Helically symmetric N-particle solutions in scalar gravity

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Within a scalar model theory of gravity, where the interaction between particles is given by the half-retarded + half-advanced solution of the scalar wave equation, we consider an N-body problem: we investigate configurations of N particles which form an equilateral N-angle and are in helical motion about their common center. We prove that there exists a unique equilibrium configuration and compute the equilibrium radius explicitly in a post-Newtonian expansion.

Self-gravitating systems with helical symmetry have recently attracted considerable interest in the context of the numerical evolution of coalescing neutron star and black hole binaries [1, 2]. Since numerical codes simulating relativistic collapse cannot evolve for large times, initial data should be close to the final plunge. A reasonable approximation to such initial data might be to consider data given by spacetimes with a helical Killing vector, the reason being that gravitational radiation tends to reduce the eccentricity of orbits. However, not only for numerical purposes, but also from a more systematic point of view, spacetimes with a helical Killing vector form a class of time independent solutions of the Einstein equations which are interesting in themselves and about which little — including their existence — is known. The simplest class of examples are N point particles of equal mass m in Newtonian theory forming an equilateral N-angle and uniformly rotating about their common center. We construct the analog of these solutions in a special relativistic scalar theory of gravity, with particles interacting via the half-retarded + half-advanced ("symmetric") solution of the wave equation. The case $N = 2$ (and allowing for different masses) has been considered by [3, 4]; the electromagnetic $N = 2$ case has been treated in the seminal paper [5]. In this paper we perform a careful study of the symmetric interaction of particles in helical motion which is absent in the literature even in the antipodal ($N = 2$) case. This serves the purpose of proving the following result: given m and the angular velocity Ω of the helical motion, there exists, like in Newtonian theory, exactly one radius $\bar{r}_{\mathrm{e}}$, for which the symmetric interaction is balanced by the centrifugal force.

Although scalar theories of gravity are known to disagree with experiment, they provide simple test models for relativistic gravity, mainly because they have just one degree of freedom, see [6]. Scalar theories derive from an action

$$S = \frac{1}{2}\int_M g^{\alpha\beta}\Phi_{,\alpha}\Phi_{,\beta}d^4x + 4\pi\int_M \rho\, F(\Phi)d^4x,$$

where $(M, g_{\alpha\beta})$ is Minkowski space and ρ the energy density of matter; we have set $G = 1$, $c = 1$. The resulting equation for Φ is $\Box\Phi = 4\pi\rho F'(\Phi)$. When $F(0)$ is chosen

to be one, the associated theory has the correct Newtonian limit. The choice $F(\Phi) = \exp\Phi$ corresponds to the model theory recently proposed in [6]. To further simplify matters, we make the choice $F(\Phi) = 1 + \Phi$, which corresponds to a first-order expansion, see [7, 8]; this leads to the linear wave equation for Φ,

$$\Box\Phi = 4\pi\rho\,. \tag{1}$$

We consider a family of N structureless point particles of equal mass m; let $\bar{x}_n(s_n)$ be the world line of the nth particle, where s_n denotes proper time; then

$$\rho(x) = m\sum_{n=0}^{N-1}\int \delta^{(4)}\big(x - \bar{x}_n(s_n)\big)\,ds_n\,,$$

so that the particle equations of motion are

$$m\frac{d}{ds_n}\big[\big(1 + \Phi|_{\bar{x}_n}\big)\dot{x}_n^\alpha\big] + (\partial^\alpha\Phi)\big|_{\bar{x}_n} = 0 \tag{2}$$

for $n = 0, 1, \ldots (N-1)$, where the field $\Phi|_{\bar{x}_n}$ acting on the nth particle is the symmetric solution of (1) generated by the remaining particles [9].

In helical symmetry, fields are invariant under the action of a helical Killing vector ξ, whose components are $\xi_t = 1$, $\xi_\phi = \Omega = \mathrm{const}$, $\xi_r = 0 = \xi_z$, and whose Lorentz norm is $\xi^2 = -1 + \Omega^2 r^2$; here, (r, ϕ, z) are cylindrical coordinates associated with $\vec{x} = (x, y, z)$. Helical motion is motion tangent to the Killing orbits, i.e., circular motion with constant angular velocity Ω in planes $z = \mathrm{const}$. When we define $\mu = \phi - \Omega t$, we find that a field ψ on $(M, g_{\alpha\beta})$ is helically symmetric, if it is of the form $\psi(\mu, r, z)$, where ψ is periodic in μ with period 2π; helical motion is any motion with $(\mu, r, z)(s) \equiv \mathrm{const}$.

