Abelian gerbes as a gauge theory of quantum mechanics on phase space

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Abstract

We construct a U(1) gerbe with a connection over a finite-dimensional, classical phase space $\mathbb{P}$. The connection is given by a triple of forms A, B, H: a potential 1-form A, a Neveu–Schwarz potential 2-form B, and a field-strength 3-form $H = \mathrm{d}B$. All three of them are defined exclusively in terms of elements already present in $\mathbb{P}$, the only external input being Planck's constant $\hbar$. U(1) gauge transformations acting on the triple A, B, H are also defined, parametrized either by a 0-form or by a 1-form. While H remains gauge invariant in all cases, *quantumness* versus *classicality* appears as a choice of 0-form gauge for the 1-form A. The fact that $[H]/2\pi\mathrm{i}$ is an integral class in de Rham cohomology is related to the discretization of symplectic area on $\mathbb{P}$. This is an equivalent, coordinate-free reexpression of Heisenberg's uncertainty principle. A choice of 1-form gauge for the 2-form B relates our construction to generalized complex structures on classical phase space. Altogether this allows one to interpret the quantum mechanics corresponding to $\mathbb{P}$ as an Abelian gauge theory.

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1. Introduction

Authoritative treatises on quantum mechanics usually place Heisenberg's principle of uncertainty at the very beginning, regarding it as a litmus test that tells the classical world from the quantum world [1]. The inequality $\Delta Q \Delta P \geqslant \hbar/2$ is a consequence of the canonical commutator $[Q, P] = \mathrm{i}\hbar$ on Hilbert space. In turn, this commutator follows from the canonical Poisson brackets $\{q, p\} = 1$ on classical phase space $\mathbb{P}$. Mathematically it is convenient to regard $\mathbb{P}$ as a finite-dimensional symplectic manifold endowed with a symplectic form ω. Then q and p are local Darboux coordinates. It would appear that Heisenberg's principle

could, after all, have a geometrical origin in classical phase space, the only nonclassical input being Planck's constant $\hbar$.

Although the previous reasoning is basically sound, it overlooks the important fact that *Darboux* coordinates q and p are being used *and* that they cannot be replaced with non-Darboux coordinates. This explicit dependence on a particular choice of (an equivalence class of) coordinates is unsatisfactory. One would much rather have a statement that holds valid regardless of the coordinates being used.

The desired coordinate-free reexpression of Heisenberg's priciple can easily be obtained, at least in the WKB approximation. The uncertainty principle implies that, along each canonically conjugate pair (q^j, p_j), symplectic area is quantized in units of $\hbar$. For all *closed* surfaces $\mathbb{S} \subset \mathbb{P}$ we have, in the WKB approximation,

$$\frac{1}{2\pi\hbar} \int_{\mathbb{S}} \omega \in \mathbb{Z}, \qquad \partial\mathbb{S} = 0. \tag{1}$$

This can be equivalently expressed in de Rham cohomology by saying that $[\omega]/2\pi\hbar$ is an integral class, which is a coordinate-free statement.

Lifting the requirement that one work in Darboux coordinates has one added bonus. Namely, one can implement the notion of *duality* as the dependence of the notion of an elementary quantum with respect to the observer. Yes, with respect to the observer, the latter understood as in general relativity: a little man carrying a ruler and a clock. One can dispense with the little man and his instruments, to conclude that an observer is a choice of local coordinates on $\mathbb{P}$. (To further pursue the analogy with the theory of relativity we note, *en passant*, that Darboux coordinates on $\mathbb{P}$ would correspond to inertial observers.) In choosing phase space as the framework for quantum mechanics we make contact with current trends [2–5].

As already mentioned, recent developments suggest the need for developing a relativity principle for the notion of a quantum [6] and the corresponding duality transformations. The latter would arise as maps between two or more, apparently different, limits of a single theory, in which limits, however, respective observers would not necessarily agree on the notion of an elementary quantum. Given a certain mechanical action S, an example of a duality is the exchange of the *semiclassical* regime, where $S/\hbar \gg 1$, with the *strong-quantum* regime, in which $S/\hbar \approx 1$. The previous $\mathbb{Z}_2$-duality may extend to larger duality groups.

Specifically, in this paper we will consider an extension of the duality group $\mathbb{Z}_2$, the latter acting as the exchange of *semiclassical* and *strong-quantum*, to the group U(1). A gauge theory on phase space $\mathbb{P}$ can be associated with this U(1). However it will differ from standard Yang–Mills theory in that there will be a triple A, B, H of forms. Namely, there will be a potential 1-form A, a potential 2-form B (also called *Neveu–Schwarz field*), and a field-strength 3-form $H = dB$. We will see that the mathematical language best suited to describe this gauge theory is that of gerbes with a connection over phase space [7]; generalized complex geometry will also enter into the picture. An interesting feature of the gerbe that we will construct over $\mathbb{P}$ is that the field-strength H will automatically encode Heisenberg's principle. As such, the volume integral of H over closed, 3-dimensional volumes will also provide a coordinate-free reexpression of the uncertainty principle. This latter reexpression will imply that of equation (1).

The purpose of constructing a gerbe over $\mathbb{P}$ goes well beyond that of reexpressing the uncertainty principle in a coordinate-free form. More importantly, we will make contact with the interesting, yet little-known formulation of quantum mechanics on phase space [2–5]. Our conclusions can be summarized by saying that we construct a U(1) gauge theory of quantum mechanics on phase space. Last but not least, in implementing the notion of duality, this

construction provides the first steps along an interesting alternative approach to a quantum theory of gravity.

2. Quantum mechanics as a U(1) gauge theory

2.1. U(1) gerbes with a connection

A comprehensive treatment of gerbes can be found in [7]; a nice review is [8]. A *U(1) gerbe* on the base manifold $\mathbb{B}$ is defined as a 2-cocycle $g \in H^2(\mathbb{B}, C^\infty(U(1)))$. The latter is the second Čech cohomology group of $\mathbb{B}$ with coefficients in the sheaf of germs of smooth, U(1)-valued functions [9]. Let $\{U_\alpha\}$ be a good cover of $\mathbb{B}$ by open sets U_α. This means that we have a collection $\{g_{\alpha_1\alpha_2\alpha_3}\}$ of maps defined on each 3-fold overlap on $\mathbb{B}$

$$g_{\alpha_1\alpha_2\alpha_3} : U_{\alpha_1} \cap U_{\alpha_2} \cap U_{\alpha_3} \longrightarrow U(1) \tag{2}$$

satisfying

$$g_{\alpha_1\alpha_2\alpha_3} = g_{\alpha_2\alpha_1\alpha_3}^{-1} = g_{\alpha_1\alpha_3\alpha_2}^{-1} = g_{\alpha_3\alpha_2\alpha_1}^{-1}, \tag{3}$$

as well as the 2-cocycle condition

$$g_{\alpha_2\alpha_3\alpha_4} g_{\alpha_1\alpha_3\alpha_4}^{-1} g_{\alpha_1\alpha_2\alpha_4} g_{\alpha_1\alpha_2\alpha_3}^{-1} = 1 \qquad \text{on} \quad U_{\alpha_1} \cap U_{\alpha_2} \cap U_{\alpha_3} \cap U_{\alpha_4}. \tag{4}$$

Now g is a 2-coboundary in Čech cohomology whenever it holds that

$$g_{\alpha_1\alpha_2\alpha_3} = \tau_{\alpha_1\alpha_2} \tau_{\alpha_2\alpha_3} \tau_{\alpha_3\alpha_1} \tag{5}$$

for a certain collection $\{\tau_{\alpha_1\alpha_2}\}$ of U(1)-valued functions $\tau_{\alpha_1\alpha_2}$ on $U_{\alpha_1} \cap U_{\alpha_2}$ such that $\tau_{\alpha_2\alpha_1} = \tau_{\alpha_1\alpha_2}^{-1}$. The collection $\{\tau_{\alpha_1\alpha_2}\}$ is called a *trivialization* of the gerbe. One can prove that over any given open set U_α of the cover $\{U_\alpha\}$ there always exists a trivialization of the gerbe. Moreover, any two trivializations $\{\tau_{\alpha_1\alpha_2}\}$, $\{\tau'_{\alpha_1\alpha_2}\}$ differ by a unitary line bundle. This is so because the quotient $\tau'_{\alpha_1\alpha_2}/\tau_{\alpha_1\alpha_2}$ satisfies the 1-cocycle condition required of line bundles. A gerbe, however, does not qualify as a manifold, since the difference between two trivializations is not a transition function, but a line bundle. To compare with fibre bundles, the total space of a bundle is always a manifold, any two local trivializations differing by a transition function.

