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Physics Letters B 647 (2007) 211–218

PHYSICS LETTERS B

www.elsevier.com/locate/physletb

Holography at string field theory level: Conformal three point functions of BMN operators

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Received 18 April 2006; accepted 25 January 2007

Available online 15 February 2007

Editor: L. Alvarez-Gaumé

Abstract

We propose a general framework for applying the pp-wave approximation to holographic calculation in the AdS/CFT correspondence. By assuming the existence and some properties of string field theory (SFT) on the $AdS_5 \times S^5$ background, we extend the holographic ansatz proposed by Gubser, Klebanov, Polyakov and by Witten to the SFT level. We extract relevant information about assumed SFT on $AdS_5 \times S^5$ from its approximation, pp-wave SFT. As an explicit example, we study conformal three point functions of BMN operators. We find a new formula which expresses a three point function as an infinite series of matrix elements of the SFT vertex. We identify a broad class of field redefinitions which do not affect the final observable. Known ambiguity in the pp-wave SFT vertex is due to a particular redefinition in this class. Under these redefinitions, matrix elements themselves change, but the sum of the series is invariant due to a non-trivial cancellation. The result agrees with that previously calculated in gauge theory.

1. Introduction

The AdS/CFT correspondence conjecture [1], which states that string theory on $AdS_5 \times S^5$ is equivalent to $\mathcal{N} = 4$ supersymmetric Yang–Mills theory, is one of the most explicit proposal of equivalence between string theory and large N gauge theory. An important theme is to understand the fundamental mechanism of the correspondence, in other words, to understand how degrees of freedom of closed strings arise from those of gauge theory. Solving this problem will considerably improve our understanding of string theory itself.

As a first step, it will be useful to calculate corresponding observables in both string theory and gauge theory independently and check the mutual agreement. However, it is difficult to carry out such comparison in general. On the string theory side quantum string theory on $AdS_5 \times S^5$ is not defined, and we cannot treat non-zero modes of closed strings. On the gauge theory side we have no general methods of computation; perturbative methods do not work since the expansion parameter $g_{\mathrm{YM}}^2 N$ is large.

A breakthrough has been made in [4]. On the string theory side the pp-wave approximation is found, which is applicable to states with large orbital angular momentum J on S^5. Quantum theory is well-defined under this approximation, so that one can in particular treat non-zero modes of closed strings. On the gauge theory side, the operators (BMN operators) corresponding to these states are proposed. The expansion parameter becomes $g_{\mathrm{YM}}^2 N/J^2$ for them and hence perturbative calculations are valid in the regime $g_{\mathrm{YM}}^2 N/J^2 \ll 1$. Using these methods many tests have been performed based on the postulated equivalence between energy in string theory and the dilatation operator in gauge theory.

However, for the holographic aspects [2,3] (in the sense that observables in gauge theory are related to behaviour of strings at the boundary of AdS space), there remains much to be uncovered. Since holography is the most characteristic feature of the AdS/CFT correspondence, it is important to apply the pp-wave approximation to study these holographic aspects. A step has been made by

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0370-2693/$ – see front matter
doi:10.1016/j.physletb.2007.01.066

Dobashi, Yoneya, and the author [5]; in order to directly apply the pp-wave approximation to holography, one should think that closed strings are in tunnelling states under the barrier of the gravitational potential in the *AdS* background.

In this Letter we further propose, under the tunnelling picture, a general framework for applying the pp-wave approximation to holographic evaluation of conformal n-point correlation functions of BMN operators. In particular, we perform an explicit calculation (on the string theory side) of three point functions of scalar BMN operators. An important problem here is known ambiguity in the three string vertex of the pp-wave string field theory (SFT) [7,8]. Our solution to this problem is as follows. In our approach a three point function is expressed as an infinite series of matrix elements of the string vertex. Although the matrix elements themselves are ambiguous, the result of the summation is unambiguous due to a non-trivial cancellation. We further identify a broad class of field redefinitions (including the known ambiguity) which does not affect the physical observables in this way. Our results are thus unambiguous, and agree with those previously calculated on the gauge theory side up to an overall factor.

Our basic strategy is as follows. We begin by assuming the existence of SFT on the $AdS_5 \times S^5$ background. We then show that there is a very straightforward extension of the holographic ansatz [2,3], from the supergravity level to the SFT level. Although full construction of SFT on $AdS_5 \times S^5$ would be a hard task, we can extract relevant information about it from pp-wave SFT, which can be considered as an approximation of its near BPS sector. To facilitate extraction of information, we introduce new coordinates (5) and basis functions (13), (14). By expanding usual bulk-boundary propagators in this basis we obtain the representation (21), (23) of the final observable as an infinite series of SFT matrix elements.

2. Extension of holographic ansatz

Let us start by making minimum assumptions on the nature of SFT on $AdS_5 \times S^5$. Firstly, we assume that the string field consists of infinitely many fields ϕ_L defined on AdS_5. The SFT action is then a functional of the fields ϕ_L. Secondly, we assume that the free part of the action is made up by usual Klein–Gordon operators (with masses m_L) on AdS_5, at least for those ϕ_L which transform as scalar on AdS_5.

