (10)$$

³Another well-known solution $z^{4-\Delta}$ corresponds to a non-normalisable wavefunction, $e^{+\frac{\Delta}{2}\tilde{x}^2} e^{-\Delta \tilde{y}}$.

We next seek for the solutions which reduce to the wavefunctions given by gaussian multiplied by hermite polynomials. To this end, let us consider the Killing vector field

$$\left(\frac{\partial}{\partial x'^{\mu}}\right)_{z'} = e^{+\tilde{y}} \left(\sqrt{1 + \tilde{x}^2} \frac{\partial}{\partial \tilde{x}^{\mu}} - \frac{\tilde{x}^{\mu}}{\sqrt{1 + \tilde{x}^2}} \frac{\partial}{\partial \tilde{y}} \right), \quad (11)$$

which corresponds to the special conformal symmetry in gauge theory. Being Killing vector field, it gives a new solution to the Klein-Gordon equation when it acts upon a solution. Furthermore, since wavefunctions have essentially the gaussian form $e^{-\frac{\Delta}{2}\tilde{x}^2}$, \tilde{x} should be considered as a small quantity of order $\frac{1}{\sqrt{\Delta}}$. Also, we have $\frac{\partial}{\partial \tilde{y}} \approx \mp \Delta$, the sign reflecting whether we apply the operator to positive or negative frequency solutions. By using these approximations we get

$$\left(\frac{\partial}{\partial x'^{\mu}}\right)_{z'} \approx e^{+\tilde{y}} \sqrt{2\Delta} \left(\sqrt{\frac{1}{2\Delta}} \frac{\partial}{\partial \tilde{x}^{\mu}} \pm \sqrt{\frac{\Delta}{2}} \tilde{x}^{\mu} \right). \quad (12)$$

The right hand side is, apart from the factor $\sqrt{2\Delta}$, the ladder operator (or the annihilation, creation operator in the first quantized sense) of the closed string zero modes in pp-wave.⁴ Therefore, we can construct the desired wavefunctions by applying (11) several times to the ground state wavefunction (8). There is also the inverted version $\left(\frac{\partial}{\partial x^{\mu}}\right)_z$, corresponding to the translational symmetry in gauge theory.

Thus we have obtained the basis of the negative and positive energy solutions

$$\left(\frac{\partial}{\partial x'^{\mu_1}}\right)_{z'} \cdots \left(\frac{\partial}{\partial x'^{\mu_n}}\right)_{z'} z^{\Delta} = \Psi_{\mu_1 \dots \mu_n}(\tilde{x}) e^{(\Delta+n)\tilde{y}}, \quad (13)$$

$$\left(\frac{\partial}{\partial x^{\mu_1}}\right)_z \cdots \left(\frac{\partial}{\partial x^{\mu_n}}\right)_z (z')^{\Delta} = \Psi_{\mu_1 \dots \mu_n}(\tilde{x}) e^{-(\Delta+n)\tilde{y}}, \quad (14)$$

respectively, where

$$\Psi_{\mu_1 \dots \mu_n}(\tilde{x}) \approx \sqrt{2\Delta}^n a_0^{\mu_1 \dagger} \dots a_0^{\mu_n \dagger} e^{-\frac{\Delta}{2}\tilde{x}^2}. \quad (15)$$

Here, $a_0^{\mu \dagger}$'s denote the creation operators of the closed string zero modes which have polarisations, labeled by μ 's, corresponding to insertion of vector impurities to BMN operators.

Our next task is to expand the boundary-bulk propagator (4) by the basis. It is readily seen that the expansion of (4) by x_r gives just the expansion by the basis,

$$\left(\frac{z}{z^2 + (x - x_r)^2}\right)^{\Delta_r} = \sum_n \frac{1}{n!} (-x_r^{\mu_1}) \dots (-x_r^{\mu_n}) \left(\frac{\partial}{\partial x^{\mu_n}}\right)_z \cdots \left(\frac{\partial}{\partial x^{\mu_1}}\right)_z \left(\frac{z}{z^2 + x^2}\right)^{\Delta_r}. \quad (16)$$

⁴The factor $e^{+\tilde{y}}$ arises because of representing the operator in the Heisenberg picture.