One can define a connection on a gerbe in a way that parallels the definition of a connection on a unitary line bundle. On a gerbe specified by the 2-cocycle $g_{\alpha_1\alpha_2\alpha_3}$, a connection is specified by a 1-form A, a 2-form B and a 3-form H satisfying

$$\begin{aligned}
H|_{U_\alpha} &= \mathrm{d}B_\alpha \\
B_{\alpha_2} - B_{\alpha_1} &= \mathrm{d}A_{\alpha_1\alpha_2} \\
A_{\alpha_1\alpha_2} + A_{\alpha_2\alpha_3} + A_{\alpha_3\alpha_1} &= g_{\alpha_1\alpha_2\alpha_3}^{-1} \, \mathrm{d}g_{\alpha_1\alpha_2\alpha_3}.
\end{aligned} \tag{6}$$

H is the curvature of the gerbe connection. The latter is called *flat* if $H = 0$.

2.2. A U(1) gerbe on phase space

In this section, we summarize the results of [10] concerning the construction of an Abelian gerbe with a connection on a $2d$-dimensional phase space $\mathbb{P}$. The latter may be the cotangent bundle to a certain configuration space $\mathbb{M}$, on which a mechanical action

$$S := \int_{\mathbb{I}} \mathrm{d}t \, L \tag{7}$$

is given as the integral of the Lagrangian L over a certain time interval $\mathbb{I} \subset \mathbb{R}$. On the open set $U_\alpha \subset \mathbb{P}$ we can pick Darboux coordinates $q^j_{(\alpha)}$, $p^{(\alpha)}_j$ such that the restriction $\omega|_{U_\alpha}$ reads

$$\omega|_{U_\alpha} = \sum_{j=1}^d dq^j_{(\alpha)} \wedge dp^{(\alpha)}_j, \tag{8}$$

or, dropping the index α,

$$\omega = \sum_{j=1}^d dq^j \wedge dp_j. \tag{9}$$

The canonical 1-form θ on $\mathbb{P}$ defined as [11]

$$\theta := -\sum_{j=1}^d p_j \, dq^j \tag{10}$$

satisfies

$$d\theta = \omega. \tag{11}$$

We will also need the integral invariant of Poincaré–Cartan, denoted as λ. If $\mathcal{H}$ denotes the Hamiltonian, λ is defined as [11]

$$\lambda := \theta + \mathcal{H} \, dt. \tag{12}$$

Then the action (7) equals (minus) the line integral of λ,

$$S = -\int_{\mathbb{I}} \lambda. \tag{13}$$

On constant-energy submanifolds of $\mathbb{P}$, or else for fixed values of the time, we have

$$d\lambda = \omega, \qquad \mathcal{H} = \text{const}. \tag{14}$$

In what follows it will be convenient to drop the index j while maintaining the index α of Čech cohomology. Let any three points $(q_{\alpha_1}, p_{\alpha_1})$, $(q_{\alpha_2}, p_{\alpha_2})$, $(q_{\alpha_3}, p_{\alpha_3})$ be given on $\mathbb{P}$, respectively covered by coordinate charts U_{α_1}, U_{α_2} and U_{α_3}. Assume that $U_{\alpha_1} \cap U_{\alpha_2} \cap U_{\alpha_3}$ is nonempty, i.e.,

$$U_{\alpha_1 \alpha_2 \alpha_3} := U_{\alpha_1} \cap U_{\alpha_2} \cap U_{\alpha_3} \neq \phi, \tag{15}$$

and let $(q_{\alpha_{123}}, p_{\alpha_{123}})$ be a variable point in this triple overlap,

$$(q_{\alpha_{123}}, p_{\alpha_{123}}) \in U_{\alpha_1 \alpha_2 \alpha_3}. \tag{16}$$

Furthermore let $\mathbb{L}_{\alpha_1 \alpha_2 \alpha_3}(\alpha_{123})$ be a closed loop within $\mathbb{P}$ as indicated in figure 1,

$$\mathbb{L}_{\alpha_1 \alpha_2 \alpha_3}(\alpha_{123}) := \mathbb{L}_{\alpha_1 \alpha_2}(\alpha_{123}) + \mathbb{L}_{\alpha_2 \alpha_3}(\alpha_{123}) + \mathbb{L}_{\alpha_3 \alpha_1}(\alpha_{123}), \tag{17}$$

where have explicitly indicated the dependence of the trajectory on the variable midpoint $(q_{\alpha_{123}}, p_{\alpha_{123}}) \in U_{\alpha_1 \alpha_2 \alpha_3}$. Altogether, the latter is traversed three times: once along the leg $\mathbb{L}_{\alpha_1 \alpha_2}$ from α_1 to α_2, once more along the leg $\mathbb{L}_{\alpha_2 \alpha_3}$ from α_2 to α_3, and finally along the leg $\mathbb{L}_{\alpha_3 \alpha_1}$ from α_3 to α_1. For ease of writing, however, we will drop α_{123} from our notation. The 2-cocycle defining a U(1) gerbe on $\mathbb{P}$ is given by the following ratio of functional integrals [10]:

$$g_{\alpha_1 \alpha_2 \alpha_3} := \frac{\tilde{g}_{\alpha_1 \alpha_2 \alpha_3}}{|\tilde{g}_{\alpha_1 \alpha_2 \alpha_3}|}, \tag{18}$$

where

$$\tilde{g}_{\alpha_1 \alpha_2 \alpha_3} \sim \int D\mathbb{L}_{\alpha_1 \alpha_2 \alpha_3} \exp\left(-\frac{i}{\hbar} \int_{\mathbb{L}_{\alpha_1 \alpha_2 \alpha_3}} \lambda\right). \tag{19}$$
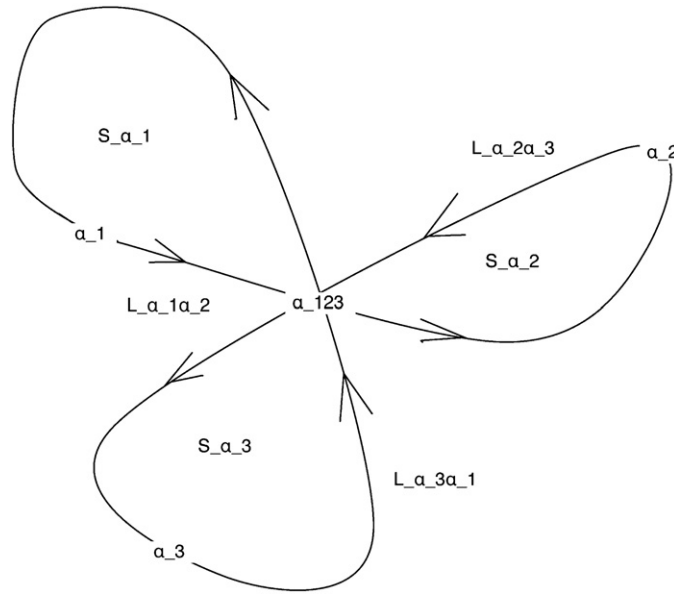
Figure 1. The closed loop $\mathbb{L}_{\alpha_1\alpha_2\alpha_3}(\alpha_{123})$ of equation (17).

The right-hand side of the above is independent of any choice of points $\alpha_1, \alpha_2, \alpha_3$, since a path integral is being taken over all possible loops as explained. The sign $\sim$ stands for *proportionality*. Indeed functional integrals are defined up to some (usually divergent) normalization factor. However all such normalizations will cancel in the ratios of functional integrals we are interested in, such as (18). The functional integral (19) extends over all the closed trajectories of the type specified in equation (17). Let us consider the sum of the surfaces (see figure 1)

$$\mathbb{S}_{\alpha_1\alpha_2\alpha_3} := \mathbb{S}_{\alpha_1\alpha_2} + \mathbb{S}_{\alpha_2\alpha_3} + \mathbb{S}_{\alpha_3\alpha_1}, \tag{20}$$

where, again to simplify the notation, the explicit dependence on the variable midpoint α_{123} has been dropped. The closed trajectory (17) bounds the surface (20), i.e., $\mathbb{L}_{\alpha_1\alpha_2\alpha_3} = \partial \mathbb{S}_{\alpha_1\alpha_2\alpha_3}$. Picking $\mathbb{S}_{\alpha_1\alpha_2\alpha_3}$ to be a constant-energy surface within $\mathbb{P}$, or else for fixed values of the time, we have by equation (14) and Stokes' theorem

$$\tilde{g}_{\alpha_1\alpha_2\alpha_3} \sim \int D\mathbb{S}_{\alpha_1\alpha_2\alpha_3} \exp\left(-\frac{i}{\hbar}\int_{\mathbb{S}_{\alpha_1\alpha_2\alpha_3}}\omega\right), \tag{21}$$

the functional integral extending over all surfaces (20).

