Light cone structure near null infinity of the Kerr metric

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Motivated by our attempt to understand the question of angular momentum of a relativistic rotating source carried away by gravitational waves, in the asymptotic regime near future null infinity of the Kerr metric, a family of null hypersurfaces intersecting null infinity in shearfree (good) cuts are constructed by means of asymptotic expansion of the eikonal equation. The geometry of the null hypersurfaces as well as the asymptotic structure of the Kerr metric near null infinity are studied. To the lowest order in angular momentum, the Bondi-Sachs form of the Kerr metric is worked out. The Newman-Unti formalism is then further developed, with which the Newman-Penrose constants of the Kerr metric are computed and shown to be zero. Possible physical implications of the vanishing of the Newman-Penrose constants of the Kerr metric are also briefly discussed.

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I. INTRODUCTION

In general relativity, the angular momentum measured at future null infinity (denoted by $\mathcal{I}^+$) of a relativistic rotating source is an enigmatic notion (see for instance [1], chapter 9). For a generic weakly asymptotically simple spacetime, the infinite dimensional supertranslation symmetries at $\mathcal{I}^+$ means that the definition of angular momentum carried over from Minkowski space is not canonically defined. The tidal force generated by the Weyl curvature distorts an outgoing null hypersurface near $\mathcal{I}^+$ in such a way that the spherical cut at which the null hypersurface intersects $\mathcal{I}^+$ acquires nontrivial shear structure in the null direction tangential to the hypersurface. This entails that, in general, it is not always possible to find a family of shearfree cuts at $\mathcal{I}^+$ like that in a stationary spacetime and we are forced to treat, subject to certain smoothness assumption, all cuts including those with complicated shear structures on equal footing. Further, unlike the energy of gravitational radiation, it does not seem to make sense to define a news function describing the angular momentum carried away by gravitational radiation.

This long standing problem emerges naturally also in a more pragmatic context when we try to understand the gravitational waveform generated by a relativistic rotating source near $\mathcal{I}^+$, mainly motivated by numerical consideration (see for instance [2]). However, rather than following the standard route of attempting to impose extra structure on $\mathcal{I}^+$ to single out preferred cuts [3,4], the generation of waveform calls for a better understanding of how a measure of rotation (angular momentum), in some sense appropriately defined, is encoded into the Bondi-Sachs metric or its variants [5,6]. Underlying these coordinates is the construction of a family of null hypersurfaces whose intersections with $\mathcal{I}^+$ generate the Bondi time coordinate.

As a preliminary step to seek further geometric insight into the problem, the present work purports to construct asymptotically, in the important example of a Kerr black hole in which angular momentum is well defined, null hypersurfaces whose intersections with $\mathcal{I}^+$ generate good cuts and try to see how angular momentum is encoded in the geometry of the null hypersurfaces. The null hypersurfaces to be constructed here are different from those considered by a number of authors [7–9]. We will make further remarks concerning this point in the next section.

The outline of the present article may be described as follows. In Sec. II, we will solve the eikonal equation for the Kerr metric asymptotically near null infinity and then go on to construct a family of null hypersurfaces intersecting $\mathcal{I}^+$. A Newman-Penrose (NP) tetrad adapted to the null hypersurfaces is then defined and used to study the geometry of the hypersurfaces as well as the asymptotic structure of the Kerr metric near null infinity. In Sec. III, the Bondi-Sachs form of the Kerr metric is worked out. The Newman-Unti formalism will then be developed and the NP constants for the Kerr metric will be calculated to be zero. Possible physical implications of the vanishing of the NP constants will also be briefly discussed. Throughout the present work, the $(+, -, -, -)$ signature will be adopted for the spacetime metric and we follow the NP notations in [10].

II. CHARACTERISTC STRUCTURE NEAR NULL INFINITY

A. Eikonal equation

Let us start by looking at the explicit construction of a null hypersurface in the Kerr metric. To this end, locally, it is sufficient to seek a smooth, real valued function u such that the eikonal equation

$$g^{ab}u_{,a}u_{,b} = 0 \tag{1}$$

is satisfied where g^{ab} is the contravariant form of the spacetime metric. The most obvious solutions to the above eikonal equation are obtained by means of separation of variables ([11], see also [12], chapter 7), as that used in integrating the geodesic equations. The detailed geometry of these null hypersurfaces parametrized by a Carter constant, in particular, the suspected singular behavior along the symmetry axis, is still under investigation and remains to be understood better. See [7,8] in this context when the Carter constant takes on the specific value a^2, where a has its standard meaning in the Kerr metric.

