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The Newtonian Limit for Perfect Fluids

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Abstract: We prove that there exists a class of non-stationary solutions to the Einstein-Euler equations which have a Newtonian limit. The proof of this result is based on a symmetric hyperbolic formulation of the Einstein-Euler equations which contains a singular parameter $\epsilon = v_T/c$, where v_T is a characteristic velocity scale associated with the fluid and c is the speed of light. The symmetric hyperbolic formulation allows us to derive ϵ independent energy estimates on weighted Sobolev spaces. These estimates are the main tool used to analyze the behavior of solutions in the limit $\epsilon \searrow 0$.

1. Introduction

The Einstein-Euler equations or, in other words, the Einstein equations coupled to a simple perfect fluid are given by the following system of equations:

$$G^{ij} = \frac{8\pi G}{c^4} T^{ij}, \tag{1.1}$$

$$\nabla_i T^{ij} = 0, \tag{1.2}$$

where the stress-energy tensor for the fluid is given by

$$T^{ij} = (\rho + c^{-2}p)v^i v^j + pg^{ij} \tag{1.3}$$

with ρ the fluid density, p the fluid pressure, and v the fluid four-velocity normalized by $v^i v_i = -c^2$, c the speed of light, and G the Newtonian gravitational constant. The study of the behavior of solutions to these equations in the limit that $\epsilon = v_T/c \searrow 0$, where v_T is a characteristic velocity scale associated with the fluid matter is known as

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the *Newtonian limit*. By suitably rescaling the gravitational and matter variables (see Sect. 2), the Einstein-Euler equations can be written as

$$G^{ij} = 2\kappa\epsilon^4 T^{ij} \quad \text{and} \quad \nabla_i T^{ij} = 0, \tag{1.4}$$

where $\kappa = 4\pi G\rho_T/v_T^2$, $v_i v^i = -\epsilon^{-2}$, ρ_T is a characteristic value for the fluid density, and $t = x^4/v_T$ is a "Newtonian" time coordinate. In the limit $\epsilon \searrow 0$, one expects that there exists a class of solutions to Einstein-Euler equations (1.4) that approach solutions of the Poisson-Euler equations

$$\partial_t\rho + \partial_I(\rho w^I) = 0, \qquad (I, J = 1, 2, 3) \tag{1.5}$$

$$\rho(\partial_t w^J + w^I\partial_I w^J) = -(\rho\partial^J\Phi + \partial^J p), \qquad (\partial^I = \delta^{IJ}\partial_J) \tag{1.6}$$

$$\Delta\Phi = \rho, \qquad (\Delta = \partial_I\partial^I) \tag{1.7}$$

of Newtonian gravity in some sense. As above, ρ and p are the fluid density and pressure, respectively, while w^I is the fluid (three) velocity. This problem has been studied since the discovery of general relativity by many people and there is a large number of results available in the literature. The majority of results are based on formal expansions in the parameter ϵ which are used to calculate the (approximate) values of physical quantities and also to investigate the behavior of the gravitational and matter fields in the limit $\epsilon \searrow 0$. For some classic and recent results of this type see [2,3,6,9,11–13,20–22,31,41] and references cited therein. The main difficulty with the formal expansions is that they leave completely unanswered the question of convergence. In the absence of a precise notion of convergence, it becomes unclear to what extent the formal expansions actually approximate relativistic solutions.

In this paper, we go beyond formal considerations and supply a precise notion of convergence for gravitating perfect fluids as $\epsilon \searrow 0$. This necessitates introducing suitable variables that are compatible with the limit $\epsilon \searrow 0$. The metric g_{ij}, which defines the gravitational field, turns out to be singular in this limit. To remedy this problem, we introduce a new gravitational density $\bar{u}^{ij}$ which is related to the metric via the formula

$$g^{ij} = \frac{\epsilon}{\sqrt{-\det(Q)}} Q^{ij}, \tag{1.8}$$

where

$$Q^{ij} = \begin{pmatrix} \delta^{IJ} & 0 \\ 0 & 0 \end{pmatrix} + \epsilon^2 \begin{pmatrix} 4\bar{u}^{IJ} & 0 \\ 0 & -1 \end{pmatrix} + 4\epsilon^3 \begin{pmatrix} 0 & \bar{u}^{I4} \\ \bar{u}^{J4} & 0 \end{pmatrix} + 4\epsilon^4 \begin{pmatrix} 0 & 0 \\ 0 & \bar{u}^{44} \end{pmatrix}. \tag{1.9}$$

From this, it not difficult to see that the density $\bar{u}^{ij}$ is equivalent to the metric g_{ij} for $\epsilon > 0$ and is well defined at $\epsilon = 0$. For the fluid, we also introduce a new velocity variable w^i according to

$$v^I = w^I \quad \text{and} \quad v^4 = 1 + \epsilon w^4. \tag{1.10}$$

For technical reasons, we only consider isentropic flow where the pressure is related to the density by an equation of state of the form $p = f(\rho)$. Moreover, to formulate a symmetric hyperbolic system for the fluid variables $\{\rho, v\}$, we need to deal with the well known problem that the system becomes singular when $\rho + c^{-2}p = 0$. This is a particular problem for fluid balls having compact support. To get around this problem, we follow Rendall [34] and use a technique of Makino [24] to regularize the fluid equations so that a class of gravitating fluid ball solutions can be constructed. Thus as in [34], we assume an equation of state of the form

$$p = K\rho^{(n+1)/n}, \tag{1.11}$$

where $K \in \mathbb{R}_{>0}$, $n \in \mathbb{N}$, and we introduce a new "density" variable α via the formula

$$\rho = \frac{1}{(4Kn(n+1))^n}\alpha^{2n}. \tag{1.12}$$

As discussed by Rendall, the type of fluid solutions obtained by this method have freely falling boundaries and hence do not include static stars of finite radius and so this method is far from ideal. However, in trying to understand the Newtonian limit and post-Newtonian approximations these solutions are almost certainly general enough to obtain a comprehensive understanding of the mathematical issues involved in the Newtonian limit and post-Newtonian approximations. We would also like to remark that the results contained in this article are largely independent of the specific structure of the fluid equations. We therefore expect that the analysis in this paper can be carried over without much difficulty to other matter models whose equations can be formulated as a symmetric hyperbolic system and have a finite propagation speed for the matter density in the limit $\epsilon \searrow 0$.

Our approach to analyze the limit $\epsilon \searrow 0$ is to use the gravitational and matter variables $\{\bar{\mathfrak{u}}^{ij}, w^i, \alpha\}$ along with a harmonic gauge to put the Einstein-Euler equations into the following form:

$$b^0(\epsilon V)\partial_t V = \frac{1}{\epsilon}c^I\partial_I V + b^I(\epsilon, V)\partial_I V + f(\epsilon, V)V + \frac{1}{\epsilon}g(V)V + h(\epsilon), \tag{1.13}$$

where V comprises both the gravitational and matter variables, and the c^I are constant matrices. This system is symmetric hyperbolic and hence by standard theory there exist local solutions. However, the difficulty in analyzing the limit $\epsilon \searrow 0$ of such solutions is that the equation contains the singular terms $\epsilon^{-1}c^I\partial_I V$ and $\epsilon^{-1}g(V)V$. Although, singular limits of symmetric hyperbolic equations have been previously analyzed in [5, 19, 37, 38], these results cannot be directly applied to the system (1.13). There are two main difficulties in adapting these results to the Einstein-Euler system. The first is that the Einstein-Euler system (6.1) must be modified by including an elliptic equation, essentially the Newtonian Poisson equation, in order to be of the canonical form required by [5, 19, 37, 38]. This results in a coupled elliptic-hyperbolic system of the form

$$B^0(\epsilon W)\partial_t W = \frac{1}{\epsilon}c^I\partial_I W + B^I(\epsilon, W)\partial_I W + F(\epsilon, W)W + H(\epsilon), \tag{1.14}$$

where W is related to V via an elliptic equation and F is a non-local functional. The second difficulty is that the initial data which must include a $1/r$ piece for the metric and hence it cannot lie in the Sobolev space H^k. This $1/r$ type fall-off behavior is crucial for obtaining the correct limit and is intimately tied to the elliptic part of our formulation of the Einstein-Euler system. The standard procedure in general relativity to deal with this type of fall off, at least for elliptic systems, is to replace the spaces H^k with the weighted Sobolev spaces H^k_δ [1,7]. However, the arguments used in [5, 19, 37, 38] fail for the weighted spaces as the weight used to define the H^k_δ spaces destroys the integration by parts argument which is used to control the singular term $\epsilon^{-1}c^I\partial_I W$ in (1.14). Indeed, using integration by parts, it follows easily from the definition of the weighted L^2_δ inner-product (see (A.4) with $\epsilon = 1$) that

$$\left\langle -\epsilon^{-1} c^I \partial_I W | W \right\rangle_{L^2_\delta} = -\frac{1}{2\epsilon} \left\langle \partial_I (\sigma^{-2\delta-3}) c^I W | W \right\rangle_{L^2}, \tag{1.15}$$

where $\sigma(x) = \sqrt{1 + |x|^2/4}$. In general, this term will blow up as $\epsilon \searrow 0$ unless $\delta = -3/2$ which coincides with the standard L^2 norm. However, to include $1/r$ fall-off, we need to consider $-1 < \delta < 0$ which introduces a singular $1/\epsilon$ term into energy estimates based on the weighted norm H^k_δ.

To overcome this problem, we introduce a sequence of weighted spaces $H^k_{\delta,\epsilon}$ (see Appendix A for a definition) by replacing the weight $\sigma(x)$ with $\sigma_\epsilon(x) = \sigma(\epsilon x)$. Under this replacement, (1.15) changes to

$$\left\langle -\epsilon^{-1} c^I \partial_I W | W \right\rangle_{L^2_{\delta,\epsilon}} \leq C \langle W | W \rangle_{L^2_{\delta,\epsilon}},$$

which is no longer singular as $\epsilon \searrow 0$. This allows us to derive ϵ independent energy estimates for solutions to the Einstein-Euler equations. These estimates can then be used to define a precise notion of convergence for gravitating perfect fluids solutions in the limit $\epsilon \searrow 0$ which is essentially a statement about the validity of the zeroth order expansion in ϵ. This is formalized in the following theorem; for a more precise version see Propositions 5.1, 6.1 and 7.8, and Theorems 7.7 and 7.12.

Theorem 1.1. *Suppose $-1 < \delta < -1/2$, $k \geq 3 + s$, $\beta^j \in \bigcap_{\ell=0}^s C^\ell([0, T^*], H^{k-\ell}_{\delta-1})$ is a harmonic gauge source function, and $\underset{o}{\alpha}, \underset{o}{w}^I \in H^k_{\delta-1}$, $\mathfrak{z}^{IJ} \in H^{k+1}_\delta$, $\mathfrak{z}_4^{IJ} \in H^k_{\delta-1}$ is the free initial data for the Einstein-Euler equations where* supp $\underset{o}{\alpha} \subset B^*_R$ *for some $R^* > 0$. Then for ϵ_0 small enough, there exists a $T \in (0, T^*]$ independent of $\epsilon \in (0, \epsilon_0]$, and maps*

$$\bar{\mathfrak{u}}^{ij}_\epsilon(t) - \bar{\mathfrak{u}}^{ij}_\epsilon(0),\ \partial_I \bar{\mathfrak{u}}^{ij}_\epsilon(t),\ \partial_t \bar{\mathfrak{u}}_\epsilon(t),\ \alpha_\epsilon(t),\ w^i_\epsilon(t) \in \bigcap_{\ell=0}^{s+1} C^\ell([0, T], H^{k-\ell}_{\delta-1,\epsilon}),$$

$$\Phi \in C^0([0, T^*], H^{k+2}_\delta) \cap C^1([0, T^*], H^{k+1}_\delta),$$

$$w^I \in C^0([0, T^*], H^k_{\delta-1}) \cap C^1([0, T^*], H^{k-1}_{\delta-1}),$$

$$\rho \in C^0([0, T^*], H^k_{\delta-1}) \cap C^1([0, T^*], H^{k-1}_{\delta-2}),$$

such that

(i)

$$(\bar{\mathfrak{u}}^{ij}_\epsilon(0)) = \begin{pmatrix} \epsilon \mathfrak{z}^{IJ} & \epsilon \mathfrak{w}^I_\epsilon \\ \epsilon \mathfrak{w}^J_\epsilon & \phi_\epsilon \end{pmatrix},$$

$$(\partial_t \bar{\mathfrak{u}}^{ij}_\epsilon(0)) = \begin{pmatrix} \mathfrak{z}_4^{IJ} & -\partial_K \mathfrak{z}^{KI} + \beta^I(0) \\ -\partial_K \mathfrak{z}^{KJ} + \beta^J(0) & -\partial_K \mathfrak{w}^K_\epsilon + \beta^4 \end{pmatrix},$$

$$w^4_\epsilon(0) = -\frac{1}{\epsilon} + \frac{-\epsilon \bar{g}_{4J} \underset{o}{w}^J - \sqrt{\epsilon^2 (\bar{g}_{4J} \underset{o}{w}^J)^2 - \bar{g}_{44}(\epsilon^2 \bar{g}_{IJ} \underset{o}{w}^I \underset{o}{w}^J + 1)}}{\epsilon \bar{g}_{44}},$$

$$w^I_\epsilon(0) = w^I(0) = \underset{o}{w},$$
$$\alpha_\epsilon(0) = \underset{o}{\alpha},$$
$$\rho(0) = \underset{o}{\rho} = (4Kn(n+1))^{-n}\underset{o}{\alpha}^{2n},$$

where $\phi_\epsilon = \phi(\epsilon, \underset{o}{\rho}, \underset{o}{w}^I, \mathfrak{z}_4^{IJ}, \beta^j(0), \mathfrak{z}^{IJ})$, *and* $\mathfrak{w}_\epsilon = \mathfrak{w}(\epsilon, \underset{o}{\rho}, \underset{o}{w}^I, \mathfrak{z}_4^{IJ}, \beta^j(0), \mathfrak{z}^{IJ})$ *is the initial data determined by the gravitational constraint equations (see Proposition 5.1), and* $\bar{g}_{ij}$ *is determined from* $\bar{\mathfrak{u}}^{ij}_\epsilon(0)$ *by the formulas* (1.8) *and* (3.1)*,*

(ii) $\{\bar{\mathfrak{u}}^{ij}_\epsilon(x^I,t), \alpha_\epsilon(x^I,t), w^i_\epsilon(x^I,t)\}$ *determines, via the formulas* (1.8)*,* (1.9)*,* (1.10)*, and* (1.12)*, a* 1*-parameter family* $(0 < \epsilon \le \epsilon_0)$ *of solutions to the Einstein-Euler equations* (1.4) *in the harmonic gauge* $\epsilon\partial_t\bar{\mathfrak{u}}^{4j}_\epsilon + \partial_I\bar{\mathfrak{u}}^{Ij}_\epsilon = \epsilon\beta^j$ *on the common spacetime region* $(x^I,t) \in D = \mathbb{R}^3 \times [0,T]$*,*

(iii) $\{\Phi(x^I,t), \rho(x^I,t), w^I(x^I,t)\}$ *solves the Euler-Poisson equations* (1.5)–(1.7) *on the spacetime region D,*

(iv) there exists a constant $R \in (R^*, \infty)$ *independent of* $\epsilon \in (0,\epsilon_0]$ *such that* supp $\alpha_\epsilon(t)$, supp $\rho(t) \subset B_R$ *for all* $(t,\epsilon) \in [0,T] \times (0,\epsilon_0]$*, and*

(v) there exists a constant $C > 0$ *independent of* $\epsilon \in (0,\epsilon_0]$ *such that*

$$\|\bar{\mathfrak{u}}^{ij}_\epsilon(t) - \delta^i_4\delta^i_4\Phi(t)\|_{L^6} + \|\partial_I\bar{\mathfrak{u}}^{ij}_\epsilon(t) - \delta^i_4\delta^j_4\partial_I\Phi(t)\|_{H^{k-1}} + \|v^I(t) - w^I(t)\|_{H^{k-1}}$$
$$+\epsilon^{-1}\|v^4(t) - 1\|_{H^{k-1}} + \|\rho_\epsilon(t) - \rho(t)\|_{H^{k-1}} + \|\partial_t\rho_\epsilon(t) - \partial_t\rho(t)\|_{H^{k-2}} \le C\epsilon$$

for all $(t,\epsilon) \in [0,T^*] \times (0,\epsilon_0]$.

We remark that the techniques of this paper can also be used to derive *convergent* expansions in ϵ of the type considered in Theorems 2 and 3 of [19] and [38], respectively. These convergent expansions in general differ from the formal post-Newtonian expansions. To get post-Newtonian expansion to a certain order in ϵ requires that the initial data must be chosen correctly. In the absence of constraints on the initial data, a general procedure for doing this is discussed in [5]. Due to the fact that there are constraints on the initial data in general relativity, this becomes a non-trivial problem called the *initialization problem*. See [18] for an extended discussion. The proof of convergence and a discussion of the initialization problem will be presented in a separate paper [27].

We note that similar results for the Vlasov-Einstein system have been derived in [36] using a zero shift maximal slicing gauge. However, unlike [36], our approach is able to handle not only higher order expansions in ϵ, but also a wide variety of matter models. We also note that in [16,18], there is another interesting proposal for analyzing the limit as $\epsilon \searrow 0$ which is based on a gauge for which the Einstein equations are again elliptic-hyperbolic but distinct from [36]. As in this article, the authors of [16,18] also propose to use the methods of [5,19,37,38]. However, the required estimates are not proven and it is yet to be verified if this approach would be successful.

We remark that the results of this and the companion paper [27] are local in time and therefore address the "near zone" problem. In the special case of spherical symmetry, the situation improves and there are some global results available on the Newtonian limit [26,32]. However, because spherically symmetric systems do not generate gravitational radiation, these results do not shed light on the "far zone" problem for post-Newtonian expansions where radiation plays a crucial role and the $\epsilon \searrow 0$ limit must be analyzed in the region "close" to future null infinity. We plan to investigate the far zone problem in the near future.

Our paper is organized as follows: in Sect. 2, we define dimensionless variables for the Einstein-Euler system. Sections 3 and 4 are devoted to introducing variables and a gauge condition that cast the Einstein-Euler equations into a form suitable for analyzing the limit $\epsilon \searrow 0$. Appropriate initial data which is regular in the limit $\epsilon \searrow 0$ is constructed in Sect. 5 while in Sect. 6 we prove a local existence theorem for the Einstein-Euler system on the weighted spaces. Finally, in Sect. 7, we show that solutions to the Einstein-Euler system converge as $\epsilon \searrow 0$ to solutions of the Poisson-Euler system of Newtonian gravity. A precise statement of convergence is contained in Theorem 7.12 which is the main result of this paper.

2. Units

Our conventions for units are as follows:

$$[x^i]=L, \quad [g_{ij}]=1, \quad [\rho]=\frac{M}{L^3}, \quad [p]=\frac{M}{LT^2}, \quad [v^i]=[c]=\frac{L}{T}, \text{ and } [G]=\frac{L^3}{MT^2}.$$

Note that with these choices the stress-energy tensor has units of an energy density, i.e. $[T^{ij}] = \frac{M}{LT^2}$. To introduce dimensionless variables, we define

$$v^i = v_T \hat{v}^i \quad \text{and} \quad \rho = \rho_T \hat{\rho},$$

where v_T and ρ_T are "typical" values for the velocity and the density, respectively. The Einstein-Euler equations then can be written as

$$\hat{G}^{ij} = 2\kappa\epsilon^4 \hat{T}^{ij} \quad \text{and} \quad \hat{\nabla}_i \hat{T}^{ij} = 0,$$

where

$$\epsilon = \frac{v_T}{c}, \quad \kappa = \frac{4\pi G \rho_T}{v_T^2}, \quad \hat{x}^i = \sqrt{\kappa}\, x^i, \quad \hat{g}_{ij} = g_{ij}, \quad \hat{p} = \frac{p}{v_T^2 \rho_T},$$

and

$$\hat{T}^{ij} = (\hat{\rho} + \epsilon^2 \hat{p})\hat{v}^i \hat{v}^j + \hat{p}\hat{g}^{ij}.$$

The normalization $v_i v^i = -c^2$, implies that

$$\hat{v}_i \hat{v}^i := \hat{g}_{ij} \hat{v}^i \hat{v}^j = -\frac{1}{\epsilon^2}.$$

Also, we can introduce a time coordinate t via

$$t = x^4 / v_T.$$

With these choices, we have

$$[\epsilon] = [\hat{v}^i] = [\hat{\rho}] = [\hat{p}] = [\hat{g}] = [\hat{x}^i] = 1, \quad [v_T] = \frac{L}{T}, \quad [t] = [T], \text{ and } [\kappa] = \frac{1}{L^2}.$$

Thus all our dynamical variables and coordinates are dimensionless and the two constants v_T and κ can be used to fix the length and time scales by using units so that

$$v_T = 1 \quad \text{and} \quad \kappa = 1.$$

In this case we can use t and x^4 interchangeably as long as we remember that they carry different units. To simplify notation, we will drop the "hats" from the hatted variables for the remainder of this article.