Helical solutions of the wave equation. We consider the wave equation (1) for a helically symmetric source $\rho(t, \vec{x}) = \rho_{\mathrm{h}}(\mu, r, z)$, where ρ_{h} is 2π-periodic in μ; we assume $\rho_{\mathrm{h}} = 0$ for $r \geq \Omega^{-1}$ so that the source is confined within the light cylinder $r = \Omega^{-1}$, where velocities are less than the speed of light. The retarded solution $\Phi_{\mathrm{ret}}(t, \vec{x})$ and the advanced solution $\Phi_{\mathrm{adv}}(t, \vec{x})$ of (1) are given by

$$\Phi_{\mathrm{ret/adv}} = -\int_{\mathbb{R}^3}\frac{\rho(t \mp |\vec{x} - \vec{x}'|, \vec{x}')}{|\vec{x} - \vec{x}'|}\,d^3x'\,. \tag{3}$$

The solutions Φ_{ret} and Φ_{adv} share the symmetries of the source, i.e., $\Phi_{\text{ret}} = \Phi_{\text{ret}}(\mu, r, z)$. To make this explicit, we first introduce cylindrical coordinates (t, r, ϕ, z) and (r', ϕ', z') associated with $(t, \vec{x})$ and $\vec{x}'$ in (3); we then find that the integrand can be regarded as a 2π-periodic function of $\sigma = \phi - \phi'$, which further entails that

$$\Phi_{\text{ret}} = - \int\limits_0^{2\pi} d\sigma \int\limits_0^{\Omega^{-1}} r' dr' \int\limits_{-\infty}^{\infty} dz' \, \frac{\rho_{\text{h}}(\mu - \sigma + \Omega|\vec{x} - \vec{x}'|, r', z')}{|\vec{x} - \vec{x}'|} \,,$$

where $|\vec{x} - \vec{x}'|^2 = r^2 + r'^2 - 2rr' \cos\sigma + (z - z')^2$. Consequently, $\Phi_{\text{ret}} = \Phi_{\text{ret}}(\mu, r, z)$ with 2π-periodicity in μ. Note that the advanced solution $\Phi_{\text{adv}}(\mu, r, z)$ arises from $\Phi_{\text{ret}}(\mu, r, z)$ by making the replacement $\Omega \to (-\Omega)$.

We proceed by defining a variable μ' via

$$\mu' = \mu - \sigma + \Omega \left[r^2 + r'^2 - 2rr' \cos\sigma + (z - z')^2\right]^{\frac{1}{2}}. \quad (4)$$

For fixed μ, r, $r' < \Omega^{-1}$, z, z', the map $\sigma \mapsto \mu'$ is monotonically decreasing and thus a diffeomorphism. This follows from a straightforward computation, where we invoke de l'Hospital's rule (for $r = r'$, $z = z'$, $\sigma = 2k\pi$, $k \in \mathbb{Z}$). Performing a change of integration variables from σ to μ', where we use that a shift by 2π in σ causes a shift by -2π in μ', we eventually arrive at

$$\Phi_{\text{ret}} = \Omega \int\limits_0^{2\pi} d\mu' \int\limits_0^{\Omega^{-1}} r' dr' \int\limits_{-\infty}^{\infty} dz' \, \frac{\rho_{\text{h}}(\mu', r', z')}{\mu - \mu' - \sigma + \Omega^2 rr' \sin\sigma} \,.$$

In this integral, σ is to be regarded as a function of the other variables, implicitly given by (4). In fact, $\sigma = \sigma(\mu - \mu', r, r', z - z')$, where $\sigma(\mu, r, r', z)$ satisfies

$$\mu - \sigma + \Omega \left[(r - r')^2 + 4rr' \sin^2\frac{\sigma}{2} + z^2\right]^{\frac{1}{2}} = 0\,. \quad (5)$$