In another work [9], Bondi-Sachs coordinates for the Kerr metric is constructed using solution to the eikonal equation obtained by Pretorious and Isarel [13]. The detailed asymptotic geometry of the Bondi-Sachs coordinates is yet to be analyzed. Further, the metric coefficients of the Bondi-Sachs metric are given in terms of implicit functions and make them difficult to implement numerically.

In the present work, we shall put forward a new construction of a family of null hypersurfaces near $\mathcal{I}^+$, based upon which Bondi-Sachs type coordinates may be constructed. One distinct feature of our construction is that the intersections of these null hypersurfaces with $\mathcal{I}^+$ generate good cuts, the existence of which is characteristic of the asymptotic structure of a stationary spacetime admitting $\mathcal{I}^+$.

To begin with, let (t, r, θ, φ) be the standard Boyer-Lindquist coordinates of the Kerr metric. Consider first in the flat space limit a light cone in Minkowski space described in terms of the oblate spheroid coordinates. The solution to the eikonal equation in this case is given by

$$u = t \pm \sqrt{r^2 + a^2\sin^2\theta}. \tag{2}$$

Apparently this solution cannot be obtained by means of the conventional method of separable of variables. In the case of the Kerr metric, to seek a solution of (1) without separation of variables and in the Minkowski space limit degenerates into (2) turns out to be quite difficult. However, for our purpose, it is sufficient to seek an asymptotic solution of (1) when r is sufficiently large.

To see the way ahead, we first look at a light cone in the Schwarzschild metric described by

$$u = t - \left(r + 2M\ln\frac{r - 2M}{2M}\right). \tag{3}$$

Asymptotically when r is sufficiently large, (3) becomes

$$u = t - \left(r + 2M\ln\frac{r}{2M} - \frac{4M^2}{r} + \cdots\cdots\right). \tag{4}$$

The term $t - r - 2M\ln\frac{r}{2M}$ survives in the asymptotic limit and this guides us to adopt the following ansatz for the solution of u in (1) in the Kerr metric when r is sufficiently large,

$$u = t - r - 2M\ln\frac{r}{2M} + \sum_{k=1}^{\infty}\frac{f_k}{r^k}. \tag{5}$$

As we envisage the Bondi-Sachs type coordinates to be constructed from the level sets of u are axisymmetric, the functions f_k, $k = 1, 2\cdots$ in (5) are then necessarily functions of θ only and independent of φ.

The eikonal equation to be solved is given as

$$r^2\left(1 - \frac{2M}{r} + \frac{a^2}{r^2}\right)\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial \theta}\right)^2 = \frac{(r^2 + a^2)^2}{r^2 - 2Mr + a^2} - a^2\sin^2\theta. \tag{6}$$

Substitute (5) into the eikonal equation above and solve the eikonal equation order by order, we obtain

$$u = t - \left(r + 2M\ln\frac{r}{2M} - \frac{4M^2 - \frac{1}{2}a^2\sin^2\theta}{r} - \frac{4M^3 - Ma^2}{r^2} + O(1/r^3)\right). \tag{7}$$

Inserting (5) into the eikonal equation in (6) enables us to solve f_k recursively. The ansatz in (5) serves to determine uniquely the lowest order terms f_1 and f_2. With f_1 and f_2 as initial conditions for the algebraic process of repeated iterations of f_k, $k \geq 2$, it may be checked that f_{k+1} is determined uniquely by f_r, $r = 1, 2\cdots k$. No freedom like, for instance, the existence of an arbitrary, nonzero constant is allowed in each order. In principle repeated iterations of f_k generate terms of any desirable order in the asymptotic expansion. However, as it occurs quite frequently in asymptotic expansion, the higher order terms inevitably become more complicated with the increase in order and no regular pattern seems to be noticeable.

For a light cone in Minkowski spacetime described in terms of oblate spheroid coordinates, we have from (2) that in the asymptotic limit $r \to \infty$,

$$u = t - \left(r + \frac{a^2\sin^2\theta}{2r} + \cdots\cdots\right). \tag{8}$$

This may also be obtained from (7) by taking the flat space limit $M \to 0$, and thereby provides a self consistency check on the validity of (7). Further, the flat space and Schwarzschild limits of (7) suggest that the constant u hypersurfaces constructed here are asymptotic parts of light cones emanating from a timelike world line.