3. Reduced Einstein Equations

To aid in deriving the appropriate symmetric hyperbolic system for the gravitational variables, we temporarily introduce a new set of coordinates related to old ones by the simple rescaling

$$\bar{x}^J = x^J, \quad \bar{x}^4 = x^4/\epsilon$$

and let

$$\partial_i = \frac{\partial}{\partial x^i}, \quad \bar{\partial}_i = \frac{\partial}{\partial \bar{x}^i}.$$

In the new coordinates, the metric $\bar{g}_{ij}$ and its inverse $\bar{g}^{ij}$ are given by

$$(\bar{g}_{ij}) = \begin{pmatrix} g_{IJ} & \epsilon g_{I4} \\ \epsilon g_{4J} & \epsilon^2 g_{44} \end{pmatrix} \quad \text{and} \quad (\bar{g}^{ij}) = \begin{pmatrix} g^{IJ} & \epsilon^{-1} g^{I4} \\ \epsilon^{-1} g^{4J} & \epsilon^{-2} g^{44} \end{pmatrix}. \tag{3.1}$$

Next, consider the metric density

$$\bar{\mathfrak{g}}^{ij} = \sqrt{|\bar{g}|}\, \bar{g}^{ij} \quad \text{where} \quad |\bar{g}| = -\det(\bar{g}_{ij}). \tag{3.2}$$

We note that the metric $\bar{g}^{ij}$ is related to the density $\bar{\mathfrak{g}}^{ij}$ by the following formula:

$$\bar{g}^{ij} = \frac{1}{\sqrt{|\bar{g}|}} \bar{\mathfrak{g}}^{ij} \quad \text{where} \quad |\bar{g}| = -\det \bar{\mathfrak{g}}^{ij}, \tag{3.3}$$

and hence

$$(g^{ij}) = \frac{1}{\sqrt{|\bar{g}|}} \begin{pmatrix} \bar{\mathfrak{g}}^{IJ} & \epsilon \bar{\mathfrak{g}}^{I4} \\ \epsilon \bar{\mathfrak{g}}^{4J} & \epsilon^2 \bar{\mathfrak{g}}^{44} \end{pmatrix}. \tag{3.4}$$

To obtain a gravitational variable that is regular and non-trivial in the limit $\epsilon \searrow 0$, we define

$$\bar{\mathfrak{u}}^{ij} = \frac{1}{4\epsilon^2} \left(\bar{\mathfrak{g}}^{ij} - \eta^{ij} \right), \tag{3.5}$$

where

$$\eta^{ij} = \begin{pmatrix} \mathbb{I}_{3\times 3} & 0 \\ 0 & -1 \end{pmatrix}$$

is the Minkowski metric density. As stated in the introduction, for $\epsilon > 0$, the metric g_{ij} can be recovered from the density $\bar{\mathfrak{u}}^{ij}$ via the formulas (1.8)–(1.9). As we shall see, even though the metric g_{ij} is singular in the limit $\epsilon \searrow 0$, the quantity $\bar{\mathfrak{u}}^{ij}$ is well defined at $\epsilon = 0$. We note that these variables are closely related to the gravitational variables discovered by Jürgen Ehlers and subsequently used in the papers [17,28,29] to construct stationary/static solutions to the Einstein equations coupled to various matter sources.

In the $(\bar{x}^i)$ coordinate system, the Christofell symbols are given by

$$\bar{\Gamma}^k_{ij} = \epsilon^2 \left(\bar{\mathfrak{g}}^{km} (2\bar{\mathfrak{g}}_{i\ell}\bar{\mathfrak{g}}_{jp} - \bar{\mathfrak{g}}_{ij}\bar{\mathfrak{g}}_{\ell p}) \bar{\partial}_m \bar{\mathfrak{u}}^{\ell p} + 2(\bar{\mathfrak{g}}_{\ell p}\delta^k_{(i}\bar{\partial}_{j)}\bar{\mathfrak{u}}^{\ell p} - 2\bar{\mathfrak{g}}_{\ell(i}\bar{\partial}_{j)}\bar{\mathfrak{u}}^{k\ell}) \right). \tag{3.6}$$

We note that Christofell symbols in the (x^i) coordinate system are related to the $\bar{\Gamma}^k_{ij}$ as follows:

$$\Gamma^A_{44} = \epsilon^{-2}\bar{\Gamma}^A_{44}, \quad \Gamma^4_{44} = \epsilon^{-1}\bar{\Gamma}^4_{44}, \quad \Gamma^4_{A4} = \bar{\Gamma}^4_{A4}, \tag{3.7}$$

$$\Gamma^4_{AB} = \epsilon\Gamma^4_{AB}, \quad \Gamma^A_{B4} = \epsilon^{-1}\bar{\Gamma}^A_{B4} \quad \text{and} \quad \Gamma^A_{BC} = \bar{\Gamma}^A_{BC}. \tag{3.8}$$

Using (3.6), a straightforward calculation shows that the Einstein tensor $\bar{G}^{ij}$ is given in terms of the density $\bar{\mathfrak{u}}^{ij}$ by

$$\mathcal{G}^{ij} := \frac{1}{2\epsilon^2}|\bar{g}|\,\bar{G}^{ij} = \bar{\mathfrak{g}}^{k\ell}\bar{\partial}^2_{k\ell}\bar{\mathfrak{u}}^{ij} + \epsilon^2\left(A^{ij} + B^{ij} + C^{ij}\right) + D^{ij}, \tag{3.9}$$

where

$$|\bar{g}| = -\det(\bar{\mathfrak{g}}^{ij}), \tag{3.10}$$

$$A^{ij} = 2\left(\tfrac{1}{2}\bar{\mathfrak{g}}_{k\ell}\bar{\mathfrak{g}}_{mn} - \bar{\mathfrak{g}}_{km}\bar{\mathfrak{g}}_{\ell n}\right)\left(\bar{\mathfrak{g}}^{ip}\bar{\mathfrak{g}}^{jq} - \tfrac{1}{2}\bar{\mathfrak{g}}^{ij}\bar{\mathfrak{g}}^{pq}\right)\bar{\partial}_p\bar{\mathfrak{u}}^{k\ell}\bar{\partial}_q\bar{\mathfrak{u}}^{mn}, \tag{3.11}$$

$$B^{ij} = 4\bar{\mathfrak{g}}_{k\ell}\left(2\bar{\mathfrak{g}}^{n(i}\bar{\partial}_m\bar{\mathfrak{u}}^{j)\ell}\bar{\partial}_n\bar{\mathfrak{u}}^{km} - \tfrac{1}{2}\bar{\mathfrak{g}}^{ij}\bar{\partial}_m\bar{\mathfrak{u}}^{kn}\bar{\partial}_n\bar{\mathfrak{u}}^{m\ell} - \bar{\mathfrak{g}}^{mn}\bar{\partial}_m\bar{\mathfrak{u}}^{ik}\bar{\partial}_n\bar{\mathfrak{u}}^{j\ell}\right), \tag{3.12}$$

$$C^{ij} = 4\left(\bar{\partial}_k\bar{\mathfrak{u}}^{ij}\partial_\ell\bar{\mathfrak{u}}^{k\ell} - \bar{\partial}_k\bar{\mathfrak{u}}^{i\ell}\bar{\partial}_\ell\bar{\mathfrak{u}}^{jk}\right), \tag{3.13}$$

$$D^{ij} := \bar{\mathfrak{g}}^{ij}\bar{\partial}^2_{k\ell}\bar{\mathfrak{u}}^{k\ell} - 2\bar{\partial}^2_{k\ell}\bar{\mathfrak{u}}^{k(i}\bar{\mathfrak{g}}^{j)\ell}. \tag{3.14}$$

To fix the gauge, we assume that

$$\bar{\partial}_i\bar{\mathfrak{u}}^{ij} = \epsilon\beta^j \tag{3.15}$$

for prescribed spacetime functions $\beta^j = \beta^j(x^I, x^4)$. For $\epsilon > 0$, $\bar{\partial}_i\bar{\mathfrak{u}}^{ij} = \epsilon\beta^j$ implies that

$$\bar{\partial}_i\bar{\mathfrak{g}}^{ij} = 4\epsilon^3\beta^j$$

or equivalently

$$\partial_k\mathfrak{g}^{k4} = 4\epsilon^3\beta^4 \quad \text{and} \quad \partial_k\mathfrak{g}^{kA} = 4\epsilon^2\beta^A,$$

where $\mathfrak{g}^{ij} = \sqrt{-\det(g_{k\ell})}g^{ij}$ is the metric density in the (x^k) coordinates. Thus (3.15) is, for $\epsilon > 0$, a generalized harmonic type gauge and is harmonic if the functions β^j are chosen to be identically zero. Clearly, if we define

$$E^{ij} := \bar{\mathfrak{g}}^{ij}\bar{\partial}_k\beta^k - 2\bar{\partial}_k\beta^{(i}\bar{\mathfrak{g}}^{j)k},$$

then (3.15) implies that

$$D^{ij} = \epsilon E^{ij}.$$

Setting

$$\mathcal{G}^{ij}_R := \mathcal{G}^{ij} - D^{ij} + \epsilon E^{ij} = \bar{\mathfrak{g}}^{k\ell}\bar{\partial}^2_{k\ell}\bar{\mathfrak{u}}^{ij} + \epsilon E^{ij} + \epsilon^2\left(A^{ij} + B^{ij} + C^{ij}\right) \tag{3.16}$$

and

$$\mathcal{T}^{ij} := \epsilon^2|\bar{g}|\,\bar{T}^{ij} = |\bar{g}|\begin{pmatrix}\epsilon^2 T^{IJ} & \epsilon^1 T^{I4}\\ \epsilon^1 T^{4J} & T^{44}\end{pmatrix}$$

the Einstein equations $G^{ij} = 2\epsilon^4 T^{ij}$ in the gauge (3.15) become

$$\mathcal{G}_R^{ij} = \mathcal{T}^{ij}, \tag{3.17}$$

which we will refer to as the *reduced Einstein equations*.

To write the reduced Einstein equations in first order form, we introduce the variables

$$\bar{\mathfrak{u}}_k^{ij} := \bar{\partial}_k \bar{\mathfrak{u}}^{ij} = \begin{cases} \partial_I \bar{\mathfrak{u}}^{ij} & \text{if } k = I \\ \epsilon \partial_4 \bar{\mathfrak{u}}^{ij} & \text{if } k = 4 \end{cases}.$$

The reduced Einstein equations then become

$$\begin{aligned} -\bar{\mathfrak{g}}^{44}\bar{\partial}_4\bar{\mathfrak{u}}_4^{ij} &= \bar{\mathfrak{g}}^{4I}\bar{\partial}_I\bar{\mathfrak{u}}_4^{ij} + \bar{\mathfrak{g}}^{IJ}\bar{\partial}_I\bar{\mathfrak{u}}_J^{ij} + \epsilon E^{ij} + \epsilon^2\left(A^{ij} + B^{ij} + C^{ij}\right) - \mathcal{T}^{ij}, \\ \bar{\mathfrak{g}}^{IJ}\bar{\partial}_4\bar{\mathfrak{u}}_J^{ij} &= \bar{\mathfrak{g}}^{IJ}\bar{\partial}_J\bar{\mathfrak{u}}_4^{ij}, \\ \bar{\partial}_4\bar{\mathfrak{u}}^{ij} &= \bar{\mathfrak{u}}_4^{ij}, \end{aligned}$$

or equivalently

$$\begin{aligned} -\bar{\mathfrak{g}}^{44}\partial_4\bar{\mathfrak{u}}_4^{ij} &= \frac{1}{\epsilon}\bar{\mathfrak{g}}^{4I}\partial_I\bar{\mathfrak{u}}_4^{ij} + \frac{1}{\epsilon}\bar{\mathfrak{g}}^{IJ}\partial_I\bar{\mathfrak{u}}_J^{ij} + E^{ij} + \epsilon\left(A^{ij} + B^{ij} + C^{ij}\right) - \frac{1}{\epsilon}\mathcal{T}^{ij}, \\ \bar{\mathfrak{g}}^{IJ}\partial_4\bar{\mathfrak{u}}_J^{ij} &= \frac{1}{\epsilon}\bar{\mathfrak{g}}^{IJ}\partial_J\bar{\mathfrak{u}}_4^{ij}, \\ \partial_4\bar{\mathfrak{u}}^{ij} &= \frac{1}{\epsilon}\bar{\mathfrak{u}}_4^{ij}. \end{aligned}$$

Next, define

$$\mathfrak{u}^{ij} := \epsilon\bar{\mathfrak{u}}^{ij} \quad \mathfrak{u}_k^{ij} := \bar{\mathfrak{u}}_k^{ij}, \tag{3.18}$$

and let

$$\mathcal{V} = \left\{(r^{ij}) \in \mathbb{M}_{4\times 4} \mid \det(\eta^{ij} + 4r^{ij}) > 0\right\}.$$

Then using vector notation

$$\mathbf{u}^{ij} := \left(\mathfrak{u}_4^{ij}, \mathfrak{u}_J^{ij}, \mathfrak{u}^{ij}\right)^T,$$

we can write the reduced Einstein equations as

$$A^4(\epsilon\mathbf{u})\partial_4\mathbf{u}^{ij} = \frac{1}{\epsilon}C^I\partial_I\mathbf{u}^{ij} + A^I(\mathbf{u})\partial_I\mathbf{u}^{ij} + \bar{F}^{ij}(\epsilon, \mathbf{u}) - \frac{1}{\epsilon}(\mathcal{T}^{ij}, 0, 0)^T, \tag{3.19}$$

where

$$A^4(\epsilon\mathbf{u}) = \begin{pmatrix} 1 - 4\epsilon\mathfrak{u}^{44} & 0 & 0 \\ 0 & \delta^{IJ} + 4\epsilon\mathfrak{u}^{IJ} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.20}$$

$$C^I = \begin{pmatrix} 0 & \delta^{IJ} & 0 \\ \delta^{IJ} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.21}$$

$$A^I(\mathbf{u}) = \begin{pmatrix} 4\mathfrak{u}^{4I} & 4\mathfrak{u}^{IJ} & 0 \\ 4\mathfrak{u}^{IJ} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.22}$$

and

$$\bar{F}^{ij}(\epsilon, \mathbf{u}) = (E^{ij} + \epsilon \bar{f}^{ij}(\epsilon \mathfrak{u}, \mathfrak{u}_k), 0, \mathfrak{u}_4^{ij})^T. \tag{3.23}$$

The functions $\bar{f}^{ij}(\epsilon\mathfrak{u}, \mathfrak{u}_k)$ are analytic for $\epsilon\mathfrak{u} \in \mathcal{V}$ and moreover are quadratic in $\mathfrak{u}_k$. Here we are using the notation

$$\mathfrak{u} = (\mathfrak{u}^{ij}) \quad \text{and} \quad \mathfrak{u}_k = (\mathfrak{u}_k^{ij}).$$

The stress-energy tensor is given in terms of the $\mathfrak{u}$ variable by

$$\begin{aligned}(T^{ij}) = \rho(v^i v^j) + \frac{1}{\sqrt{|\bar{g}|}}\begin{pmatrix}\delta^{IJ}p & 0\\ 0 & 0\end{pmatrix} + \frac{\epsilon}{\sqrt{|\bar{g}|}}\begin{pmatrix}4\mathfrak{u}^{IJ}p & 0\\ 0 & 0\end{pmatrix}\\ +\epsilon^2\left(p(v^i v^j) + \frac{p}{\sqrt{|\bar{g}|}}\begin{pmatrix}0 & 4\mathfrak{u}^{I4}\\ 4\mathfrak{u}^{4J} & -1+4\epsilon\mathfrak{u}^{44}\end{pmatrix}\right),\end{aligned} \tag{3.24}$$

and hence

$$\frac{1}{\epsilon}(\mathcal{T}^{ij}) = \begin{pmatrix}0 & 0\\ 0 & \epsilon^{-1}\rho\end{pmatrix} + \mathcal{S}^{ij}, \tag{3.25}$$

where

$$\begin{aligned}(\mathcal{S}^{ij}) = \rho\begin{pmatrix}0 & |\bar{g}|v^I v^4\\ |\bar{g}|v^J v^4 & \epsilon^{-1}\left((|\bar{g}|-1)(v^4)^2 + ((v^4)^2-1)\right)\end{pmatrix}\\ +\epsilon|\bar{g}|\begin{pmatrix}(\rho+\epsilon^2 p)v^I v^J + |\bar{g}|^{-1/2}p(\delta^{IJ}+4\epsilon\mathfrak{u}^{IJ}) & \epsilon p v^I v^4 + 4\epsilon|\bar{g}|^{-1/2}p\mathfrak{u}^{I4}\\ \epsilon p v^J v^4 + 4\epsilon|\bar{g}|^{-1/2}p\mathfrak{u}^{4J} & p(v^4)^2 + |\bar{g}|^{-1/2}p(-1+4\epsilon\mathfrak{u}^{44})\end{pmatrix}.\end{aligned} \tag{3.26}$$

We remark that if $v^4 - 1 = \mathrm{O}(\epsilon)$, then S^{ij} is regular in ϵ as is easily seen from the above formula and the expansion

$$|\bar{g}| = 1 + 4\epsilon\eta_{ij}\mathfrak{u}^{ij} + f(\epsilon\mathfrak{u}), \tag{3.27}$$

where $f(\epsilon\mathfrak{u})$ is analytic for $\epsilon\mathfrak{u} \in \mathcal{V}$ and also satisfies $f(y) = \mathrm{O}(|y|^2)$ as $y \to 0$.

4. Regularized Euler Equations

There are various approaches to symmetric hyperbolic formulations of the relativistic Euler equations [4, 14, 15, 34, 40]. We use the approach of [4] which is based on fluid projection and the introduction of a Makino variable.

In the coordinates $(\bar{x}^i)$, the Euler equations are given by

$$\bar{\nabla}_i \bar{T}^{ij} = 0, \tag{4.1}$$

where $\bar{T}^{ij} = (\rho + \epsilon^2 p)\bar{v}^i\bar{v}^j + p\bar{g}^{ij}$ and the fluid velocity $\bar{v}^i$ is normalized according to

$$\bar{v}_i\bar{v}^i = -\frac{1}{\epsilon^2}. \tag{4.2}$$

Differentiating (4.2) yields

$$\bar{v}_i\bar{\nabla}_j\bar{v}^i = 0 \tag{4.3}$$

which implies

$$\bar{v}^j \bar{v}_i \bar{\nabla}_j \bar{v}^i = 0. \tag{4.4}$$

Writing out (4.1) explicitly, we have

$$(\bar{\partial}_i \rho + \epsilon^2 \bar{\partial}_i p)\bar{v}^i \bar{v}^j + (\rho + \epsilon^2 p)(\bar{v}^j \bar{\nabla}_i \bar{v}^i + \bar{v}^i \bar{\nabla}_i \bar{v}^j) + \bar{g}^{ij} \bar{\partial}_i p = 0. \tag{4.5}$$

The operator

$$L_i^j = \delta_i^j + \epsilon^2 \bar{v}^j \bar{v}_i$$

projects into subspace orthogonal to the fluid velocity $\bar{v}^i$, i.e. $L_i^j L_k^i = L_k^j$ and $L_i^j \bar{v}^i = 0$. Using L_k^j to project the Euler equations (4.5) into components parallel and orthogonal to $\bar{v}^i$ yields, after using the relations (4.2)–(4.4), the following system:

$$\bar{v}^i \bar{\partial}_i \rho + (\rho + \epsilon^2 p) L_j^i \bar{\nabla}_i \bar{v}^j = 0, \tag{4.6}$$

$$M_{ij} \bar{v}^k \bar{\nabla}_k \bar{v}^j + \frac{1}{\rho + \epsilon^2 p} L_j^i \bar{\partial}_i p = 0, \tag{4.7}$$

where

$$M_{ij} = \bar{g}_{ij} + 2\epsilon^2 \bar{v}_i \bar{v}_j.$$

As discussed in the introduction, we introduce a new density variable α via the formula (1.12). Multiplying (4.6) by the square of the function

$$h(\epsilon\alpha) = \left(1 + \frac{1}{4n(n+1)}(\epsilon\alpha)^2\right),$$

gives

$$h^2 \bar{v}^i \bar{\partial}_i \alpha + h^2(\rho + \epsilon^2 p)\frac{d\alpha}{d\rho} L_j^i \bar{\nabla}_i \bar{v}^j = 0, \tag{4.8}$$

$$M_{ij} \bar{v}^k \bar{\nabla}_k \bar{v}^j + \frac{s^2}{\rho + \epsilon^2 p}\frac{dp}{d\alpha} L_i^j \bar{\partial}_j \alpha = 0, \tag{4.9}$$

where

$$s^2 = \frac{dp}{d\rho} = \frac{1}{4n^2}\alpha^2$$

is the square of the speed of sound. A simple calculation shows that

$$\frac{s^2}{\rho + \epsilon^2 p}\frac{dp}{d\alpha} = h^2(\rho + \epsilon^2 p)\frac{d\alpha}{d\rho} = q,$$

where

$$q = q(\epsilon, \alpha) = \frac{1}{2nh(\epsilon\alpha)}\alpha.$$

This shows that the system (4.8)–(4.9) is *symmetric*, and moreover at a point where $\alpha = 0$ and hence $p = \rho = 0$, it is *regular* unlike (4.6)–(4.7). This is the point of introducing the Makino variable α. Also note that the pressure is given in terms of the Makino variable by

$$p = \frac{K}{(4Kn(n+1))^{n+1}}\alpha^{2n+2}. \tag{4.10}$$

Define

$$w^I := \bar{v}^I, \quad \text{and} \quad w^4 := \bar{v}^4 - \frac{1}{\epsilon}$$

so that

$$v^I = w^I, \quad \text{and} \quad v^4 = 1 + \epsilon w^4. \tag{4.11}$$

Using vector notation

$$\mathbf{w} = (\alpha, w^i)^T,$$

we can write (4.8) and (4.9) as

$$a^4 \partial_4 \mathbf{w} = a^I \partial_I \mathbf{w} + b, \tag{4.12}$$

where

$$a^4 = \begin{pmatrix} h^2(1+\epsilon w^4) & \epsilon q L^4_j \\ \epsilon q L^4_i & M_{ij}(1+\epsilon w^4) \end{pmatrix}, \tag{4.13}$$

$$a^I = \begin{pmatrix} -h^2 w^I & -qL^I_j \\ -qL^I_j & -M_{ij} w^I \end{pmatrix}, \tag{4.14}$$

and

$$b = \begin{pmatrix} -qL^i_j \bar{\Gamma}^j_{i\ell} \bar{v}^\ell \\ -M_{ij} \bar{\Gamma}^j_{k\ell} \bar{v}^k \bar{v}^\ell \end{pmatrix}. \tag{4.15}$$