One can compute the functional integral (19) and its U(1)-phase (18) in the stationary-phase approximation (for $\hbar \to 0$) [12]. Then the 2-cocycle $g^{(0)}_{\alpha_1\alpha_2\alpha_3}$ defining a U(1) gerbe on $\mathbb{P}$ turns out to be [10]

$$g^{(0)}_{\alpha_1\alpha_2\alpha_3} = \exp\left(-\frac{i}{\hbar}\int_{\mathbb{L}^{(0)}_{\alpha_1\alpha_2\alpha_3}}\lambda\right), \tag{22}$$

the superindex $^{(0)}$ standing for *evaluation at the extremal*, that is, at that closed loop $\mathbb{L}^{(0)}_{\alpha_1\alpha_2\alpha_3}$ of the type (17) that renders the integral of λ extremal. Equivalently, we can express $g^{(0)}_{\alpha_1\alpha_2\alpha_3}$ in terms of an integral over an extremal surface, as in equation (21):

$$g^{(0)}_{\alpha_1\alpha_2\alpha_3} = \exp\left(-\frac{\mathrm{i}}{\hbar}\int_{\mathbb{S}^{(0)}_{\alpha_1\alpha_2\alpha_3}}\omega\right). \tag{23}$$

Equation (22) and its equivalent (23) give the stationary-phase approximation $g^{(0)}_{\alpha_1\alpha_2\alpha_3}$ to the 2-cocycle $g_{\alpha_1\alpha_2\alpha_3}$. The latter is a function of the variable midpoint (16) through the extremal integration path $\mathbb{L}^{(0)}_{\alpha_1\alpha_2\alpha_3}$ or its equivalent extremal integration surface $\mathbb{S}^{(0)}_{\alpha_1\alpha_2\alpha_3}$, even if we no longer indicate this explicitly. Henceforth we will also drop the superindex $^{(0)}$, with the understanding that we are always working in the stationary-phase approximation. The stationary-phase method is equivalent to the quantum-mechanical WKB approximation. Its role is that of minimizing the symplectic area of the surface $\mathbb{S}_{\alpha_1\alpha_2\alpha_3}$. Now, in the WKB method, the absolute value of $\int_{\mathbb{S}}\omega/\hbar$ is proportional to the number of quantum-mechanical states contributed by the surface $\mathbb{S}$ [12]. Hence the stationary-phase approximation applied here picks out those surfaces that contribute the least number of quantum-mechanical states. Moreover, since we are considering constant-energy surfaces $\mathbb{S}$, those states are stationary.

Using equations (6) and (22) one finds for the 1-form A

$$A = -\frac{\mathrm{i}}{\hbar}\lambda. \tag{24}$$

For the 2-form B one finds, on constant-energy submanifolds of phase space,

$$B_{\alpha_2} - B_{\alpha_1} = -\frac{\mathrm{i}}{\hbar}\omega_{\alpha_1\alpha_2}. \tag{25}$$

The above equation is interpreted as follows. Given the coordinate patches U_{α_1} and U_{α_2} such that $U_{\alpha_1}\cap U_{\alpha_2}$ is nonempty, let $\omega_{\alpha_1\alpha_2}$ denote the restriction of ω to $U_{\alpha_1}\cap U_{\alpha_2}$. Then a knowledge of B on the patch U_{α_1} gives us the value of B on the patch U_{α_2}. Finally we have the 3-form

$$H = \mathrm{d}B. \tag{26}$$

2.3. The uncertainty principle from the gerbe field-strength

In the WKB approximation it is well known that the symplectic area of any open surface $\mathbb{S}_{\alpha_1\alpha_2\alpha_3}$ is quantized according to the rule [12]

$$\frac{1}{\hbar}\int_{\mathbb{S}_{\alpha_1\alpha_2\alpha_3}}\omega = 2\pi\left(n_{\alpha_1\alpha_2\alpha_3} + \frac{1}{2}\right), \qquad n_{\alpha_1\alpha_2\alpha_3}\in\mathbb{Z}. \tag{27}$$

Consider now two *open*, constant-energy, symplectically minimal surfaces $\mathbb{S}^{(1)}\subset\mathbb{P}$ and $\mathbb{S}^{(2)}\subset\mathbb{P}$ such that $\partial\mathbb{S}^{(1)} = -\partial\mathbb{S}^{(2)}$. Join them along their common boundary to form the *closed* surface $\mathbb{S} := \mathbb{S}^{(1)} - \mathbb{S}^{(2)}$. The latter bounds a 3-dimensional volume $\mathbb{V}$. We have in [10] analysed the conditions under which equation (27) can be recast as the quantization condition

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{V}}H \in \mathbb{Z}, \qquad \partial\mathbb{V} = \mathbb{S}. \tag{28}$$

Equation (28) is an equivalent, coordinate-free rendering of Heisenberg's uncertainty principle, one that makes no use of Darboux coordinates on $\mathbb{P}$. Yet, equation (28) is not an equation in de Rham cohomology because the volumes $\mathbb{V}$ integrated over have a boundary. Rather, we would like an equation such as

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{V}}H \in \mathbb{Z}, \qquad \partial\mathbb{V} = 0 \tag{29}$$

to hold for any 3-dimensional $\mathbb{V}$ *without* boundary. Now starting from (28) it does not follow that (29) holds true in general. However, (29) does follow from (28) in one particular case: that of all closed, 3-dimensional $\mathbb{V}$ that can be obtained by gluing two $\mathbb{V}^{(1)}$ and $\mathbb{V}^{(2)}$ along a common boundary $\partial\mathbb{V}^{(1)} = \mathbb{S} = -\partial\mathbb{V}^{(2)}$.

Now in the theory of gerbes [7] one proves that a U(1) gerbe over a compact manifold is characterized by an integral de Rham cohomology class $[H]/2\pi\mathrm{i}$. To the extent that phase space $\mathbb{P}$ is the cotangent bundle to configuration space $\mathbb{M}$, hence noncompact even when $\mathbb{M}$ is compact, we cannot identify the gerbe constructed in section 2.2 by its characteristic cohomology class $[H]/2\pi\mathrm{i}$. Cohomology with compact support within $\mathbb{P}$ is the closest one can get. Alternatively, in the particular case mentioned above, one can regard $[H]/2\pi\mathrm{i}$ as being a consequence of Heisenberg's uncertainty principle.

2.4. A local U(1) invariance on classical phase space

By equation (13) we can perform the transformation

$$\lambda \longrightarrow \lambda + \mathrm{d}f, \qquad f \in C^{\infty}(\mathbb{P}), \tag{30}$$

where f is an arbitrary function on $\mathbb{P}$ with the dimensions of an action, without altering the classical mechanics defined by ω. Since the classical action S is given by the line integral (13), the transformation (30) amounts to shifting S by a constant C,

$$S \longrightarrow S + C, \qquad C := -\int_{\mathbb{I}} \mathrm{d}f. \tag{31}$$

The way the transformation (30) acts on the quantum theory is well known. In the WKB approximation, the wavefunction reads

$$\psi_{\mathrm{WKB}} = R \exp\left(\frac{\mathrm{i}}{\hbar} S\right) \tag{32}$$

for some amplitude R. Thus the transformation (30) multiplies the WKB wavefunction ψ_{WKB} and, more generally, any wavefunction ψ, by the *constant* phase factor $\exp(\mathrm{i}C/\hbar)$:

$$\psi \longrightarrow \exp\left(\frac{\mathrm{i}}{\hbar} C\right) \psi. \tag{33}$$

Gauging the rigid symmetry (33) one obtains the transformation law

$$\psi \longrightarrow \Psi_f := \exp\left(-\frac{\mathrm{i}}{\hbar} f\right) \psi, \qquad f \in C^{\infty}(\mathbb{P}), \tag{34}$$

f being an arbitrary function on phase space, with the dimensions of an action. Now equation (34) implies that, if the original wavefunction ψ depends only on the coordinates q, its transform Ψ_f under an arbitrary $f \in C^{\infty}(\mathbb{P})$ generally depends also on the momenta p. According to standard lore this is prohibited by Heisenberg's uncertainty principle. Moreover, even if wavefunctions can be defined on phase space, the local transformations (34) need not be a symmetry of our theory. We address these two points separately in subsections 2.5 and 2.8.