B. NP Tetrad and asymptotic structure near $\mathcal{I}^+$

To study further the geometry of the null hypersurfaces constructed as well as the asymptotic structure of the Kerr metric near future null infinity, it will be helpful to define an NP tetrad adapted to the constant u null hypersurfaces. The null hypersurfaces described in (7) are outgoing. The dual ingoing null hypersurfaces are given as

$$v = t + \left(r + 2M \ln\frac{r}{2M} - \frac{4M^2 - \frac{1}{2}a^2\sin^2\theta}{r} - \frac{4M^3 - Ma^2}{r^2} + \cdots\cdots \right). \tag{9}$$

Naturally, we choose two legs of the null tetrad to be parallel to the gradient vectors of u and v. In terms of the Boyer-Lindquist coordinates, the NP tetrad may then be constructed as

$$l_a = (du)_a = (1, -h_1, -h_2, 0) \qquad n_a = \frac{1}{g^{00} - g^{11}h_1^2 - g^{22}h_2^2}(1, h_1, h_2, 0)$$
$$m_a = \left(g_{03}\frac{i}{\sin\theta}\sqrt{\frac{\rho^2}{2\Sigma^2}}, -g_{11}\sqrt{\frac{-h_2^2}{2g_{11}h_2^2 + 2g_{22}h_1^2}}, g_{22}\sqrt{\frac{-h_1^2}{2g_{11}h_2^2 + 2g_{22}h_1^2}}, g_{33}\frac{i}{\sin\theta}\sqrt{\frac{\rho^2}{2\Sigma^2}} \right) \tag{10}$$

where $\Sigma^2 = (r^2 + a^2)^2 - \Delta a^2\sin^2\theta$ and $\rho^2 = r^2 + a^2\cos^2\theta$. h_1, h_2 are functions to be determined by the solution (7) and may be solved asymptotically order by order. With a view to compute the NP constants for the Kerr metric later, we compute h_1, h_2 to sufficiently high order so that the eikonal equation in (1) is solved up to $1/r^7$. The results are

$$\begin{aligned} h_1 &= 1 + \frac{2M}{r} + \frac{4M^2 - \frac{1}{2}a^2\sin^2\theta}{r^2} + \frac{8M^3 - 2Ma^2}{r^3} + \frac{16M^4 - 8M^2a^2 + \frac{3}{8}a^4\sin^4\theta}{r^4} \\ &\quad + \frac{32M^5 - 24M^3a^2 + 2Ma^4 + \frac{1}{4}Ma^4\sin^4\theta}{r^5} + \frac{64M^6 - 64M^4a^2 + 12M^2a^4 - \frac{5}{16}a^6\sin^6\theta}{r^6} \\ &\quad + \frac{128M^7 - 160M^5a^2 + 48M^3a^4 - 2Ma^6 - \frac{1}{2}Ma^6\sin^6\theta}{r^7} + O(1/r^8) \\ h_2 &= \frac{a^2\sin\theta\cos\theta}{r} - \frac{\frac{1}{2}a^4\sin^3\theta\cos\theta}{r^3} - \frac{\frac{1}{4}Ma^4\sin^3\theta\cos\theta}{r^4} + \frac{\frac{3}{8}a^6\sin^5\theta\cos\theta}{r^5} + \frac{\frac{1}{2}Ma^6\sin^5\theta\cos\theta}{r^6} + O(1/r^7). \end{aligned} \tag{11}$$

The NP tetrad defined in (10) is different from the standard Kinnersley tetrad as l_a is hypersurface forming. In the limit $a \to 0$, the tetrad degenerates to the standard NP tetrad in the Schwarzschild metric ([12], chapter 3).

The spin coefficients of the tetrad defined in (10) may further be given in the asymptotic limit $r \to \infty$ as