In this expansion only positive energy solutions, which are exponentially decreasing, appear. Hence, it is well-behaved when $\tilde{y} \rightarrow +\infty$. On the other hand, it seems that the propagator becomes singular when $\tilde{y} \rightarrow -\infty$, at first sight. Actually, this is not the case. Interestingly, the expansion above converges only in certain region, which is given by $\tilde{y} > \log|x_r|$. In the opposite region $\tilde{y} < \log|x_r|$, the propagator has another expansion. We first write the propagator in the inverted frame, and then perform similar expansion. The result is

$$\left(\frac{z}{z^2 + (x - x_r)^2}\right)^{\Delta_r} = \frac{1}{|x_r|^{2\Delta_r}} \sum_n \frac{1}{n!} \left(-\frac{x_r^{\mu_1}}{|x_r|^2}\right) \cdots \left(-\frac{x_r^{\mu_n}}{|x_r|^2}\right) \left(\frac{\partial}{\partial x'^{\mu_n}}\right)_{z'} \cdots \left(\frac{\partial}{\partial x'^{\mu_1}}\right)_{z'} z'^{\Delta_r}, \quad (17)$$

in which only exponentially increasing solutions appear. Thus the expansion of the propagator is actually quite well-behaved. The existence of the critical time, $\log|x_r|$, is essential to the following calculations. The origin of the circle of convergence is that the propagator is a rational function. We would not see the existence of the critical time if we used the approximate gaussian wavefunctions instead of the exact basis.

We now compute the right hand side of the holographic ansatz (3) by the expansions (16) and (17). There are critical time $\log|x_1|$, $\log|x_2|$, $\log|x_3|$, corresponding to the three scalar fields. We hereafter fix the radial order to be $|x_1| < |x_2| < |x_3|$. Then we have four regions, $-\infty < \tilde{y} < \log|x_1|$, $\log|x_1| < \tilde{y} < \log|x_2|$, $\log|x_2| < \tilde{y} < \log|x_3|$, $\log|x_3| < \tilde{y} < \infty$. In each of these regions, we have expansion labeled by three integers corresponding to the three fields. In each term in the expansion, the scalar fields behave exponentially in \tilde{y} . Since S_{int} is linear in each of the fields, the integrand itself is an exponential function. We define the matrix elements L 's between the members of the basis (13)(14) by

$$\begin{aligned} & S_{int} \left[\Psi_{\lambda_1 \dots \lambda_l} e^{\pm 1(\Delta_1 + l)\tilde{y}}, \Psi_{\mu_1 \dots \mu_m} e^{\pm 2(\Delta_2 + m)\tilde{y}}, \Psi_{\nu_1 \dots \nu_n} e^{\pm 3(\Delta_3 + n)\tilde{y}} \right] \\ &= L_{\lambda_1 \dots \lambda_l; \mu_1 \dots \mu_m; \nu_1 \dots \nu_n}^{\pm 1 \pm 2 \pm 3} e^{(\pm 1(\Delta_1 + l) \pm 2(\Delta_2 + m) \pm 3(\Delta_3 + n))\tilde{y}}. \end{aligned} \quad (18)$$

The signs $(\pm)_r$ for the fields ϕ_r reflect whether the negative or positive energy solutions are considered. Then, by performing the integral in each region, we obtain,

$$\begin{aligned} & \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle \\ & \equiv \sum_{l, m, n=0}^{\infty} \frac{(-1)^{l+m+n}}{l!m!n!} \left(\frac{x_1^{\lambda_1}}{|x_1|} \cdots \frac{x_1^{\lambda_l}}{|x_1|}\right) \left(\frac{x_2^{\mu_1}}{|x_2|} \cdots \frac{x_2^{\mu_m}}{|x_2|}\right) \left(\frac{x_3^{\nu_1}}{|x_3|} \cdots \frac{x_3^{\nu_n}}{|x_3|}\right) \\ & \left[\left(-M_{\lambda; \mu; \nu}^{---} + M_{\lambda; \mu; \nu}^{--+}\right) |x_3|^{-\Delta_1 - \Delta_2 - \Delta_3} \left(\frac{|x_1|}{|x_2|}\right)^l \left(\frac{|x_2|}{|x_3|}\right)^{l+m} \right. \\ & \left. + \left(-M_{\lambda; \mu; \nu}^{--+} + M_{\lambda; \mu; \nu}^{+++}\right) |x_2|^{-\Delta_1 - \Delta_2 + \Delta_3} |x_3|^{-2\Delta_3} \left(\frac{|x_1|}{|x_2|}\right)^l \left(\frac{|x_2|}{|x_3|}\right)^n \right] \end{aligned}$$