From (3.3), (3.5), (3.18), and (3.27), we find that

$$\bar{g}_{ij} = \eta_{ij} + f_{ij}(\epsilon \mathfrak{u}), \tag{4.16}$$

where the $f_{ij}(y)$ are analytic and satisfy $f_{ij}(y) = \mathrm{O}(|y|)$ as $y \to 0$. Also, (3.6) shows that

$$\bar{\Gamma}^k_{ij} = \epsilon \left[\eta^{km} \left(2\eta_{i\ell}\eta_{jp} - \eta_{ij}\eta_{\ell p} \right) \epsilon \mathfrak{u}^{lp}_m + 2 \left(\eta_{\ell p} \delta^k_{(i} \epsilon \mathfrak{u}^{\ell p}_{j)} - 2\eta_{\ell(i} \epsilon \mathfrak{u}^{k\ell}_{j)} \right) \right] + \epsilon f^k_{ij}(\epsilon \mathfrak{u}, \epsilon \mathfrak{u}_m) \tag{4.17}$$

for functions $f^k_{ij}(\epsilon\mathfrak{u}, \epsilon\mathfrak{u}_m)$ that are analytic for $\epsilon\mathfrak{u} \in \mathcal{V}$, linear in the $\epsilon\mathfrak{u}_m$, and satisfy $f^k_{ij}(0, y) = 0$. So then

$$M_{ij} = \bar{g}_{ij} + 2\epsilon^2 \bar{g}_{ik} \bar{g}_{j\ell} \bar{v}^k \bar{v}^\ell = \delta_{ij} + m_{ij}(\epsilon\mathfrak{u}, \epsilon w^k) \tag{4.18}$$

and

$$L_i^j = \delta_i^j + \epsilon^2 \bar{g}_{ik}\bar{v}^k\bar{v}^j = \delta_i^j - \delta_i^4\delta_4^j + \ell_i^j(\epsilon \mathfrak{u}, \epsilon w^k), \tag{4.19}$$

where $\ell_i^j(\epsilon\mathfrak{u}, \epsilon w^k)$ and $m_{ij}(\epsilon\mathfrak{u}, \epsilon w^k)$ are analytic for $\epsilon\mathfrak{u} \in \mathcal{V}$ and $\ell_i^j(0,0) = m_{ij}(0,0) = 0$. Using (4.16)–(4.19), the matrices a^i and the vector b can be written as

$$a^4 = \begin{pmatrix} 1 & 0 \\ 0 & \delta_{ij} \end{pmatrix} + \hat{a}^4(\epsilon\mathfrak{u}, \epsilon\mathbf{w}), \tag{4.20}$$

$$a^I = \begin{pmatrix} -w^I & -\frac{\alpha}{2n}\delta_j^I \\ -\frac{\alpha}{2n}\delta_i^I & -\delta_{ij}w^I \end{pmatrix} + w^I\hat{a}(\epsilon\mathfrak{u}, \epsilon\mathbf{w}) + \alpha\hat{a}^I(\epsilon\mathfrak{u}, \epsilon\mathbf{w}), \tag{4.21}$$

and

$$b = \begin{pmatrix} 0 \\ -\eta^{im}\left(2\eta_{4\ell}\eta_{4p} + \eta_{\ell p}\right)\mathfrak{u}_m^{lp} - 2\left(\eta_{\ell p}\delta_4^i\mathfrak{u}_4^{\ell p} - 2\eta_{\ell 4}\mathfrak{u}_4^{i\ell}\right) \end{pmatrix} + \begin{pmatrix} \alpha\hat{b}_1(\epsilon\mathfrak{u}, \epsilon\mathbf{w}) \cdot \epsilon\mathfrak{u}_k \\ \hat{b}_2(\epsilon\mathfrak{u}, \epsilon\mathbf{w}) \cdot \mathfrak{u}_k \end{pmatrix}. \tag{4.22}$$

Note that (i) $\hat{a}^4$, $\hat{a}$, $\hat{a}^I$, $\hat{b}_1$, and $\hat{b}_2$ are analytic in all their variables provided that $\epsilon\mathfrak{u} \in \mathcal{V}$, (ii) $\hat{a}^4$, $\hat{a}$ and $\hat{a}^I$ are symmetric, and (iii) $\hat{a}^4(0,0) = 0$, $\hat{a}^I(0,0) = 0$, $\hat{a}(0,0) = 0$, $\hat{b}_1(0,0) = 0$, and $\hat{b}_2(0,0) = 0$. Consequently the system (4.12) is *symmetric hyperbolic* on a region where $(\epsilon\mathfrak{u}, \epsilon\mathbf{w})$ is small enough to ensure that a^4 is positive definite. This can always be arranged by taking ϵ small enough and since we are interested in the limit $\epsilon \searrow 0$ no generality is lost in assuming this.

It is important to realize that the derivation above of (4.12) required that both the Euler equations (4.1) and the fluid velocity normalization (4.2) are satisfied. Alternatively, we can first assume that (4.12) is satisfied and then show that (4.1) and (4.2) are also satisfied. To see this, define

$$\mathcal{N} := \epsilon\bar{v}_i\bar{v}^i + 1/\epsilon = \epsilon\bar{g}_{44}(1/\epsilon + w^4)^2 + 1/\epsilon + 2\bar{g}_{4J}(1+\epsilon w^4)w^J + \epsilon\bar{g}_{IJ}w^Iw^J. \tag{4.23}$$

Clearly, $\mathcal{N} = 0$ is equivalent to $\bar{v}^i\bar{v}_i = -1/\epsilon^2$ for $\epsilon > 0$. Furthermore, any solution of (4.12) also solves (4.6)–(4.7) for any $\epsilon > 0$. So assuming that $\bar{v}$ is a solution to the system (4.6)–(4.7), contracting (4.7) with $\bar{v}^i$ yields

$$(1 + 2\epsilon^2\bar{v}^i\bar{v}_i)\bar{v}^k\bar{\partial}_k(\bar{v}^i\bar{v}_i) = 0.$$

For $(2\epsilon\mathcal{N} - 1) \neq 0$, this implies

$$(1 + \epsilon w^4)\partial_4\mathcal{N} = -w^I\partial_I\mathcal{N}. \tag{4.24}$$

Clearly, this is a symmetric hyperbolic equation for $\mathcal{N}$ whenever $0 < 1/C \leq (1+\epsilon w^4) \leq C$ for some constant C. This can always be arranged at $x^4 = 0$ by choosing ϵ small enough. Therefore, if initially $\mathcal{N}\big|_{x^4=0} = 0$, then $\mathcal{N} = 0$ for as long as $(1 + \epsilon w^4)$ stays absolutely bounded and bounded away from zero. Consequently, choosing initial data for the system (4.12) such that $\mathcal{N}\big|_{x^4=0} = 0$ will guarantee that the solution will satisfy the full Euler equations (4.5) in an open neighborhood of the hypersurface $x^4 = 0$. In particular, if $\{\alpha, w^i\}$ is a solution to (4.12) with initial data satisfying $\mathcal{N}|_{x^4=0}$, then α is a solution to the equation

$$\partial_4\alpha + X^I\partial_I\alpha + Y\alpha = 0, \tag{4.25}$$

where

$$X^I := \frac{w^I}{1+\epsilon w^4}, \qquad \text{and} \qquad Y := \frac{\bar{\nabla}_i \bar{v}^i}{2n(1+\epsilon w^4)h^3(\epsilon\alpha)}. \tag{4.26}$$

Observe that

$$Y = \bar{Y}(\epsilon w^4, \epsilon\alpha)(\epsilon\partial_t w^4 + \partial_I w^I) + \hat{Y}(\epsilon \mathfrak{u}, \epsilon w^4, \epsilon \mathfrak{u}_k, \epsilon w^I, \epsilon),$$

where $\bar{Y}(0,0) - 1/(2n) = 0$, $\hat{Y}(0,\ldots,0) = 0$ and $\bar{Y}(\epsilon w^4, \epsilon\alpha^4)$, $\hat{Y}(\epsilon\mathfrak{u}, \epsilon w^4, \epsilon\mathfrak{u}_k, \epsilon w^I, \epsilon\alpha^4)$ are analytic on the region $\epsilon\mathfrak{u} \in \mathcal{V}$ and $1+\epsilon w^4 > 0$.

5. Newtonian Initial Data

Let $S_0 \cong \mathbb{R}^3$ be the hypersurface defined by $S_0 := \{(x^I, 0) \mid (x^I) \in \mathbb{R}^3\}$. The covector $n_i = \delta_i^4$ is conormal to S_0 implying that constraint equations for the initial data on S_0 are given by $n_i G^{ij} = 2\kappa\epsilon^4 n_i T^{ij}$. Defining

$$\mathcal{C}^J := \epsilon^{-1}(\mathcal{G}^{4J} - \kappa\mathcal{T}^{4J}) \quad \text{and} \quad \mathcal{C}^4 := \mathcal{G}^{44} - \kappa\mathcal{T}^{44},$$

we find that $\mathcal{C}^j = 0$ is equivalent to $n_i G^{ij} = 2\kappa\epsilon^4 n_i T^{ij}$ for $\epsilon > 0$. Also, by defining

$$\mathcal{H}^j := \bar{\partial}_i \bar{\mathfrak{u}}^{ij} - \epsilon\beta^j, \tag{5.1}$$

the generalized harmonic gauge (3.15) can be written as $\mathcal{H}^j = 0$.

As will be seen in the proof of the next proposition the equations $\mathcal{C}^j = 0$ are regular at $\epsilon = 0$. So to find appropriate initial data that is well defined at $\epsilon = 0$, we solve the regularized constraint equations $\mathcal{C}^j = 0$. Moreover, we must also ensure that the harmonic gauge condition $\mathcal{H}^j = 0$ and the fluid normalization $\mathcal{N} = 0$ are satisfied. To solve the constraints $\mathcal{C}^j = 0$, $\mathcal{H}^j = 0$, and $\mathcal{N} = 0$, we use a implicit function technique based on the work of Lottermoser [23]. We assume that the fluid velocity can be written as (4.10) which is consistent with the expected behavior of the fluid velocity as $\epsilon \searrow 0$. We will not assume that the density and pressure are related by the equation of state (1.11). Instead, we will consider them as independent prescribed fields for the purpose of finding solutions to the constraint equations. We do this so that the following proposition remains valid for other equations of state.

Proposition 5.1. *Suppose $-1 < \delta < 0$, $k > 3/2$, $R > 0$ and $(\tilde{\rho}, \tilde{p}, \tilde{w}^I, \tilde{\mathfrak{z}}_4^{IJ}, \tilde{\beta}^j, \tilde{\mathfrak{z}}^{IJ}) \in (H^{k-2}_{\delta-2})^2 \times H^k_{\delta-1} \times (H^{k-1}_{\delta-1})^2 \times B_R(H^k_\delta)$. Then there exists an $\epsilon_0 > 0$, an open neighborhood U of $(\tilde{\rho}, \tilde{p}, \tilde{w}^I, \tilde{\mathfrak{z}}_4^{IJ}, \tilde{\beta}^j, \tilde{\mathfrak{z}}^{IJ})$, and analytic maps $(-\epsilon_0, \epsilon_0) \times U \to H^k_{\delta-1}$: $(\epsilon, \rho, p, w^I, \mathfrak{z}_4^{IJ}, \beta^j, \mathfrak{z}^{IJ}) \mapsto w^4$, $(-\epsilon_0, \epsilon_0) \times U \to H^k_\delta$: $(\epsilon, \rho, p, w^I, \mathfrak{z}_4^{IJ}, \beta^j, \mathfrak{z}^{IJ}) \mapsto \phi$, $(-\epsilon_0, \epsilon_0) \times U \to H^k_\delta$: $(\epsilon, \rho, p, w^I, \mathfrak{z}_4^{IJ}, \beta^j, \mathfrak{z}^{IJ}) \mapsto \mathfrak{w}^I$ such that for each $(\rho, p, w^I, \mathfrak{z}_4^{IJ}, \beta^j, \mathfrak{z}^{IJ}) \in U$, $(\epsilon, \rho, p, w^I, w^4, \bar{\mathfrak{u}}_4^{ij}, \beta^j, \bar{\partial}_4\bar{\mathfrak{u}}^{ij})$ is a solution to the three constraints*

$$\mathcal{C}^j = 0, \quad \mathcal{H}^j = 0, \quad \textit{and} \quad \mathcal{N} = 0, \tag{5.2}$$

where

$$(\bar{\mathfrak{u}}^{ij}) = \begin{pmatrix} \epsilon\mathfrak{z}^{IJ} & \epsilon\mathfrak{w}^I \\ \epsilon\mathfrak{w}^J & \phi \end{pmatrix}, \tag{5.3}$$

$$(\partial_t \bar{\mathfrak{u}}^{ij}) = \begin{pmatrix} \mathfrak{z}_4^{IJ} & -\partial_K\mathfrak{z}^{KI} + \beta^I \\ -\partial_K\mathfrak{z}^{KJ} + \beta^J & -\partial_K\mathfrak{w}^K + \beta^4 \end{pmatrix} \qquad (t = x^4), \tag{5.4}$$

and

$$w^4 = -\frac{1}{\epsilon} + \frac{-\epsilon\bar{g}_{4J}w^J - \sqrt{\epsilon^2(\bar{g}_{4J}w^J)^2 - \bar{g}_{44}(\epsilon^2\bar{g}_{IJ}w^I w^J + 1)}}{\epsilon\bar{g}_{44}}. \tag{5.5}$$

Moreover, if we let $\phi_0 = \phi|_{\epsilon=0}$, $\mathfrak{w}_0^I = \mathfrak{w}^I|_{\epsilon=0}$, *and* $w_0^4 = w^4|_{\epsilon=0}$, *then* ϕ_0, $\mathfrak{w}_0^I$, *and* w_0^4 *satisfy the equations*

$$\Delta\phi_0 = \kappa\rho, \quad \Delta\mathfrak{w}_0^I = \partial_I\beta^4 - \partial_L\mathfrak{z}_4^{LI} + \kappa\rho w^I, \quad \textit{and} \quad w_0^4 = 0,$$

respectively.

Proof. Let $\bar{\mathfrak{u}}^{44} = \phi$, $\bar{\mathfrak{u}}^{IJ} = \epsilon\mathfrak{z}^{IJ}$, $\bar{\mathfrak{u}}^{I4} = \epsilon\mathfrak{w}^I$, and $\bar{\partial}_4\bar{\mathfrak{u}}^{IJ} = \epsilon\mathfrak{z}_4^{IJ}$. Solving $\mathcal{H}^j\big|_{S_0} = 0$ yields

$$\bar{\partial}_4\bar{\mathfrak{u}}^{44} = \epsilon\left(-\partial_I\mathfrak{w}^I + \beta^4\right) \quad \text{and} \quad \bar{\partial}_4\bar{\mathfrak{u}}^{4J} = \epsilon(-\partial_I\mathfrak{z}^{IJ} + \beta^J), \tag{5.6}$$

while solving $\mathcal{N}\big|_{S_0} = 0$ gives

$$w^4 = -\frac{1}{\epsilon} + \frac{-\epsilon\bar{g}_{4J}w^J - \sqrt{\epsilon^2(\bar{g}_{4J}w^J)^2 - \bar{g}_{44}(\epsilon^2\bar{g}_{IJ}w^I w^J + 1)}}{\epsilon\bar{g}_{44}}. \tag{5.7}$$

From (3.3) and (3.5), it is not difficult to verify that

$$w^4 = \epsilon^{-1}f(\epsilon w^I, \epsilon^3\mathfrak{z}, \epsilon^3\mathfrak{w}, \epsilon^2\phi),$$

where $f(\mathbf{y})$ ($\mathbf{y} = (y_1, \ldots, y_4)$) is analytic in a neighborhood of $(0, 0, 0, 0)$ and moreover $f(\mathbf{y}) = \mathrm{O}(|\mathbf{y}|^2)$ as $\mathbf{y} \to 0$.

Using the relation (5.6) to eliminate $\bar{\partial}_4\bar{\mathfrak{u}}^{44}$ and $\bar{\partial}_4\bar{\mathfrak{u}}^{4J}$ in favour of $\mathfrak{w}^I$ and $\mathfrak{z}^{IJ}$, we find that

$$\begin{aligned}
\bar{\mathfrak{g}}^{k\ell}\bar{\partial}^2_{k\ell}\bar{\mathfrak{u}}^{44} + D^{44} &= \Delta\phi - \epsilon\partial^2_{KL}\mathfrak{z}^{KL} + 4\epsilon^2 h^4, \\
\bar{\mathfrak{g}}^{k\ell}\bar{\partial}^2_{k\ell}\bar{\mathfrak{u}}^{4J} + D^{4J} &= \epsilon\left(\Delta\mathfrak{w}^J - \partial_J\beta^4 + \partial_L\mathfrak{z}_4^{LJ} + 4\epsilon h^J\right),
\end{aligned}$$

where

$$\begin{aligned}
h^4 &= \epsilon\mathfrak{z}^{KL}\partial_{KL}\phi + \epsilon\phi\partial^2_{KL}\mathfrak{z}^{KL} - 2\epsilon^2\mathfrak{w}^L\partial^2_{KL}\mathfrak{w}^K, \\
h^J &= \epsilon^2\mathfrak{z}^{KL}\partial^2_{KL}\mathfrak{w}^J + \epsilon^2\mathfrak{w}^J\partial^2_{KL}\mathfrak{z}^{KL} - \epsilon^2\mathfrak{w}^L\partial^2_{KL}\mathfrak{z}^{KJ} - \epsilon\phi\partial_k\mathfrak{z}_4^{KJ} - \epsilon^2\mathfrak{z}^{JL}\partial_L\beta^4.
\end{aligned}$$

Using this and Eqs. (3.9), (3.10)–(3.14), (3.24)–(3.26), and (4.10)–(4.11), we see that

$$\begin{aligned}
\mathcal{C}^I = \Delta\mathfrak{w}^J + \partial_L\mathfrak{z}_4^{LJ} + \epsilon h^J + \epsilon f^I(&\epsilon^3\mathfrak{z}, \epsilon^3\mathfrak{w}, \epsilon^2\phi, \epsilon D\mathfrak{z}, \epsilon D\mathfrak{w}, D\phi, \epsilon\mathfrak{z}_4, \\
&\epsilon(-\partial_k\mathfrak{w}^I + \beta^4), \epsilon(-\partial_K\zeta^{KL} + \beta^L)) - \kappa S^{4I},
\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}
\mathcal{C}^4 = \Delta\phi - \kappa\rho - \kappa\epsilon(2w^4 + \epsilon(w^4)^2)\rho - \epsilon\partial^2_{KL}\mathfrak{z}^{KL} + 4\epsilon^2 h^4 + \epsilon^2 f^4(&\epsilon^3\mathfrak{z}, \epsilon^3\mathfrak{w}, \epsilon^2\phi, \\
&\epsilon D\mathfrak{z}, \epsilon D\mathfrak{w}, D\phi, \epsilon\mathfrak{z}_4, \epsilon(-\partial_k\mathfrak{w}^I + \beta^4), \epsilon(-\partial_K\zeta^{KL} + \beta^L)) - \epsilon\kappa S^{44},
\end{aligned} \tag{5.9}$$

where the functions $f^I(\mathbf{y})$ ($\mathbf{y} = (y_1, \dots, y_9)$) are analytic in a neighborhood of $\{(0,0,0)\} \times U$, where U is any open set and are quadratic in $(y_4, \dots, y_9)$. Note that

$$\mathcal{S}^{44} = \rho S_1^4(\epsilon, w^I, \epsilon^2\mathfrak{z}, \epsilon^2\mathfrak{w}, \epsilon\phi) + pS_1^4(\epsilon, w^I, \epsilon^2\mathfrak{z}, \epsilon^2\mathfrak{w}, \epsilon\phi)$$

and

$$\mathcal{S}^{4I} = \rho w^I + \epsilon\rho S_1^I(\epsilon, w^I, \epsilon^2\mathfrak{z}, \epsilon^2\mathfrak{w}, \epsilon\phi) + \epsilon p S_2^I(\epsilon, w^I, \epsilon^2\mathfrak{z}, \epsilon^2\mathfrak{w}, \epsilon\phi),$$

where the functions $S^j(\mathbf{y})$ ($\mathbf{y} = (y_1, \dots, y_7)$) are analytic in a neighborhood of $U \times \{(0,0,0)\}$ for any open set U.

Using Lemma A.8 and Proposition 3.6 of [17], we see from the above considerations that for any $R > 0$ there exists an $\epsilon_0 > 0$ such that the maps

$$(-\epsilon_0, \epsilon_0) \times B_R(H^k_{\delta-1}) \times B_R(H^k_\delta)^3 \longrightarrow H^k_\delta \;:\; (\epsilon, w^I, \mathfrak{z}, \mathfrak{w}, \phi) \longmapsto w^4$$

and

$$(-\epsilon_0, \epsilon_0) \times (H^{k-2}_{\delta-2})^2 \times B_R(H^k_{\delta-1}) \times (H^{k-1}_{\delta-1})^2 \times B_R(H^k_\delta)^3 \longrightarrow H^{k-2}_{\delta-2} \;:\; (\epsilon, \rho, p, w^I, \mathfrak{z}_4, \beta, \mathfrak{z}, \mathfrak{w}, \phi) \longmapsto \mathcal{C}^j$$

are analytic. Since

$$\mathcal{C}^I|_{\epsilon=0} = \Delta\mathfrak{w}^I - \partial_I\beta^4 + \partial_L\mathfrak{z}_4^{LI} - \kappa\rho w^I, \qquad \mathcal{C}^4|_{\epsilon=0} = \Delta\phi - \kappa\rho \tag{5.10}$$

and for $-1 < \delta < 0$ the Laplacian $\Delta : H^k_\delta \to H^{k-2}_{\delta-2}$ is an isomorphism (see [1], Proposition 2.2), we can use the analytic version of the implicit function theorem (see [10] Theorem 15.3) to conclude, shrinking ϵ_0 if necessary, that there exists an open neighborhood U of any point in $(H^{k-2}_{\delta-2})^2 \times B_R(H^k_{\delta-1}) \times (H^{k-1}_{\delta-1})^2 \times B_R(H^k_\delta)$ and analytic maps

$$(-\epsilon_0, \epsilon_0) \times U \longrightarrow H^k_\delta \;:\; (\epsilon, \rho, p, w^I, \mathfrak{z}_4, \beta, \mathfrak{z}) \longmapsto \phi$$

and

$$(-\epsilon_0, \epsilon_0) \times U \longrightarrow H^k_\delta \;:\; (\epsilon, \rho, p, w^I, \mathfrak{z}_4, \beta, \mathfrak{z}) \longmapsto \mathfrak{z}$$

such that the constraints are satisfied, i.e.

$$\mathcal{C}^j(\epsilon, \rho, p, w^I, \mathfrak{z}_4, \beta, \mathfrak{z}, \mathfrak{w}(\epsilon, \rho, p, w^I, \mathfrak{z}_4, \mathfrak{z}), \phi(\epsilon, \rho, p, w^I, \mathfrak{z}_4, \mathfrak{z})) = 0$$

for all $(\epsilon, \rho, p, w^I, \mathfrak{z}_4, \beta, \mathfrak{z}) \in (-\epsilon_0, \epsilon_0) \times U$. $\square$