$$\begin{aligned} &\kappa = \kappa' = 0 \qquad \rho = -\frac{1}{r} + \frac{a^2\sin^2\theta}{2r^3} + \frac{Ma^2\sin^2\theta}{2r^4} - \frac{3a^4\sin^4\theta}{8r^5} - \frac{7Ma^4\sin^4\theta}{8r^6} + \frac{a^4\sin^4\theta(5a^2\sin^2\theta - 16M^2)}{16r^7} + O(1/r^8) \\ &\sigma = -\frac{3Ma^2\sin^2\theta}{2r^4} - \frac{5iMa^3\cos\theta\sin^2\theta}{r^5} + O(1/r^6) \qquad \tau = -\frac{3iMa\sin\theta}{\sqrt{2}r^3} + O(1/r^4) \qquad \epsilon = O(1/r^5) \\ &\beta = \frac{\cot\theta}{2\sqrt{2}r} - \frac{1}{4\sqrt{2}\sin\theta}(-a^2\cos^3\theta + 2a^2\cos\theta + 6iMa\sin^2\theta)\frac{1}{r^3} + O(1/r^4) \qquad \rho' = \frac{1}{2r} - \frac{M}{r^2} - \frac{a^2\sin^2\theta}{4r^3} + O(1/r^4) \\ &\sigma' = \frac{3Ma^2\sin^2\theta}{4r^4} + O(1/r^5) \qquad \tau' = -\frac{3iMa}{\sqrt{2}r^3}\sin\theta + O(1/r^4) \qquad \epsilon' = -\frac{M}{2r^2} + O(1/r^4) \\ &\beta' = \frac{\cot\theta}{2\sqrt{2}r} - \frac{1}{4\sqrt{2}\sin\theta}(-a^2\cos^3\theta + 2a^2\cos\theta + 6iMa\sin^2\theta)\frac{1}{r^3} + O(1/r^4). \end{aligned} \tag{12}$$

The Weyl curvature components may also be given asymptotically as

$$\Psi_0 = \frac{3Ma^2\sin^2\theta}{r^5} + \frac{15iMa^3\sin^2\theta\cos\theta}{r^6} + O(1/r^7) \qquad \Psi_1 = \frac{3iMa\sin\theta}{\sqrt{2}r^4} - \frac{6\sqrt{2}Ma^2\sin\theta\cos\theta}{r^5} + O(1/r^6)$$
$$\Psi_2 = -\frac{M}{r^3} - \frac{3iMa\cos\theta}{r^4} + O(1/r^5) \qquad \Psi_3 = -\frac{3iMa\sin\theta}{2\sqrt{2}r^4} + O(/r^5) \qquad \Psi_4 = \frac{3Ma^2\sin^2\theta}{4r^5} + O(1/r^6).$$

From

$$g_{ab} = l_a n_b + l_b n_a - m_a\bar{m}_b - \bar{m}_a m_b$$

together with (10) and (11), the asymptotic form of the Kerr metric near future null infinity may be worked out to be

$$\begin{aligned} ds^2 = &\left\{1 - \frac{2M}{r} + \frac{2Ma^2\cos^2\theta}{r^3} + O(1/r^4)\right\}du^2 + \left\{1 - \frac{a^2\sin^2\theta}{2r^2} + O(1/r^3)\right\}2dudr + \left\{\frac{a^2\sin\theta\cos\theta}{r} + O(1/r^2)\right\}2dud\theta \\ &+ \left\{\frac{2Ma\sin^2\theta}{r} + O(1/r^2)\right\}2dud\varphi + O(1/r^4)dr^2 + \left\{\frac{a^2\sin\theta\cos\theta}{r} + O(1/r^2)\right\}2drd\theta \\ &+ \left\{\frac{2Ma\sin^2\theta}{r} + O(1/r^2)\right\}2drd\varphi - r^2\left\{1 + \frac{a^2\cos^2\theta}{r^2} + O(1/r^3)\right\}d\theta^2 + \left\{\frac{2Ma^3\sin^3\theta\cos\theta}{r^2} + O(1/r^3)\right\}2d\theta d\varphi \\ &- r^2\left\{\sin^2\theta + \frac{a^2\sin^2\theta}{r^2} + O(1/r^3)\right\}d\varphi^2. \end{aligned} \tag{13}$$

By the standard choice of conformal factor $\Omega = 1/r$, the metric in (13) may be conformally compactified. It may be checked that the gradient of Ω at $\mathcal{I}^+$ is nonzero and the second derivative of Ω vanishes at $\mathcal{I}^+$. The structure of $\mathcal{I}^+$ for the Kerr metric is Minkowskian in the sense that the null generators are complete and the topology is that of a light cone with its apex taken away (i.e. topologically $S^2 \times R$) [14,15]. Further, a constant u hypersurface intersects $\mathcal{I}^+$ in a unit two sphere.

From (13), we observe that the zero and first order of the metric coincide with that of Minkowski and Schwarzschild, respectively. Angular momentum appears in the terms of the order $O(1/r^2)$. This is reminiscent of the asymptotic behavior of the Kerr metric near spatial infinity.

The non-null character of dr in (13) means that the coordinates (θ, φ) are not constant along a null generator of a constant u hypersurface. The presence of angular momentum generates rotation of a constant r spherical section during its motion along a constant u null hypersurface, taking along with it also the symmetry axis. This suggests that the coordinates inherited from that of Boyer-Lindquist may not be the natural one to work with near null infinity. This motivates us to further develop the Newman-Unti (NU) formalism which describes a null generator of a constant u hypersurface in terms of its natural affine parameter and the angular coordinates are those pulling back from null infinity.