$$+ \left(-M_{\lambda;\mu;\nu}^{-++} + M_{\lambda;\mu;\nu}^{+++} \right) |x_1|^{-\Delta_1+\Delta_2+\Delta_3} |x_2|^{-2\Delta_2} |x_3|^{-2\Delta_3} \left[\left(\frac{|x_1|}{|x_2|} \right)^{m+n} \left(\frac{|x_2|}{|x_3|} \right)^n \right], \quad (19)$$

where, we have introduced an abbreviation

$$\frac{L_{\lambda_1 \dots \lambda_l; \mu_1 \dots \mu_m; \nu_1 \dots \nu_n}^{\pm_1 \pm_2 \pm_3}}{\pm_1 (\Delta_1 + l) \pm_2 (\Delta_2 + m) \pm_3 (\Delta_3 + n)} = M_{\lambda;\mu;\nu}^{\pm_1 \pm_2 \pm_3}.$$

We use \equiv to signify that the equality holds up to a Δ -dependent overall factor. Three terms come from critical time $\log |x_3|$, $\log |x_2|$, $\log |x_1|$, respectively. They do not mix in general since the differences of Δ 's are non-integral for generic operators.

On the other hands, dependence of the three point function on x_r is fixed by conformal symmetry to be,

$$\frac{C}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3} |x_2 - x_3|^{\Delta_2+\Delta_3-\Delta_1} |x_3 - x_1|^{\Delta_3+\Delta_1-\Delta_2}}, \quad (20)$$

where C is a constant. The expansion (19) should agree with the expansion of (20) by the two parameters $\frac{|x_1|}{|x_2|} < 1$ and $\frac{|x_2|}{|x_3|} < 1$, provided that conformal symmetry is properly realized in the interaction action i. e. in the matrix elements L 's. We can read off many identities embodying the conformal symmetry of the matrix elements by fully comparing the two expansions. In particular, only the second term should be non-vanishing in (19). By exploiting these identities one may gain some insights on SFT on $AdS_5 \times S^5$. In this letter, we shall instead concentrate on deriving the factor C , by comparing the leading term in the expansion of (20) and the $l = 0, n = 0$ part of the second term in (19). We obtain,

$$C \equiv \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{x_2^{\mu_1}}{|x_2|} \dots \frac{x_2^{\mu_m}}{|x_2|} \left(-\frac{L_{:\mu_1 \dots \mu_m:}^{-++}}{-\Delta_1 - (\Delta_2 + m) + \Delta_3} + \frac{L_{:\mu_1 \dots \mu_m:}^{-+-}}{-\Delta_1 + (\Delta_2 + m) + \Delta_3} \right). \quad (21)$$

This formula is our main result. It is interesting that the simple observable C should be written by an infinite series of the matrix elements with excited zero-modes. We emphasize that no approximations (such as the pp-wave approximation) are involved in this expression.

A couple of comments are in order. Firstly, we have performed a consistency check for this expression using toy models which have local interactions. The matrix elements $L_{:\mu_1 \dots \mu_m:}^{-\pm\pm}$ are calculated and then the series is evaluated. The results agree with those which are obtained by direct integration over z, x , although we do not give explicit formulae here. Secondly, we see that there may be some subtlety for BPS operators since the denominators may vanish for some m due to their integral Δ 's.

From now on, we shall consider the application of (21). To be specific, we present the calculations for the BMN operators ⁵ $\mathcal{O}_I, \mathcal{O}_{II}, \mathcal{O}_{III}$ corresponding to the states in string theory,

$$|\mathcal{O}_I\rangle = a_I^{\alpha_1^\dagger} a_{I-m_I}^{\beta_1^\dagger} |0; J_I\rangle, \quad |\mathcal{O}_{II}\rangle = |0; J_{II}\rangle, \quad |\mathcal{O}_{III}\rangle = a_{III m_{III}}^{\alpha_{III}^\dagger} a_{III -m_{III}}^{\beta_{III}^\dagger} |0; J_{III}\rangle, \quad (22)$$