We call $\sigma(\mu, r, r', z)$ the retarded angle associated with μ (and the particular choice of r, r', z). The following properties of σ are immediate from the above discussion:

$$\frac{d\sigma}{d\mu} > 0\,, \quad \sigma(\mu + 2\pi, r, r', z) = \sigma(\mu, r, r', z) + 2\pi\,. \quad (6)$$

Finally, regarding the retarded solution $\Phi_{\text{ret}}(\mu, r, z)$ as the convolution of an integration kernel with the source $\rho_{\text{h}}(\mu', r', z')$ yields the so-called retarded kernel

$$K_{\text{ret}}(\mu, r, r', z) = \Omega \, \frac{1}{\mu - \sigma + \Omega^2 rr' \sin\sigma}\,, \quad (7)$$

where $\sigma = \sigma(\mu, r, r', z)$. As a consequence of (6), $K_{\text{ret}}(\mu, r, r', z)$ is 2π-periodic in μ.

As noted above, the advanced kernel $K_{\text{adv}}(\mu, r, r', z)$ is given in analogy to (7), where $\Omega \to (-\Omega)$ and $\sigma \to \sigma_{\text{adv}}$;

$$\mu - \sigma_{\text{adv}} - \Omega \left[(r - r')^2 + 4rr' \sin^2\frac{\sigma_{\text{adv}}}{2} + z^2\right]^{\frac{1}{2}} = 0\,.$$

From (5) we conclude that $[-\sigma(-\mu, r, r', z)]$ satisfies this equation, hence $\sigma_{\text{adv}}(\mu, r, r', z) = -\sigma(-\mu, r, r', z)$. By periodicity of K_{ret} we thus infer the important relation

$$K_{\text{adv}}(\mu, r, r', z) = K_{\text{ret}}(2\pi - \mu, r, r', z)\,. \quad (8)$$

Symmetric solution for a point source. The simplest source that is compatible with helical symmetry is a point mass m in circular motion. Let $(\bar{\mu}, \bar{r} < \Omega^{-1}, \bar{z} = 0)$ be the position of the point particle; then the density is

$$\rho_{\text{h}}(\mu, r, z) = m \left(1 - \Omega^2 \bar{r}^2\right)^{\frac{1}{2}} \delta(\mu - \bar{\mu}) \frac{\delta(r - \bar{r})}{\bar{r}} \delta(z)\,,$$

and the associated retarded potential reads

$$\Phi_{\text{ret}}(\mu, r, z) = m \left(1 - \Omega^2 \bar{r}^2\right)^{\frac{1}{2}} K_{\text{ret}}(\mu - \bar{\mu}, r, \bar{r}, z)\,.$$

The radial component of the force at $(\mu, r, z) \neq (\bar{\mu}, \bar{r}, \bar{z})$ is given by $\partial_r \Phi_{\text{ret}}$; when $r = \bar{r}$, $z = 0$ it simplifies to

$$\left[\partial_r \Phi_{\text{ret}}\right]\big|_{r=\bar{r}} = \frac{m}{2} \left(1 - \Omega^2 \bar{r}^2\right)^{\frac{1}{2}} \partial_{\bar{r}} K_{\text{ret}}(\mu - \bar{\mu}, \bar{r}, \bar{r}, 0)\,,$$

where we have used the symmetry of the retarded kernel in r and $\bar{r}$, i.e., $K_{\text{ret}}(\mu, r, \bar{r}, z) = K_{\text{ret}}(\mu, \bar{r}, r, z)$. Consequently, the fields Φ_{ret} and $[\partial_r \Phi_{\text{ret}}]|_{r=\bar{r}}$ at positions $\mu \neq \bar{\mu}$, $r = \bar{r}$, $z = 0$ are completely described by $K_{\text{ret}}(\mu, \bar{r}, \bar{r}, 0)$ and its derivatives.