These two assumptions enable us to propose an extension of the original holographic ansatz. We work with Euclideanised AdS_5 given in Poincaré coordinates, $ds^2 = R^2(dz^2 + (dx^\mu)^2)/z^2$, where the radius R is given by $R^4/\alpha'^2 \sim g_{\mathrm{YM}}^2 N$ and μ runs through 0 to 3. Originally the correspondence are made between supergravity fields and BPS operators in gauge theory. Under the first assumption, we first extend the original correspondence to the correspondence between general fields ϕ_L in the string field and non-BPS operators $\mathcal{O}_L$ (with definite conformal dimensions $\Delta_L = 2 + \sqrt{4 + m_L^2}$). Then we simply extrapolate the original ansatz [2,3] to the SFT level,[1]

$$\left\langle e^{-\int J_L(x)\cdot\mathcal{O}_L(x)\,d^4x}\right\rangle = e^{-S[\phi_{cl}]}. \tag{1}$$

Here S is the SFT action and its classical solution ϕ_{cl} is fixed by its asymptotic behaviour near the boundary, $\phi_L \approx z^{4-\Delta_L} J_L(x)$ at $z \approx 0$. This asymptotic behaviour solves the Klein–Gordon equation and hence, by our second assumption, is consistent with the free equation of motion of SFT. The important advantage of the original ansatz is that the resulting n-point functions automatically satisfy the correct conformal transformation law. This property still holds for our extended ansatz, because it follows from the asymptotic behaviour of the fields and the conformal symmetry of the action. We believe this to be an important justification for our extension.

3. Extraction of information; free part and interaction part of SFT

To use the extended ansatz for actual calculations, we need relevant information about SFT on $AdS_5 \times S^5$. The only information of the free part of the SFT action is the masses m_L. For fields ϕ_L corresponding to BMN operators, the BMN formula [4]

$$\Delta - J = \sum_m N_m \sqrt{1 + \frac{R^4 m^2}{J^2 \alpha'^2}} \tag{2}$$

determines Δ_L hence m_L.

Extraction of the information of the interaction part requires much more work, partly due to its strong non-locality; we cannot assume the local interaction form contrary to the supergravity case (for example in [6]). Therefore we will employ below a more direct method based on expansion by appropriate basis functions.

We concentrate on three point functions of scalar BMN operators $\mathcal{O}_r$ ($r = 1, 2, 3$). Corresponding to $\mathcal{O}_r$, we have three scalar fields ϕ_r on AdS_5. The holographic ansatz (1) reads, for three point functions,

$$\big\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\big\rangle = S_{\mathrm{int}}[\phi_1, \phi_2, \phi_3]\big|_{\phi_r(z,x) = K_r(z,x;x_r)}. \tag{3}$$

[1] The right-hand side actually should be considered as a tree level approximation (in string theory) of the path integral of e^{-S}.

Here S_{int} is the cubic interaction part of the SFT action, which can be considered as a functional of the three fields ϕ_r for our purpose. The boundary-bulk propagators K_r for the fields ϕ_r are given by[2]

$$K_r(z,x;x_r) = \frac{\Gamma(\Delta_r)}{\sqrt{\pi}^4\Gamma(\Delta_r-2)}\left(\frac{z}{z^2+(x-x_r)^2}\right)^{\Delta_r}. \tag{4}$$

Thus what we want is the value of the interaction action evaluated at the propagators.

Now, what we have instead is the three string vertex, which expresses the joining (or splitting) amplitudes of strings; it gives the coefficients before $A^\dagger AA$ (or $A^\dagger A^\dagger A$) terms in the interaction Hamiltonian, where A, $A^\dagger$ respectively are annihilation, creation operators of closed strings (in the second-quantised sense). The interaction action is given by time integral of the interaction Lagrangian, which in turn is given by the Legendre transformation of the interaction Hamiltonian. In this way we should be able to calculate the right-hand side of (3) from the string vertex.

We should, however, clarify some points. We have to understand what is meant by 'time' in our case. We should also decide appropriate annihilation and creation operators (or basis functions for the positive and negative energy solutions to Klein–Gordon equations). In order to extract information from pp-wave SFT, we should choose the basis and the time coordinate which reduce, under the pp-wave approximation, to those used in pp-wave SFT.

4. Basis expansion

In flat space, the natural basis consists of functions like e^{ikx}. Now we are working in AdS space, so there is an external potential. Since it has the harmonic oscillator form under the pp-wave approximation, the natural basis functions chosen in pp-wave SFT are the wavefunctions for the harmonic oscillator: a Gaussian (the ground state) and Gaussian multiplied by Hermite polynomials (excited states). We shall seek the exact basis in AdS space which reduce to those Gaussian wavefunctions in the pp-wave approximation. The necessity of the exact basis will become clear later.