respectively. These states satisfy the level matching condition. $|0; J_r\rangle$ ($r = I, II, III$) denote the first-quantised vacuum states of the closed strings with angular momentum $J_r > 0$ satisfying $J_I + J_{II} = J_{III}$. a_{rm}^\dagger 's denote creation operators of the m -th mode of the r -th string. α 's and β 's take one of the four values corresponding to insertion of scalar impurities. We take $m_I, m_{III} > 0$. We put operators $\mathcal{O}_I, \mathcal{O}_{II}, \bar{\mathcal{O}}_{III}$ respectively at the points x_1, x_2, x_3 (satisfying $|x_1| < |x_2| < |x_3|$). ⁶

By angular momentum conservation, the second term in (21) involves particles which have negative J . The contribution of them should be negligible, as is usual in the physics of the infinite momentum frame. For the first term, we substitute the matrix elements L defined via the Lagrangian by the matrix elements of the Hamiltonian, since Legendre transformation between them involves only an overall factor, at least in the leading order. Then (21) becomes, using (15),

$$C \equiv \sum_{n=0}^{\infty} \frac{(-1)^n x_2^{\mu_1}}{n! |x_2|} \cdots \frac{x_2^{\mu_n}}{|x_2|} \frac{\langle \mathcal{O}_I, \mathcal{O}_{II}, \bar{\mathcal{O}}_{III} | \sqrt{2\Delta}^n a_{II0}^{\mu_1} \cdots a_{II0}^{\mu_n} | V \rangle}{\Delta_I + (\Delta_{II} + n) - \Delta_{III}}, \quad (23)$$

where $|V\rangle$ denotes the three string vertex.

For the class of interaction vertices with prefactors quadratic in a^\dagger , such as those given in [8] or [9], the matrix elements can be manipulated as follows defining quantities E and F ,

$$\langle \mathcal{O}_I, \mathcal{O}_{II}, \bar{\mathcal{O}}_{III} | a_{II0}^{\mu_1} \cdots a_{II0}^{\mu_n} | V \rangle = \langle 0 | a_{II0}^{\mu_1} \cdots a_{II0}^{\mu_n} (E + F N_{00}^{II} a_{II0}^\dagger a_{II0}^\dagger) e^{\frac{1}{2} a_{II0}^{\nu\dagger} N_{00}^{II} a_{II0}^{\nu\dagger}} | 0 \rangle, \quad (24)$$

where N_{00}^{II} is a Neumann coefficient. E is the matrix element without any zero mode insertions, while F comes from zero modes in the prefactor. Substituting into (23), we get

$$C \equiv \sum_{\frac{n}{2}=0,1,\dots} \frac{1}{\frac{n}{2}!} \frac{E}{\Delta_I + (\Delta_{II} + n) - \Delta_{III}} (\Delta N_{00}^{II})^{\frac{n}{2}} + \sum_{\frac{n}{2}=1,2,\dots} \frac{1}{\frac{n}{2}!} \frac{nF}{\Delta_I + (\Delta_{II} + n) - \Delta_{III}} (\Delta N_{00}^{II})^{\frac{n}{2}}. \quad (25)$$

⁵Here, we refer to those mixed with double trace operators to have definite conformal dimensions.

⁶There are six possibilities regarding the radial order of the points where $\mathcal{O}_I, \mathcal{O}_{II}, \bar{\mathcal{O}}_{III}$ are inserted. All discussions work the same if we put the operators $(\mathcal{O}_I, \mathcal{O}_{II}, \bar{\mathcal{O}}_{III})$ on $(x_3, x_2, x_1), (x_2, x_1, x_3), (x_3, x_1, x_2)$, respectively. However, if we put them on $(x_1, x_3, x_2), (x_2, x_3, x_1)$, both terms in (21) involve particles with negative J . In order to treat these terms, we should carefully work out the transformation law between matrix elements in lightcone frame and in ordinary temporal frame. This issue will be discussed elsewhere.