To obtain the kernel $K_{\text{ret}}(\mu, \bar{r}, \bar{r}, 0)$ we first investigate the retarded angle $\sigma(\mu, \bar{r}, \bar{r}, 0)$. It ensues from (5) that $\mu = 0$ corresponds to $\sigma = 0$ and thus $\mu = 2\pi$ to $\sigma = 2\pi$ by (6). Hence, for $\mu \in [0, 2\pi)$, $\sin\frac{\sigma}{2}$ is non-negative and the defining equation for $\sigma(\mu, \bar{r}, \bar{r}, 0)$ thus becomes

$$\mu - \sigma + 2\Omega \bar{r} \sin\frac{\sigma}{2} = 0\,. \quad (9)$$

We find $\sigma = \mu$ in the limit $\bar{r} \to 0$; for $\bar{r} > 0$, however, $\sigma > \mu$, unless $\sigma = 0 = \mu$ or $\sigma = 2\pi = \mu$; the difference between the angle μ and the retarded angle σ is largest when $\sigma = \pi$. Keeping $\mu \in (0, 2\pi)$ fixed (so that σ is in the same interval) we obtain

$$\partial_{\bar{r}} \sigma = \frac{2\Omega \sin\frac{\sigma}{2}}{1 - \Omega \bar{r} \cos\frac{\sigma}{2}} > 0\,; \quad (10)$$

hence, in addition to being increasing in μ, $\sigma(\mu, \bar{r}, \bar{r}, 0)$ is increasing, with range $(\mu, \sigma_1(\mu) < 2\pi)$, also when viewed as a function of $\bar{r}$.

Finally, the results (9) and (10) lead to

$$K_{\text{ret}}(\mu, \bar{r}, \bar{r}, 0) = -\frac{1}{2\bar{r} \sin\frac{\sigma}{2}} \frac{1}{1 - \Omega \bar{r} \cos\frac{\sigma}{2}} \quad (11)$$

$$\partial_{\bar{r}} K_{\text{ret}} = \frac{(1 - \Omega\bar{r})^2 + 4\Omega\bar{r} \sin^2\frac{\sigma}{4}}{2\bar{r}^2 \sin\frac{\sigma}{2}} \left(\frac{1}{1 - \Omega\bar{r} \cos\frac{\sigma}{2}}\right)^3\,.$$
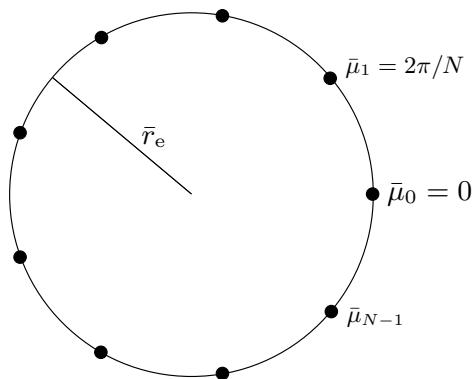
FIG. 1: N point particles of equal mass m in uniform circular motion — the mutual interaction is given by the symmetric potential (half-retarded + half-advanced potential). We prove that there exists a unique radius $\bar{r}_e$ such that the configuration is in equilibrium.

The kernel K_{ret} is manifestly negative and monotonically increasing in $\bar{r}$, since $\partial_{\bar{r}} K_{\text{ret}}$ is positive; for $\bar{r} \to 0$ both expressions diverge, while the limit is finite for $\bar{r} \to \Omega^{-1}$.

Since $K_{\text{ret}}(\pi, \bar{r}, \bar{r}, 0) = K_{\text{adv}}(\pi, \bar{r}, \bar{r}, 0)$ by (8), the retarded and advanced potential at the antipodal point $(\mu = \bar{\mu} + \pi, r = \bar{r}, z = 0)$ are equal; the same is true for the radial components of the forces. The tangential components, however, are not equal, but opposite, since $\partial_\mu K_{\text{ret}}|_{\mu=\pi} = -\partial_\mu K_{\text{adv}}|_{\mu=\pi}$ by (8). Therefore, if we consider the *symmetric solution* $\Phi(\mu, r, z)$ of (1), i.e.,

$$\Phi(\mu, r, z) = \frac{1}{2}\left[\Phi_{\text{ret}}(\mu, r, z) + \Phi_{\text{adv}}(\mu, r, z)\right], \qquad (12)$$

then the μ-derivatives of the two terms cancel, so that $\partial_\mu \Phi = 0$ at the antipodal point of a point mass. This fact is a necessary prerequisite for a system of two (or more) particles in circular motion to be in equilibrium. Henceforth we only consider symmetric potentials (12).