To find the exact basis we introduce new coordinates $\tilde{y}, \tilde{x}^\mu$ on AdS_5,

$$\tilde{y} = \log\sqrt{z^2+(x^\mu)^2}, \qquad \tilde{x}^\mu = \frac{x^\mu}{z}. \tag{5}$$

The metric becomes

$$ds^2 = \frac{dz^2+(dx^\mu)^2}{z^2} = \left(1+\tilde{x}^2\right)d\tilde{y}^2 + \left(d\tilde{x}^\mu\right)^2 - \frac{(\tilde{x}^\mu d\tilde{x}^\mu)^2}{1+\tilde{x}^2}, \tag{6}$$

hence this coordinates capture qualitative features of the pp-wave background. We have only made a coordinate transformation in (5) and there are no approximations involved; we are still quantitatively working in AdS_5. The isometry $z'=\alpha z$, $x'=\alpha x$ corresponding to a dilatation transformation in gauge theory is realised as a translation of $\tilde{y}$, $\tilde{y}' = \tilde{y} + \log\alpha$. Also the isometry corresponding to inversion,

$$z' = \frac{z}{z^2+x^2}, \qquad x'^\mu = \frac{x^\mu}{z^2+x^2}, \tag{7}$$

is realised as the $\tilde{y}$-reversal, $\tilde{y}' = -\tilde{y}$. We therefore consider $\tilde{y}$ as Euclidean time. This identification is natural in the tunnelling picture of [5], and further enables us to directly identify energy in string theory and the dilatation operator in gauge theory.

Let us then rewrite in this coordinates a well-known solution z^Δ to the Klein–Gordon equation,

$$z^\Delta = \left(\frac{1}{\sqrt{1+\tilde{x}^2}}\right)^\Delta e^{\Delta\tilde{y}}. \tag{8}$$

For large Δ the Lorentzian in the above expression can be approximated by a Gaussian,

$$\left(\frac{1}{\sqrt{1+\tilde{x}^2}}\right)^\Delta \approx e^{-\frac{\Delta}{2}\tilde{x}^2}. \tag{9}$$

Thus we have found the exact solution which reduces under the pp-wave approximation to the ground state wavefunction, large J implying large Δ.[3] From the dependence on $\tilde{y}$ we see that it is the negative energy solution (in the sense of Euclidean field theory) of energy Δ. We get, by using inversion (7), the corresponding positive energy solution

$$(z')^\Delta = \left(\frac{z}{z^2+x^2}\right)^\Delta = \left(\frac{1}{\sqrt{1+\tilde{x}^2}}\right)^\Delta e^{-\Delta\tilde{y}}. \tag{10}$$

[2] The prefactor depending only on Δ_r will be neglected in this Letter.

[3] Another well-known solution $z^{4-\Delta}$ corresponds to a non-normalisable wavefunction, $e^{+\frac{\Delta}{2}\tilde{x}^2}e^{-\Delta\tilde{y}}$.

We next seek solutions which reduce to wavefunctions given by Gaussian multiplied by Hermite polynomials. To this end we consider the Killing vector field

$$\left(\frac{\partial}{\partial x'^{\mu}}\right)_{z'} = e^{+\tilde{y}}\left(\sqrt{1+\tilde{x}^2}\frac{\partial}{\partial\tilde{x}^{\mu}} - \frac{\tilde{x}^{\mu}}{\sqrt{1+\tilde{x}^2}}\frac{\partial}{\partial\tilde{y}}\right), \tag{11}$$

which corresponds to the special conformal transformation in gauge theory. Being a Killing vector field, it generates a new solution when it acts on a solution to the Klein–Gordon equation. Since wavefunctions have essentially the Gaussian form $e^{-\frac{\Delta}{2}\tilde{x}^2}$, $\tilde{x}$ should be considered as a small quantity of order $\frac{1}{\sqrt{\Delta}}$. Also we have $\frac{\partial}{\partial\tilde{y}} \approx \mp\Delta$, the sign reflecting whether we apply the operator to positive or negative frequency solutions. Under these approximations we get

$$\left(\frac{\partial}{\partial x'^{\mu}}\right)_{z'} \approx e^{+\tilde{y}}\sqrt{2\Delta}\left(\sqrt{\frac{1}{2\Delta}}\frac{\partial}{\partial\tilde{x}^{\mu}} \pm \sqrt{\frac{\Delta}{2}}\tilde{x}^{\mu}\right). \tag{12}$$

Upper (lower) sign refers to positive (negative) solutions. The right-hand side is, apart from the factor $\sqrt{2\Delta}$, the ladder operator (or the annihilation, creation operator in the first quantised sense) for closed string zero modes in pp-wave.[4] Therefore, the desired wavefunctions are constructed by applying (11) several times to the ground state wavefunction (8). There is also the inverted version $(\frac{\partial}{\partial x^{\mu}})_z$ which corresponds to the translational symmetry in gauge theory.

Thus we have obtained the basis of negative and positive energy solutions

$$\left(\frac{\partial}{\partial x'^{\mu_1}}\right)_{z'}\cdots\left(\frac{\partial}{\partial x'^{\mu_n}}\right)_{z'} z^{\Delta} = \Psi_{\mu_1\ldots\mu_n}(\tilde{x})e^{(\Delta+n)\tilde{y}}, \tag{13}$$

$$\left(\frac{\partial}{\partial x^{\mu_1}}\right)_{z}\cdots\left(\frac{\partial}{\partial x^{\mu_n}}\right)_{z} (z')^{\Delta} = \Psi_{\mu_1\ldots\mu_n}(\tilde{x})e^{-(\Delta+n)\tilde{y}} \tag{14}$$

respectively where

$$\Psi_{\mu_1\ldots\mu_n}(\tilde{x}) \approx \sqrt{2\Delta}^{n} a_0^{\mu_1\dagger}\ldots a_0^{\mu_n\dagger} e^{-\frac{\Delta}{2}\tilde{x}^2}. \tag{15}$$

Here $a_0^{\mu\dagger}$ denotes a creation operator for a closed string zero mode. It has polarisation, labelled by μ, corresponding to insertion of a vector impurity to BMN operators.