2.5. Probability distributions on phase space

Concerning the first objection raised above one should observe that phase-space quantum mechanics, while respecting the constraints imposed by Heisenberg's principle, is almost as old as quantum mechanics itself. We refer the reader to [2–5] for details and further references. We will henceforth call the objects $\Psi_f = \Psi_f(q, p)$ introduced in (34) *probability distributions*; they are defined on $\mathbb{P}$. For simplicity, in what follows we will omit the subscript $_f$ from Ψ_f.

Specifically, in [3, 4] it has been shown that the usual Schrödinger equation for the usual wavefunction $\psi = \psi(q)$ on $\mathbb{M}$,

$$\mathcal{H}(q, -\mathrm{i}\hbar\partial_q)\psi(q) = E\psi(q), \tag{35}$$

implies the following Schrödinger-like equation for the probability distribution $\Psi = \Psi(q, p)$ on $\mathbb{P}$:

$$\mathcal{H}\left(\frac{q}{2} + \mathrm{i}\hbar\partial_p, \frac{p}{2} - \mathrm{i}\hbar\partial_q\right)\Psi(q, p) = E\Psi(q, p). \tag{36}$$

Modulo an irrelevant canonical transformation, the Hamiltonian in (36) also coincides with that in the first entry of [5]. Moreover, the quantum operators

$$Q_{A_0'} := \frac{q}{2} + \mathrm{i}\hbar\partial_p, \qquad P_{A_0'} := \frac{p}{2} - \mathrm{i}\hbar\partial_q \tag{37}$$

satisfy the usual Heisenberg algebra

$$[Q_{A_0'}, P_{A_0'}] = \mathrm{i}\hbar, \tag{38}$$

so equation (36) can be rewritten as

$$\mathcal{H}(Q_{A_0'}, P_{A_0'})\Psi(q, p) = E\Psi(q, p). \tag{39}$$

A computation shows that $\Psi(q, p)$ in (36) and $\psi(q)$ in (35) are related as per equation (34), where the argument $f(q, p)$ of this latter exponential now equals

$$f_{A_0'}(q, p) := \tfrac{1}{2}pq = \tfrac{1}{2}p_j q^j. \tag{40}$$

In other words, the Schrödinger equation (36) follows from (35) if and only if

$$\Psi(q, p) = \exp\left(-\frac{\mathrm{i}}{2\hbar}pq\right)\psi(q). \tag{41}$$

A straightforward computation shows that (41) corresponds to the choice

$$\phi = (2\pi\hbar)^{d/2}\delta(q) \tag{42}$$

in equation (17) of [4].

That $|\Psi|^2$ is a joint probability distribution in the limit $\hbar \to 0$ follows from paragraph 6 of [4]: if $\Psi = U_\phi\psi$ for

$$\phi(q) = \frac{1}{(\pi\hbar)^{d/4}}\exp\left(-\frac{1}{2\hbar}|q|^2\right), \tag{43}$$

then we have

$$\lim_{\hbar\to 0}\int |\Psi(q, p)|^2 \, \mathrm{d}p = |\psi(q)|^2 \tag{44}$$

$$\lim_{\hbar\to 0}\int |\Psi(q, p)|^2 \, \mathrm{d}q = |\hat{\psi}(p)|^2, \tag{45}$$

where $\hat{\psi}(p)$ denotes the Fourier transform of $\psi(q)$.

The reason for the subindex A_0' in (37)–(40) is the following. Consider the *symplectic exterior derivative* on phase space,

$$\mathrm{d}' := -\mathrm{d}q\,\partial_q + \mathrm{d}p\,\partial_p. \tag{46}$$

Consider also the following connection A_0' on phase space:

$$A_0' := -\frac{\mathrm{i}}{\hbar}\mathrm{d}f_{A_0'} = \frac{1}{2\mathrm{i}\hbar}(p\,\mathrm{d}q + q\,\mathrm{d}p). \tag{47}$$

Let us now covariantize d' as

$$d' \longrightarrow D'_{A'_0} := d' + A'_0. \tag{48}$$

We see that the operators of equation (37) are the result of gauging the symplectic derivative d' by the connection A'_0:

$$i\hbar D'_{A'_0} = dq \left(\frac{p}{2} - i\hbar \partial_q \right) + dp \left(\frac{q}{2} + i\hbar \partial_p \right). \tag{49}$$

Covariantizing the symplectic derivative as per equation (49) is equivalent to the symplectic transformation considered in [3, 4] that renders the quantum theory manifestly symmetric under the symplectic exchange of q and p. This latter symmetry is conspicuously absent in the usual formulation of quantum mechanics based on equation (35).

One can consider more general covariantizations of the symplectic derivative (46). Given a solution $\psi = \psi(q)$ of the usual Schrödinger equation (35), and given a function $f_{A'} \in C^\infty(\mathbb{P})$, define $\Psi = \Psi(q, p)$ as in equation (34). We can require the latter to satisfy a phase-space Schrödinger equation, that we can determine as follows. One picks a certain connection

$$A' = \frac{1}{i\hbar}[A'_q(q, p) \, dq + A'_p(q, p) \, dp] \tag{50}$$

that one takes to covariantize the symplectic derivative d' of (46),

$$D'_{A'} := d' + A'. \tag{51}$$

The components $A'_q = A'_q(q, p)$ and $A'_p = A'_p(q, p)$ are unknown functions of q, p. However they are not totally unconstrained, because the position and momentum operators

$$Q_{A'} := A'_p + i\hbar \partial_p, \qquad P_{A'} := A'_q - i\hbar \partial_q \tag{52}$$

will enter the Hamiltonian $\mathcal{H}(Q_{A'}, P_{A'})$ obtained from $\mathcal{H}(Q = q, P = -i\hbar \partial_q)$ by the replacements $Q \to Q_{A'}, P \to P_{A'}$:

$$\mathcal{H}(Q_{A'}, P_{A'}) = \frac{1}{2m} P_{A'}^2 + V(Q_{A'}) = \frac{1}{2m}(A'_q - i\hbar \partial_q)^2 + V(A'_p + i\hbar \partial_p). \tag{53}$$

As such, the operators (52) must satisfy the canonical commutation relations (38). This requires that the following *integrability condition* hold:

$$\frac{\partial A'_p}{\partial q} + \frac{\partial A'_q}{\partial p} = 1. \tag{54}$$

Note the positive sign, instead of negative, between the two summands on the left-hand side of (54). This is ultimately due to the fact that we are covariantizing the symplectic derivative d' rather than the usual exterior derivative $d = dq \partial_q + dp \partial_p$. A computation shows that the phase-space Schrödinger equation

$$\mathcal{H}(Q_{A'}, P_{A'})\Psi(q, p) = E\Psi(q, p) \tag{55}$$

is equivalent to the usual Schrödinger equation (35) if, and only if, A'_q, A'_p and $f_{A'}$ are related as

$$A'_q = \partial_q f_{A'}, \qquad A'_p = q - \partial_p f_{A'}. \tag{56}$$

When equation (56) holds, the integrability condition (54) is automatically satisfied. We conclude that picking one $f_{A'} \in C^\infty(\mathbb{P})$ and defining the connection A' as per equations (50) and (56), we arrive at the phase-space wave equation (55). Alternatively, given a connection (50) and a phase-space wave equation (55), we can find a function $f_{A'} \in C^\infty(\mathbb{P})$, defined by (56) up to integration constants, such that the corresponding probability distribution $\Psi(q, p)$ is related to the wavefunction $\psi(q)$ as per equation (34), where $f = f_{A'}$.

Equation (56) above gives us a whole $C^\infty(\mathbb{P})$'s worth of phase-space Schrödinger equations, one per each choice of a function $f_{A'}$. The latter may well be termed the *generating function* for the transformation (34) between configuration-space and phase-space probability distributions and their corresponding Schrödinger equations.

Given a connection A' as per equations (50) and (56), how is A' related to the potential 1-form A on the gerbe, equation (24)? The answer to this question will be given in subsection 2.8; it necessitates the notion of gauge transformations on the gerbe, which we introduce in subsections 2.6 and 2.7.

2.6. Gauge transformations by 0-forms

Given an arbitrary function $f \in C^\infty(\mathbb{P})$, the triple of forms A, B, H on the gerbe transform under the local U(1) group of equation (34) as

$$\delta_0 A := -\frac{\mathrm{i}}{\hbar}\,\mathrm{d}f, \qquad \delta_0 B = 0, \qquad \delta_0 H = 0, \qquad f \in C^\infty(\mathbb{P}). \tag{57}$$

The gauge transformations equation (57) are formally identical to the U(1) gauge transformations of electromagnetism. There are, however, three key differences:

(i) the Noether charge of electromagnetism may, but need not, be present here. Should electric charges e exist, one could introduce an *electromagnetic* potential A_e and its corresponding field-strength $F_e := \mathrm{d}A_e$ on $\mathbb{M} \times \mathbb{I}$. This however would be an additional U(1) symmetry, implemented by a fibre bundle instead of a gerbe;

(ii) the covariant derivative of electromagnetism is $\mathrm{d} + eA_e$, while that considered here is $\mathrm{d}' + A'$;

(iii) the 2-form $\mathrm{d}A$ on phase space is, by equation (6), not a field strength but the defining equation of the Neveu–Schwarz 2-form potential B.