6. Local Existence for the Einstein-Euler System

The combined systems (3.19) and (4.12) can be written as

$$b^0(\epsilon U, \epsilon V)\partial_t V = \frac{1}{\epsilon}c^I \partial_I V + b^I(\epsilon, U, V)\partial_I V + f(\epsilon, U, V)V + \frac{1}{\epsilon}g(V)V + h_\epsilon \qquad (t = x^4), \tag{6.1}$$

where

$$U := (0, 0, \underset{o}{\mathfrak{u}}^{ij}, 0, 0)^T, \qquad \underset{o}{\mathfrak{u}}^{ij} := \mathfrak{u}^{ij}|_{t=0} = \epsilon\bar{\mathfrak{u}}^{ij}|_{t=0}, \tag{6.2}$$

$$V := (\mathfrak{u}_4^{ij}, \mathfrak{u}_J^{ij}, \delta\mathfrak{u}^{ij}, \alpha, w^i)^T, \qquad \delta\mathfrak{u}^{ij} := \mathfrak{u}^{ij} - \underset{o}{\mathfrak{u}}^{ij}, \tag{6.3}$$

$$b^0(\epsilon U, \epsilon V) := \begin{pmatrix} A^4(\epsilon\mathfrak{u}) & 0 \\ 0 & a^4(\epsilon\mathfrak{u}, \epsilon w^i, \epsilon\alpha) \end{pmatrix}, \tag{6.4}$$

$$c^I := \begin{pmatrix} C^I & 0 \\ 0 & 0 \end{pmatrix}, \tag{6.5}$$

$$b^I(\epsilon, U, V) := \begin{pmatrix} A^I(\mathfrak{u}) & 0 \\ 0 & a^I(\epsilon, \epsilon\mathfrak{u}, w^i, \alpha) \end{pmatrix}, \tag{6.6}$$

$$f(\epsilon, U, V)V := \begin{pmatrix} \epsilon\bar{f}^{ij}(\epsilon\mathfrak{u}, \mathfrak{u}_k) - S^{ij} + 4\epsilon\delta\mathfrak{u}^{ij}\bar{\partial}_k\beta^k - 8\epsilon\bar{\partial}_k\beta^{(i}\delta\mathfrak{u}^{j)k} \\ 0 \\ \mathfrak{u}_4^{ij} \\ b(\epsilon, \epsilon\mathfrak{u}, \mathfrak{u}_k, w^i, \alpha) \end{pmatrix}, \tag{6.7}$$

$$g(V)V := (-\delta_4^i\delta_4^j\rho(\alpha), 0, \dots, 0)^T, \tag{6.8}$$

and

$$h_\epsilon := \begin{pmatrix} 4\epsilon\underset{o}{\mathfrak{u}}^{ij}\bar{\partial}_k\beta^k - 8\epsilon\bar{\partial}_k\beta^{(i}\underset{o}{\mathfrak{u}}^{j)k} + \eta^{ij}\bar{\partial}_k\beta^k - 2\bar{\partial}_k\beta^{(i}\eta^{j)k} \\ 0 \end{pmatrix}. \tag{6.9}$$

For initial data, we will use the following notation: given a function z that depends on time t, we define

$$\underset{o}{z} := z|_{t=0}.$$

To fix a region on which the system (6.1) is well defined, we note from (3.20), (4.20), and the invertibility of the Lorentz metric (η^{ij}) that there exists a constant $K_0 > 0$ such that

$$-\det(\eta^{ij} + 4\epsilon\mathfrak{u}^{ij}) > 1/16, \quad 1 + \epsilon w^4 > 1/16, \tag{6.10}$$

$$A^4(\epsilon\mathfrak{u}) \geq \frac{1}{16}\mathbb{I}, \quad a^4(\epsilon\mathfrak{u}, \epsilon w, \epsilon\alpha) \geq \frac{1}{16}\mathbb{I} \tag{6.11}$$

and

$$|A^4(\epsilon\mathfrak{u})| \leq 16, \quad |a^4(\epsilon\mathfrak{u}, \epsilon w, \epsilon\alpha)| \leq 16 \tag{6.12}$$

for all $|\epsilon\mathfrak{u}| \leq 2K_0$, $|\epsilon w^i| \leq 2K_0$, $|\epsilon\alpha| \leq 2K_0$. The choice of the bounds $1/16$ and 16 is somewhat arbitrary and they can be replaced by any number of the form $1/M$ and M for any $M > 1$ without changing any of the arguments presented in the following sections. However, since we are interested in the limit $\epsilon \searrow 0$, we lose nothing by assuming $M = 16$.

Proposition 6.1. *Suppose* $-1 < \delta < 0$, $k \geq 3 + s$, $\underset{o}{\alpha}, \underset{o}{w}^I \in H^k_{\delta-1}$, $\mathfrak{z}^{IJ} \in H^{k+1}_{\delta}$, $\mathfrak{z}^{IJ}_4 \in H^k_{\delta-1}$, $\beta^j \in C^1([-T, T], H^k_{\delta-1})$. *Let* $\underset{o}{\bar{\mathfrak{u}}}{}^{ij}_\epsilon$, $\partial_t \underset{o}{\bar{\mathfrak{u}}}{}^{ij}_\epsilon$ *and* $\underset{o}{w}{}^4_\epsilon$ *be the initial data constructed in Proposition 5.1 which, by choosing* $\epsilon_0 \leq 1$ *small enough, satisfies*

$$|\epsilon \underset{o}{w}^i|, |\epsilon \underset{o}{\alpha}|, |\epsilon \underset{o}{\bar{\mathfrak{u}}}{}^{ij}_\epsilon| \leq K_0 \quad \text{for all } \epsilon \in (0, \epsilon_0].$$

Then

(i) *for each* $\epsilon \in (0, \epsilon_0]$, *there exists* $T_1(\epsilon), T_2(\epsilon) > 0$ *and a unique solution*

$$V_\epsilon \in \bigcap_{\ell=0}^{s+1} C^\ell((-T_1(\epsilon), T_2(\epsilon)), H^{k-\ell}_{\delta-1})$$

to the system (6.1) *with initial data*

$$\underset{o}{V}_\epsilon = (\epsilon \partial_t \underset{o}{\bar{\mathfrak{u}}}{}^{ij}_\epsilon, \partial_J \underset{o}{\bar{\mathfrak{u}}}{}^{ij}_\epsilon, 0, \underset{o}{\alpha}, \underset{o}{w}^i).$$

(ii) *The identities*

$$\partial_t \bar{\mathfrak{u}}^{ij}_\epsilon = \frac{\mathfrak{u}^{ij}_{4,\epsilon}}{\epsilon}, \quad \text{and} \quad \mathfrak{u}^{ij}_{J,\epsilon} = \partial_J \bar{\mathfrak{u}}^{ij}_\epsilon$$

hold where by definition $\bar{\mathfrak{u}}^{ij}_\epsilon = \epsilon^{-1}\mathfrak{u}^{ij}_\epsilon$, $\mathfrak{u}^{ij}_\epsilon = \underset{o}{\mathfrak{u}}_\epsilon + \delta\mathfrak{u}^{ij}_\epsilon$, *and* $\underset{o}{\mathfrak{u}}_\epsilon = \epsilon \underset{o}{\bar{\mathfrak{u}}}_\epsilon$.

(iii) *The triple* $\{\bar{\mathfrak{u}}^{ij}_\epsilon, w^i_\epsilon, \alpha_\epsilon\}$ *determines, via the formulas* (1.12), (3.4), (3.5), *and* (4.11), *a solution to the full Einstein-Euler system* (1.1)–(1.2) *that satisfies the constraints*

$$\epsilon \partial_t \bar{\mathfrak{u}}^{4j}_\epsilon + \partial_I \bar{\mathfrak{u}}^{Ij}_\epsilon = \epsilon \beta^j \quad \text{and} \quad v^i v_i = -\frac{1}{\epsilon^2}.$$

(iv) *For some constant* $C > 0$ *independent of* ϵ, *the initial data* $\underset{o}{V}_\epsilon$ *satisfies the estimate*

$$\|\underset{o}{V}_\epsilon - \underset{o}{V}_0\|_{H^k_{\delta-1,\epsilon}} \leq C\|\underset{o}{V}_\epsilon - \underset{o}{V}_0\|_{H^k_{\delta-1}} \leq C\epsilon$$

while

$$\|\partial_t V_\epsilon(0)\|_{H^{k-1}_{\delta-1,\epsilon}} \leq \|\partial_t V_\epsilon(0)\|_{H^{k-1}_{\delta-1}} \leq C$$

for all $\epsilon \in (0, \epsilon_0]$.

(v) *If* $\sup_{0 \leq t < T_2(\epsilon)} \|V_\epsilon(t)\|_{W^{1,\infty}} < \infty$ *and for all* $(x, t) \in \mathbb{R}^3 \times [0, T_2(\epsilon))$, $|\epsilon \delta \mathfrak{u}_\epsilon(x, t)| < K_0$, $|\epsilon w^i(x, t)| < 2K_0$, *and* $|\epsilon \alpha_\epsilon(x, t)| < 2K_0$, *then there exists a* $T_* > T_2(\epsilon)$ *such that the solution* V_ϵ *can be continued to the interval* $(-T_1(\epsilon), T_*)$.

Proof. (i) Follows directly from Theorem B.5, Proposition B.6, and Corollary B.7, where we use the initial data from Proposition 5.1.

(ii) This follows from standard arguments on reductions of 2nd order hyperbolic equations to 1st order symmetric hyperbolic systems. See [39], Sect. 16.3 for details.

(iii) By part (ii), the triplet $\{\bar{\mathfrak{u}}^{ij}_\epsilon, w^i_\epsilon, \alpha_\epsilon\}$ satisfies the reduced Einstein equations (3.17) and the fluid equations (4.12). By construction, $\{\bar{\mathfrak{u}}^{ij}_\epsilon|_{t=0}, w^i_\epsilon|_{t=0}, \alpha_\epsilon|_{t=0}\}$ satisfies the constraints $\mathcal{N}|_{t=0} = 0$, $\mathcal{H}^j|_{t=0} = 0$, and $(\mathcal{G}^{4i} - \mathcal{T}^{4i})|_{t=0} = 0$. The reduced Einstein equations (3.17) can be written in terms of the Einstein density $\mathcal{G}^{ij}$ as

$$\mathcal{G}^{ij} - \bar{\mathfrak{g}}^{ij}\bar{\partial}_k\mathcal{H}^k + 2\bar{\partial}_k\mathcal{H}^{(i}\bar{\mathfrak{g}}^{j)k} = \mathcal{T}^{ij}.$$

Using $(\mathcal{G}^{4i} - \mathcal{T}^{4i})|_{t=0} = 0$, we see that

$$\left(-\bar{\mathfrak{g}}^{4j}\bar{\partial}_k\mathcal{H}^k + 2\bar{\partial}_k\mathcal{H}^{(4}\bar{\mathfrak{g}}^{j)k}\right)_{t=0} = 0. \tag{6.13}$$

A straightforward calculation then shows that this implies that $\partial_t\mathcal{H}^j|_{t=0} = 0$. As discussed in Sects. 4 (see (4.24)), $\mathcal{N}$ satisfies a linear symmetric hyperbolic system and hence by uniqueness, it follows that $\mathcal{N} = 0$ for all $(x^I, t) \in \mathbb{R}^3 \times (-T_1(\epsilon), T_2(\epsilon)$. Thus $\{w^i_\epsilon, \alpha_\epsilon\}$ determine a solution, via the formulas (4.11), to the Euler equation which are equivalent to $\bar{\nabla}_i\mathcal{T}^{ij} = 0$. So taking the divergence of (6.13) while using $\bar{\nabla}_i\mathcal{T}^{ij} = \bar{\nabla}_i\mathcal{G}^{ij} = 0$ shows that $\mathcal{H}^j$ satisfies an equation of the form

$$\bar{\mathfrak{g}}^{ik}\bar{\partial}^j_{ik}\mathcal{H}^j + Q^{jp}_q(\bar{\mathfrak{g}}, \bar{\partial}_k\bar{\mathfrak{g}})\bar{\partial}_p\mathcal{H}^q = 0,$$

where the Q^{jp}_q are analytic in $\bar{\mathfrak{g}}$ and $\bar{\partial}_k\bar{\mathfrak{g}}$. Clearly, this is a linear, 2^{nd} order hyperbolic equation for $\mathcal{H}^j$. Since $\mathcal{H}^j|_{t=0} = \partial_t\mathcal{H}^j|_{t=0} = 0$, we must have $\mathcal{H}^j = 0$ for all $(x^I, t) \in \mathbb{R}^3 \times (-T_1(\epsilon), T_2(\epsilon))$.

(iv) We know from Proposition 5.1 that the map $(0, \epsilon_0] \ni \epsilon \to \underset{o}{V}_\epsilon \in H^k_{\delta-1,\epsilon}$ is analytic which implies the estimate $\|\underset{o}{V}_\epsilon - \underset{o}{V}_0\|_{H^k_{\delta-1}} \leq C\epsilon$ for some fixed constant $C > 0$. So then

$$\|\underset{o}{V}_\epsilon - \underset{o}{V}_0\|_{H^k_{\delta-1,\epsilon}} \leq \|\underset{o}{V}_\epsilon - \underset{o}{V}_0\|_{H^k_{\delta-1}} \leq C\epsilon$$

by Lemma A.11. Since $\{\bar{\mathfrak{u}}_\epsilon, w^i, \alpha_\epsilon\}$ solves the reduced Einstein equations (3.17), we have that

$$\begin{aligned}&\epsilon\bar{\mathfrak{g}}^{44}_\epsilon\partial_t\bar{\partial}_4\bar{\mathfrak{u}}^{Ij}_\epsilon + 8\epsilon^2\partial_L\bar{\partial}_4\bar{\mathfrak{u}}^{Ij}_\epsilon + \bar{\mathfrak{g}}^{KL}_\epsilon\partial^2_{KL}\bar{\mathfrak{u}}^{Ij}_\epsilon + \epsilon^2 f^{Ij}(\epsilon^2\bar{\mathfrak{u}}_\epsilon, \bar{\partial}_4\bar{\mathfrak{u}}_\epsilon, \partial_L\bar{\mathfrak{u}}_\epsilon)\\ &\quad= \epsilon^2 S^{Ij}(\epsilon^2\bar{\mathfrak{u}}_\epsilon, \alpha_\epsilon, w^i_\epsilon),\end{aligned}$$

where the f^{IJ} are analytic and quadratic in $\partial_4\bar{\mathfrak{u}}_\epsilon$ and $\partial_k\bar{\mathfrak{u}}_\epsilon$ while S^{IJ} are also analytic and linear in α_ϵ and w^i_ϵ. Evaluating this equation at $t = 0$, and using the following facts from Proposition 5.1,

$$\epsilon^{-1}\|\underset{o}{\bar{\mathfrak{u}}}^{Ij}_\epsilon\|_{H^{k+1}_\delta} + \|\underset{o}{\bar{\mathfrak{u}}}^{44}_\epsilon\|_{H^{k+1}_\delta} + \|\partial_t\underset{o}{\bar{\mathfrak{u}}}^{ij}\|_{H^k_{\delta-1}} + \|\underset{o}{\alpha}_\epsilon\|_{H^k_{\delta-1}} + \|\underset{o}{w}^i\|_{H^k_{\delta-1}} \leq C, \tag{6.14}$$

we find upon solving for $\partial_t\bar{\partial}_4\bar{\mathfrak{u}}^{Ij}$ that

$$\|\partial_t\bar{\partial}_4\bar{\mathfrak{u}}^{Ij}(0)\|_{H^{k-1}_{\delta-1}} \leq C \quad \forall\, \epsilon \in (0, \epsilon_0] \tag{6.15}$$

by the calculus inequalities of Appendix A. But from part (iii), we get that $\bar{\partial}_4\bar{\mathfrak{u}}^{44}_\epsilon + \partial_I\bar{\mathfrak{u}}^{I4}_\epsilon = 0$ and hence differentiating this with respect to t and evaluating at $t = 0$ yields

$$\|\partial_t\bar{\partial}_4\bar{\mathfrak{u}}^{44}_\epsilon(0)\|_{H^{k-1}_{\delta-1}} = \|\partial_I\partial_t\underset{o}{\bar{\mathfrak{u}}}^{I4}_\epsilon\|_{H^{k-1}_{\delta-1}} \leq C \quad \forall\, \epsilon \in (0, \epsilon_0]. \tag{6.16}$$

From the estimates (6.14), the fluid equations (4.12) and similar arguments as above show that

$$\|\partial_t \alpha_\epsilon(0)\|_{H^{k-1}_{\delta-1}} \leq C + \|\partial_t w^i_\epsilon(0)\|_{H^{k-1}_{\delta-1}} \leq C \quad \forall\, \epsilon \in (0, \epsilon_0]. \tag{6.17}$$

Estimates (6.14)–(6.17) and Lemma A.11 then imply that $\|\partial_t V_\epsilon(0)\|_{H^{k-1}_{\delta-1,\epsilon}} \leq \|\partial_t V_\epsilon(0)\|_{H^{k-1}_{\delta-1}} \leq C$ for all $\epsilon \in (0, \epsilon_0]$.

(v) This is just a statement of the continuation principle of Theorem B.6. □

7. The Newtonian Limit

Let $\{V_\epsilon, 0 < \epsilon \leq \epsilon_0\}$ be the sequence of solutions from Theorem 6.1 where we will always assume that

$$-1 < \delta < -1/2 \quad \text{and} \quad \operatorname{supp} \underset{o}{\alpha} \subset B_R \text{ for some } R > 0.$$

If we let $T_m(\epsilon)$ denote the maximal time of existence for the solution V_ϵ, then

$$V_\epsilon \in \bigcap_{\ell=0}^{s+1} C^\ell([0, T_m(\epsilon)), H^{k-\ell}_{\delta-1}) \subset \bigcap_{\ell=0}^{s+1} C^\ell([0, T_m(\epsilon)), H^{k-\ell}_{\delta-1,\epsilon}). \tag{7.1}$$

So $\alpha_\epsilon \in \bigcap_{\ell=0}^{s+1} C^\ell([0, T_m(\epsilon)), H^{k-\ell}_{\delta-1})$ and hence Proposition 3.6 of [17] and Lemma A.8 imply that

$$\rho_\epsilon = \rho(\alpha_\epsilon) \in \bigcap_{\ell=0}^{s+1} C^\ell([0, T_m(\epsilon)), H^{k-\ell}_{\delta-2}).$$

Using Proposition 2.2 of [1], we can solve the equation

$$\Delta \Phi_\epsilon = \rho_\epsilon \tag{7.2}$$

to find

$$\Phi_\epsilon \in \bigcap_{\ell=0}^{s+1} C^\ell([0, T_m(\epsilon)), H^{k+2-\ell}_{\delta}).$$

To obtain the Newtonian limit, we use Φ_ϵ to take care of the singular term $\epsilon^{-1} g(V_\epsilon)V_\epsilon$ in (6.1) by introducing the new variable

$$W_\epsilon := (\mathrm{u}^{ij}_{4,\epsilon}, u^{ij}_{J,\epsilon}, \delta \mathrm{u}^{ij}_\epsilon, \alpha_\epsilon, w^i_\epsilon) \quad u^{ij}_{J,\epsilon} := \mathrm{u}^{ij}_{J,\epsilon} - \delta^i_4 \delta^j_4 \partial_J \Phi_\epsilon. \tag{7.3}$$

Observe that

$$V_\epsilon = W_\epsilon + d\Phi_\epsilon,$$

where

$$d\Phi_\epsilon := (0, \delta^i_4 \delta^j_4 \partial_J \Phi_\epsilon, 0, 0, 0).$$

Noting that

$$b^0(\epsilon U_\epsilon, \epsilon V_\epsilon) = b^0(\epsilon U_\epsilon, \epsilon W_\epsilon) \quad \text{and} \quad b^I(\epsilon, U_\epsilon, V_\epsilon) = b^I(\epsilon, U_\epsilon, W_\epsilon), \tag{7.4}$$

W_ϵ satisfies the equation

$$b^0(\epsilon U_\epsilon, \epsilon W_\epsilon)\partial_t W_\epsilon = \frac{1}{\epsilon} c^I \partial_I W_\epsilon + b^I(\epsilon, U_\epsilon, W_\epsilon)\partial_I W_\epsilon + f(\epsilon, U_\epsilon, W_\epsilon + d\Phi_\epsilon)W_\epsilon + H_\epsilon, \tag{7.5}$$

where

$$H_\epsilon := h_\epsilon - b^0(\epsilon U_\epsilon, \epsilon W_\epsilon)\partial_t d\Phi_\epsilon + b^I(\epsilon, U_\epsilon, W_\epsilon)\partial_I d\Phi_\epsilon + f(\epsilon, U_\epsilon, W_\epsilon + d\Phi_\epsilon)d\Phi_\epsilon.$$

By construction the initial data $\underset{o}{V_\epsilon}$ is bounded in $H^k_{\delta-1}$ as $\epsilon \searrow 0$. Therefore by Lemma A.11, there exists a constant K_1 such that

$$\|W_\epsilon|_{t=0}\|_{H^k_{\delta-1,\epsilon}} \le K_1 \quad \text{for all } \epsilon \in (0, \epsilon_0]. \tag{7.6}$$

Also by definition of W_ϵ and Lemma A.7,

$$\max\{\|\delta \mathrm{u}_\epsilon\|_{L^\infty}, \|\alpha_\epsilon\|_{L^\infty}, \|w^i_\epsilon\|_{L^\infty}\} \le \|W_\epsilon\|_{C^1_b} \le C_{\mathrm{Sob}}\|W_\epsilon\|_{H^k_{\delta-1,\epsilon}}, \tag{7.7}$$

where C_{Sob} is the constant from Lemma A.7 that is ϵ independent. Shrinking ϵ_0 if necessary, we can always assume that

$$2\epsilon_0 C_{\mathrm{Sob}} K_1 < K_0. \tag{7.8}$$

Define

$$\tau_\epsilon := \min\left\{\sup\left\{\tau > 0 \,|\, \sup_{0\le t\le\tau}\|W_\epsilon(t)\|_{H^k_{\delta-1,\epsilon}} \le 2K_1 \quad \text{and} \quad \sup_{0\le t\le\tau}\|V_\epsilon\|_{H^k_{\delta-1,\epsilon}} < \infty\right\}, 1\right\}. \tag{7.9}$$

From the continuation principle in Theorem 6.1, it is clear that τ_ϵ satisfies

$$0 < \tau_\epsilon \le T_m(\epsilon).$$

7.1. Energy estimates. We will now use energy estimates on the $H^k_{\delta-1,\epsilon}$ spaces to show that τ_ϵ is bounded below by a constant independent of ϵ. The strategy we use is that of [5, 19] adapted to the $H^k_{\delta,\epsilon}$ spaces. All of the results below will be derived under the assumption that the 1-parameter family V_ϵ of solutions has the additional regularity

$$V_\epsilon \in \bigcap_{\ell=0}^{s+1} C^\ell([0, \tau_\epsilon], H^{k+1-\ell}_{\delta-1}).$$

It is then not difficult to use a solution of this type to approximate solutions of the regularity type (7.1) and thereby show that all of the following results also hold for solutions with the regularity (7.1). Since these sort of approximation arguments are standard, we will leave the details to the interested reader.