Let us show the validity of using the pp-wave approximation in the above series. The validity holds if the fluctuation $\langle x \rangle$ of the oscillator is sufficiently small compared to the radius of the AdS space, $\langle x \rangle \ll R$. In terms of the excitation number n in (25), this condition reads $n \ll J$. Therefore, we should check that the leading contributions to the series should come from the terms satisfying $n \ll J$. Since we wish to compare the results to perturbative calculation in gauge theory, we shall work in the regime $\frac{g_{\text{YM}}^2 N}{J^2} \ll 1$. Since N_{00}^{II} is then of the order $\sqrt{\frac{g_{\text{YM}}^2 N}{J^2}}$ ⁷ [10], we have $\Delta N_{00}^{\text{II}} \sim \sqrt{g_{\text{YM}}^2 N} \gg 1$. Now, the leading contribution will come from terms for which the factors $(\Delta N_{00}^{\text{II}})^{\frac{n}{2}}$ and $\frac{n!}{2}$ are comparable, that is, terms with $n \sim \sqrt{g_{\text{YM}}^2 N}$. Although $\sqrt{g_{\text{YM}}^2 N}$ is large, it is much smaller than J in the regime we are working. Thus the use of pp-wave approximation is validated.

Performing the summation presents some interesting features. Firstly, $n = 0$ in the first term seems, at first sight, to make the only leading contribution because of the small denominator $\Delta_{\text{I}} + \Delta_{\text{II}} - \Delta_{\text{III}} \sim O(\frac{g_{\text{YM}}^2 N}{J^2})$. However, the numerator E is of the same order by non-trivial cancellation, hence the contribution from the second term should also be evaluated. Also other contributions from the first term become subleading. Since the summation of the second term starts with $\frac{n}{2} = 1$, the denominator can be replaced with n to the leading order. Then, the infinite series sums up essentially to the exponential function except for the missing $n = 0$ term, $F e^{\Delta N_{00}^{\text{II}}} - F$. Now, the exponent is large quantity ($\sim \sqrt{g_{\text{YM}}^2 N}$) with negative sign, since $N_{00}^{\text{II}} < 0$.⁸ Therefore the first term in this expression is extremely small and should be neglected in our approximation.

Thus finally we have found

$$C \equiv \frac{E}{\Delta_{\text{I}} + \Delta_{\text{II}} - \Delta_{\text{III}}} - F. \quad (26)$$

For the three string vertex given in [8], we get using the asymptotic form of Neumann coefficients [10]

$$\begin{aligned} E &= \frac{R^4}{J_{\text{III}}^2 \alpha'^2} \frac{(\sin m_{\text{III}} \pi y)^2}{\pi^2 y} \left(-2 \delta^{(\alpha_1 \alpha_{\text{III}} \delta^{\beta_1}) \beta_{\text{III}}} + \frac{1}{2} \delta^{\alpha_1 \beta_1} \delta^{\alpha_{\text{III}} \beta_{\text{III}}} \right) \\ F &= \frac{(\sin m_{\text{III}} \pi y)^2}{\pi^2 y} \left(\frac{-4 m_{\text{III}} (m_{\text{I}}/y)}{(m_{\text{III}}^2 - (m_{\text{I}}/y)^2)^2} \delta^{[\alpha_1 \alpha_{\text{III}} \delta^{\beta_1}] \beta_{\text{III}}} - 2 \frac{m_{\text{III}}^2 + (m_{\text{I}}/y)^2}{(m_{\text{III}}^2 - (m_{\text{I}}/y)^2)^2} \delta^{(\alpha_1 \alpha_{\text{III}} \delta^{\beta_1}) \beta_{\text{III}}} \right. \\ &\quad \left. - \frac{1}{2} \frac{m_{\text{III}}^2 + (m_{\text{I}}/y)^2}{(m_{\text{III}}^2 - (m_{\text{I}}/y)^2)^2} \delta^{\alpha_1 \beta_1} \delta^{\alpha_{\text{III}} \beta_{\text{III}}} \right), \end{aligned} \quad (27)$$

⁷This fact signifies the strong non-locality of the interaction. For local interaction, we would have Neumann coefficients (for zero modes) of order 1.