Altogether, we conclude that A is not an electromagnetic potential, nor is the corresponding U(1) that of electromagnetic gauge invariance.

2.7. Gauge transformations by 1-forms

The gauge transformations (57) by no means exhaust all possibilities for U(1) transforming the connection on the gerbe. On phase space let us consider an arbitrary 1-form $\varphi \in \Omega^1(\mathbb{P})$ with the dimensions of an action. We define a second set of U(1) gauge transformations:

$$\delta_1 A := -\frac{\mathrm{i}}{\hbar}\varphi, \qquad \delta_1 B = -\frac{\mathrm{i}}{\hbar}\,\mathrm{d}\varphi, \qquad \delta_1 H = 0, \qquad \varphi \in \Omega^1(\mathbb{P}). \tag{58}$$

Since $\delta_1 H = 0$, Heisenberg's uncertainty principle remains invariant under δ_1 transformations. We observe that δ_1 is parametrized by a 1-form φ while δ_0 had a 0-form $f_{A'}$ as its gauge parameter. The δ_1 gauge transformation law of the wavefunction is

$$\psi \longrightarrow \Psi_\varphi := \exp\left(-\frac{\mathrm{i}}{\hbar}\varphi\right)\psi, \qquad \varphi \in \Omega^1(\mathbb{P}). \tag{59}$$

Now the probability distribution Ψ_φ is no longer a function, but a nonhomogeneous differential form on phase space; we will return to this fact in section 4. By equations (11) and (58), the δ_1 gauge transformation law of B when $\varphi = -\theta$ is

$$\delta_1 B = \frac{\mathrm{i}}{\hbar}\omega, \tag{60}$$

which amounts to gauging the Neveu–Schwarz field by the symplectic form,

$$B \longrightarrow B + \frac{\mathrm{i}}{\hbar}\omega. \tag{61}$$

The combination $B + i\omega/\hbar$ is ubiquitous in generalized complex geometry [13]. For a brief introduction to generalized complex manifolds see, e.g., section 3 of [14]. A generalized complex structure is determined by a canonical line subbundle of the complex differential forms. This line bundle is generated by $\exp(B + i\omega/\hbar)$ in the general case, which reduces to $\exp(i\omega/\hbar)$ in the symplectic case. Of course, phase space qualifies as generalized complex already from the start, because phase space is symplectic. However one need not resort to this trivial fact in order to establish a link with generalized complex manifolds. Our gerbe on $\mathbb{P}$ makes the gauging (61) possible, thus providing the necessary link. The combination $B + i\omega/\hbar$ is a doublet whose real part is gauge dependent but whose imaginary part is gauge-invariant.

2.8. U(1) gauge invariance and symplectic covariance

We can now answer the question posed at the end of subsection 2.5, namely: given a connection A' as per equations (50) and (56), can one δ_0- and/or δ_1-transform the potential 1-form A on the gerbe so that $A' = A + \delta A$?

Consider δ_1-transformations first. We are looking for a 1-form $\varphi = \varphi_q \, dq + \varphi_p \, dp$ such that $A + \delta_1 A = A + \varphi/(i\hbar)$ will equal the given A' of equations (50) and (56). One immediately verifies that

$$\varphi_q(q, p) := p + \partial_q f_{A'}, \qquad \varphi_p(q, p) := q - \partial_p f_{A'} \tag{62}$$

meets our requirements; hence, any A' is δ_1-gauge equivalent to the potential 1-form A on the gerbe.

However, δ_0-gauge transformations are more restrictive. In this case we have to set $\varphi_q = \partial_q F(q, p)$ and $\varphi_p = \partial_p F(q, p)$ for a certain function $F \in C^\infty(\mathbb{P})$. The latter is to be determined by integration of the system of equations

$$\partial_q F = p + \partial_q f_{A'}, \qquad \partial_p F = q - \partial_p f_{A'}, \tag{63}$$

for a given generating function $f_{A'} \in C^\infty(\mathbb{P})$. A solution to (63) can exist only when

$$\partial_{q^j} \partial_{p_k} f_{A'} = 0, \qquad \forall j, k = 1, \ldots d. \tag{64}$$

The general solution to (64) is the sum of a function of coordinates only and a function of momenta only,

$$f_{A'}(q, p) = g(q) + h(p). \tag{65}$$

So only when the generating function $f_{A'}(q, p)$ of the given connection A' satisfies condition (65) can one find a δ_0-gauge transformation that will render A' gauge equivalent to the potential 1-form A on the gerbe (24).

This brings us back to the second objection raised after equation (34), that we can finally answer in the affirmative. The local transformations (34) *are* a symmetry of our theory, in the sense already explained in subsection 2.5. Namely, the transformation (34) from $\psi(q)$ to $\Psi(q, p)$ must be accompanied by the corresponding covariantization (51) of the symplectic derivative d' within the Schrödinger equation. Since the connection A' and the potential 1-form A on the gerbe are gauge equivalent (this is always the case under δ_1, and also under δ_0 whenever condition (65) holds), this can be understood as a covariantization of the symplectic derivative d' within the Hamiltonian operator, by means of the potential 1-form A on the gerbe. Therefore from now on we can replace equation (51) with the following covariant derivative:

$$D'_A := d' + A, \tag{66}$$

where A is the potential 1-form on the gerbe.

We conclude that gauging the rigid symmetry (33), i.e., allowing for the local transformations (34), one arrives naturally at a phase-space formulation of quantum mechanics.

In other words, U(1) gauge invariance on the gerbe is equivalent to symplectic covariance, the latter understood as in [3, 4]: as the possibility of U(1)-rotating the Schrödinger equation from configuration space into phase space, and also within the latter itself, with a point-dependent rotation parameter.

2.9. Semiclassical versus strong-quantum duality

One might be troubled by the fact that our starting point is the WKB approximation (32) for the wavefunction. In other words, how much do our conclusions depend on ψ having the explicit functional dependence of equation (32)? The answer reads: none of our conclusions depends on the explicit functional dependence of equation (32). This is borne out by the fact that any quantum amplitudes one may be interested in will be given as phase-space functional integrals with respect to Feynman's kernel $\exp(\mathrm{i}S/\hbar)$. Alternatively, the semiclassical versus strong-quantum duality to be introduced next will reassure us that our results also hold beyond the WKB approximation.

By equation (32), the transformation (34) allows one to arbitrarily shift the zero point for the mechanical action S, point by point on phase space. This renders statements like $S/\hbar \gg 1$ observer-dependent on phase space. In particular, the semiclassical regime $S/\hbar \gg 1$ can be mapped into the strong-quantum regime $S/\hbar \approx 1$ (or even $S/\hbar \ll 1$) by means of a gauge transformation, and vice versa. Therefore gauge transformations allow one to implement the relativity in the notion of a quantum mentioned in section 1 and explicitly suggested in [6], section 6. Duality transformations are precisely gauge transformations. Dualities leave the uncertainty principle invariant because $\delta_0 H = 0 = \delta_1 H$.

The Schrödinger equation is the equation of motion corresponding to the field-theory action

$$\mathcal{S}[\Psi] := \int_{\mathbb{P} \times \mathbb{I}} \mathrm{d}q \, \mathrm{d}p \, \mathrm{d}t \, \mathcal{L}, \qquad \mathcal{L} := \mathrm{i}\hbar \Psi^* \frac{\partial \Psi}{\partial t} - \Psi^* \mathcal{H} \Psi. \tag{67}$$

Above, $\Psi = \Psi(q, p, t) = \Psi(q, p) \exp(-\mathrm{i}Et/\hbar)$ is regarded as a field on $\mathbb{P} \times \mathbb{I}$. Our notation stresses the formal difference between the field-theory action $\mathcal{S}[\Psi]$ just defined and the mechanical action S we started off with. However it must be realized that there is no new physics in $\mathcal{S}[\Psi]$ as compared with S. This is best appreciated in equation (67): the field-theory momentum $\partial \mathcal{L}/\partial \dot{\Psi}$ equals $\mathrm{i}\hbar \Psi^*$, so the defining equation for $\mathcal{L}$ in fact mimics the usual relation $L = p\dot{q} - \mathcal{H}$, where now $\Psi^* \mathcal{H} \Psi$ plays the role of a field-theory Hamiltonian. To reiterate: equation (67) is no more than a useful device for reexpressing the *quantum* theory corresponding to the mechanical action S, in the language of the *classical* field theory $\mathcal{S}[\Psi]$.