Our next task is to expand the boundary-bulk propagator (4) by the basis. It is readily seen that the expansion of (4) in x_r gives precisely the expansion in the basis (14),

$$K_r = \left(\frac{z}{z^2+(x-x_r)^2}\right)^{\Delta_r} = \sum_n \frac{1}{n!}(-x_r^{\mu_1})\cdots(-x_r^{\mu_n})\left(\frac{\partial}{\partial x^{\mu_n}}\right)_z\cdots\left(\frac{\partial}{\partial x^{\mu_1}}\right)_z\left(\frac{z}{z^2+x^2}\right)^{\Delta_r}. \tag{16}$$

In this expansion only positive energy solutions appear, which are exponentially decreasing (see (14)). Hence, the propagator is well-behaved for $\tilde{y}\to+\infty$, but it seems to be singular, for $\tilde{y}\to-\infty$. Actually, this is not the case. The expansion above converges only in certain region, namely $\tilde{y} > \log|x_r|$, and interestingly the propagator has another complimentary expansion for $\tilde{y} < \log|x_r|$. To obtain this expansion, we first write down the propagator in the inverted frame (7) and then perform a similar expansion, this time in the basis (13)

$$K_r = \frac{1}{|x_r|^{2\Delta_r}}\sum_n \frac{1}{n!}\left(-\frac{x_r^{\mu_1}}{|x_r|^2}\right)\cdots\left(-\frac{x_r^{\mu_n}}{|x_r|^2}\right)\left(\frac{\partial}{\partial x'^{\mu_n}}\right)_{z'}\cdots\left(\frac{\partial}{\partial x'^{\mu_1}}\right)_{z'} z^{\Delta_r}. \tag{17}$$

Here only exponentially increasing solutions appear. Thus the expansion of the propagator actually works quite well. The existence of the critical time $\log|x_r|$, or the natural switching between the positive and negative energy basis, is essential to the following calculations. Note that we could see the existence of the critical time only because we use the exact basis instead of the approximate Gaussian form.

5. Calculation of the holographic observable

Now we calculate $S_{\text{int}}[K_1,K_2,K_3]$ in (3). We first define the matrix elements L between three members in the basis (13), (14), by considering the value of the cubic interaction Lagrangian (with $\tilde{y}$ as time)

$$L_{\text{int}}\left[\Psi_{\lambda_1\ldots\lambda_l}e^{\pm_1(\Delta_1+l)\tilde{y}}, \Psi_{\mu_1\ldots\mu_m}e^{\pm_2(\Delta_2+m)\tilde{y}}, \Psi_{\nu_1\ldots\nu_n}e^{\pm_3(\Delta_3+n)\tilde{y}}\right] = L^{\pm_1\pm_2\pm_3}_{\lambda_1\ldots\lambda_l:\mu_1\ldots\mu_m:\nu_1\ldots\nu_n}e^{(\pm_1(\Delta_1+l)\pm_2(\Delta_2+m)\pm_3(\Delta_3+n))\tilde{y}}. \tag{18}$$

[4] The factor $e^{+\tilde{y}}$ arises because the operator is represented in the Heisenberg picture.

Signs $(\pm)_r$ for fields ϕ_r reflect whether negative or positive energy solutions are considered. Dependence of the right-hand side on $\tilde{y}$ follows from linearity of L_{int} in each field ϕ_r. The matrix elements L carry the same amount of information as the functional S_{int}.

We then rewrite $S_{\text{int}}[K_1, K_2, K_3]$ in terms of these matrix elements using expansions (16), (17). Corresponding to three propagators, there are three critical times $\log|x_1|, \log|x_2|, \log|x_3|$. We hereafter fix their radial order to be $|x_1| < |x_2| < |x_3|$. In each of four regions, $-\infty < \tilde{y} < \log|x_1|$, $\log|x_1| < \tilde{y} < \log|x_2|$, $\log|x_2| < \tilde{y} < \log|x_3|$, $\log|x_3| < \tilde{y} < \infty$, we have an expansion labelled by three integers corresponding to K_r. By performing $\tilde{y}$-integral in each region we obtain