III. NU FORMALISM AND NP CONSTANTS

Suppose $l^a = (\frac{\partial}{\partial\lambda})^a$ such that λ is an affine parameter of a null generator of a constant u hypersurface. In terms of the Boyer-Lindquist coordinates, we have from (10) and (11) that

$$\begin{aligned} l^a = \Bigg(&1 + \frac{2M}{r} + \frac{4M^2}{r^2} + \frac{8M^3 - 2Ma^2\cos^2\theta}{r^3} + O(1/r^4), 1 + \frac{a^2\sin^2\theta}{2r^2} + \frac{Ma^2\sin^2\theta}{r^3} - \frac{a^4(1 + 6\cos^2\theta - 7\cos^4\theta)}{8r^4} \\ &- \frac{Ma^4(1-\cos^4\theta)}{2r^5} + \frac{a^6(1 + 5\cos^2\theta + 3\cos^4\theta - 9\cos^6\theta) - 8M^2a^4\sin^4\theta}{16r^6} + O(1/r^7), \frac{a^2\sin\theta\cos\theta}{r^3} \\ &- \frac{a^4\sin\theta\cos\theta(1 - \frac{1}{2}\sin^2\theta)}{r^5} - \frac{Ma^4\sin^3\theta\cos\theta}{4r^6} + O(1/r^7), \frac{2Ma}{r^3} + \frac{4M^2a}{r^4} + \frac{8M^3a - 2Ma^3(1 + \cos^2\theta)}{r^5} + O(1/r^6)\Bigg). \end{aligned} \tag{14}$$

From the definition of l^a and (14), we may obtain the following coordinate transformations:

$$\begin{aligned} r = &\lambda - \frac{a^2\sin^2\tilde\theta}{2\lambda} - \frac{Ma^2\sin^2\tilde\theta}{2\lambda^2} - \frac{a^4(1 - 6\cos^2\tilde\theta + 5\cos^4\tilde\theta)}{8\lambda^3} - \frac{Ma^4(3 - 10\cos^2\tilde\theta + 7\cos^4\tilde\theta)}{8\lambda^4} \\ &- \frac{a^4\sin^2\tilde\theta[16M^2\sin^2\tilde\theta + 5a^2(1 - 14\cos^2\tilde\theta + 21\cos^4\tilde\theta)]}{80\lambda^5} + O(1/\lambda^6) \\ \theta = &\tilde\theta - \frac{a^2\sin\tilde\theta\cos\tilde\theta}{2\lambda^2} + \frac{3a^4\sin\tilde\theta\cos\tilde\theta\cos2\tilde\theta}{8\lambda^4} - \frac{Ma^4\sin^3\tilde\theta\cos(\tilde\theta)}{4\lambda^5} + O(1/\lambda^6) \qquad \varphi = \tilde\varphi - \frac{Ma}{\lambda^2} - \frac{4M^2a}{3\lambda^3} + O(1/\lambda^4), \end{aligned} \tag{15}$$

where $(\tilde\theta, \tilde\varphi)$ are the angular coordinates pulled back from that of a cut at $\mathcal{I}^+$.

Before we move on, we digress at this point to work out the Bondi-Sachs form of the Kerr metric, which is important for the understanding of the characteristic structure of the Kerr metric. Define the luminosity parameter $\bar r$ in the standard way (see for instance [16]) by

$$\partial_\lambda \bar r = -\rho\bar r. \tag{16}$$

In view of (15) and the explicit expression of ρ given in terms of the Boyer-Lindquist coordinates in (12), it may be inferred that

$$\rho = -\frac{1}{\lambda} - \frac{9M^2a^4\sin^4\tilde\theta}{20\lambda^7} + O(1/\lambda^8). \tag{17}$$

Integrating (16) with the help of (17), we then find

$$\bar{r} = \lambda - \frac{3M^2a^4\sin^4\tilde{\theta}}{40\lambda^5} + O(1/\lambda^6). \tag{18}$$