⁸The sign differs from the literature since our convention (12) of the a^\dagger includes a factor of i .

where $y = J_{\text{I}}/J_{\text{III}}$. Anti-symmetric and traceless symmetric pieces are denoted by

$$\begin{aligned}\delta^{[\alpha_{\text{I}}\alpha_{\text{III}}]\beta_{\text{III}}}&= \frac{1}{2}(\delta^{\alpha_{\text{I}}\alpha_{\text{III}}}\delta^{\beta_{\text{I}}\beta_{\text{III}}} - \delta^{\beta_{\text{I}}\alpha_{\text{III}}}\delta^{\alpha_{\text{I}}\beta_{\text{III}}}), \\ \delta^{(\alpha_{\text{I}}\alpha_{\text{III}})\beta_{\text{III}}}&= \frac{1}{2}(\delta^{\alpha_{\text{I}}\alpha_{\text{III}}}\delta^{\beta_{\text{I}}\beta_{\text{III}}} + \delta^{\beta_{\text{I}}\alpha_{\text{III}}}\delta^{\alpha_{\text{I}}\beta_{\text{III}}}) - \frac{1}{4}\delta^{\alpha_{\text{I}}\beta_{\text{I}}}\delta^{\alpha_{\text{III}}\beta_{\text{III}}},\end{aligned}$$

respectively. Substituting (27) into (26), we see that the the gauge theory results [11][12] are reproduced.

At this point, let us clarify the issue of ambiguities of the three string vertex. We consider unitary transformations, $H_{free} + H'_{int} = (1 + D + \dots)^{-1}(H_{free} + H_{int})(1 + D + \dots)$, or,

$$H'_{int} = H_{int} + [H_{free}, D], \quad (28)$$

where H_{int} and D have the order of the string coupling constant. H'_{int} has the same symmetry as the original H_{int} by construction. In particular, it satisfies the constraints of supersymmetry. These transformations can be considered as field redefinitions, of which D 's are the generators. In usual field theory, it is guaranteed that the physical observables do not change under these redefinitions, provided the locality of the transformations. We can actually show that also in our case the observable C does not depend on the transformation (28) for a broad class of D . Indeed, for any D which is given by the overlap part with a polynomial prefactor, $|D\rangle = P(a^\dagger)e^{\frac{1}{2}a^\dagger N a^\dagger}|0\rangle$, we get from (23)

$$C' = C + \langle 0|e^{-\sqrt{2\Delta}\frac{x_2^\mu}{|x_2|}a^\mu}|D\rangle = C + P\left(-\sqrt{2\Delta}\frac{x_2^\mu}{|x_2|}\right)e^{\Delta N_{00}^{\text{II}}}. \quad (29)$$

Thus, C does not change up to exponentially negligible term, which should be neglected in our approximation. Hence, our method does not suffer from these ambiguities.⁹ The form of D given above closely resembles that of the generator of a local field redefinition in ordinary field theory, the overlap part and the polynomial of a^\dagger 's respectively corresponding to the δ -function part and the polynomial of derivatives.

A special case of (28) is the three point vertex given in [9], which can be written as $[H_{free}, D]$ with D the overlap part itself. For the vertex, E -term and F -term in (26) cancel so that it gives a null contribution to C , as expected from the above argument. Previously, the following relation has been proposed[13]

$$C = \frac{E}{\Delta_{\text{I}} + \Delta_{\text{II}} - \Delta_{\text{III}}}, \quad (30)$$

⁹This property makes it natural to call these observables as on-shell, although $\Delta_1 + \Delta_2 \neq \Delta_3$. This is also natural from the viewpoint of the holographic ansatz since they are given by the path-integral with a fixed asymptotic behaviour of the fields. We note that for ordinary transition amplitudes in time-independent systems, it is the condition $\Delta_1 + \Delta_2 = \Delta_3$ which guarantees that they do not change upon field redefinitions.

which lacks the F -term, so that the results of which are affected by the ambiguities. Recently, Dobashi and Yoneya [14] [15] have succeeded in reproducing the gauge theory results using (30) (with some refinements) by choosing a vertex which is a special linear combination of the vertices of [8] and [9].¹⁰ In our perspective, their success follows from the fact that $F = 0$ holds for their choice.

We should be able to perform many tests of the AdS/CFT correspondence by further applying the framework proposed in this letter. It will be interesting to study (a) the subleading order of our approximation which has rather intricate structures, (b) general BMN operators such as those which have vector impurities, for which one should use different propagators and x_r -dependences (20), (c) the four (or more) point functions.

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¹⁰Bosonic part of this vertex is first discussed in [16]. Very recently, the work [17] appeared along this line.

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