$$\begin{aligned}
\big\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\big\rangle &= S_{\text{int}}[K_1, K_2, K_3] \\
&\equiv \sum_{l,m,n=0}^{\infty} \frac{(-1)^{l+m+n}}{l!m!n!}\left(\frac{x_1^{\lambda_1}}{|x_1|}\cdots\frac{x_1^{\lambda_l}}{|x_1|}\right)\left(\frac{x_2^{\mu_1}}{|x_2|}\cdots\frac{x_2^{\mu_m}}{|x_2|}\right)\left(\frac{x_3^{\nu_1}}{|x_3|}\cdots\frac{x_3^{\nu_n}}{|x_3|}\right) \\
&\quad\times\Bigg[\big(-M^{---}_{\lambda:\mu:\nu} + M^{--+}_{\lambda:\mu:\nu}\big)|x_3|^{-\Delta_1-\Delta_2-\Delta_3}\left(\frac{|x_1|}{|x_2|}\right)^{l}\left(\frac{|x_2|}{|x_3|}\right)^{l+m} \\
&\quad+\big(-M^{--+}_{\lambda:\mu:\nu} + M^{-++}_{\lambda:\mu:\nu}\big)|x_2|^{-\Delta_1-\Delta_2+\Delta_3}|x_3|^{-2\Delta_3}\left(\frac{|x_1|}{|x_2|}\right)^{l}\left(\frac{|x_2|}{|x_3|}\right)^{n} \\
&\quad+\big(-M^{-++}_{\lambda:\mu:\nu} + M^{+++}_{\lambda:\mu:\nu}\big)|x_1|^{-\Delta_1+\Delta_2+\Delta_3}|x_2|^{-2\Delta_2}|x_3|^{-2\Delta_3}\left(\frac{|x_1|}{|x_2|}\right)^{m+n}\left(\frac{|x_2|}{|x_3|}\right)^{n}\Bigg],
\end{aligned} \tag{19}$$

where we have introduced an abbreviation

$$\frac{L^{\pm_1\pm_2\pm_3}_{\lambda_1\ldots\lambda_l:\mu_1\ldots\mu_m:\nu_1\ldots\nu_n}}{\pm_1(\Delta_1+l)\pm_2(\Delta_2+m)\pm_3(\Delta_3+n)} = M^{\pm_1\pm_2\pm_3}_{\lambda:\mu:\nu}.$$

We have used the symbol $\equiv$ to signify that the equality holds up to a Δ-dependent overall factor. Three terms in (19) come from critical time $\log|x_3|, \log|x_2|, \log|x_1|$, respectively. They do not mix in general since differences of Δ_r are non-integral for generic operators.

Now let us recall that the dependence of the three point function on x_r is fixed by conformal symmetry to be,

$$\frac{C}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}|x_3-x_1|^{\Delta_3+\Delta_1-\Delta_2}}, \tag{20}$$

where C is a constant. Provided that the conformal symmetry is properly realised in S_{int}, or the matrix elements L, (19) should agree with the expansion of (20) in the two parameters $|x_1|/|x_2| < 1$ and $|x_2|/|x_3| < 1$. We can read off many identities between the matrix elements, imposed by the conformal symmetry, by fully comparing the two expansions. In particular, only the second term in (19) should be non-vanishing. Further study of these identities may give us important insights on SFT on $AdS_5 \times S^5$. Here, we shall instead concentrate on deriving the factor C. By comparing the leading term in the expansion of (20) and the $l = 0$, $n = 0$ part of the second term in (19), we obtain

$$C \equiv \sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\frac{x_2^{\mu_1}}{|x_2|}\cdots\frac{x_2^{\mu_m}}{|x_2|}\left(-\frac{L^{--+}_{:\mu_1\ldots\mu_m:}}{-\Delta_1-(\Delta_2+m)+\Delta_3} + \frac{L^{-++}_{:\mu_1\ldots\mu_m:}}{-\Delta_1+(\Delta_2+m)+\Delta_3}\right). \tag{21}$$

This formula is one of our main results. It is interesting that the simple observable C is written as an infinite series of the matrix elements with excited zero-modes. This is in contrast with other approaches [12–14] in which a gauge theory observable is compared to a single matrix element. As we will see, this feature is essential for the invariance of C under field redefinitions. We also remark that no approximations (such as the pp-wave approximation) are involved in this expression.

A couple of comments are in order. Firstly, we have performed consistency checks for this expression using toy models with local interactions such as $S_{\text{int}} \sim \int \phi_1\phi_2\phi_3\, d^4x\, dz$. For these models matrix elements $L^{-\pm+}_{:\mu_1\ldots\mu_m:}$ can be computed in terms of Euler Γ-functions. Summation of the infinite series (21) then exactly yields the standard result [6] obtained by direct integration over z, x. Secondly, we see that BPS operators are rather singular (compared to general non BPS ones) since the denominators in (21) may vanish for some m due to integral Δ_r.

6. Application of pp-wave approximation

To be specific, we present our calculation for the BMN operators[5] $\mathcal{O}_{\text{I}}, \mathcal{O}_{\text{II}}, \mathcal{O}_{\text{III}}$ corresponding to the states in string theory (satisfying level matching conditions),

$$|\mathcal{O}_{\text{I}}\rangle = a^{\alpha_{\text{I}}\dagger}_{\text{I}\,m_{\text{I}}}a^{\beta_{\text{I}}\dagger}_{\text{I}-m_{\text{I}}}|0; J_{\text{I}}\rangle, \quad |\mathcal{O}_{\text{II}}\rangle = |0; J_{\text{II}}\rangle, \quad |\mathcal{O}_{\text{III}}\rangle = a^{\alpha_{\text{III}}\dagger}_{\text{III}\,m_{\text{III}}}a^{\beta_{\text{III}}\dagger}_{\text{III}-m_{\text{III}}}|0; J_{\text{III}}\rangle, \tag{22}$$

[5] We refer to BMN operators which are mixed with appropriate double trace operators; this mixing is necessary in order that the BMN operators have definite conformal dimensions.