Substituting (15) and (18) into the metric in (13) or alternatively expressing the NP tetrad in ((10)) in terms of the coordinates $(u, \bar{r}, \tilde{\theta}, \tilde{\varphi})$, we may derive the Bondi-Sachs form of the Kerr metric given as

$$\begin{aligned} ds^2 = {} & \Big(1 - \frac{2M}{\bar{r}} + \frac{Ma^2(2\cos^2\tilde{\theta} - \sin^2\tilde{\theta})}{\bar{r}^3} + O(1/\bar{r}^4)\Big)du^2 + \Big(1 - \frac{15a^4\sin^2\tilde{\theta}(5 + 8\cos 2\tilde{\theta} - \cos^2\tilde{\theta})}{8\bar{r}^4} + O(1/\bar{r}^5)\Big)2dud\bar{r} \\ & - \Big(\frac{3Ma^2\sin\tilde{\theta}\cos\tilde{\theta}}{\bar{r}^2} + O(1/\bar{r}^3)\Big)2dud\tilde{\theta} + \Big(\frac{2Ma\sin^2\tilde{\theta}}{\bar{r}} + O(1/\bar{r}^2)\Big)2dud\tilde{\varphi} - \Big(\bar{r}^2 - \frac{Ma^2\sin^2\tilde{\theta}}{\bar{r}} + O(1/\bar{r}^2)\Big)d\tilde{\theta}^2 \\ & - \Big(\frac{6M^2a^3\sin^3\tilde{\theta}\cos\tilde{\theta}}{\bar{r}^3} + O(1/\bar{r}^4)\Big)2d\tilde{\theta}d\tilde{\varphi} - \Big(\bar{r}^2\sin^2\tilde{\theta} + \frac{Ma^2\sin^4\tilde{\theta}}{\bar{r}} + O(1/\bar{r}^2)\Big)d\tilde{\varphi}^2. \end{aligned}$$

In principle, with more involved calculations, higher order terms of the Bondi-Sachs form of the metric may be generated using the same algorithm.

Now we return to our discussion of the NU formalism for the Kerr metric, and we shall adopt the coordinates $\{u, \lambda, \tilde{\theta}, \tilde{\phi}\}$ again in our subsequent discussions. The NP tetrad given in (10) and (11) are not parallelly transported along a generator of a constant u hypersurface. This manifests in the nonzeroness of τ' and the imaginary part of ϵ in (12). In the next step, with l^a kept fixed, we shall rotate the NP tetrad defined in (10) into one which is parallelly transported along a null generator of a constant u hypersurface, again in an order by order fashion. To this end, we first rotate m^a by a phase angle, i.e. $m^a \to e^{i\chi}m^a$ where χ is a real valued function of λ, $\tilde{\theta}$, $\tilde{\phi}$. The spin coefficient ϵ transforms accordingly as

$$\epsilon \to \epsilon + \frac{1}{2}il^a\nabla_a\chi.$$

For m^a to be parallelly transported along a null generator with tangent vector l^a, the vanishing of ϵ requires

$$\chi = \chi_0(\tilde{\theta}, \tilde{\varphi}) + O(1/\lambda^5), \tag{19}$$

where χ_0 is an arbitrary, real valued function defined on a unit two sphere and it signifies the $SO(2)$ degrees of freedom in the definition of m^a and $\bar{m}^a$ with l^a, n^a fixed. By further stipulating that asymptotically the angular part of m^a should take the form $\frac{1}{\sqrt{2}r}(\partial_{\tilde{\theta}} + i\csc\tilde{\theta}\partial_{\tilde{\varphi}})$, we may choose χ_0 to be zero and conclude from (19) that

$$\chi = O(\lambda^{-5}). \tag{20}$$

This will be sufficient for the calculation of the NP constants to be considered in a moment.

For n^a to be parallelly transported, we need to perform the null rotation

$$\begin{aligned} & l^a \to l^a, \qquad m^a \to m^a + bl^a, \\ & n^a \to n^a + \bar{b}m^a + b\bar{m}^a + b\bar{b}l^a \end{aligned} \tag{21}$$

where b is a complex valued function of λ, $\tilde{\theta}$, $\tilde{\varphi}$ and $\bar{b}$ denotes its complex conjugation. Subject to (21), ϵ remains unchanged due to $\kappa = 0$, while τ' transforms as

$$\tau' \to \tau' - 2\bar{b}\epsilon - l^a\nabla_a\bar{b}. \tag{22}$$

With the help of (12) and (20), from (22) we may infer that, for τ' to vanish, we require

$$b = \frac{3iMa\sin\tilde{\theta}}{2\sqrt{2}\lambda^2} - \frac{Ma^2\sin\tilde{\theta}\cos\tilde{\theta}}{\sqrt{2}\lambda^3} + O(1/\lambda^4). \tag{23}$$

The constant of integration is set to zero in (23) so that asymptotically the component of m^a in the $\frac{\partial}{\partial\lambda}$ direction starts from the order of $O(1/\lambda)$.