The action $\mathcal{S}[\Psi]$ is invariant under the global U(1) transformations (33) because C is time independent. In order to render $\mathcal{S}[\Psi]$ invariant under time-independent but (q, p)-dependent U(1) transformations we can profit from the connection on our gerbe. Let us covariantize phase-space derivatives as per (66). Then the gauged field-theory action

$$\mathcal{S}[\Psi; A] := \int_{\mathbb{P} \times \mathbb{I}} \mathrm{d}q \, \mathrm{d}p \, \mathrm{d}t \, \mathcal{L}(A), \quad \mathcal{L}(A) := \mathrm{i}\hbar \Psi^* \frac{\partial \Psi}{\partial t} - \Psi^* \mathcal{H}(A)\Psi \tag{68}$$

is invariant under the local U(1) transformations of equation (34). The notation $\mathcal{H}(A)$ stresses that symplectic derivatives on phase space $\mathbb{P}$ are to be gauged as per equation (66). That is, the Hamiltonian operator will be as in equation (53), where A' is gauge equivalent to A.

The Noether charge associated with δ_0 gauge transformations is the inverse Planck constant $\hbar^{-1}$. Now the electromagnetic 1-form A_e on spacetime is the photon field. Hence the 1-form potential A on the gerbe might be called *the quanton*, because the gauge property it carries is *quantumness* as opposed to *classicality*: the property of being quantum as opposed to

classical. Related analyses concerning the meaning of quantum versus classical were carried out in [15–22], albeit under different, apparently unrelated guises.

Combining the results of subsections 2.8 and 2.9 we conclude that *semiclassical* versus strong-quantum duality is equivalent to the possibility of U(1)-rotating the Schrödinger equation in(to) phase space, with a point-dependent rotation parameter.

3. Examples

Following the referee's advice, in this section we work out some applications of the formalism of previous sections. Our examples are drawn from the realms of quantum gravity and branes. We would also like to thank the referee for drawing our attention to [23], where issues closely related to those treated here are analysed.

3.1. A fundamental length scale

Quantum gravity effects are due to arise at scales of the order of the Planck length L_P. Let ds denote the Lorentz-invariant interval on a spacetime manifold $\mathbb{M}$. It has been argued that the existence of a fundamental length scale L_P implies modifying the spacetime interval according to the rule

$$\mathrm{d}s^2 \longrightarrow \mathrm{d}s^2 + L_P^2, \tag{69}$$

so L_P effectively becomes the shortest possible distance. In [24] it has been proved that modifying the spacetime interval according to (69) is equivalent to requiring invariance of a field theory under the following exchange of short and long distances:

$$\mathrm{d}s \longleftrightarrow \frac{L_P^2}{\mathrm{d}s}. \tag{70}$$

This equivalence has been proved in [24] for the quantum theory of a scalar field; interesting applications of the transformation (70) to other quantum field theories have been worked out in [25]. Not only field theory: also strings, thanks to T-duality [26], are symmetric under the exchange of short and long distances.

We will show how to implement the projective transformation

$$\frac{S}{\hbar} \longleftrightarrow \frac{\hbar}{S} \tag{71}$$

within the Feynman kernel $\exp(\mathrm{i}S/\hbar)$. The precise nature of the action S is immaterial. It can be, e.g., the action for a mechanical system with a finite number of degrees of freedom, or the field theory actions studied in [24, 25], or any other. The key property of the duality (71) is the fact that it maps the semiclassical regime $S \gg \hbar$ into the strong quantum regime $S \ll \hbar$, and vice versa, thus implementing the relativity in the notion of a quantum alluded to above as a duality. We will prove that the dualities (70) and (71) are equivalent: whenever the one holds, so does the other, and vice versa. In order to establish this equivalence we proceed as follows.

The generating function for the Bessel functions $J_n(x)$ of integer order n (see, e.g., [27]) is $\exp(w(v - v^{-1})/2)$:

$$\mathrm{e}^{\frac{w}{2}(v-\frac{1}{v})} = \sum_{n=-\infty}^{\infty} v^n J_n(w), \qquad 0 < |v| < \infty. \tag{72}$$

The choice of variables

$$w := \frac{S}{\hbar}, \qquad v - v^{-1} := 2\mathrm{i}, \tag{73}$$

gives the expansion

$$e^{i\frac{S}{\hbar}} = \sum_{n=-\infty}^{\infty} i^n J_n\left(\frac{S}{\hbar}\right). \tag{74}$$

The above is an infinite sum of terms, each one of which satisfies the Bessel equation of order n,

$$\frac{d^2}{dw^2}J_n(w) + \frac{1}{w}\frac{d}{dw}J_n(w) + \left(1 - \frac{n^2}{w^2}\right)J_n(w) = 0, \qquad n = 0, \pm 1, \pm 2, \ldots. \tag{75}$$

Some insight into the physical meaning of expansion (74) can be gained from the following observation. Consider the time-independent Schrödinger wave equation for a free nonrelativistic particle of mass m on an auxiliary copy of the plane $\mathbb{R}^2$,

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi, \tag{76}$$

which we solve by separation of variables in polar coordinates r, φ. Then the Laplacian operator ∇^2 on this auxiliary $\mathbb{R}^2$ has the radial piece

$$\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + \left(1 - \frac{l^2}{\rho^2}\right), \qquad l = 0, \pm 1, \pm 2, \ldots, \tag{77}$$

where the radial variable r (with dimensions of length) is related to the dimensionless ρ through

$$\rho := \lambda^{-1}r, \qquad \lambda := \frac{\hbar}{\sqrt{2mE}}. \tag{78}$$

The solutions to the radial piece obtained by equating (77) to zero are Bessel functions $J_l(\rho)$ of integer order $l \in \mathbb{Z}$. The eigenfunctions $Y_l(\varphi)$ of the angular-momentum operator corresponding to (76) and (77) are

$$Y_l(\varphi) = e^{il\varphi}, \qquad l = 0, \pm 1, \pm 2, \ldots. \tag{79}$$

The general solution to the Schrödinger equation (76) is a linear combination

$$\sum_{l=-\infty}^{\infty} c_l J_l(\rho) Y_l(\varphi), \tag{80}$$

the $c_l \in \mathbb{C}$ being certain coefficients.

Now, it is well known that given a classical action S on spacetime $\mathbb{M}$, a semiclassical wavefunction ψ for the corresponding quantum-mechanical problem is obtained as $\psi = e^{iS/\hbar}$, where S satisfies the Hamilton–Jacobi equation [1]. For a stationary state we can separate out the time dependence $e^{-iEt/\hbar}$. Consider the auxiliary plane $\mathbb{R}^2$ spanned by the dimensionless polar coordinates ρ, φ. The previous argument shows that the nth term in expansion (74) is a partial-wave contribution to the Feynman wave $e^{iS/\hbar}$, where the dimensionless radial coordinate ρ on the auxiliary $\mathbb{R}^2$ is identified with the action S as measured in units of the quantum $\hbar$, i.e.,

$$\rho = \frac{|S|}{\hbar} = |w|. \tag{81}$$

To complete the picture, we identify the angular variable φ on the auxiliary $\mathbb{R}^2$ as

$$\varphi = 2\pi \text{Re}(e^{-iEt/\hbar}), \tag{82}$$

while the angular momentum $l \in \mathbb{Z}$ in equation (75) is identified with the index $n \in \mathbb{Z}$. To summarize, Feynman's time-dependent exponential of the action becomes

$$e^{\frac{i}{\hbar}(S - Et)} = \sum_{l=-\infty}^{\infty} i^l J_l\left(\frac{S}{\hbar}\right) \exp[2\pi i l \mathrm{Re}(e^{-iEt/\hbar})]. \tag{83}$$

This is an infinite sum over all possible angular momenta $l \in \mathbb{Z}$ of the auxiliary particle on the auxiliary $\mathbb{R}^2$. We stress the fact that the auxiliary particle of mass m is not to be confused with the physical degrees of freedom of the action S under consideration. Nor is the auxiliary plane $\mathbb{R}^2$ to be confused with the spacetime $\mathbb{M}$ where S is defined. However, the introduction of this auxiliary particle on $\mathbb{R}^2$ turns out to be a useful device for our purposes.