Equilibrium configuration for N point masses. We now consider the helical configuration depicted in Fig. 1: let $n = 0, 1, \ldots, (N-1)$ be N point masses of equal mass m, equidistantly distributed along a circle of radius $\bar{r}$ at $z = 0$ and uniformly rotating about their common center — the nth particle's position is thus given by $(\bar{\mu}_n, \bar{r}, 0)$ with $\bar{\mu}_n = 2\pi n/N$. Let $\Phi_n(\mu, r, z)$ denote the symmetric potential generated by the nth particle. At the position $(\mu, r, z) = (0, \bar{r}, 0)$ of the first point mass the total potential Φ is then given as $\sum_{n \geq 1} \Phi_n(0, \bar{r}, 0)$, hence

$$\Phi(0, \bar{r}, 0) = \frac{m}{2}\left(1 - \Omega^2 \bar{r}^2\right)^{\frac{1}{2}} \sum_{n=1}^{N-1} (K_{\text{ret}} + K_{\text{adv}})(\bar{\mu}_n, \bar{r}, \bar{r}, 0).$$

Making use of (8) results in

$$\Phi(0, \bar{r}, 0) = m\left(1 - \Omega^2 \bar{r}^2\right)^{\frac{1}{2}} \sum_{n=1}^{N-1} K_{\text{ret}}(\bar{\mu}_n, \bar{r}, \bar{r}, 0)$$

$$\left[\partial_r \Phi\right]\big|_{\bar{r}} = \frac{m}{2}\left(1 - \Omega^2 \bar{r}^2\right)^{\frac{1}{2}} \sum_{n=1}^{N-1} \partial_{\bar{r}} K_{\text{ret}}(\bar{\mu}_n, \bar{r}, \bar{r}, 0) \quad (13)$$

for the potential and the radial component of the force at $(0, \bar{r}, 0)$. The tangential component of the force, i.e., $\partial_\mu \Phi(\mu, \bar{r}, 0)|_{\mu=0}$, vanishes, since $\partial_\mu K_{\text{adv}}|_\mu = -\partial_\mu K_{\text{ret}}|_{2\pi-\mu}$ and thus $\partial_\mu K_{\text{adv}}|_{\mu_n} = -\partial_\mu K_{\text{ret}}|_{\mu_{N-n}}$; likewise, $\partial_z \Phi(0, \bar{r}, z)|_{z=0} = 0$, which is a simple consequence of the mirror symmetry in z. The equation of motion (2) for the first particle thus reduces to

$$\left[\partial_r \Phi(0, r, 0)\right]\big|_{r=\bar{r}} - \left(1 + \Phi(0, \bar{r}, 0)\right)\frac{\Omega^2 \bar{r}}{1 - \Omega^2 \bar{r}^2} = 0, \quad (14)$$

where we have used the independence of (proper) time of Φ at the particle's position and $m\ddot{x}_0^\alpha = (\partial^\alpha \xi^2)/(2\xi^2)$, where $\xi^2 = -1 + \Omega^2 r^2$, for the centrifugal term. The equations for the remaining $(N-1)$ particles are identical, since the symmetry of the configuration entails that none of the particles is distinguished. Hereby, the system of $3N$ equations (2) reduces to one single equation (14).

We conclude that the configuration of Fig. 1 is in equilibrium, if condition (14) holds, i.e., if the radial force acting on each particle is balanced by the centrifugal force. As follows from (11), the potential $\Phi(0, \bar{r}, 0)$ is negative and monotonically increasing for all $\Omega \bar{r} < 1$; it diverges for $\bar{r} \to 0$ and converges to zero when $\Omega \bar{r} \to 1$. Consequently, there exists a unique radius $\bar{r}_0$ such that $(1 + \Phi(0, \bar{r}, 0))$ is negative for all $\bar{r} < \bar{r}_0$, and positive for $\bar{r} > \bar{r}_0$. The term $\left[\partial_r \Phi(0, r, 0)\right]\big|_{r=\bar{r}}$ is positive for all $\Omega \bar{r} < 1$; it diverges as $\bar{r} \to 0$ and it goes to zero when $\Omega \bar{r} \to 1$. Combining the results it follows that the function on the l.h.s. of (14) is positive for all $\bar{r} \leq \bar{r}_0$ and goes to $-\infty$ as $\Omega \bar{r} \to 1$. We thus conclude that this function assumes the value zero at least once in the interval $\bar{r} \in (\bar{r}_0, \Omega^{-1})$, so that there exists at least one radius $\bar{r}_e$ for which condition (14) is satisfied and the configuration is in equilibrium. In the following we prove that radius $\bar{r}_e$ is unique by showing that the l.h.s. of (14) is decreasing for $\bar{r} \in (\bar{r}_0, \Omega^{-1})$.