For the parallelly transported NP tetrad on a constant u hypersurface, the corresponding spin coefficients and the peeling off behavior of the Weyl curvature components may then be worked out to be

$$\begin{aligned} & \kappa = \epsilon = \kappa' = \tau' = 0 \qquad \rho = -\frac{1}{\lambda} - \frac{9M^2a^4\sin^4\tilde{\theta}}{20\lambda^7} + O(1/\lambda^8) \qquad \sigma = -\frac{3Ma^2\sin^2\tilde{\theta}}{2\lambda^4} - \frac{5iMa^3\sin^2\tilde{\theta}\cos\tilde{\theta}}{r^5} + O(1/\lambda^6) \\ & \tau = -\frac{3iMa\sin\tilde{\theta}}{2\sqrt{2}\lambda^3} + \frac{2\sqrt{2}Ma^2\sin\tilde{\theta}\cos\tilde{\theta}}{\lambda^4} + O(1/\lambda^5) \qquad \beta = \frac{\cot\tilde{\theta}}{2\sqrt{2}\lambda} - \frac{3iMa\sin\tilde{\theta}}{2\sqrt{2}\lambda^3} + O(1/\lambda^4) \\ & \rho' = \frac{1}{2\lambda} - \frac{M}{\lambda^2} - \frac{3iMa\cos\tilde{\theta}}{2\lambda^3} + O(1/\lambda^4) \qquad \sigma' = \frac{Ma^2\sin^2\tilde{\theta}}{4\lambda^4} + O(1/\lambda^5) \qquad \epsilon' = -\frac{M}{2\lambda^2} - \frac{3iMa\cos\tilde{\theta}}{4\lambda^3} + O(1/\lambda^4) \\ & \beta' = \frac{\cot\tilde{\theta}}{2\sqrt{2}\lambda} + O(1/\lambda^4) \end{aligned} \tag{24}$$

and

$$\Psi_0 = \frac{3Ma^2\sin^2\tilde{\theta}}{\lambda^5} + \frac{15iMa^3\sin^2\tilde{\theta}\cos\tilde{\theta}}{\lambda^6} + O(1/\lambda^7) \qquad \Psi_1 = \frac{3iMa\sin\tilde{\theta}}{\sqrt{2}\lambda^4} - \frac{6\sqrt{2}Ma^2\sin\tilde{\theta}\cos\tilde{\theta}}{\lambda^5} + O(1/\lambda^6)$$
$$\Psi_2 = -\frac{M}{\lambda^3} - \frac{3iMa\cos\tilde{\theta}}{\lambda^4} + O(1/\lambda^5) \qquad \Psi_3 = -\frac{3iMa\sin\tilde{\theta}}{2\sqrt{2}\lambda^4} + O(1/\lambda^5) \qquad \Psi_4 = \frac{3Ma^2\sin^2\tilde{\theta}}{4\lambda^5} + O(1/\lambda^6). \tag{25}$$

We may see from the spin coefficient ρ in (24) that, once the Boyer-Lindquist coordinates is chosen, the scaling freedom for the affine parameter is also determined. From the asymptotic behavior of the spin coefficient σ, it may also be seen that the asymptotic shear defined by $\Omega^{-2}\sigma$ responsible for the news vanishes on the unit sphere at which a constant u hypersurface intersects $\mathcal{I}^+$. This existence of this kind of good cuts is characteristic of the asymptotic structure of a stationary, weakly asymptotically simple spacetime in which the gravitational radiation field defined by $\Omega^{-1}\Psi_4$ vanishes [17].

Define $\hat{\tau} = \Omega^{-3}\tau$ and $\hat{\Psi}_1 = \Omega^{-4}\Psi_1$. The angular momentum of the Kerr metric may be expressed as

$$Ma = -\frac{\sqrt{2}}{3i\pi^2}\int \hat{\tau} d\hat{S} = \frac{\sqrt{2}}{3i\pi^2}\int \hat{\Psi}_1 d\hat{S}, \tag{26}$$

where the integration is over a unit two sphere at $\mathcal{I}^+$. Using the NP equations and the explicit expressions of the spin coefficients given in (24), it may be deduced that the above angular momentum expressions are all special cases of the the linkage expression [18] (or equivalently the Komar integral) written in terms of the NP tetrad chosen here. From (26), we also see that the angular momentum is a measure of nonintegrability of the timelike two plane spanned by the null vectors l^a, n^a (see also [19] in this connection).