respectively. Here $|0; J_r\rangle$ ($r = \mathrm{I}, \mathrm{II}, \mathrm{III}$) denote the first-quantised vacuum states of the closed strings with angular momentum $J_r > 0$ satisfying $J_\mathrm{I} + J_\mathrm{II} = J_\mathrm{III}$. We denote by $a_{rm}^\dagger$ a creation operator of the mth mode of the rth string. Indices α and β take one of the four values corresponding to insertion of scalar impurities and we take $m_\mathrm{I}, m_\mathrm{III} > 0$. We put operators $\mathcal{O}_\mathrm{I}, \mathcal{O}_\mathrm{II}, \bar{\mathcal{O}}_\mathrm{III}$ respectively at the points x_1, x_2, x_3 (satisfying $|x_1| < |x_2| < |x_3|$).[6]

Because of angular momentum conservation, the second term of (21) involves particles with negative J. We neglect its contribution, as is usual in the physics of the infinite momentum frame. Since we have chosen the basis which reduces to that used in pp-wave SFT, it should be possible to replace matrix elements L in the first term of (21) by corresponding matrix elements of the pp-wave SFT, for the near BPS sector.[7] Then (21) becomes

$$C \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x_2^{\mu_1}}{|x_2|} \cdots \frac{x_2^{\mu_n}}{|x_2|} \frac{\langle \mathcal{O}_\mathrm{I}, \mathcal{O}_\mathrm{II}, \bar{\mathcal{O}}_\mathrm{III}|(2\Delta_\mathrm{II})^{n/2} a_{\mathrm{II}0}^{\mu_1} \ldots a_{\mathrm{II}0}^{\mu_n}|V\rangle}{\Delta_\mathrm{I} + (\Delta_\mathrm{II} + n) - \Delta_\mathrm{III}}, \tag{23}$$

where $|V\rangle$ denotes the three string vertex. The factor $(2\Delta_\mathrm{II})^{n/2}$ follows from (15).

For the class of interaction vertices with prefactors quadratic in $a^\dagger$, such as those given in [7] or [8], the matrix elements in (23) can be manipulated as follows,

$$\langle \mathcal{O}_\mathrm{I}, \mathcal{O}_\mathrm{II}, \bar{\mathcal{O}}_\mathrm{III}|a_{\mathrm{II}0}^{\mu_1} \ldots a_{\mathrm{II}0}^{\mu_n}|V\rangle = \langle 0|a_{\mathrm{II}0}^{\mu_1} \ldots a_{\mathrm{II}0}^{\mu_n}\big(E + F N_{00}^{\mathrm{II}\mathrm{II}} a_{\mathrm{II}0}^{\mu\dagger} a_{\mathrm{II}0}^{\mu\dagger}\big) e^{\frac{1}{2} a_{\mathrm{II}0}^{\nu\dagger} N_{00}^{\mathrm{II}\mathrm{II}} a_{\mathrm{II}0}^{\nu\dagger}}|0\rangle. \tag{24}$$

Here $N_{00}^{\mathrm{II}\mathrm{II}}$ is a Neumann coefficient, and we have introduced quantities E and F: E is the matrix element without any zero mode insertions, while F comes from zero modes in the prefactor. Substituting into (23), we get

$$C \equiv \sum_{n/2=0,1,\ldots} \frac{1}{\frac{n}{2}!} \frac{E}{\Delta_\mathrm{I} + (\Delta_\mathrm{II} + n) - \Delta_\mathrm{III}} \big(\Delta_\mathrm{II} N_{00}^{\mathrm{II}\mathrm{II}}\big)^{n/2} + \sum_{n/2=1,2,\ldots} \frac{1}{\frac{n}{2}!} \frac{nF}{\Delta_\mathrm{I} + (\Delta_\mathrm{II} + n) - \Delta_\mathrm{III}} \big(\Delta_\mathrm{II} N_{00}^{\mathrm{II}\mathrm{II}}\big)^{n/2}. \tag{25}$$

The validity of the pp-wave approximation made in the above series should be examined carefully. The approximation is valid when the fluctuation $\langle x\rangle$ of the oscillator is sufficiently small compared to the radius, $\langle x\rangle \ll R$. In terms of the excitation number n in (25) this condition reads $n \ll J$. Therefore, what we should check is whether the leading contribution to the series comes from the terms satisfying $n \ll J$. We shall work in the regime $g_\mathrm{YM}^2 N/J^2 \ll 1$, since we wish to compare the results to perturbative calculations in gauge theory. Since $N_{00}^{\mathrm{II}\mathrm{II}}$ is then of the order $\sqrt{g_\mathrm{YM}^2 N/J^2}$ [8] [9], we have $\Delta_\mathrm{II} N_{00}^{\mathrm{II}\mathrm{II}} \sim \sqrt{g_\mathrm{YM}^2 N} \gg 1$. Now, the leading contribution will come from terms for which the factors $(\Delta_\mathrm{II} N_{00}^{\mathrm{II}\mathrm{II}})^{n/2}$ and $\frac{n}{2}!$ are comparable, that is, terms with $n \sim \sqrt{g_\mathrm{YM}^2 N}$. Although $\sqrt{g_\mathrm{YM}^2 N}$ is large, it is much smaller than J in our regime. Thus the use of pp-wave approximation is correct.