Initially one may assume that the polar coordinates ρ, φ of equations (78)–(82) cover all of the auxiliary $\mathbb{R}^2$ and nothing else. However, there is no reason for ρ, φ to be global coordinates. More generally, ρ, φ could be *local* coordinates on a certain auxiliary surface $\mathbb{S}$ other than $\mathbb{R}^2$. For example, imagine that $\mathbb{S}$ is the Riemann sphere $\mathbb{CP}^1$, and let us consider the local holomorphic coordinate on $\mathbb{CP}^1$ given by

$$z := \rho\, e^{i\varphi}. \tag{84}$$

Now the point at infinity on $\mathbb{CP}^1$ is not covered by the coordinate (84). However we may reach this point by introducing the new holomorphic coordinate $\tilde{z}$ on $\mathbb{CP}^1$

$$\tilde{z} := -\frac{1}{z} = \tilde{\rho}\, e^{i\tilde{\varphi}}, \tag{85}$$

where

$$\tilde{\rho} = \frac{\hbar}{S}, \qquad \tilde{\varphi} = -(\varphi + \pi). \tag{86}$$

This leads one to the Feynman-like exponential

$$\exp\left(i\frac{\hbar}{S}\right) \tag{87}$$

as a candidate for describing the strong quantum regime of a theory whose auxiliary surface $\mathbb{S}$ is $\mathbb{CP}^1$. Then the new choice of variables in equation (72)

$$w := \frac{\hbar}{S}, \qquad v - v^{-1} := 2i \tag{88}$$

leads to the expansion

$$e^{i\frac{\hbar}{S}} = \sum_{n=-\infty}^{\infty} i^n J_n\left(\frac{\hbar}{S}\right). \tag{89}$$

The semiclassical regime of (89) is mapped into the strong quantum regime of (74), and vice versa.

For a small trajectory of order Δq, the time-independent piece of the action $S = \int p\, dq$ can be approximated by $p\Delta q$. Under the duality (70), where $\Delta q \to L_P^2/\Delta q$, the action transforms as

$$S = p\Delta q \longrightarrow \frac{pL_P^2}{\Delta q} = \frac{p^2 L_P^2}{S}. \tag{90}$$

Given that $|z| = |S|/\hbar$, equation (90) is the dimensionful equivalent of the projective transformation (85) on $\mathbb{CP}^1$.

To summarize, the statement that the duality (70) holds is equivalent to the statement that one can projectively transform the coordinate z on the auxiliary surface $\mathbb{S}$ as per equation (85). In turn, this latter statement is equivalent to the existence of the duality (71) between the semiclassical and the strong-quantum regimes.

3.2. Born–Infeld branes

Another example where the duality (71) arises is the following. Consider the U(1) Born–Infeld Lagrangian

$$\mathcal{L}_{\mathrm{BI}} = \det(\eta_{\mu\nu} + bF_{\mu\nu})^{1/2}. \tag{91}$$

Above, $\eta_{\mu\nu}$ is the Mikowski metric on spacetime and $F_{\mu\nu}$ is the field strength of a U(1)-valued gauge field A_μ, while b is a constant. For example, when one couples Born–Infeld electrodynamics to a point particle of mass m and electric charge e, we have that $b = e/(ma_{\max})$, where $a_{\max}$ is the maximal acceleration possible [28]. Now, in natural units, the inverse $a_{\max}^{-1}$ of a *maximal* acceleration is a *minimal* length, that we can identify (possibly up to numerical factors) with the *Planck length* L_P on spacetime. Hence the existence of a maximal acceleration is equivalent to the existence of a minimal length scale. On the other hand, in section 3.1 we have proved that the following two statements are equivalent:

 (i) there exists a fundamental length scale L_P on spacetime;
 (ii) one can perform the exchange (71).

It follows that the Born–Infeld Lagrangian (91) exhibits the duality (71).

That the Born–Infeld Lagrangian must exhibit the duality (71) is easy to understand from an alternative standpoint. The (bosonic piece of the) Lagrangian for branes contains the Born–Infeld term. Moreover, branes are solitonic solutions to the supergravity equations of motion [6]. The latter contain Einstein's equations for the gravitational field. Upon quantization we expect a fundamental length scale to arise. The duality (71) then follows by the previous arguments.

3.3. Conclusions

The duality (70) arises naturally in the geometrical setup of quantum gravity. On the contrary its close cousin (71), though equivalent, may on first sight appear puzzling. After all, Planck's quantum of action $\hbar$ is a fundamental constant in units of which all physical observables with a discrete spectrum are quantized, while the duality between short and long distances is of a more geometrical nature. It may thus cause some concern to even consider physical processes in which the measurable action S becomes much smaller than the quantum of action $\hbar$. However, a moment's reflection shows that similar objections might be raised to the transformation (70), given that L_P is also a fundamental constant for every given spacetime dimension. Therefore, if it makes sense to consider the *geometrical* duality (70), it makes no less sense to consider its close cousin, the *physical* duality (71). Then the same reasoning that led one to require invariance under (70) will also apply to require invariance under (71).

That rendering the notion of a quantum *relative* provides a dual approach to a quantum theory of relativity is more than just a pun on words. The above line of reasoning establishes the following chain of equivalences: by [24], the existence of a fundamental length scale L_P is equivalent to invariance under the duality (70). In turn, the latter is equivalent to the duality (71). Hence the semiclassical versus strong-quantum duality (71) is equivalent to the existence of the Planck length. The existence of a fundamental length scale is a hallmark of any quantum theory of gravity.

4. Discussion

Gerbes, both Abelian and non-Abelian, have attracted considerable attention recently; close cousins of gerbes were also studied earlier [29, 30]. In this paper we have used Abelian gerbes in order to develop a U(1) gauge theory of quantum mechanics.

Our input is the phase space $\mathbb{P}$ of a given mechanical action S for a finite number of degrees of freedom. Our output is the 2-cocycle defining a gerbe over $\mathbb{P}$, as well as the potential 1-form A, the Neveu–Schwarz 2-form B and the field-strength 3-form H specifying a connection. It should be emphasized that no additional data are required in order to define the gerbe and the connection A, B, H: the action S, the phase space $\mathbb{P}$ and Planck's constant $\hbar$ suffice.

A gerbe with a connection A, B, H places a set of gauge fields at our disposal. Gauge fields were invented for covariantizing derivatives. What derivatives are to be covariantized? Now, the quantum theory corresponding to the mechanical action S is governed by Schrödinger's equation. The latter can be obtained as the classical equation of motion for an auxiliary, field-theory action $\mathcal{S}[\Psi]$ whose field variable is the probability distribution Ψ satisfying the Schrödinger equation on phase space. This auxiliary, *classical* action $\mathcal{S}[\Psi]$, defined on phase space $\mathbb{P}$ and globally U(1)-invariant, encapsulates the *quantum* dynamics corresponding to the mechanical action S. Moreover, at least in the WKB approximation, the time-independent probability distribution has $\exp(\mathrm{i}S/\hbar)$ as its U(1)-phase. Provided that all symplectic derivatives contained in $\mathcal{S}[\Psi]$ are gauged by the potential 1-form A, the covariantized action $\mathcal{S}[\Psi; A]$ becomes invariant under *local* U(1) transformations. In particular, the field-theory action $\mathcal{S}[\Psi; A]$ enjoys the desired property of allowing one to locally exchange the semiclassical and the strong-quantum regimes corresponding to the mechanical action S.

The previous reasoning motivates us to call the quantum mechanics corresponding to the mechanical action S an Abelian gauge theory of quantum mechanics. This latter gauge theory is described by the *classical* field-theory action $\mathcal{S}[\Psi; A]$ on phase space $\mathbb{P}$; the corresponding Noether charge is the inverse Planck constant $\hbar^{-1}$. The physical property carried by this Noether charge could well be termed *quantumness*: the property of being *quantum* as opposed to *classical*. Electric charges e, if at all present, would carry attached an additional U(1) gauge invariance, with an additional potential 1-form A_e and an additional field-strength 2-form $F_e = \mathrm{d}A_e$ on a fibre bundle over phase space. None of the latter belongs to our gerbe over phase space.

This U(1) symmetry on the gerbe can also be taken to be generated by 1-forms. It turns out that the corresponding gauge variation of the Neveu–Schwarz field B is equivalent to gauging it by the symplectic form ω on phase space, i.e., to performing the transformation $B \to B + \mathrm{i}\omega/\hbar$. This property allows us to make contact with generalized complex manifolds, of which $\mathbb{P}$ is an instance. Indeed the combination $B + \mathrm{i}\omega/\hbar$ is ubiquitous in generalized complex geometry. The field-strength H is always gauge invariant, regardless of the generator picked for U(1) gauge transformations (0-forms or 1-forms). The gauge invariance of H is essential since we have proved that this 3-form in fact encodes Heisenberg's uncertainty principle. One can thus say that H stands for Heisenberg.