The proof would be trivial if the radial force $\left[\partial_r \Phi(0, r, 0)\right]\big|_{r=\bar{r}}$ were decreasing in $\bar{r}$ (since the second term on the l.h.s. of (14) is manifestly decreasing for $\bar{r} > \bar{r}_0$). However, whether monotonicity of $\left[\partial_r \Phi(0, r, 0)\right]\big|_{r=\bar{r}}$ actually holds, is unclear in general. Namely, it can be shown numerically that $\psi(\mu, \bar{r}) = \partial_{\bar{r}}[(1 - \Omega^2 \bar{r}^2)^{1/2} \partial_{\bar{r}} K_{\text{ret}}(\mu, \bar{r}, \bar{r}, 0)]$ does not have a sign: there exists a connected domain D in the set $(0, 2\pi) \times (0, \Omega^{-1})$ such that ψ is positive when $(\mu, \bar{r}) \in D$ and negative when $(\mu, \bar{r}) \notin \bar{D}$ — this is in stark contrast to the Newtonian case, where ψ is negative for

all $(\mu, \bar{r})$. The main properties of D are the following: $\min_D \bar{r} \approx 3/4 \, \Omega^{-1}$, hence negativity holds for small $\bar{r}$, where velocities are small compared to c so that Newtonian gravity is a good approximation to scalar gravity; $\max_D \mu \approx 1/4$, hence the radial force is decreasing at least when the number of particles is sufficiently small, i.e., when $\bar{\mu}_n > 1/4 \; \forall n$; typical values of ψ on D are by several orders of magnitude larger than typical values of $|\psi|$ on $\big[(0, 2\pi) \times (0, \Omega^{-1})\big] \backslash \bar{D}$ — this complicates matters when one seeks to prove that the sum over all $\bar{\mu}_n$ is negative. Despite this last remark, numerical evidence suggests that $\big[\partial_r \Phi(0, r, 0)\big]\big|_{r=\bar{r}}$ is in fact decreasing irrespective of the number of particles; a rigorous proof, however, seems difficult to obtain.

In our proof we therefore proceed along different lines. The derivative of the function on the l.h.s. of (14) reads

$$
m \sum_{n=1}^{N-1} \left\{ \partial_{\bar{r}} \left[\frac{1}{2} \left(1 - \Omega^2 \bar{r}^2\right)^{\frac{1}{2}} \partial_{\bar{r}} \, K_{\text{ret}}(\bar{\mu}_n, \bar{r}, \bar{r}, 0) \right] \right.
$$
$$
\left. - \partial_{\bar{r}} \left[\left(1 - \Omega^2 \bar{r}^2\right)^{\frac{1}{2}} K_{\text{ret}}(\bar{\mu}_n, \bar{r}, \bar{r}, 0) \right] \frac{\Omega^2 \bar{r}}{1 - \Omega^2 \bar{r}^2} \right\}
$$
$$
- \left(1 + \Phi(0, \bar{r}, 0)\right) \partial_{\bar{r}} \left(\frac{\Omega^2 \bar{r}}{1 - \Omega^2 \bar{r}^2} \right) . \qquad (15)
$$

Since the last line is clearly negative when $\bar{r} > \bar{r}_0$, in order to show that the whole function is negative, it suffices to prove that each of the terms in braces is negative individually. To this end let $\sigma = \sigma(\bar{\mu}_n, \bar{r}, \bar{r}, 0)$ for some n; then each individual term in braces has the form

$$
\frac{\left(1 - \Omega^2 \bar{r}^2\right)^{-\frac{1}{2}}}{4 \bar{r}^3 \left(1 - \Omega^2 \bar{r}^2 \cos \frac{\sigma}{2}\right)^5 \sin \frac{\sigma}{2}} \; P\!\left(\Omega \bar{r}, \cos \frac{\sigma}{2}\right) , \qquad (16)
$$

where $P(\bar{v}, \cos \frac{\sigma}{2})$ is a complicated polynomial of degree eight in $\bar{v} = \Omega \bar{r} < 1$ and of degree four in $\cos \frac{\sigma}{2}$. In a second step we replace $\cos \frac{\sigma}{2}$ by a variable δ defined through $\cos \frac{\sigma}{2} = \bar{v}^{-1}[1 - (1 - \bar{v}^2)\delta]$. Since $-1 \leq \cos \frac{\sigma}{2} \leq 1$, the permitted range of δ is

$$
\frac{1}{2} < \frac{1}{1 + \bar{v}} \leq \delta \leq \frac{1}{1 - \bar{v}} . \qquad (17)
$$