The next lemma contains the basic energy estimate which is the key to deriving estimates independent of ϵ. We note that this type of estimate has been derived previously for the standard Sobolev spaces in [5, 19]. It also makes clear why we need to introduce the variables W_ϵ and Φ_ϵ to put the Einstein-Euler equations into the form (7.5).

Lemma 7.1. *Suppose* $\epsilon \geq 0$, $a^0 \in C^1([0,\tau], W^{1,\infty})$, $a^I \in C^0([0,\tau], W^{1,\infty})$, $g \in C^0([0,\tau], L^2_{\lambda,\epsilon})$, *and that* $w \in C^1([0,\tau], H^1_{\lambda,\epsilon})$ *is a solution to the linear equation*

$$a^0 \partial_t w = a^I \partial_I w + g.$$

Then there exists a constant $C > 0$ *independent of* ϵ *such that*

$$\frac{d}{dt}\langle w | a^0 w\rangle_{L^2_{\lambda,\epsilon}} \leq C\left[(\|\mathrm{div}\, a\|_{L^\infty} + \epsilon\|\vec{a}\|_{L^\infty})\,\|w\|^2_{L^2_{\lambda,\epsilon}} + \|g\|_{L^2_{\lambda,\epsilon}}\|w\|_{L^2_{\lambda,\epsilon}}\right],$$

where $\mathrm{div}\, a = \partial_t a^0 + \partial_I a^I$ *and* $\vec{a} = (a^1, a^2, a^3)$.

Proof. Let $\bar{\sigma} = \sigma_\epsilon^{-2\lambda-3}$. Then $\|\bar{\sigma}^{-1}\partial_j \bar{\sigma}\|_{L^\infty} \leq \epsilon C$ for some constant $C > 0$ that is independent of ϵ. Using this, the proof follows by a standard integration by parts argument as in the proof of Lemma B.4. □

To continue, we estimate, in terms of K_1, how much the support of α_ϵ can change as $\epsilon \searrow 0$.

Lemma 7.2.

$$\mathrm{supp}\, \alpha_\epsilon(t) \subset B_{R+32K_1}$$

for all $(t,\epsilon) \in [0,\tau_\epsilon] \times (0,\epsilon_0]$.

Proof. Letting X^I, $\bar{Y}$ and $\hat{Y}$ be as in Sect. 4 (see (4.26)), we define

$$X^I_\epsilon(t) := X^I(\epsilon w^4_\epsilon(t), w^J_\epsilon(t))$$

and

$$\begin{aligned} Y^I_\epsilon(t) :=\ & \bar{Y}(\epsilon w^4_\epsilon(t), \epsilon\alpha_\epsilon(t))\left(\epsilon\partial_t w^4_\epsilon(t) + \partial_I w^I_\epsilon(t)\right) \\ & + \hat{Y}(\epsilon(\underset{o}{\mathfrak{u}}_\epsilon + \delta\mathfrak{u}(t)), \epsilon w^4(t), \epsilon \mathfrak{u}_k(t), \epsilon w^J(t), \epsilon\alpha(t)). \end{aligned}$$

Using (6.10), (7.7), (7.8), and (7.9), we obtain the bound

$$\|X^I_\epsilon(t)\|_{L^\infty} \leq 32K_1 \quad \forall\, (t,\epsilon) \in [0,\tau_\epsilon] \times (0,\epsilon_0]. \tag{7.10}$$

From Lemmas (A.7) and (A.10), and (7.1), it follows that $X^I_\epsilon \in C^0([0,\tau_\epsilon], C^1_b)$ and $Y^I_\epsilon \in C^0([0,\tau_\epsilon], C^0_b)$. Therefore the vector field X^I_ϵ can be integrated to get a C^1 flow $\psi^I_\epsilon(t,x)$ that is well defined for all $(t,x) \in [0,\tau_\epsilon] \times \mathbb{R}^3$. For each $x \in \mathbb{R}^3$, define $\alpha^x_\epsilon(t) := \alpha_\epsilon(t, \psi_\epsilon(t,x))$. Then $\partial_t \psi^I_\epsilon(t,x) = X^I_\epsilon(t, \psi_\epsilon(t,x))$ together with the evolution equation (4.25) implies that

$$\frac{d}{dt}\alpha^x_\epsilon(t) + Y(t, \psi_\epsilon(t,x))\alpha^x_\epsilon(t) = 0.$$

By assumption $\mathrm{supp}\,\alpha_0 \subset B_R$ and hence $\alpha^x_\epsilon(0) = \underset{o}{\alpha}(x) = 0$ for $x \in E_R := \mathbb{R}^3 \backslash B_R$. Therefore

$$\alpha_\epsilon(t, \psi_\epsilon(t,x)) = 0 \quad \text{all } x \in E_R \tag{7.11}$$

by the uniqueness of solutions to ODEs. But

$$|\psi_\epsilon(t,x)-x| \le \int_0^{\tau_\epsilon} |\partial_t \psi_\epsilon(t,x)| = \int_0^{\tau_\epsilon} |X_\epsilon(t,\psi_\epsilon(t,x))| \le 32K_1\tau_\epsilon \le 32K_1$$

by (7.10) and $0 < \tau_\epsilon \le 1$. From this, (7.11), and the fact that for each t the map $\mathbb{R}^3 \ni x \mapsto \psi_\epsilon(t,x) \in \mathbb{R}^3$ defines a C^1 diffeomorphism, it follows that $\operatorname{supp}\alpha_\epsilon(t) \subset B_{R+32K_1}$ for all $(t,\epsilon) \in [0,\tau_\epsilon] \times (0,\epsilon_0]$. □

Next, we estimate $\|\Phi_\epsilon\|_{H^{k+2}_\delta}$ in terms of $\|W_\epsilon\|_{H^k_{\delta-1,\epsilon}}$.

Lemma 7.3. *Let* $\bar{R} = R + 32K_1$ *and*

$$C_1 = (1+\bar{R})^{-(\delta-2)-3/2}\sqrt{1+(1+\bar{R})^{2k}}.$$

Then there exists a constant $C > 0$ *such that*

$$\|\Phi_\epsilon(t)\|_{H^{k+2}_\delta} \le CC_1\|W_\epsilon(t)\|^n_{H^k_{\delta-1,\epsilon}}$$

for all $(t,\epsilon) \in [0,\tau_\epsilon] \times (0,\epsilon_0]$.

Proof. By Lemma 7.2, the $\operatorname{supp}\alpha_\epsilon(t) \subset B_{R+32K_1}$ for all $(t,\epsilon) \in [0,\tau_\epsilon]\times(0,\epsilon_0]$. Letting $\bar{R} = R + 32K_1$, it follows directly from the definition of the weighted norms that

$$\|u\|_{L^2} \le \|u\|_{L^2_{\eta,\epsilon}} \le (1+\bar{R})^{-\eta-3/2}\|u\|_{L^2}$$

for all functions u whose support is contained in $B_{\bar{R}}$ and for any $\epsilon \in (0,1]$ and $-\eta-3/2 \ge 0$. Therefore

$$\|\rho_\epsilon\|_{H^k_{\delta-2}} \le CC_1\|\rho_\epsilon\|_{H^k_{\delta-1,\epsilon}},$$

where $C > 0$ is a constant independent of ϵ and

$$C_1 = (1+\bar{R})^{-(\delta-2)-3/2}\sqrt{1+(1+\bar{R})^{2k}}.$$

Since $\Delta : H^{k+2}_\delta \to H^k_{\delta-2}$ is an isomorphism and $\Delta\Phi_\epsilon = \rho_\epsilon$, we have $\|\Phi_\epsilon\|_{H^{k+2}_\delta} \le C\|\rho_\epsilon\|_{H^k_{\delta-2}}$, and hence, by Lemma A.8 (see also (1.12) and (7.3)) and the above estimate that

$$\|\Phi_\epsilon\|_{H^{k+2}_\delta} \le CC_1\|\rho_\epsilon\|_{H^k_{\delta-2,\epsilon}} \le CC_1\|\alpha_\epsilon\|^n_{H^k_{\delta-1,\epsilon}} \le CC_1\|W_\epsilon\|^n_{H^k_{\delta-1,\epsilon}}.$$

□

We note that for the remainder of this section, all of the constants appearing in the estimates may depend on the fixed constant K_1. We will often use C to denote constants that depend on K_1 and that may change from line to line.

Let $W^\alpha_\epsilon = D^\alpha W_\epsilon$ ($|\alpha| \ge 0$), $b^0_\epsilon = b^0(\epsilon U_\epsilon, \epsilon W_\epsilon)$, $b^I_\epsilon = b^I(\epsilon, U_\epsilon, V_\epsilon)$ and $f_\epsilon = f(\epsilon, U_\epsilon, W_\epsilon + d\Phi_\epsilon)W_\epsilon$. The evolution equation (7.5) implies that

$$\partial_t W_\epsilon = (b^0_\epsilon)^{-1}\left(\frac{1}{\epsilon}c^I + b^I_\epsilon\right)\partial_I W_\epsilon + (b^0_\epsilon)^{-1} f_\epsilon + (b^0_\epsilon)^{-1} H_\epsilon. \quad (7.12)$$

Differentiating this equation yields

$$b^0_\epsilon \partial_t W^\alpha_\epsilon = \frac{1}{\epsilon} c^I \partial_I W^\alpha_\epsilon + b^I_\epsilon \partial_I W^\alpha_\epsilon + q^\alpha \quad |\alpha| \geq 0, \tag{7.13}$$

where

$$q^\alpha = b^0_\epsilon [D^\alpha, (b^0_\epsilon)^{-1}(\epsilon^{-1} c^I + b^I_\epsilon)]\partial_I W_\epsilon + b^0_\epsilon D^\alpha((b^0_\epsilon)^{-1} f_\epsilon) + b^0_\epsilon D^\alpha((b^0_\epsilon)^{-1} H_\epsilon). \tag{7.14}$$

From Lemma A.11, we know, since $-1 < \delta < -1/2$, that $\|\epsilon \underset{o}{\bar{\mathrm{u}}}_\epsilon\|_{H^{k+1}_{\delta,\epsilon}} \leq \epsilon^{|\delta+1/2|} \|\underset{o}{\bar{\mathrm{u}}}_\epsilon\|_{H^{k+1}_\delta}$. Since $\|\underset{o}{\bar{\mathrm{u}}}_\epsilon\|_{H^{k+1}_\delta}$ is uniformly bounded in ϵ, we get, by Lemmas A.7 and A.11, that

$$\|U_\epsilon\|_{C^{1,\infty}_b} \leq C_{\text{Sob}} \|U_\epsilon\|_{H^{k+1}_{\delta,\epsilon}} \leq C\epsilon^{|\delta+1/2|} \tag{7.15}$$

for some constant $C > 0$ independent of ϵ. So

$$\|b^i_\epsilon(t)\|_{W^{1,\infty}} \leq C \quad \forall\, (t, \epsilon) \in [0, \tau_\epsilon] \times (0, \epsilon_0] \tag{7.16}$$

by (7.4), (7.7), (7.9) and (7.15) . Also, note that

$$\|d\Phi_\epsilon\|_{L^\infty} + \|Dd\Phi_\epsilon\|_{L^\infty} \leq C\|\Phi_\epsilon\|_{H^{k+2}_\delta} \leq C \quad \text{and} \quad \|\partial_t d\Phi_\epsilon\|_{L^\infty} \leq C\|\Phi_\epsilon\|_{H^{k+1}_\delta}$$

by (A.3), (A.24) and Lemmas 7.3 and A.7. The evolution equation (7.12) then implies that

$$\|\partial_t b^0_\epsilon\|_{L^\infty} = \|\epsilon Db^0(\epsilon U_\epsilon, \epsilon W_\epsilon) \cdot \partial_t W_\epsilon\|_{L^\infty} \leq C(1 + \|\partial_t d\Phi_\epsilon\|_{H^{k+1}_\delta}). \tag{7.17}$$

Together (7.16) and (7.17) establish the existence of a constant $C > 0$ such that

$$\|\operatorname{div} b_\epsilon(t)\|_{L^\infty} \leq C(1 + \|\partial_t \Phi_\epsilon(t)\|_{H^{k+1}_\delta}) \quad \forall\, (t, \epsilon) \in [0, \tau_\epsilon] \times (0, \epsilon_0]. \tag{7.18}$$

Differentiating $(b^0_\epsilon)^{-1}$ yields

$$\partial_J (b^0_\epsilon)^{-1} = -\epsilon (b^0_\epsilon)^{-1}(Db^0(\epsilon U_\epsilon, \epsilon W_\epsilon) \cdot (\partial_J U_\epsilon, \partial_J W_\epsilon))(b^0_\epsilon)^{-1}.$$

This along with (7.15), (7.16), (A.3), (A.24), and Lemmas A.7 and A.9 can be used to control the singular term in (7.14) and results in the following estimate (see also Appendix B.2)

$$\|q^\alpha(t)\|_{L^2_{\delta-1-|\alpha|,\epsilon}} \leq P_\alpha(\|W_\epsilon(t)\|_{H^k_{\delta-1,\epsilon}}, \|\Phi_\epsilon(t)\|_{H^{k+2}_\delta}, \|\partial_t \Phi_\epsilon(t)\|_{H^{k+1}_\delta}) \quad \forall\, t \in [0, \tau_\epsilon] \tag{7.19}$$

where $P_\alpha(y_1, y_2, y_3)$ is a polynomial that is independent of ϵ and satisfies $P(0) = 0$. Note that in deriving this result, we have used the estimate

$$\|d\Phi_\epsilon\|_{H^k_{\delta-1,\epsilon}} + \|Dd\Phi_\epsilon\|_{H^k_{\delta-2,\epsilon}} \leq C\|\Phi_\epsilon\|_{H^{k+2}_\delta} \quad \text{and} \quad \|\partial_t d\Phi_\epsilon\|_{H^k_{\delta-1,\epsilon}} \leq C\|\partial_t \Phi_\epsilon\|_{H^{k+1}_\delta} \tag{7.20}$$

for some C independent of ϵ which follows from (A.3), (A.24), and Lemma A.11.

Define

$$|\!|\!| W_\epsilon |\!|\!|^2_{k,\delta-1,\epsilon} := \sum_{|\alpha| \leq k} \langle \partial^\alpha W_\epsilon | b^0_\epsilon \partial^\alpha W_\epsilon \rangle_{L^2_{\delta-|\alpha|,\epsilon}}.$$

Then

$$\frac{1}{4}\|W_\epsilon(t)\|_{H^k_{\delta-1,\epsilon}} \leq |||W_\epsilon(t)|||_{k,\delta-1,\epsilon} \leq 4\|W_\epsilon(t)\|_{H^k_{\delta-1,\epsilon}} \quad \forall\, t \in [0,\tau_\epsilon] \tag{7.21}$$

by (6.11) and (6.12). Lemma 7.1 combined with the estimates (7.16), (7.18), and (7.19) implies that

$$\frac{d}{dt}|||W_\epsilon|||^2_{k,\delta-1,\epsilon} \leq P(|||W_\epsilon|||_{k,\delta-1,\epsilon}, \|\Phi_\epsilon\|_{H^{k+2}_\delta}, \|\partial_t\Phi_\epsilon\|_{H^{k+1}_\delta})|||W_\epsilon|||_{k,\delta-1,\epsilon}$$

or equivalently

$$\frac{d}{dt}|||W_\epsilon(t)|||_{k,\delta-1,\epsilon} \leq P(|||W_\epsilon(t)|||_{k,\delta-1,\epsilon}, \|\Phi_\epsilon(t)\|_{H^{k+2}_\delta}, \|\partial_t\Phi_\epsilon(t)\|_{H^{k+1}_\delta}) \quad \forall\, t \in [0,\tau_\epsilon] \tag{7.22}$$

for a ϵ independent polynomial $P(y_1, y_2, y_3)$ satisfying $P(0) = 0$. By Lemma 7.3, $\|\Phi_\epsilon\|_{H^{k+2}_\delta}$ can be bounded by a polynomial of $\|W_\epsilon\|_{H^k_{\delta-1,\epsilon}}$ that is independent of ϵ and vanishes for $\|W_\epsilon\|_{H^k_{\delta-1,\epsilon}} = 0$. The differential inequality (7.22) shows that if we can do the same for $\|\partial_t\Phi_\epsilon\|_{H^{k+1}_\delta}$ then we get an estimate for $|||W_\epsilon(t)|||_{k,\delta-1,\epsilon}$ independent of ϵ.

Lemma 7.4. *There exists a polynomial $P(y)$ with coefficients independent of ϵ such that $P(0) = 0$ and*

$$\|\partial_t\Phi_\epsilon(t)\|_{H^{k+1}_\delta} \leq P(\|W_\epsilon(t)\|_{H^k_{\delta-1,\epsilon}})$$

for all $(t,\epsilon) \in [0,\tau_\epsilon] \times (0,\epsilon_0]$.

Proof. By (4.12), $\mathbf{w}_\epsilon := (\alpha_\epsilon, w^i_\epsilon)^T$ satisfies an equation of the form

$$a^4(\epsilon U_\epsilon, \epsilon W_\epsilon)\partial_t\mathbf{w}_\epsilon = a^I(\epsilon U_\epsilon, \epsilon W_\epsilon)\partial_t\mathbf{w}_\epsilon + b_1(\epsilon U_\epsilon, \epsilon W_\epsilon)W_\epsilon + b_2(\epsilon U_\epsilon, \epsilon W_\epsilon)d\Phi_\epsilon$$

and so

$$\partial_t\mathbf{w}_\epsilon = (a^4)^{-1}a^I\partial_I\mathbf{w}_\epsilon + (a^4)^{-1}b_1W_\epsilon + (a^4)^{-1}b_2 d\Phi_\epsilon.$$

Thus

$$\begin{aligned}\|\partial_t\mathbf{w}_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} \leq{}& \|(a^4)^{-1}a^I\|_{H^{k-1}_{1,\epsilon}}\|DW_\epsilon\|_{H^{k-1}_{\delta-2,\epsilon}} \\ &+\|(a^4)^{-1}b_1\|_{H^{k-1}_{0,\epsilon}}\|W_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} + \|(a^4)^{-1}b_2\|_{H^{k-1}_{0,\epsilon}}\|d\Phi_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}}\end{aligned}$$

by Lemma A.8. Also by (7.15), (A.3), (A.24), and Lemmas A.7 and A.9, we have that

$$\|(a^4)^{-1}a^I\|_{H^{k-1}_{1,\epsilon}} \leq P(\|W_\epsilon\|_{H^k_{\delta-1,\epsilon}}), \quad \|DW_\epsilon\|_{H^{k-1}_{\delta-2,\epsilon}} \leq \|W_\epsilon\|_{H^k_{\delta-1,\epsilon}},$$
$$\|(a^4)^{-1}b_1\|_{H^{k-1}_{0,\epsilon}} \leq P(\|W_\epsilon\|_{H^k_{\delta-1,\epsilon}}), \quad \text{and} \quad \|(a^4)^{-1}b_2\|_{H^{k-1}_{0,\epsilon}} \leq P(\|W_\epsilon\|_{H^k_{\delta-1,\epsilon}})$$

for some polynomial $P(y)$ that is independent of ϵ. The above two inequalities along with (7.20) and Lemma 7.3 show that

$$\|\partial_t\alpha_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} \leq \|\partial_t\mathbf{w}_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} \leq P(\|W_\epsilon\|_{H^k_{\delta-1,\epsilon}})$$

for a polynomial $P(y)$ independent of ϵ and satisfying $P(0) = 0$. Using Lemma A.8, the above estimate implies that

$$\|\partial_t \rho_\epsilon\|_{H^{k-1}_{\delta-2,\epsilon}} \leq P\left(\|W_\epsilon\|_{H^k_{\delta-1,\epsilon}}\right),$$

where as above $P(y)$ is a polynomial that is independent of ϵ. Since $\Delta\partial_t\Phi_\epsilon = \partial_t\rho_\epsilon$, the same arguments used in the proof of Lemma 7.3 can be used to conclude

$$\|\partial_t \Phi_\epsilon\|_{H^{k+1}_{\delta}} \leq C\|\partial_t\rho_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} \leq P\left(\|W_\epsilon\|_{H^k_{\delta-1,\epsilon}}\right).$$

□

Lemmas 7.3 and 7.4 combined with the estimate (7.22) yield

$$\frac{d}{dt}\|\!|W_\epsilon(t)\|\!|_{k,\delta-1,\epsilon} \leq P(\|\!|W_\epsilon(t)\|\!|_{k,\delta-1,\epsilon})\|\!|W_\epsilon(t)\|\!|_{k,\delta-1,\epsilon} \quad \forall\, t \in [0, \tau_\epsilon] \tag{7.23}$$

for a polynomial $P(y)$ that is independent of ϵ and whose coefficients depend only on K_1. By Gronwall's inequality there exists a time $T^* \in (0, 1)$, independent of ϵ, such that if $y(t) \geq 0$ is C^1 and satisfies $dy/dt \leq P(y)y$, then $y(t) \leq e^{K_3 t}y(0)$, where K_3 is a constant that depends on K_1. Therefore

$$\|\!|W_\epsilon(t)\|\!|_{k,\delta-1,\epsilon} \leq e^{K_3 t}\|\!|W_\epsilon(0)\|\!|_{k,\delta-1,\epsilon} \text{ for all } (t,\epsilon) \in [0, \min\{T^*, \tau_\epsilon\}] \times (0, \epsilon_0]. \tag{7.24}$$

Shrinking T^* if necessary, we conclude that

$$\|\!|W_\epsilon(t)\|\!|_{k,\delta-1,\epsilon} \leq \frac{3}{2}K_1 \quad \text{for all } (t,\epsilon) \in [0, \min T^*, \tau_\epsilon] \times (0, \epsilon_0]. \tag{7.25}$$

Note also that

$$\|V_\epsilon(t)\|_{H^k{}_{\delta-1,\epsilon}} \leq C \quad \text{for all } (t,\epsilon) \in [0, \min\{T^*, \tau_\epsilon\}] \times (0, \epsilon_0] \tag{7.26}$$

by 7.20, 7.21 and Lemma 7.3. Therefore by the definition of τ_ϵ, we must have $0 < T^* < \tau_\epsilon$ for all $0 < \epsilon \leq \epsilon_0$.