A. NP constants of the Kerr metric

With the NU framework we have developed and the calculations we have done on various quantities, we are now in a position to further compute the NP constants for the Kerr metric.

Consider the NP constants [17] defined as

$$G_m = \int_0^{2\pi}\int_0^{\pi} {}_2\bar{Y}_{2m}\Psi_0^1 \sin\tilde{\theta} d\tilde{\theta} d\tilde{\phi}, \quad m = 0, \pm 1, \pm 2 \tag{27}$$

where ${}_2Y_{2m}$ are the spin weight 2 spherical harmonics. Ψ_0^1 is defined by the asymptotic expansion of Ψ_0 as

$$\Psi_0 = \frac{\Psi_0^0}{\lambda^5} + \frac{\Psi_0^1}{\lambda^6} + O(1/\lambda^7).$$

Axisymmetry of the Kerr metric means that $G_{\pm 1}$ and $G_{\pm 2}$ vanish trivially. With

$$\Psi_0^1 = 15iMa^3\sin^2\tilde{\theta}\cos\tilde{\theta}$$

according to (25), it may be calculated easily from (27) that $G_0 = 0$ and therefore all NP constants vanish in a Kerr metric.

Alternatively, with the definitions of multipole moments defined in [20], we may also work out from (25) the explicit expressions for the monopole ($\mathcal{M}$), dipole ($\mathcal{D}$) and quadruple ($\mathcal{Q}$) moments and may be given, respectively, as

$$\mathcal{M} = -\frac{1}{4}\int_0^{\pi}(\Psi_2^0 + \bar{\Psi}_2^0)\sin\tilde{\theta} d\tilde{\theta} = M$$
$$\mathcal{D} = -\frac{1}{2\sqrt{2}}\int_0^{\pi}\Psi_1^0 P_1^1(\cos\tilde{\theta})\sin\tilde{\theta} d\tilde{\theta} = iMa$$
$$\mathcal{Q} = -\frac{5}{24}\int_0^{\pi}\Psi_0^0 P_2^2(\cos\tilde{\theta})\sin\tilde{\theta} d\tilde{\theta} = -2Ma^2,$$

where P_l^m are the standard Legendre polynomials. The NP constant G_0 may also be calculated from the formula [17]

$$G_0 = 2\sqrt{30\pi}(2\mathcal{D}^2 - \mathcal{M}\mathcal{Q}) \tag{28}$$

and again we obtain zero. This gives a consistency check on the calculations of the NP constants using the definition in (27). In principle, higher multipole moments of the Kerr metric may also be obtained similarly at the cost of more complex calculations of the higher order terms of Ψ_0.

B. Physical implications.

The vanishing of the NP constants of the Kerr metric is puzzling in connection with the no hair theorem for a black hole [21]. The no hair theorem asserts that a Kerr black hole is the unique final state for gravitational collapse, like for instance in the merger of binary black holes with nonzero residual angular momentum. Certainly we do not expect a generic initial data set which lead to eventual gravitational collapse will have vanishing NP constants (see for instance [22]). But then how do we reconcile this with the vanishing of the NP constants for the Kerr metric?

One way out, without compromising the no hair theorem, is that at the initial stage of a black hole evolution, the structure of null infinity is not smooth enough, i.e. the conformal completion of the physical spacetime is C^k, $k < 5$. The NP constants are then not well defined at this early time of the evolution. When the evolution enters a stage in which the Weyl curvature falls off sufficiently rapid so that $\mathcal{I}^+$ becomes smooth enough, the NP constants begin to set in. Another possibility we should not overlook is that perhaps some hypotheses of the no hair theorem may not be applicable for a generic collapse situation. Certainly there are other possibilities we may think of. The vanishing of the NP constants for the Kerr metric together with the no hair theorem set a very stringent constraint for black hole evolution. It is also worth understanding better to what extent the NP constants constrain the dynamics of gravitational collapse.

IV. CONCLUDING REMARKS

One obvious shortcoming of the present work is that the construction is valid only in a neighborhood of null infinity. Unless we have an analytic solution to the eikonal equation which matches to that given here near null infinity, it is difficult to extend the asymptotic coordinates to cover entirely the Kerr metric exterior to the event horizon. Still we hope the present work will provide a small step towards our understanding of the gravitational waveform of a relativistic rotating source. Further, the vanishing of the NP constants of the Kerr metric also requires better understanding from the perspective of black hole evolution and gravitational wave physics of a spacetime. This will hopefully enables us to gain deeper insight into the physical meaning of these mysterious constants.

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