Using δ leads to a simple representation of $P(\bar{v}, \cos \frac{\sigma}{2})$:

$$
P = -\frac{3 - 8\delta + \delta^2(5 - 4\bar{v}^2) + \delta^3(2 + 4\bar{v}^2) + 2\delta^4 \bar{v}^4}{(1 - \bar{v}^2)^{-4}} .
$$

The roots of the polynomial P are explicitly given by $\bar{v}^2 = \delta^{-2}(1 - \delta \pm \sqrt{\Delta})$, where the discriminant Δ reads

$$
\Delta = -2 \left(\delta - \frac{1}{2}\right) \left(\delta - [\sqrt{2} - 1]\right) \left(\delta + [\sqrt{2} + 1]\right) . \qquad (18)
$$

Evidently, Δ is non-negative if and only if $\delta \leq -1 - \sqrt{2}$ or $\delta \in [\sqrt{2} - 1, \frac{1}{2}]$. As a consequence, the roots of P

lie outside of the admissible domain (17) of the variables $(\bar{v}, \delta)$. Since in addition $P < 0$ for $\bar{v} \to 0$ and $\delta = 1$, it follows that P is negative everywhere on the admissible $(\bar{v}, \delta)$-domain, or, equivalently,

$$
P\!\left(\Omega \bar{r}, \cos \frac{\sigma}{2}\right) < 0 \qquad \forall (\bar{r}, \sigma) \in (0, \Omega^{-1}) \times [0, 2\pi) . \quad (19)
$$

With $P < 0$ the expression (16) is negative, which completes the proof of the claim.

Post-Newtonian expansion. For a given number of particles, the equilibrium radius $\bar{r}_{\text{e}}$ of the configuration in Fig. 1 is a function of the angular velocity Ω and the mass. This functional dependence cannot be made explicit, since this would involve, among other things, an explicit knowledge of the retarded angle (9). (For a two-particle system, $\bar{r}_{\text{e}}$ can be given as a (non-explicit) function of the orbital velocity $\Omega \bar{r}_{\text{e}}$, see [3], which, of course, does not lead to an explicit solution for $\bar{r}_{\text{e}}$.)

It is feasible, however, to analyze the equilibrium condition (14) by means of a post-Newtonian approximation scheme. With the support of a computer algebra program necessary manipulations can be done in a straightforward way and we eventually obtain a post-Newtonian expansion of $\bar{r}_{\text{e}}$; here, we merely state some results.

Let ω be the angular velocity as measured in standard units, i.e., $[\omega] = s^{-1}$; clearly, $\omega = \Omega c$, where c is the speed of light; furthermore, let G be the gravitational constant and $M = Nm$ the total mass of the N-particle configuration of Fig. 1. We define a quanitity R (with unit length) and a dimensionless quantity x according to

$$
R = \left(GM\omega^{-2}\right)^{\frac{1}{3}} , \qquad x = \left(GM\omega c^{-3}\right)^{\frac{1}{3}} .
$$

In terms of R, x the post-Newtonian expansion of $\bar{r}_{\text{e}}$ is:

N	$\bar{r}_{\text{e}}$
2	$\frac{1}{2}R \left[1 + x^2/12 - 7x^4/72 + O(x^6)\right]$
3	$\frac{1}{\sqrt{3}}R \left[1 + 7x^2/72 - 0.065x^4 + O(x^6)\right]$
⋮	⋮
10^5	$1.228R \left[1 + 0.273x^2 - 0.465x^4 + O(x^6)\right]$

For highly relativistic configurations numerical investigations indicate that $\bar{r}_{\text{e}} \propto \Omega^{-1}$ as $\Omega \to \infty$, so there does not exist an innermost circular orbit.

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