Differentiating (7.12) with respect to t, shows that $\dot{W}_\epsilon := \partial_t W_\epsilon$ and $d\dot{\Phi}_\epsilon := \partial_t d\Phi_\epsilon$ satisfy the equation

$$\begin{aligned} b^0(\epsilon U_\epsilon, \epsilon W_\epsilon)\partial_t \dot{W}_\epsilon = &\ \frac{1}{\epsilon}c^I\partial_I\dot{W}_\epsilon + b^I(\epsilon, U_\epsilon, W_\epsilon)\partial_I\dot{W}_\epsilon \\ &+ \bar{f}_1(\epsilon, U_\epsilon, W_\epsilon, DW_\epsilon, d\Phi_\epsilon, Dd\Phi_\epsilon, d\dot{\Phi}_\epsilon)\dot{W}_\epsilon \\ &+ \bar{f}_2(\epsilon, U_\epsilon, W_\epsilon, d\Phi_\epsilon, Dd\Phi, d\dot{\Phi}_\epsilon, Dd\dot{\Phi}_\epsilon, \partial_t d\dot{\Phi}_\epsilon) + \partial_t h \end{aligned}$$

for analytic functions f_1, f_2 with f_2 linear in the last 3 variables. This equation has the same structure (7.5) and it is not difficult to show that the arguments used to derive (7.24) can also be used to obtain the estimate

$$\|\dot{W}_\epsilon(t)\|_{H^{k-1}_{\delta-1,\epsilon}} \leq C \quad \forall\, (\epsilon, t) \in (0, \epsilon_0] \times [0, T^*] \tag{7.27}$$

under the assumption that $\|\dot{W}_\epsilon(0)\|_{H^{k-1}_{\delta-1,\epsilon}}$ is bounded as $\epsilon \searrow 0$. But this is clear from Proposition 6.1 and Lemma 7.3 and so the estimate holds. We have proved the following proposition.

Proposition 7.5. *For $\epsilon_0 > 0$ small enough, there exists a $T^* > 0$ independent of $\epsilon \in (0, \epsilon_0]$ such that the one parameter family of solutions V_ϵ exists, for all $\epsilon \in (0, \epsilon_0]$, on a common time interval $[0, T^*]$. Moreover, there exist constants $C > 0$, $\bar{R} > 0$ such that*

$$\max\{\|\delta \mathrm{u}_\epsilon\|_{L^\infty}, \|\alpha_\epsilon\|_{L^\infty}, \|w^i_\epsilon\|_{L^\infty}\} \le \frac{K_0}{\epsilon_0} \quad \|V_\epsilon(t)\|_{H^k_{\delta-1,\epsilon}} \le C, \quad \|\partial_t V_\epsilon(t)\|_{H^{k-1}_{\delta-1,\epsilon}} \le C,$$
$$\|\Phi_\epsilon(t)\|_{H^{k+2}_\delta} \le C, \quad \|\partial_t \Phi_\epsilon(t)\|_{H^{k+1}_\delta} \le C,$$

and $\operatorname{supp} \alpha_\epsilon(t) \subset B_{\bar{R}}$ *for all* $(\epsilon, t) \in (0, \epsilon_0] \times [0, T^*]$.

7.2. Properties of the limit equations. To fully understand the limit equations of Sect. 7.3, we first need to consider the following system:

$$\partial_t \hat{\alpha} = -\hat{w}^I \partial_I \hat{\alpha} - \frac{\hat{\alpha}}{2n} \partial_I \hat{w}^I, \tag{7.28}$$
$$\partial_t \hat{w}^J = -\frac{\hat{\alpha}}{2n} \partial^J \hat{\alpha} - \hat{w}^I \partial_I \hat{w}^J - \partial^J \hat{\Phi}, \tag{7.29}$$
$$\Delta \hat{\Phi} = \hat{\rho}, \tag{7.30}$$

with initial data

$$\hat{\alpha}(0) = \underset{o}{\alpha} \quad \text{and} \quad \hat{w}^I(0) = \underset{o}{w}^I, \tag{7.31}$$

where $\underset{o}{\alpha}$ and $\underset{o}{w}^I$ are as defined in Proposition 6.1. This system is precisely the Poisson-Euler equation written using the Makino variable $\hat{\rho} = \frac{1}{(4Kn(n+1))^{-n}} \hat{\alpha}^{2n}$. Indeed, a straightforward calculation shows that $(\hat{\rho}, \hat{w}^I)$ satisfy the Poisson-Euler equations of Newtonian gravity

$$\partial_t \hat{\rho} + \partial_I (\hat{\rho} \hat{w}^I) = 0, \tag{7.32}$$
$$\hat{\rho}(\partial_t \hat{w}^J + \hat{w}^I \partial_I \hat{w}^J) = -(\hat{\rho} \partial^J \hat{\Phi} + \partial^J \hat{p}), \tag{7.33}$$
$$\Delta \hat{\Phi} = \hat{\rho}, \tag{7.34}$$

where $\hat{p} = K \hat{\rho}^{(n+1)/n}$.

Proposition 7.6. *There exist a $T > 0$ and a solution*

$$\hat{\alpha}, \hat{w}^I \in C^0([0, T], H^k_{\delta-1}) \cap C^1([0, T], H^{k-1}_{\delta-1}),$$
$$\hat{\Phi} \in C^0([0, T], H^{k+2}_\delta) \cap C^1([0, T], H^{k+1}_\delta), \quad \partial_t \hat{\Phi} \in C^0([0, T], H^{k+1}_{\delta-1})$$

to the initial value problem (7.28)–(7.31)*, where $\hat{\alpha}(t)$ has compact support for all $t \in [0, T]$. Moreover*

(i) this solution is unique in the class

$$\tilde{\alpha}, \tilde{w} \in C^0([0, T], H^k) \cap C^1(\mathbb{R}^n \times [0, T]) \quad \tilde{\Phi} \in C^0([0, T], H^{k+2}_\delta) \cap C^1(\mathbb{R}^n \times [0, T]),$$

where $\tilde{\alpha}(t)$ has compact support for all $t \in [0, T]$, and *(ii) the solution also satisfies*

$$\hat\alpha, \hat w^I \in \cap_{\ell=0}^{s+1} C^\ell([0,T], H^{k-\ell}_{\delta-1}),$$
$$\hat\Phi \in \cap_{\ell=0}^{s+1} C^\ell([0,T], H^{k+2-\ell}_{\delta}), \quad \partial_t\hat\Phi \in \cap_{\ell=0}^{s} C^\ell([0,T], H^{k+1-\ell}_{\delta-1}).$$

Proof. Writing the system (7.28)–(7.30) as

$$\partial_t \begin{pmatrix}\hat\alpha \\ \hat w^J\end{pmatrix} = \begin{pmatrix} -\hat w^I & -\frac{\hat\alpha}{2n}\delta^I_J \\ -\frac{\hat\alpha}{2n}\delta^{IJ} & -\hat w^I \end{pmatrix}\partial_I \begin{pmatrix}\hat\alpha \\ \hat w^J\end{pmatrix} - \frac{1}{(4Kn(n+1))^n}\begin{pmatrix} 0 \\ \partial^J(\Delta^{-1}\alpha^{2n})\end{pmatrix},$$

we see that this system is symmetric hyperbolic with a non-local source term. Since $\Delta : H^{k+2}_\delta \to H^k_{\delta-2}$ is an isomorphism, it is not difficult to adapt the approximation scheme and energy estimates of Appendices B.1 and B.2 to this system. Then as in Appendix B.3, this is enough to produce an existence theorem. Consequently, there exists a $T > 0$ and a solution

$$\hat\alpha, \hat w^I \in C^0([0,T], H^k_{\delta-1}) \cap C^1([0,T], H^{k-1}_{\delta-1}). \tag{7.35}$$

Therefore

$$\hat\rho \in C^0([0,T], H^k_{\delta-2}) \cap C^1([0,T], H^{k-1}_{\delta-2}), \tag{7.36}$$

and hence $\tilde\Phi = \Delta^{-1}\hat\rho \in C^0([0,T], H^{k+2}_\delta) \cap C^1([0,T], H^{k+1}_\delta)$.

Differentiating (7.34) with respect to t and using (7.32) yields

$$\Delta\partial_t\hat\Phi = -\partial_I(\hat\rho\hat w^I). \tag{7.37}$$

But, (7.35) implies that $\hat\rho\hat w^I \in C^0([0,T], H^k_{\delta-2})$ and hence $\Delta^{-1}(\hat\rho\hat w^I) \in C^0([0,T], H^{k+2}_\delta)$. Taking the divergence then gives $\partial_I(\Delta^{-1}\hat\rho\hat w^I) \in C^0([0,T], H^{k+1}_{\delta-1})$. However, (7.37) implies that $\partial_t\hat\Phi = -\Delta^{-1}\partial_I(\hat\rho\hat w^I) = -\partial_I(\Delta^{-1}(\hat\rho\hat w^I))$ and so $\partial_t\hat\Phi \in C^0([0,T], H^{k+1}_{\delta-1})$.

The statement about compact support follows from the symmetric hyperbolic equation satisfied by $\hat\alpha$ and the property of finite propagation speed. Uniqueness follows from a slight modification of standard arguments, see [39] Proposition 1.3, Sect. 16.1. □

7.3. Convergence as $\epsilon \searrow 0$. In this section, we identify the limit of the relativistic solutions as $\epsilon \searrow 0$. To accomplish this, we adapt the arguments of [37], Sect. III. Define

$$\tilde V := (\tilde{\mathrm u}^{ij}_4, \tilde{\mathrm u}^{ij}_J, \delta\tilde{\mathrm u}^{ij}, \tilde\alpha, \tilde w^i)^T,$$

$$\tilde a^I := \begin{pmatrix} -\tilde w^I & \frac{\tilde\alpha}{2n}\delta^I_j \\ -\frac{\tilde\alpha}{2n}\delta^I_i & -\delta_{ij}w^I\end{pmatrix},$$

$$\tilde b^I := \begin{pmatrix} A^I(\delta\tilde{\mathrm u}^{ij}) & 0 \\ 0 & \tilde a^I\end{pmatrix},$$

$$\tilde{\mathcal S}^{ij} := \rho\begin{pmatrix} 0 & \tilde w^I \\ \tilde w^J & 4\eta_{ij}\delta\tilde{\mathrm u}^{ij} + 2\tilde w^4\end{pmatrix},$$

$$\tilde b := \begin{pmatrix} 0 \\ -\eta^{im}\left(2\eta_{4\ell}\eta_{4p} + \eta_{\ell p}\right)\tilde{\mathrm u}^{\ell p}_m - 2\left(\eta_{\ell p}\delta^i_4\bar{\mathrm u}^{\ell p}_4 - 2\eta_{\ell 4}\bar{\mathrm u}^{i\ell}_4\right)\end{pmatrix},$$

$$\tilde f(\tilde V)\tilde V := (-\tilde{\mathcal S}^{ij}, 0, \tilde{\mathrm u}^{ij}_4, \tilde b)^T,$$

and

$$\tilde{h} := (\eta^{ij}\partial_I\beta^I - 2\partial_I\beta^{(i}\eta^{j)I}, 0, \ldots, 0)^T.$$

Theorem 7.7. *For any* $r > 0$*,* Φ_ϵ *and* V_ϵ *converge in* $C^0([0, T^*], H^{k+1-r}_{\mathrm{loc}})$ *and* $C^0([0, T^*], H^{k-r}_{\mathrm{loc}})$ *as* $\epsilon \searrow 0$ *to* $\tilde{\Phi} \in C^1(\mathbb{R}^3 \times [0, T^*]) \cap C^0([0, T^*], H^{k+2}_\delta)$ *and the unique solution* $\tilde{V} \in C^1(\mathbb{R}^3 \times [0, T^*]) \cap C^0([0, T^*], H^k)$ *of the system*

$$\mathbb{P}\left(\partial_t\tilde{V} - \tilde{b}^I\partial_I\tilde{V} - \tilde{f}(\tilde{V})\tilde{V} - \tilde{h}\right) = 0,$$

$$c^I\partial_I(\tilde{V} - d\tilde{\Phi}) = 0,$$

$$\tilde{V}(0) = \underset{o}{V_0}(0),$$

$$\Delta\tilde{\Phi} = \tilde{\rho},$$

where $\mathbb{P}$ *is the projection onto the* L^2 *orthogonal complement of* $\{c^I\partial_I W = 0 \,|\, W \in H^1\}$*. Moreover,*

(i) there exists a $\bar{R} > 0$ *such that* $\operatorname{supp}\tilde{\alpha}(t) \subset B_{\bar{R}}$ *for all* $t \in [0, T^*]$*,*
(ii) there exists a $\omega \in C^0([0, T^*], H^k_{\mathrm{loc}})$ *such that* $\partial_I\omega \in C^0([0, T^*], H^{k-1})$ *and*

$$\partial_t\tilde{V} - \tilde{b}^I\partial_I\tilde{V} - \tilde{f}(\tilde{V})\tilde{V} - \tilde{h} - c^I\partial_I\omega = 0, \tag{7.38}$$

(iii) and for $\delta_1 \geq -1/2$*, there exists a* $\tilde{\bar{\mathrm{u}}} \in C^0([0, T], L^6_{\delta_1})$ *such that*

$$\tilde{\mathrm{u}}^{ij}_J = \partial_J\tilde{\bar{\mathrm{u}}}^{ij}.$$

Proof. By assumption $-1 < \delta < -1/2$, and so it follows directly from the definition of the weighted norms that for every $\ell \geq 0$,

$$\|u\|_{H^\ell} \leq \|u\|_{H^\ell_{\delta-1,\epsilon}} \quad \text{for all } u \in H^\ell_{\delta-1,\epsilon}. \tag{7.39}$$

So by Proposition 7.5,

$$V_\epsilon \in C^0([0, T^*], H^k) \cap C^1([0, T^*], H^{k-1}) \subset C^0([0, T^*], H^k_{\delta-1,\epsilon}) \cap C^1([0, T^*], H^{k-1}_{\delta-1,\epsilon})$$

and $\Phi_\epsilon \in C^0([0, T^*], H^{k+2}_\delta) \cap C^1([0, T^*], H^{k+1}_\delta)$ are uniformly bounded for $\epsilon \in (0, \epsilon_0]$. Therefore by the Banach-Alaoglu theorem there exists subsequences of Φ_ϵ and V_ϵ, which we still denote by Φ_ϵ and V_ϵ, and $\tilde{\Phi} \in L^{1,\infty}([0, T^*], H^{k+1}_\delta) \cap \mathrm{Lip}([0, T^*], H^k_\delta)$, $\tilde{V} \in L^{1,\infty}([0, T^*], H^k) \cap \mathrm{Lip}([0, T^*], H^{k-1})$ such that Φ_ϵ and V_ϵ converge weakly to $\tilde{\Phi}$ and $\tilde{V}$, respectively, as $\epsilon \searrow 0$.

By Proposition 7.5, the support of α_ϵ is uniformly bounded in ϵ and hence the support of the weak limit $\tilde{\alpha}$ must also be bounded. From Proposition 6.1, we have that $\mathrm{u}^{ij}_{J,\epsilon} = \partial_J\bar{\mathrm{u}}^{ij}_\epsilon$. So by Lemmas A.7 and A.11, and (7.15), we find that for $\delta_1 \geq -1/2 \geq \delta$,

$$\|\bar{\mathrm{u}}^{ij}_\epsilon\|_{L^6_{\delta_1}} \leq C\|\bar{\mathrm{u}}^{ij}_\epsilon\|_{L^6_\delta} \leq C\|\bar{\mathrm{u}}^{ij}_\epsilon\|_{L^6_{\delta,\epsilon}} \leq C\left(\|\mathrm{u}^{ij}_{J,\epsilon}\|_{L^2_{\delta-1,\epsilon}} + \|\mathrm{u}^{ij}_\epsilon\|_{L^2_{\delta,\epsilon}}\right) \leq C(1 + \|V_\epsilon\|_{H^k_{\delta-1,\epsilon}})$$

for a constant C independent of ϵ. It follows that $\bar{\mathrm{u}}^I_\epsilon$ converges weakly to a $\tilde{\bar{\mathrm{u}}}^{ij} \in L^{1,\infty}([0, T^*], L^6_{\delta_1})$ for which $\partial_J\tilde{\bar{\mathrm{u}}}^{ij} = \tilde{\mathrm{u}}_J$.

Now, V_ϵ satisfies

$$b^0(\epsilon U_\epsilon, \epsilon V_\epsilon)\partial_t V_\epsilon - \frac{1}{\epsilon}c^I\partial_I(V_\epsilon - d\Phi_\epsilon) + b^I(\epsilon, U_\epsilon, V_\epsilon)\partial_I V_\epsilon$$
$$-f(\epsilon, U_\epsilon, V_\epsilon)V_\epsilon - h(\epsilon U_\epsilon) = 0, \tag{7.40}$$

and hence it follows from the boundedness of Φ_ϵ and V_ϵ that

$$\|c^I\partial_I(V_\epsilon - d\Phi_\epsilon)\|_{H^{k-1}} \leq \|c^I\partial_I(V_\epsilon - d\Phi_\epsilon)\|_{H^{k-1}_{\delta-1,\epsilon}} \leq C\epsilon.$$

Letting $\epsilon \searrow 0$ yields

$$c^I\partial_I(\tilde{V} - d\tilde{\Phi}) = 0.$$

Next, applying the projection $\mathbb{P}$ (note that $V_\epsilon - d\Phi_\epsilon \in H^1$) to (7.40) gives

$$\mathbb{P}(b^0(\epsilon U_\epsilon, \epsilon V_\epsilon)\partial_t V_\epsilon - b^I(\epsilon, U_\epsilon, V_\epsilon)\partial_I V_\epsilon - f(\epsilon, U_\epsilon, V_\epsilon)V_\epsilon - h(\epsilon U_\epsilon)) = 0$$

or equivalently

$$\mathbb{P}b^0_\epsilon\mathbb{P}\partial_t V_\epsilon + \mathbb{P}b^0_\epsilon(\mathbb{I} - \mathbb{P})\partial_t V_\epsilon - \mathbb{P}(b^I_\epsilon\partial_I V_\epsilon - f_\epsilon - h_\epsilon) = 0,$$

where we set $b^0_\epsilon = b^0(\epsilon U_\epsilon, \epsilon V_\epsilon)$, $b^I_\epsilon = b^I(\epsilon, U_\epsilon, V_\epsilon)\partial_I V_\epsilon$, $f_\epsilon = f(\epsilon, U_\epsilon, V_\epsilon)V_\epsilon$, and $h_\epsilon = h(\epsilon U_\epsilon)$. Suppose $\psi \in C^\infty_0$ and let $\langle u|v\rangle = \int_{\mathbb{R}^3} uv\, d^3x$ be the standard L^2 norm. Then

$$\langle\psi|\mathbb{P}b^0_\epsilon(\mathbb{I} - \mathbb{P})\partial_t V_\epsilon\rangle = \langle(\mathbb{I} - \mathbb{P})b^0_\epsilon\mathbb{P}\psi|\partial_t V_\epsilon\rangle \tag{7.41}$$

as $\mathbb{P}$ is a self-adjoint projection operator. Since the imbedding $H^k(B_R) \to H^{k-r}(B_R)$ $(r > 0)$ is compact for any ball B_R, V_ϵ and Φ_ϵ converge in $C^0([0, T^*], H^{k-r}_{\text{loc}})$ and $C^0([0, T^*], H^{k+2-r}_{\text{loc}})$ to $\tilde{V}$ and $\tilde{\Phi}$, respectively, as $\epsilon \searrow 0$. Using this strong convergence and (7.15), we find that $(\mathbb{I} - \mathbb{P})b^0_\epsilon\mathbb{P}\psi \to (\mathbb{I} - \mathbb{P})\mathbb{P}\psi = 0$ in L^2 as $\epsilon \searrow 0$ and hence $\left\langle\psi|\mathbb{P}b^0_\epsilon(\mathbb{I} - \mathbb{P})\partial_t V_\epsilon\right\rangle \to 0$ by (7.41) and the fact that $\|\partial_t V_\epsilon\|_{L^2}$ is uniformly bounded in ϵ. Therefore, we have established that

$$\mathbb{P}b^0_\epsilon(\mathbb{I} - \mathbb{P})\partial_t V_\epsilon \longrightarrow 0 \quad \text{weakly in } L^2 \text{ as } \epsilon \searrow 0.$$

The remainder of the proof follows from a straightforward adaptation of the proof of Theorem 2 in [37]. □

From the block diagonal form of the matrix c^I, it is clear that ω can be written as

$$\omega = (\omega^{ij}_4, \omega^{ij}_I, 0, \ldots, 0)^T.$$

Using this, we can write the system (7.38) as

$$\partial_t \tilde{\alpha} = -\tilde{w}^I \partial_I \tilde{\alpha} - \frac{\tilde{\alpha}}{2n} \partial_I \tilde{w}^I, \tag{7.42}$$

$$\partial_t \tilde{w}^J = -\frac{\tilde{\alpha}}{2n} \partial^J \tilde{\alpha} - \tilde{w}^I \partial_I \tilde{w}^J - \delta^{IJ} \left(\tilde{\mathrm{u}}_I^{44} + \delta_{KL} \tilde{\mathrm{u}}_I^{KL} \right), \tag{7.43}$$

$$\partial_t \tilde{w}^4 = -\tilde{w}^I \partial_I \tilde{w}^4 - \left(\tilde{\mathrm{u}}_4^{44} + \delta_{IJ} \tilde{\mathrm{u}}_4^{IJ} \right), \tag{7.44}$$

$$\partial_t \tilde{\mathrm{u}}_4^{ij} = 4\delta\tilde{\mathrm{u}}^{4I} \partial_I \tilde{\mathrm{u}}_4^{ij} + 4\delta\tilde{\mathrm{u}}^{IJ} \partial_I \tilde{\mathrm{u}}_J^{ij} + \eta^{ij} \partial_I \beta^I - 2\partial_I \beta^{(i} \eta^{j)I} - \tilde{\mathcal{S}}^{ij} + \partial^I \omega_I^{ij}, \tag{7.45}$$

$$\partial_t \tilde{\mathrm{u}}_I^{ij} = 4\delta\tilde{\mathrm{u}}^{IJ} \partial_J \tilde{\mathrm{u}}_4^{ij} + \partial_I \omega_4^{ij}, \tag{7.46}$$

$$\partial_t \delta\tilde{\mathrm{u}}^{ij} = \tilde{\mathrm{u}}_4^{ij}, \tag{7.47}$$

$$\partial_J \tilde{\mathrm{u}}_4^{ij} = 0, \tag{7.48}$$

$$\partial^J \tilde{\mathrm{u}}_J^{ij} = \delta_4^i \delta_4^j \Delta\tilde{\Phi}, \tag{7.49}$$

$$\Delta\tilde{\Phi} = \tilde{\rho}, \tag{7.50}$$

with initial conditions

$$\tilde{\mathrm{u}}_4^{ij}(0) = 0, \quad \tilde{\mathrm{u}}_I^{iJ}(0) = 0, \quad \tilde{\mathrm{u}}_I^{44} = \partial_I \phi \quad (\phi := \Delta^{-1} \tilde{\rho}(0)), \tag{7.51}$$

$$\tilde{\alpha}(0) = \underset{o}{\alpha}, \quad \tilde{w}^I = \underset{o}{w}^I, \quad \tilde{w}^4 = 0. \tag{7.52}$$