The summation in (25) presents interesting features. Firstly, $n/2 = 0$ in the first term seems, at first sight, to make the only leading contribution because of the small denominator $\Delta_\mathrm{I} + \Delta_\mathrm{II} - \Delta_\mathrm{III} \sim \mathrm{O}(g_\mathrm{YM}^2 N/J^2)$. However the numerator E is of the same order by non-trivial cancellation, hence the contribution from the second term cannot be neglected. (Other contributions from the first term are sub-leading.) Then the series adds up to an exponential function minus the missing $\frac{n}{2} = 0$ term, $F e^{\Delta_\mathrm{II} N_{00}^{\mathrm{II}\mathrm{II}}} - F$. Now the exponent is a large quantity ($\sim \sqrt{g_\mathrm{YM}^2 N}$) with negative sign.[9] Therefore the first term in this expression is extremely small and should be neglected in our approximation. Thus we have

$$C \equiv \frac{E}{\Delta_\mathrm{I} + \Delta_\mathrm{II} - \Delta_\mathrm{III}} - F. \tag{26}$$

7. Comparison to gauge theory

For the vertex in [7], we obtain

$$E = \frac{R^4}{J_\mathrm{III}^2 \alpha'^2} \frac{(\sin m_\mathrm{III} \pi y)^2}{\pi^2 y} \left(-2\delta^{(\alpha_\mathrm{I}\alpha_\mathrm{III}}\delta^{\beta_\mathrm{I})\beta_\mathrm{III}} + \frac{1}{2}\delta^{\alpha_\mathrm{I}\beta_\mathrm{I}}\delta^{\alpha_\mathrm{III}\beta_\mathrm{III}}\right),$$

[6] There are six possibilities regarding the radial order of the insertion points of the operators. Discussions below work the same if we put $(\mathcal{O}_\mathrm{I}, \mathcal{O}_\mathrm{II}, \bar{\mathcal{O}}_\mathrm{III})$ on (x_3, x_2, x_1), (x_2, x_1, x_3), (x_3, x_1, x_2). However, if we put them on (x_1, x_3, x_2), (x_2, x_3, x_1) both terms in (21) involve particles with negative J. In order to treat these cases it would be necessary to carefully work out the transformation law between matrix elements in light cone frame and those in ordinary temporal frame.

[7] Matrix elements L are defined via the Lagrangian (in (18)) while matrix elements in pp-wave SFT are those of the Hamiltonian. This difference causes no trouble, since the Legendre transformation between them involves only an overall factor.

[8] This fact signifies the strong non-locality of the interaction. For local interactions, we would have Neumann coefficients (for zero modes) of order 1.

[9] We have $N_{00}^{\mathrm{II}\mathrm{II}} < 0$, which differs from the literature since our convention (12) of $a^\dagger$ includes a factor of i.

$$F = \frac{(\sin m_{\mathrm{III}}\pi y)^2}{\pi^2 y}\left(\frac{-4m_{\mathrm{III}}(m_{\mathrm{I}}/y)}{(m_{\mathrm{III}}^2-(m_{\mathrm{I}}/y)^2)^2}\delta^{[\alpha_{\mathrm{I}}\alpha_{\mathrm{III}}}\delta^{\beta_{\mathrm{I}}]\beta_{\mathrm{III}}} - 2\frac{m_{\mathrm{III}}^2+(m_{\mathrm{I}}/y)^2}{(m_{\mathrm{III}}^2-(m_{\mathrm{I}}/y)^2)^2}\delta^{(\alpha_{\mathrm{I}}\alpha_{\mathrm{III}}}\delta^{\beta_{\mathrm{I}})\beta_{\mathrm{III}}} \right.$$
$$\left. -\frac{1}{2}\frac{m_{\mathrm{III}}^2+(m_{\mathrm{I}}/y)^2}{(m_{\mathrm{III}}^2-(m_{\mathrm{I}}/y)^2)^2}\delta^{\alpha_{\mathrm{I}}\beta_{\mathrm{I}}}\delta^{\alpha_{\mathrm{III}}\beta_{\mathrm{III}}}\right) \tag{27}$$

with $y = J_{\mathrm{I}}/J_{\mathrm{III}}$. Here we have used the asymptotic form of Neumann coefficients [9]. Anti-symmetric and traceless symmetric pieces are respectively given by

$$\delta^{[\alpha_{\mathrm{I}}\alpha_{\mathrm{III}}}\delta^{\beta_{\mathrm{I}}]\beta_{\mathrm{III}}} = \frac{1}{2}\left(\delta^{\alpha_{\mathrm{I}}\alpha_{\mathrm{III}}}\delta^{\beta_{\mathrm{I}}\beta_{\mathrm{III}}} - \delta^{\beta_{\mathrm{I}}\alpha_{\mathrm{III}}}\delta^{\alpha_{\mathrm{I}}\beta_{\mathrm{III}}}\right),$$
$$\delta^{(\alpha_{\mathrm{I}}\alpha_{\mathrm{III}}}\delta^{\beta_{\mathrm{I}})\beta_{\mathrm{III}}} = \frac{1}{2}\left(\delta^{\alpha_{\mathrm{I}}\alpha_{\mathrm{III}}}\delta^{\beta_{\mathrm{I}}\beta_{\mathrm{III}}} + \delta^{\beta_{\mathrm{I}}\alpha_{\mathrm{III}}}\delta^{\alpha_{\mathrm{I}}\beta_{\mathrm{III}}}\right) - \frac{1}{4}\delta^{\alpha_{\mathrm{I}}\beta_{\mathrm{I}}}\delta^{\alpha_{\mathrm{III}}\beta_{\mathrm{III}}}.$$

Substituting (27) into (26), it is easy to see that the gauge theory results [10,11] are reproduced.