The question may arise, are gerbes over phase space really necessary? Wouldn't line bundles suffice, as in geometric quantization? After all, the potential 1-form A on the gerbe has been proved to be gauge equivalent to a connection A' on a line bundle, on which the construction of subsection 2.5 is based. The answer is that gerbes are necessary in order to implement dualities, something that geometric quantization was *not* designed to implement. Line bundles are such that, across overlapping coordinate patches, sections will transform according to some transition function. As a transformation rule, the latter falls well short of our goal of implementing dualities. On the contrary, the transformation rule on a gerbe is such that the difference between two trivializations is not a transition function, but a line bundle. This is precisely the situation we have dealt with in subsection 2.7. As we have seen, this fact allows one to gauge transform any connection A' into the potential 1-form A on the gerbe, thus allowing for the desired duality transformations.

There is one additional reason for considering gerbes on phase space, rather than bundles. On the latter, gauge transformations are generated only by 0-forms, while both 0-forms and 1-forms can be taken to generate gauge transformations on the former. Now gauge transformations by 0-forms amount to canonical transformations on phase space. However they are nontrivial in that they give rise to new (though equivalent) Schrödinger equations. Gauge transformations by 1-forms are specific to gerbes. They are more interesting because they turn the wavefunction into a (generally nonhomogeneous) differential form on phase space. This is reminiscent of gravity theories. Indeed, an interesting spinoff of our analysis is that it takes the first steps along an alternative approach to a quantum theory of gravity. There exist numerous approaches to a quantum theory of spacetime and gravitation [31]. It is not unusual to adopt the standpoint that one must first have a classical theory, which one later quantizes. The corresponding notion of a quantum is usually universal in the sense that it is observer independent, even in those cases in which spacetime itself arises as a secondary, not a primary concept. In rendering the notion of a quantum relative to the observer, as done here, one takes the view that one is also approaching quantum gravity, although from an dual perspective. Namely, one quantizes gravity inasmuch as one relativizes the notion of a quantum. This alternative viewpoint has been analysed in [10]; it boils down to the following idea. Since quantization is effected through the exponential of ($i/\hbar$ times) the classical action S, if we are given the liberty to pick the origin of actions at will, on a point-by-point basis, then the notion of a quantum will also vary on a point-by-point basis.

One further consequence of our analysis is the following. WKB quantization is enough if one picks the appropriate U(1) gauge; corrections may however arise in other gauges. This is in good agreement with the assertion made in [32], section 7, where it was stated that one can always find a set of coordinates in which the quantum system under consideration will be semiclassical. To the extent that a gerbe does not qualify as a manifold, the above statement from [32] must be slightly modified so as to read: one can always find a local choice of gauge in which the quantum system under consideration will be semiclassical.

Finally we would like to add that the ideas put forward here may shed light on the string-theory landscape and on the correspondence between gauge theory and quantum gravity. Our conclusions also contribute towards a modern geometric view of quantum mechanics, a beautiful presentation of which has been given in [33].

Acknowledgments

References

[1] Landau L and Lifshitz E 2000 *Quantum Mechanics* (*Course of Theoretical Physics* vol 3) (Oxford: Butterworth-Heinemann)
[2] de Gosson M 2003 *Phys. Lett.* A **317** 365
     de Gosson M 2005 *Preprints* math.SG/0504013, math-ph/0602055
     de Gosson M 2006 *Symplectic Geometry and Quantum Mechanics* (Basel: Birkhäuser)
[3] de Gosson M 2005 *J. Phys. A: Math. Gen.* **38** L325
[4] de Gosson M 2005 *J. Phys. A: Math. Gen.* **38** 9263

[5] Zachos C 2002 *Int. J. Mod. Phys*. A **17** 297
    Zachos C, Fairlie D and Curtright T (ed) 2006 *Quantum Mechanics in Phase Space* (*World Scientific Series in 20th Century Physics* vol 34) (Singapore: World Scientific)
[6] Vafa C 1997 *Preprint* hep-th/9702201
[7] Giraud J 1971 *Cohomologie Non-Abélienne* (*Grundlehren der Mathematischen Wissenschaften* vol 179) (Berlin: Springer)
    Brylinski J 1993 *Loop Spaces, Characteristic Classes and Geometric Quantization* (*Progress in Mathematics* vol 107) (Boston: Birkhäuser)
    Breen L 1994 *Classification of 2-Gerbes and 2-Stacks* (*Astérisque* vol 225) (Paris: Société Mathématique de France)
    Murray M 1996 *J. Lond. Math. Soc*. **54** 403
[8] Picken R 2005 A cohomological description of abelian bundles and gerbes *Twenty Years of Bialowieza: A Mathematical Anthology* (*World Sci. Monogr. Ser. Math*. vol 8) (Hackensack, NJ: World Scientific) pp 217–28
[9] Bott R and Tu W 1986 *Differential Forms in Algebraic Topology* (Berlin: Springer)
    For a physics-oriented presentation see, e.g., Alvarez O 1985 *Commun. Math. Phys*. **100** 279
[10] Isidro J M 2005 *Preprints* hep-th/0510075, hep-th/0512241
[11] Mangiarotti L and Sardanashvily G 2000 *Connections in Classical and Quantum Field Theory* (Singapore: World Scientific)
[12] Zinn-Justin J 2005 *Path Integrals in Quantum Mechanics* (Oxford: Oxford University Press)
[13] Hitchin N 2002 *Preprint* math.DG/0209099
    Gualtieri M 2004 *Preprint* math.DG/0401221
[14] Isidro J M 2006 *Int. J. Mod. Phys*. A **21** 1189
[15] Faraggi A and Matone M 1997 *Phys. Rev. Lett*. **78** 163
[16] Padmanabhan T 1997 *Phys. Rev. Lett*. **78** 1854
    Padmanabhan T 1998 *Phys. Rev*. D **57** 6206
[17] Pinzul A and Stern A 2004 *Phys. Lett*. B **593** 279
[18] Minic D and Tze C 2003 *Phys. Rev*. D **68** 061501
    Minic D and Tze C 2004 *Phys. Lett*. B **581** 111 (*Preprint* hep-th/0401028)
    Jejjala V and Minic D 2006 *Preprint* hep-th/0605105
[19] Bracken A 2006 *Preprint* quant-ph/0601086
[20] Frasca M 2006 *Preprint* hep-th/0603182
[21] Singh T 2006 *Preprint* hep-th/0605112
[22] Matone M, Pasti P, Shadchin S and Volpato R 2006 *Preprint* hep-th/0607133
[23] Borel A 1986 Hermann Weyl and Lie groups *Hermann Weyl Centenary Lectures* (Berlin: Springer) pp 53–82
    Weil A 1964 Sur certains groupes d' oprateurs unitaires *Acta Math*. **111** 143–211
    Weil A 1979 *Collected Papers* vol III (Berlin: Springer) pp 1–69
    Lepowski J 1988 Perspectives on vertex operators and the monster *The Mathematical Heritage of Hermann Weyl* (*Proc. Symposia in Pure Math*. vol 48) ed R Wells Jr pp 181–97
[24] Padmanabhan T 1997 *Phys. Rev. Lett*. **78** 1854
    Padmanabhan T 1998 *Phys. Rev*. D **57** 6206
[25] Srinivasan K, Sriramkumar L and Padmanabhan T 1998 *Phys. Rev*. D **58** 044009
    Shankaranarayanan S and Padmanabhan T 2001 *Int. J. Mod. Phys*. **10** 351
    Padmanabhan T 2002 *Mod. Phys. Lett*. A **17** 1147
    Smailagic A, Spallucci E and Padmanabhan T 2003 *Preprint* hep-th/0308122
[26] Giveon T, Porrati M and Rabinovici E 1994 *Phys. Rep*. **244** 77
[27] Courant R and Hilbert D 1989 *Methods of Mathematical Physics* (*Wiley Classics Edition* vol 1) (New York: Wiley)
[28] Schuller F 2002 *Ann. Phys*. **299** 174
[29] Orland P 1982 *Nucl. Phys*. B **205** 107
[30] Teitelboim C 1986 *Phys. Lett*. B **167** 63
    Teitelboim C 1986 *Phys. Lett*. B **167** 69
[31] For a review see, e.g., Ashtekar A 2005 *New J. Phys*. **7** 198
[32] Isidro J M 2001 *Int. J. Mod. Phys*. A **16** 3853
[33] Giachetta G, Mangiarotti L and Sardanashvily G 2005 *Geometric and Algebraic Topological Methods in Quantum Mechanics* (Singapore: World Scientific)