Equation (7.48) immediately implies that

$$\tilde{\mathrm{u}}_4^{ij} = 0, \tag{7.53}$$

and hence, by uniqueness and the fact that $\delta\mathrm{u}^{ij}(0) = 0$, it follows from (7.47) that

$$\delta\tilde{\mathrm{u}}^{ij} = 0. \tag{7.54}$$

Since $\tilde{\mathrm{u}}_J^{ij} = \partial_J \tilde{\tilde{\mathrm{u}}}^{ij}$, we get from (7.49) that $\Delta\tilde{\tilde{\mathrm{u}}}^{ij} = \delta_4^i \delta_4^j \Delta\tilde{\Phi}$. But $\tilde{\tilde{\mathrm{u}}}^{ij} \in L^6_{\delta_1}$ and $\Delta\tilde{\Phi} \in L^2_{\delta-2}$ and so by Theorem 1.2 and Proposition 1.6 of [1], we find that $\tilde{\tilde{\mathrm{u}}}^{ij} \in H^k_{\delta_2}$ for $0 > \delta_2 > \delta_1 \geq -1/2 > \delta > -1$. Since the Laplacian $\Delta : H^k_{\delta_2} \to H^{k-2}_{\delta_2 - 1}$ is injective for $\delta_2 < 0$ (see [1], Proposition 2.2), we must have $\tilde{\tilde{\mathrm{u}}}^{ij} = \delta_4^i \delta_4^j \tilde{\Phi}$ and hence

$$\tilde{\mathrm{u}}_J^{ij} = \delta_4^i \delta_4^j \partial_J \tilde{\Phi}. \tag{7.55}$$

Substituting (7.53)–(7.55) into (7.42)–(7.49) yields

$$\partial_t \tilde{\alpha} = -\tilde{w}^I \partial_I \tilde{\alpha} - \frac{\tilde{\alpha}}{2n} \partial_I \tilde{w}^I, \tag{7.56}$$

$$\partial_t \tilde{w}^J = -\frac{\tilde{\alpha}}{2n} \partial^J \tilde{\alpha} - \tilde{w}^I \partial_I \tilde{w}^J - \partial^J \tilde{\Phi}, \tag{7.57}$$

$$\Delta\tilde{\Phi} = \tilde{\rho}, \tag{7.58}$$

and

$$\partial_t \tilde{w}^4 = -\tilde{w}^I \partial_I \tilde{w}^4, \tag{7.59}$$
$$\partial^I \omega_I^{ij} = \eta^{ij}\partial_I \beta^I - 2\partial_I \beta^{(i}\eta^{j)I} + \tilde{\mathcal{S}}^{ij}, \tag{7.60}$$
$$\partial_I \omega_4^{Jk} = 0, \tag{7.61}$$
$$\partial_t \partial_I \tilde{\Phi} = \partial_I \omega_4^{44}. \tag{7.62}$$

Since $\tilde{w}^4(0) = 0$, uniqueness of solutions to hyperbolic equations implies that

$$\tilde{w}^4 = 0. \tag{7.63}$$

Proposition 7.6 and (7.56)–(7.58) imply that $\{\tilde{\Phi}, \tilde{w}^I, \tilde{\alpha}\}$ must satisfy

$$\tilde{\alpha}, \tilde{w}^I \in C^0\left([0, T^*], H^k_{\delta-1}\right) \cap C^1\left([0, T^*], H^{k-1}_{\delta-1}\right) \tag{7.64}$$

and

$$\tilde{\Phi} \in C^0\left([0, T^*], H^{k+2}_{\delta}\right) \cap C^1\left([0, T^*], H^{k+1}_{\delta}\right), \tag{7.65}$$
$$\partial_t \tilde{\Phi} \in C^0\left([0, T^*], H^{k+1}_{\delta-1}\right) \cap C^1\left([0, T^*], H^{k}_{\delta-1}\right). \tag{7.66}$$

We then get from (7.61) and (7.62) that

$$\omega_4^{44} = \partial_t \tilde{\Phi} \in C^1([0, T^*], H^k_{\delta-1}) \tag{7.67}$$

and

$$\omega_4^{4J} = 0. \tag{7.68}$$

Equations (7.54) and (7.63) imply that $\tilde{\mathcal{S}}^{ij}$ can be written as $\tilde{\mathcal{S}}^{ij} = 2\delta_I^{(i}\delta_4^{j)}\tilde{w}^I$. We then find from (7.60) that

$$\omega_I^{ij} = \partial_I \Omega^{ij}, \tag{7.69}$$

where

$$\Omega^{ij} = \Delta^{-1}(\eta^{ij}\partial_I \beta^I - 2\partial_I \beta^{(i}\eta^{j)I} + 2\delta_I^{(i}\delta_4^{j)}\tilde{w}^I). \tag{7.70}$$

Note that

$$\Omega^{ij} \in C^1([0, T^*], H^{k+1}_{\delta}),$$

since $\partial_I \beta^j \in C^1([0, T^*], H^{k-1}_{\delta-2})$ and $\tilde{\mathcal{S}}^{ij} \in C^1([0, T^*], H^{k-1}_{\delta-2})$ by (7.64). Therefore

$$\omega_I^{ij} = \partial_I \Omega^{ij} \in C^1([0, T^*], H^k_{\delta-1}). \tag{7.71}$$

We collect the above results in the following proposition.

Proposition 7.8. *The limit solution* $\{\tilde{V}, \tilde{\Phi}\}$ *from Theorem 7.7 satisfies*

$$\begin{aligned}
&\delta\tilde{u}^{ij} = \tilde{u}_4^{ij} = \tilde{w}^4 = 0,\\
&\tilde{\Phi} \in C^0([0,T^*], H_\delta^{k+2}) \cap C^1([0,T^*], H_\delta^{k+1}),\\
&\quad \partial_t\Phi \in C^0([0,T^*], H_{\delta-1}^{k+1}) \cap C^1([0,T^*], H_{\delta-1}^{k}),\\
&\tilde{u}_J^{ij} = \delta_4^i\delta_4^j\partial_J\tilde{\Phi} \in C^1([0,T^*], H_{\delta-1}^{k+1}) \cap C^1([0,T^*], H_{\delta-1}^{k}),\\
&\tilde{\alpha}, \tilde{w}^I \in C^0([0,T^*], H_{\delta-1}^{k}) \cap C^1([0,T^*], H_{\delta-1}^{k-1}),
\end{aligned}$$

while $\{\tilde{\Phi}, \tilde{\alpha}, \tilde{w}^I\}$ *solves Eqs.* (7.56)–(7.58). *Moreover, the* ω *from Theorem 7.7 is given by*

$$\omega = (\omega_4^{ij}, \omega_I^{ij}, 0, \ldots, 0)^T,$$

where

$$\begin{aligned}
\omega_4^{ij} &= \delta_4^i\delta_4^j\partial_t\tilde{\Phi} \in C^1([0,T^*], H_{\delta-1}^{k}),\\
\omega_I^{ij} &= \partial_I\Delta^{-1}\left(\eta^{ij}\partial_I\beta^I - 2\partial_I\beta^{(i}\eta^{j)I} + 2\delta_I^{(i}\delta_4^{j)}\tilde{w}^I\right) \in C^1([0,T^*], H_{\delta-1}^{k}).
\end{aligned}$$

7.4. Error estimate. To get an error estimate which measures the difference between the relativistic and Newtonian solutions, we adapt the arguments of [37], Sect. IV. Define

$$Z_\epsilon := V_\epsilon - \tilde{V} + d\Phi_\epsilon - d\tilde{\Phi} - \epsilon\omega \quad \text{and} \quad \gamma_\epsilon := \alpha_\epsilon - \tilde{\alpha}.$$

A simple but useful observation is that

$$\|\gamma_\epsilon\|_{H_{\delta-1,\epsilon}^{k-1}} = \|\alpha_\epsilon - \tilde{\alpha}\|_{H_{\delta-1,\epsilon}^{k}} \le \|Z_\epsilon\|_{H_{\delta-1,\epsilon}^{k-1}} \quad \text{and} \quad \|w_\epsilon^i - \tilde{w}^i\|_{H_{\delta-1,\epsilon}^{k-1}} \le \|Z_\epsilon\|_{H_{\delta-1,\epsilon}^{k-1}}. \tag{7.72}$$

Lemma 7.9. *There exists an* ϵ *independent constant* $C > 0$ *such that*

$$\begin{aligned}
&\|d\Phi_\epsilon(t) - d\Phi(t)\|_{H_{\delta-1,\epsilon}^{k-1}} + \|Dd\Phi_\epsilon(t) - Dd\Phi(t)\|_{H_{\delta-2,\epsilon}^{k-1}} \le \|\Phi_\epsilon(t) - \tilde{\Phi}(t)\|_{H_\delta^{k+1}}\\
&\quad \le C\|Z_\epsilon(t)\|_{H_{\delta-1,\epsilon}^{k-1}}\\
&\quad \|\partial_t d\Phi_\epsilon(t) - \partial_t d\tilde{\Phi}(t)\|_{H_{\delta-1,\epsilon}^{k-1}} \le \|\partial_t\Phi_\epsilon(t) - \partial_t\tilde{\Phi}(t)\|_{H_\delta^k} \le C\|Z_\epsilon(t)\|_{H_{\delta-1,\epsilon}^{k-1}} + C\epsilon
\end{aligned}$$

and

$$\|\partial_t\gamma_\epsilon\|_{H_{\delta-1,\epsilon}^{k-2}} \le C\|Z_\epsilon(t)\|_{H_{\delta-1,\epsilon}^{k-1}} + C\epsilon$$

for all $(t,\epsilon) \in [0,T^*] \times (0,\epsilon_0]$.

Proof. Since the support of $\alpha_\epsilon(t)$ and $\tilde{\alpha}(t)$ are both bounded for all $(t,\epsilon) \in [0,T^*] \times (0,\epsilon_0]$, there exists a ϵ independent constant $C > 0$ such that

$$C^{-1}\|\rho_\epsilon - \tilde{\rho}\|_{H_{\delta-2}^{k-1}} \le \|\rho_\epsilon - \tilde{\rho}\|_{H_{\delta-1,\epsilon}^{k-2}} \le C\|\rho_\epsilon - \tilde{\rho}\|_{H_{\delta-2}^{k-1}}.$$

Also, $\Delta\Phi_\epsilon = \rho_\epsilon$, $\Delta\tilde{\Phi} = \tilde{\rho}$, and $\Delta : H_\delta^{k+1} \to H_{\delta-1}^{k-1}$ is an isomorphism, and therefore

$$\|\Phi_\epsilon - \tilde{\Phi}\|_{H_\delta^{k+1}} \le \|\rho_\epsilon - \tilde{\rho}\|_{H_{\delta-2}^{k-1}} \le C\|\rho_\epsilon - \tilde{\rho}\|_{H_{\delta-1,\epsilon}^{k-1}} \le C\|\gamma_\epsilon\|_{H_{\delta-1,\epsilon}^{k-1}} \le C\|Z_\epsilon\|_{H_{\delta-1,\epsilon}^{k-1}} \tag{7.73}$$

by Proposition 7.5 and Lemma A.10. From (4.25) and (7.56), it follows that γ_ϵ satisfies

$$\partial_t \gamma_\epsilon = -X^I \partial_I \gamma_\epsilon - Y\gamma_\epsilon + \big(X^I - \tilde{w}^I\big)\partial_I \tilde{\alpha} + \Big(Y - \frac{\partial_I \tilde{w}^I}{2n}\Big)\tilde{\alpha}, \tag{7.74}$$

where X^I and Y are given by (4.26). But $X^I = X^I(\epsilon w_\epsilon^4, w_\epsilon^I)$ and $\tilde{w}^I = X^I(0, \tilde{w}^I)$, and hence

$$\|X^I - \tilde{w}^I\|_{H^{k-2}_{\delta-1,\epsilon}} \leq C\|w_\epsilon^I - \tilde{w}^I\|_{H^{k-2}_{\delta-1,\epsilon}} \leq C\|Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} \tag{7.75}$$

by (7.72), (A.24), Lemma A.10 and Proposition 7.5. Next,

$$\begin{aligned} Y - \frac{\partial_I \tilde{w}^I}{2n} = \bigg(\bar{Y}(\epsilon V_\epsilon) - \frac{1}{2n}\bigg)\Big(\epsilon \partial_t w_\epsilon^4 + \partial_I w_\epsilon^I\Big) + \frac{1}{2n}\epsilon \partial_t w_\epsilon^4 \\ + \frac{1}{2n}\Big(\partial_I w_\epsilon^I - \partial_I \tilde{w}^I\Big) + \hat{Y}(\epsilon U_\epsilon, \epsilon V_\epsilon), \end{aligned}$$

where $\hat{Y}(0) = 0$ and $\bar{Y}(0) - 1/(2n) = 0$. Using (7.15), (A.3), (A.24), Proposition 7.5, and Lemmas A.7–A.10, we can estimate each of the above terms as follows:

$$\begin{aligned} &\| \bigg(\bar{Y}(\epsilon V_\epsilon) - \frac{1}{2n}\bigg)(\epsilon \partial_t w_\epsilon^4 + \partial_I w_\epsilon^I)\|_{H^{k-2}_{\delta,\epsilon}} \leq \| \bigg(\bar{Y}(\epsilon V_\epsilon) - \frac{1}{2n}\bigg)\|_{H^{k-2}_{\delta-1,\epsilon}} \\ &\quad \times \Big(\epsilon\|\partial_t w_\epsilon^4\|_{H^{k-2}_{\delta-1,\epsilon}} + \|w_\epsilon^I\|_{H^{k-2}_{\delta-1,\epsilon}}\Big) \\ &\quad \leq C\epsilon\|V_\epsilon\|_{H^k_{\delta-1,\epsilon}}\Big(\epsilon\|\partial_t V_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} + \|V_\epsilon\|_{H^k_{\delta-1,\epsilon}}\Big) \leq C\epsilon, \\ &\quad \|\frac{1}{2n}\epsilon\partial_t w_\epsilon^4\|_{H^{k-2}_{\delta,\epsilon}} \leq C\epsilon\|\partial_t V_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} \leq C\epsilon, \\ &\quad \|\frac{1}{2n}(\partial_I w_\epsilon^I - \partial_I \tilde{w}^I)\|_{H^{k-2}_{\delta,\epsilon}} \leq C\|Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}}, \end{aligned}$$

and

$$\|\hat{Y}(\epsilon U_\epsilon, \epsilon V_\epsilon)\|_{H^{k-2}_{\delta,\epsilon}} \leq C\epsilon\Big(\|U_\epsilon\|_{H^k_{\delta,\epsilon}} + \|V_\epsilon\|_{H^k_{\delta-1,\epsilon}}\Big) \leq C\epsilon.$$

Therefore

$$\|Y - \frac{\partial_I \tilde{w}^I}{2n}\|_{H^{k-2}_{\delta,\epsilon}} \leq C\|Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} + C\epsilon. \tag{7.76}$$

We can also estimate X^I and Y as follows:

$$\|X^I\|_{H^{k-2}_{\delta-1,\epsilon}} \leq C\|V_\epsilon\|_{H^k_{\delta-1,\epsilon}} \leq C, \tag{7.77}$$

$$\|Y\|_{H^{k-2}_{\delta,\epsilon}} \leq C(\|U_\epsilon\|_{H^k_{\delta,\epsilon}} + \|V_\epsilon\|_{H^k_{\delta-1,\epsilon}} + \|\partial_t V_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}}) \leq C. \tag{7.78}$$

The estimates (7.72), (7.75), (7.76), (7.77), (7.78) along with Lemma A.8 imply via Eq. (7.74) that

$$\|\partial_t \gamma_\epsilon\|_{H^{k-2}_{\delta-1,\epsilon}} \leq C\|Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} + C\epsilon. \tag{7.79}$$

Since $\Delta\partial_t\Phi_\epsilon = \delta\rho_\epsilon$ and $\Delta\partial_t\tilde{\Phi} = \partial_t\tilde{\rho}$, the same arguments used to establish the estimate (7.73) can be used in conjunction with (7.79) to show

$$\|\partial_t\Phi_\epsilon - \partial_t\tilde{\Phi}\|_{H^k_\delta} \le C\|Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} + C\epsilon. \tag{7.80}$$

Finally from (7.73), (7.80), and Lemma A.11, we get the desired estimates

$$\|d\Phi_\epsilon - d\Phi\|_{H^{k-1}_{\delta-1,\epsilon}} + \|Dd\Phi_\epsilon - Dd\Phi\|_{H^{k-1}_{\delta-2,\epsilon}} \le \|\Phi_\epsilon - \tilde{\Phi}\|_{H^{k+1}_\delta} \le C\|Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}}$$

and

$$\|\partial_t d\Phi_\epsilon - \partial_t d\tilde{\Phi}\|_{H^{k-1}_{\delta-1,\epsilon}} \le \|\partial_t\Phi_\epsilon - \partial_t\tilde{\Phi}\|_{H^k_\delta} \le C\|Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} + C\epsilon$$

for some constant C independent of ϵ. □

Lemma 7.10. *There exists a constant $C > 0$ such that*

$$\|\partial_t\alpha_\epsilon - \partial_t\tilde{\alpha}\|_{H^{k-2}_{\delta-1,\epsilon}} + \|V_\epsilon(t) - \tilde{V}(t)\|_{H^{k-1}_{\delta-1,\epsilon}} \le C\epsilon \quad \textit{for all } (t,\epsilon) \in [0,T^*] \times (0,\epsilon_0].$$

Proof. From the evolution equation (6.1), we find that Z_ϵ satisfies the equation

$$b^0_\epsilon\partial_t Z_\epsilon = \frac{1}{\epsilon}c^I\partial_I Z_\epsilon + b^I_\epsilon\partial_I Z_\epsilon + F_\epsilon, \tag{7.81}$$

where $b^0_\epsilon = b^0(\epsilon U_\epsilon, \epsilon V_\epsilon)$, $b^I_\epsilon = b(\epsilon, U_\epsilon, V_\epsilon)$ and

$$\begin{aligned} F_\epsilon = &-b^0_\epsilon\partial_t(d\tilde{\Phi} - d\Phi_\epsilon) - \epsilon b^0_\epsilon\partial_t\omega + b^I_\epsilon(\partial_I d\tilde{\Phi} - \partial_I d\Phi_\epsilon) + \epsilon b^I_\epsilon\partial_I\omega \\ &-(b^0_\epsilon - 1\!\!1)\partial_t\tilde{V} + (\tilde{b}^I - b^I_\epsilon)\partial_I\tilde{V} + f(\epsilon, U_\epsilon, V_\epsilon)V_\epsilon - \tilde{f}(\tilde{V})\tilde{V} + h_\epsilon - \tilde{h}. \end{aligned} \tag{7.82}$$

Using (7.15), (A.3), (A.24), Lemmas 7.9, A.7–A.9, and Propositions 7.5 and 7.8, we get the following estimates:

$$\|b^0_\epsilon - 1\!\!1\|_{H^k_{\delta,\epsilon}} \le C\epsilon(\|U_\epsilon\|_{H^k_{\delta,\epsilon}} + \|V_\epsilon\|_{H^k_{\delta-1,\epsilon}}) \le C\epsilon, \tag{7.83}$$

$$\begin{aligned} \|b^0_\epsilon\partial_t(d\tilde{\Phi} - d\Phi_\epsilon)\|_{H^{k-1}_{\delta-1,\epsilon}} &\le \|(b^0_\epsilon - 1\!\!1)\partial_t(d\tilde{\Phi} - d\Phi_\epsilon)\|_{H^{k-1}_{\delta-1,\epsilon}} + \|\partial_t(d\tilde{\Phi} - d\Phi_\epsilon)\|_{H^{k-1}_{\delta-1,\epsilon}} \\ &\le C(\|b^0_\epsilon - 1\!\!1\|_{H^{k-1}_{\delta,\epsilon}} + 1)\|\partial_t(d\tilde{\Phi} - d\Phi_\epsilon)\|_{H^{k-1}_{\delta-1,\epsilon}} \le C\|Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} + C\epsilon, \end{aligned} \tag{7.84}$$

$$\|\epsilon b^0_\epsilon\partial_t\omega\|_{H^{k-1}_{\delta-1,\epsilon}} \le \epsilon C(\|b^0_\epsilon - 1\|_{H^{k-1}_{\delta,\epsilon}} + 1)\|\partial_t\omega\|_{H^{k-1}_{\delta-1,\epsilon}} \le C\epsilon, \tag{7.85}$$

$$\|\epsilon b^I_\epsilon\partial_I\omega\|_{H^{k-1}_{\delta-1,\epsilon}} \le C\epsilon\|b^I_\epsilon\|_{H^{k-1}_{\delta,\epsilon}}\|\partial_I\omega\|_{H^{k-1}_{\delta-2,\epsilon}} \le C\epsilon\|b^I_\epsilon\|_{H^{k-1}_{\delta,\epsilon}}\|\omega\|_{H^k_{\delta-1,\epsilon}} \le C\epsilon, \tag{7.86}$$

$$\|h_\epsilon - \tilde{h}\|_{H^{k-1}_{\delta-1,\epsilon}} \le C\epsilon. \tag{7.87}$$

To estimate the term $b^I_\epsilon - \tilde{b}$, we first note that

$$b^I_\epsilon - \tilde{b} = \begin{pmatrix} A^I(\underset{o}{\mathfrak{u}}{}^{iJ}_\epsilon) + A^I(\delta\mathfrak{u}^{iJ}_\epsilon) & 0 \\ 0 & a^I(\epsilon\mathfrak{u}_\epsilon, \epsilon w^i_\epsilon, \epsilon\alpha_\epsilon, w^I_\epsilon, \alpha_\epsilon) - a^I(0,0,0,\tilde{w}^i, \tilde{\alpha}) \end{pmatrix},$$

where the map a^I is analytic. Next, the estimate (7.15) implies that

$$\|\mathfrak{u}^{ij}_\epsilon\|_{H^{k-1}_\delta,\epsilon} \le \|\underset{o}{\mathfrak{u}}{}^{ij}_\epsilon\|_{H^{k-1}_{\delta,\epsilon}} + C\|\delta\mathfrak{u}^{ij}_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} \le C + C\|Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}}. \tag{7.88}$$