8. Invariance under field redefinitions

Now we clarify the issue of ambiguity in the three string vertex. We consider a unitary transformation $H_{\mathrm{free}} + H'_{\mathrm{int}} + \cdots = (1 + D + \cdots)^{-1}(H_{\mathrm{free}} + H_{\mathrm{int}} + \cdots)(1 + D + \cdots)$, or,

$$H'_{\mathrm{int}} = H_{\mathrm{int}} + [H_{\mathrm{free}}, D]. \tag{28}$$

Here we have expanded in the string coupling constant and retained terms of the first order, H_{int}, H'_{int} and D. The transformed Hamiltonian H'_{int} has by construction the same symmetry, in particular supersymmetry, as the original one H_{int}. This transformation can be considered as a field redefinition, of which D is the generator. In usual field theory it is guaranteed that physical observables do not change under these redefinitions, provided that the transformations are local. The situation is the same here. The observable C is invariant under the transformation (28) for a broad class of D. Indeed, for any D given by the overlap part with a polynomial prefactor, $|D\rangle = P(a^\dagger)e^{\frac{1}{2}a^\dagger N a^\dagger}|0\rangle$, we get from (23)

$$C' = C + \left\langle 0|e^{-\sqrt{2\Delta_{\mathrm{II}}}\frac{x_2^\mu}{|x_2|}a^\mu}|D\right\rangle = C + P\left(-\sqrt{2\Delta_{\mathrm{II}}}\,\frac{x_2^\mu}{|x_2|}\right)e^{\Delta_{\mathrm{II}}N_{00}^{\mathrm{II}\,\mathrm{II}}}. \tag{29}$$

Thus we find that C does not change, in the regime $g_{\mathrm{YM}}^2 N \gg 1$, up to an exponentially negligible term. Thus our method does not suffer from these ambiguity and gives unique results.[10] The form of D given above closely resembles that of the generator of a local field redefinition in ordinary field theory, the overlap part and the polynomial in $a^\dagger$ respectively corresponding to the δ-function part and the polynomial in spacetime derivatives.

The three point vertex given in [8] can be considered as a particular case of this ambiguity (28) with D the overlap part itself. Hence it gives null contribution to C; in terms of (26), the E-term and the F-term cancel each other. Therefore our methods give the same answer for any vertex of the form

$$H_{\mathrm{int}} = H_{\mathrm{int}}^A + \alpha H_{\mathrm{int}}^B, \tag{30}$$

where H_{int}^A and H_{int}^B respectively refer to the vertices of [7] and [8].

Previously, the following relation has been proposed [12]

$$C = \frac{E}{\Delta_{\mathrm{I}} + \Delta_{\mathrm{II}} - \Delta_{\mathrm{III}}}. \tag{31}$$

The right-hand side is identical to the $n = 0$ term in our rule (23). Lacking contributions from $n \geqslant 1$ terms (the F-term in (26)), results of (31) are affected by the ambiguity. Recently, Dobashi and Yoneya [13,14] have taken a different approach (initiated in [5]) from ours and constructed a vertex which reproduces known gauge theory results while using (31).[11] Their vertex is the $\alpha = 1$ case of the linear combination (30).[12] In our framework, their vertex reproduces gauge theory results with (31) since the choice $\alpha = 1$ leads to $F = 0$ (in (26)).

The framework proposed in this Letter should provide us with many further tests of the AdS/CFT correspondence. It will be interesting to study (a) the sub-leading order of our approximation which has rather intricate structures, (b) general BMN operators

[10] This property makes it natural to call these observables on-shell, although $\Delta_1 + \Delta_2 \neq \Delta_3$. This is also natural in the light of the holographic ansatz since they are given by the path-integral with fixed asymptotic behaviour of the fields. We note that for ordinary transition amplitudes in a time-independent system, it is the condition $\Delta_1 + \Delta_2 = \Delta_3$ which guarantees invariance under field redefinitions.

[11] In [13,14], also a refinement to (31) is introduced for impurity non-conserving cases, $\Delta_{\mathrm{I}} + \Delta_{\mathrm{II}} - \Delta_{\mathrm{III}} \sim \mathcal{O}(1)$, in order to reproduce the gauge theory results. It would be interesting to consider whether our approach leads naturally to their prescription.

[12] The bosonic part of this vertex (for scalar impurities) is first discussed in [15]. See also [16].

such as those which have vector impurities, for which one should use different propagators and x_r-dependences (20), (c) the four (or more) point functions.

Acknowledgements

I would like to thank T. Yoneya, S. Dobashi for many discussions and encouragement, T. Sato, Y. Aisaka, K. Hashimoto, T. Muramatsu, T. Shimada, M. Kato, Y. Kazama for discussions and encouragement. I would like to thank T. Eguchi, S. Iso, H. Kawai, S. Sekino, N. Sasakura for useful comments. This work was supported in part by JSPS Research Fellowships for Young Scientists (#16-10606).

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