From Proposition 5.2 and Lemma A.11, we see that the $\underset{o}{\mathfrak{u}}{}^{iJ}_{\epsilon}$ can be estimated by

$$\|\underset{o}{\mathfrak{u}}{}^{iJ}_{\epsilon}\|_{H^{k+1}_{\delta,\epsilon}} = \|\epsilon\underset{o}{\bar{\mathfrak{u}}}{}^{iJ}_{\epsilon}\|_{H^{k+1}_{\delta,\epsilon}} \leq \epsilon^{|\delta+1/2|}\|\underset{o}{\bar{\mathfrak{u}}}{}^{iJ}_{\epsilon}\|_{H^{k+1}_{\delta}} \leq C\epsilon^{|\delta+1/2|+1}. \tag{7.89}$$

Also, from Proposition 7.8 and Lemma 7.9, we obtain

$$\begin{aligned}\|\partial_t\alpha_\epsilon - \partial_t\tilde{\alpha}\|_{H^{k-2}_{\delta-1,\epsilon}} + \|V_\epsilon - \tilde{V}\|_{H^{k-1}_{\delta-1,\epsilon}} &\leq \|Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} + \|d\Phi_\epsilon - d\tilde{\Phi}\|_{H^{k-1}_{\delta-1,\epsilon}} \\ +\epsilon\|\omega\|_{H^{k-1}_{\delta-1,\epsilon}} &\leq C\|Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} + C\epsilon.\end{aligned} \tag{7.90}$$

The three estimates (7.88)–(7.90) along with Lemmas A.9 and A.10, and Propositions 7.5 and 7.8, show that

$$\|A^I(\underset{o}{\mathfrak{u}}{}^{iJ}_{\epsilon}) + A^I(\delta\mathfrak{u}^{iJ}_{\epsilon})\|_{H^{k-1}_{\delta,\epsilon}} \leq C\|\underset{o}{\mathfrak{u}}{}^{iJ}_{\epsilon}\|_{H^{k}_{\delta,\epsilon}} + C\|\delta\mathfrak{u}^{iJ}_{\epsilon}\|_{H^{k-1}_{\delta-1,\epsilon}} \leq C\epsilon + C\|Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}},$$

and

$$\begin{aligned}&\|a^I(\epsilon\mathfrak{u}_\epsilon, \epsilon w^i_\epsilon, \epsilon\alpha_\epsilon, w^I_\epsilon, \alpha_\epsilon) - a^I(0,0,0,\tilde{w}^i,\tilde{\alpha})\|_{H^{k-1}_{\delta,\epsilon}} \\ &\quad\leq C\left(\epsilon\|\mathfrak{u}^{ij}_\epsilon\|_{H^{k-1}_{\delta,\epsilon}} + \|\alpha_\epsilon - \tilde{\alpha}\|_{H^{k-1}_{\delta-1,\epsilon}} + \|w^i_\epsilon - \tilde{w}^i\|_{H^{k-1}_{\delta-1,\epsilon}} \leq C\epsilon + C\|Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}}\right).\end{aligned}$$

Therefore

$$\|b^I_\epsilon - \tilde{b}\|_{H^{k-1}_{\delta-1,\epsilon}} \leq C\epsilon + C\|Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}},$$

and hence

$$\|(\tilde{b} - b^I_\epsilon)\partial_I\tilde{V}\|_{H^{k-1}_{\delta-1,\epsilon}} \leq C\|\tilde{b} - b^I_\epsilon\|_{H^{k-1}_{\delta,\epsilon}}\|D\tilde{V}\|_{H^{k-1}_{\delta-2,\epsilon}} \leq C\epsilon + \|Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}}. \tag{7.91}$$

Next, we notice that

$$f(\epsilon, U_\epsilon, V_\epsilon)V_\epsilon - \tilde{f}(\tilde{V})\tilde{V} = -\rho_\epsilon\mathcal{F}_\epsilon + \hat{f}(V_\epsilon)V_\epsilon - \hat{f}(\tilde{V})V_\epsilon + \epsilon\bar{f}(\epsilon, U_\epsilon, V_\epsilon)V_\epsilon,$$

where

$$\mathcal{F}_\epsilon := -4\rho_\epsilon(\epsilon\delta^i_4\delta^j_4\eta_{pq}\underset{o}{\bar{\mathfrak{u}}}{}^{pq}_{\epsilon}, 0, \ldots, 0)^T$$

and $\hat{f}$ and $\bar{f}$ are analytic. We obtain

$$\|\hat{f}(V_\epsilon)V_\epsilon - \hat{f}(\bar{V})V_\epsilon + \epsilon\bar{f}(\epsilon, U_\epsilon, V_\epsilon)V_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} \leq C\|Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} + C\epsilon \tag{7.92}$$

by the arguments used above. Also, the boundedness of the support of $\alpha_\epsilon(t)$ implies that

$$\|\mathcal{F}_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} \leq C\epsilon\|\rho_\epsilon\eta_{ij}\underset{o}{\bar{\mathfrak{u}}}{}^{ij}_{\epsilon}\|_{H^{k-1}_{\delta}} \leq C\epsilon\|\rho_\epsilon\|_{H^{k-1}_{\delta}}\|\underset{o}{\bar{\mathfrak{u}}}\|_{H^{k-1}_{\delta}} \leq C\epsilon\|\rho_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}}\|\underset{o}{\bar{\mathfrak{u}}}\|_{H^{k-1}_{\delta}} \leq C\epsilon. \tag{7.93}$$

So then

$$\|f(\epsilon, U_\epsilon, V_\epsilon)V_\epsilon - \tilde{f}(\tilde{V})\tilde{V}\|_{H^{k-1}_{\delta-1,\epsilon}} \leq C\|Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} + C\epsilon \tag{7.94}$$

by (7.92) and (7.93). Combining the estimates (7.83)–(7.87), (7.91), (7.92), and (7.94) yields

$$\|F_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} \le C\|Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} + C\epsilon. \tag{7.95}$$

Letting $Z^\alpha_\epsilon = D^\alpha Z_\epsilon$ and differentiating Eq. (7.81) yields

$$b^0_\epsilon \partial_t Z^\alpha_\epsilon = \frac{1}{\epsilon} c^I \partial_I Z^\alpha_\epsilon + b^I_\epsilon \partial_I Z^\alpha_\epsilon + q^\alpha \qquad 0 \le |\alpha| \le k-1,$$

where

$$q^\alpha = -[D^\alpha, b^0_\epsilon]\partial_t Z^\alpha_\epsilon + [D^\alpha, b^I_\epsilon]\partial_I Z^\alpha_\epsilon + D^\alpha F_\epsilon.$$

Using the estimates above along with Propositions 7.5 and 7.8 and the calculus inequalities from Appendix A, we find

$$\begin{aligned}
\|\partial_t Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} &\le C\|Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} + C\epsilon,\\
\|[D^\alpha, b^0_\epsilon]\partial_t Z^\alpha_\epsilon\|_{L^2_{\delta-1-|\alpha|,\epsilon}} &\le C\|b^0_\epsilon - \mathbb{I}\|_{H^{k-1}_{\delta,\epsilon}}\|\partial_t Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} \le C\|Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} + C\epsilon,\\
\|[D^\alpha, b^I_\epsilon]\partial_I Z^\alpha_\epsilon\|_{L^2_{\delta-1-|\alpha|,\epsilon}} &\le C\|b^I_\epsilon\|_{H^{k-1}_{\delta,\epsilon}}\|DZ_\epsilon\|_{H^{k-2}_{\delta-2,\epsilon}} \le C\|Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}},
\end{aligned}$$

and hence

$$\|q^\alpha\|_{H^{k-1}_{\delta-1,\epsilon}} \le C\|Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} + C\epsilon.$$

Combining this estimate with the estimates

$$\|\partial_t b^0_\epsilon + \partial_I b^I_\epsilon\|_{L^\infty} \le C, \quad \|b^I_\epsilon\|_{L^\infty} \le C,$$

and Lemma 7.1 shows that

$$\frac{d}{dt}\left\langle Z^\alpha_\epsilon | b^0_\epsilon Z^\alpha_\epsilon \right\rangle_{L^2_{\delta-1-|\alpha|,\epsilon}} \le C(\|Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} + \epsilon)\|Z_\epsilon\|_{H^{k-1}_{\delta-1,\epsilon}} \quad 0 \le |\alpha| \le k-1.$$

Summing over α and using Gronwall's inequality, we get

$$\|Z_\epsilon(t)\|_{H^{k-1}_{\delta-1,\epsilon}} \le C\|Z_\epsilon(0)\|_{H^{k-1}_{\delta-1,\epsilon}} + C\epsilon \quad \text{for all } (t,\epsilon) \in [0, T^*] \times (0, \epsilon_0].$$

This estimate and (7.90) then prove the proposition since $\|Z_\epsilon(0)\|_{H^{k-1}_{\delta-1,\epsilon}} \le C\epsilon$ by Proposition 6.1. □

We are now ready to prove a precise error estimate for the difference between the relativistic and Newtonian solutions.

Proposition 7.11. *Suppose* $-1 < \delta < -1/2$ *and* $k \ge 3$. *Then there exists a constant* $C > 0$ *such that*

$$\begin{aligned}
&\|\bar{\mathfrak{u}}^{ij}_\epsilon(t) - \delta^i_4\delta^i_4\tilde{\Phi}(t)\|_{L^6_{\delta,\epsilon}} + \|\partial_I\bar{\mathfrak{u}}^{ij}_\epsilon(t) - \delta^i_4\delta^j_4 d\tilde{\Phi}(t)\|_{H^{k-1}_{\delta-1,\epsilon}} + \|v^I(t) - \tilde{w}^I(t)\|_{H^{k-1}_{\delta-1,\epsilon}}\\
&\quad + \epsilon^{-1}\|v^4(t) - 1\|_{H^{k-1}_{\delta-1,\epsilon}} + \|\rho_\epsilon(t) - \tilde{\rho}(t)\|_{H^{k-1}_{\delta-1,\epsilon}} + \|\partial_t\rho_\epsilon(t) - \partial_t\tilde{\rho}(t)\|_{H^{k-2}_{\delta-1,\epsilon}} \le C\epsilon
\end{aligned}$$

for all $(t,\epsilon) \in [0, T^*] \times (0, \epsilon_0]$.

Proof. From the evolution equations and Proposition 7.8, we have

$$\epsilon\partial_t\left(\bar{\mathrm{u}}^{ij}_\epsilon - \delta^i_4\delta^j_4\tilde{\Phi}\right) = \mathrm{u}^{ij}_4 - \epsilon\omega^{ij}_4,$$

and hence integrating yields

$$\epsilon\|\bar{\mathrm{u}}^{ij}_\epsilon(t) - \delta^i_4\delta^j_4\tilde{\Phi}(t)\|_{L^2_{\delta,\epsilon}} \leq \epsilon\|\underset{o}{\bar{\mathrm{u}}}^{ij}_\epsilon - \delta^i_4\delta^j_4\phi\|_{L^2_{\delta,\epsilon}} + \int_0^t \|\mathrm{u}^{ij}_4(s) - \epsilon\omega^{ij}_4(s)\|_{L^2_{\delta,\epsilon}}\,ds. \tag{7.96}$$

But

$$\int_0^t \|\mathrm{u}^{ij}_4(s) - \epsilon\omega^{ij}_4(s)\|_{L^2_{\delta,\epsilon}}\,ds \leq \int_0^t \|V_\epsilon(s) - \tilde{V}(s)\|_{H^{k-1}_{\delta-1,\epsilon}} + \epsilon\|\omega^{ij}(s)\|_{H^{k-1}_{\delta-1,\epsilon}}\,ds \tag{7.97}$$

and

$$\epsilon\|\underset{o}{\bar{\mathrm{u}}}^{ij}_\epsilon - \delta^i_4\delta^j_4\phi\|_{L^2_{\delta,\epsilon}} \leq C\epsilon^{3/2} \tag{7.98}$$

by the calculus inequalities of Appendix A and Proposition 5.1. Also, by Lemma A.4 and $\mathrm{u}_{I,\epsilon} = \partial_I\bar{\mathrm{u}}_\epsilon$, we have

$$\begin{aligned}\|\bar{\mathrm{u}}^{ij}_\epsilon - \delta^i_4\delta^j_4\tilde{\Phi}\|_{L^6_{\delta,\epsilon}} \leq C\|\mathrm{u}^{ij}_{I,\epsilon} - \delta^i_4\delta^j_4 d\tilde{\Phi}\|_{L^2_{\delta-1,\epsilon}} &+ \epsilon\|\bar{\mathrm{u}}^{ij}_\epsilon(t) - \delta^i_4\delta^j_4\tilde{\Phi}(t)\|_{L^2_{\delta,\epsilon}} \\ &+ \epsilon\|\bar{\mathrm{u}}^{ij}_\epsilon(t) - \delta^i_4\delta^j_4\tilde{\Phi}(t)\|_{L^2_{\delta,\epsilon}}.\end{aligned} \tag{7.99}$$

Recall that $\rho_\epsilon = (4Kn(n+1))^{-n}\alpha^{2n}_\epsilon$ and $\tilde{\rho} = (4Kn(n+1))^{-n}\tilde{\alpha}^{2n}$. Since $\|\alpha_\epsilon\|_{H^k_{\delta-1,\epsilon}}$ is bounded as $\epsilon \searrow 0$, we obtain

$$\|\rho_\epsilon - \tilde{\rho}\|_{H^{k-1}_{\delta-1,\epsilon}} \leq C\|\alpha_\epsilon - \tilde{\alpha}\|_{H^{k-1}_{\delta-1,\epsilon}} \leq C\|V_\epsilon - \tilde{V}\|_{H^{k-1}_{\delta-1,\epsilon}} \tag{7.100}$$

by Lemma A.10. We also have that $\|\partial_t\alpha_\epsilon\|_{H^{k-2}_{\delta-1,\epsilon}}$ is bounded as $\epsilon \searrow 0$, so the formulas

$$\partial_t\rho_\epsilon = \frac{2n}{(4Kn(n+1))^n}\alpha^{2n-1}_\epsilon\partial_t\alpha_\epsilon, \quad \partial_t\tilde{\rho} = \frac{2n}{(4Kn(n+1))^n}\tilde{\alpha}^{2n-1}\partial_t\tilde{\alpha},$$

and the calculus inequalities of Appendix A imply that

$$\begin{aligned}\|\partial_t\rho_\epsilon - \partial_t\tilde{\rho}\|_{H^{k-2}_{\delta-1,\epsilon}} &\leq C(\|\alpha_\epsilon - \tilde{\alpha}\|_{H^{k-2}_{\delta-1,\epsilon}} + \|\partial_t\alpha_\epsilon - \partial_t\tilde{\alpha}\|_{H^{k-2}_{\delta-1,\epsilon}}) \\ &\leq C(\|V_\epsilon - \tilde{V}\|_{H^{k-1}_{\delta-1,\epsilon}} + \|\partial_t\alpha_\epsilon - \partial_t\tilde{\alpha}\|_{H^{k-2}_{\delta-1,\epsilon}}).\end{aligned} \tag{7.101}$$

Finally, from the definition of V_ϵ and $\tilde{V}$, we have

$$\begin{aligned}&\|\partial_I\bar{\mathrm{u}}^{ij}_\epsilon(t) - \delta^i_4\delta^j_4 d\tilde{\Phi}(t)\|_{H^{k-1}_{\delta-1,\epsilon}} + \|v^I(t) - \tilde{w}^I(t)\|_{H^{k-1}_{\delta-1,\epsilon}} + \epsilon^{-1}\|v^4(t) - 1\|_{H^{k-1}_{\delta-1,\epsilon}} \\ &\leq C\|V_\epsilon - \tilde{V}\|_{H^{k-1}_{\delta-1,\epsilon}}.\end{aligned} \tag{7.102}$$

The proof now follows as a direct consequence of Lemma 7.10 and (7.96)–(7.102). □

In the above error estimate, the norm itself depends on ϵ. We now show how to choose norms independent of ϵ which are compatible with the error estimate above. First, for any $\eta \in \mathbb{R}$ define a norm by

$$\|u\|_{\ell,p,\eta} := \sum_{|\alpha|\leq\ell} \|D^\alpha u\|_{L^p_\eta}.$$

Recalling that $-1 < \delta < -1/2$, fix $\eta \in [\delta, -1/2]$. Then from (A.24) and Lemma A.11, we get that

$$\|u\|_{\ell,2,\eta-1} \leq C\epsilon^{\eta+1/2}\|u\|_{H^\ell_{\delta-1,\epsilon}} \quad \text{and} \quad \|u\|_{0,6,\eta} \leq C\epsilon^{\eta+1/2}\|u\|_{L^6_{\delta,\epsilon}} \tag{7.103}$$

for some constant $C > 0$ independent of ϵ. Combining (7.103) with Corollary 7.11 yields the following theorem which is our main result.

Theorem 7.12. *Suppose* $-1 < \delta < -1/2$, $-\delta \leq \eta \leq -1/2$ *and* $k \geq 3$. *Then there exists a constant* $C > 0$ *such that*

$$\begin{aligned}
&\|\bar{\mathrm{u}}^{ij}_\epsilon(t) - \delta^i_4\delta^i_4\tilde{\Phi}(t)\|_{0,6,\eta} + \|\partial_I\bar{\mathrm{u}}^{ij}_\epsilon(t) - \delta^i_4\delta^j_4\partial_I\tilde{\Phi}(t)\|_{k-1,2,\eta-1} \\
&\quad+\|v^I(t) - \tilde{w}^I(t)\|_{k-1,2,\eta-1} + \epsilon^{-1}\|v^4(t) - 1\|_{k-1,2,\eta-1} \\
&\quad+\|\rho_\epsilon(t) - \tilde{\rho}(t)\|_{k-1,2,\eta-1} + \|\partial_t\rho_\epsilon(t) - \partial_t\tilde{\rho}(t)\|_{k-2,2,\eta-1} \leq C\epsilon^{\eta+3/2}
\end{aligned}$$

for all $(t, \epsilon) \in [0, T^*] \times (0, \epsilon_0]$.

Note that for $\eta = -1/2$, we have

$$\|u\|_{0,6,-1/2} = \|u\|_{L^6} \quad \text{and} \quad \|u\|_{\ell,2,-3/2} = \|u\|_{H^\ell},$$

where $\|u\|_{H^\ell}$ is the standard Sobolev norm. So the above theorem shows that the difference between the relativistic and Newtonian solutions is of order ϵ with respect to the norms $\|\cdot\|_{L^6}$ and $\|\cdot\|_{H^{k-1}}$.

A. Weighted Calculus Inequalities

In this and the following sections C will denote a constant that may change value from line to line but whose exact value is not needed.

Let V be a finite dimensional vector space with inner product $(\cdot|\cdot)$ and corresponding norm $|\cdot|$. For $u \in L^p_{\mathrm{loc}}(\mathbb{R}^n, V)$, $1 \leq p \leq \infty$, $\delta \in \mathbb{R}$, and $\epsilon \in \mathbb{R}_{\geq 0}$, the *weighted* L^p *norm* of u is defined by

$$\|u\|_{L^p_{\delta,\epsilon}} := \begin{cases} \|\sigma_\epsilon^{-\delta-n/p}u\|_{L^p} & \text{if } 1 \leq p < \infty \\ \|\sigma_\epsilon^{-\delta}u\|_{L^\infty} & \text{if } p = \infty \end{cases}, \tag{A.1}$$

where $\sigma_\epsilon(x) := \sqrt{1 + \frac{1}{4}|\epsilon x|^2}$. The *weighted Sobolev norms* are then defined by

$$\|u\|_{W^{k,p}_{\delta,\epsilon}} := \begin{cases} \left(\displaystyle\sum_{|\alpha|\leq k} \|D^\alpha u\|^p_{L^p_{\delta-|\alpha|,\epsilon}}\right)^{1/p} & \text{if } 1 \leq p < \infty \\ \displaystyle\sum_{|\alpha|\leq k} \|D^\alpha u\|_{L^\infty_{\delta-|\alpha|,\epsilon}} & \text{if } p = \infty \end{cases}, \tag{A.2}$$

where $k \in \mathbb{N}_0$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ is a multi-index and $D^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$. Here

$$\partial_i = \frac{\partial}{\partial x^i},$$

where $(x^1, \ldots, x^n)$ are the standard Cartesian coordinates on $\mathbb{R}^n$.

The weighted Sobolev spaces are then defined as

$$W^{k,p}_{\delta,\epsilon} = \left\{ u \in W^{k,p}_{\text{loc}}(\mathbb{R}^n, V) \mid \|u\|_{W^{k,p}_{\delta,\epsilon}} < \infty \right\}.$$

Directly from this definition, we observe the simple but useful inequality

$$\|\partial_j u\|_{W^k_{\delta,\epsilon}} \leq \|u\|_{W^{k+1}_{\delta+1,\epsilon}}. \tag{A.3}$$

We note that $W^{k,p}_{\delta,0}$ are the standard Sobolev spaces and for $\epsilon > 0$, the $W^{k,p}_{\delta,\epsilon}$ are equivalent to the radially weighted Sobolev spaces [1,7]. For $p = 2$, we use the alternate notation $H^k_{\delta,\epsilon} := W^{k,2}_{\delta,\epsilon}$. The spaces $L^2_{\delta,\epsilon}$ and $H^k_{\delta,\epsilon}$ are Hilbert spaces with inner products

$$\langle u|v\rangle_{L^2_{\delta,\epsilon}} := \int_{\mathbb{R}^n} (u|v)\sigma_\epsilon^{-2\delta-n} d^n x, \tag{A.4}$$

and

$$\langle u|v\rangle_{H^k_{\delta,\epsilon}} := \sum_{|\alpha|\leq k} \left\langle D^\alpha u | D^\alpha v\right\rangle_{L^2_{\delta-|\alpha|,\epsilon}}, \tag{A.5}$$

respectively. When $\epsilon = 1$, we will also use the notation $W^{k,p}_\delta = W^{k,p}_{\delta,1}$ and $H^k_\delta = H^k_{\delta,1}$.

Let B_R be the open ball of radius R and a_R and A_R denote the annuli $B_{2R} \setminus B_R$ and $B_{4R} \setminus B_R$, respectively. Let $\{\phi_j\}_{j=0}^\infty$ be a smooth partition of unity satisfying

$$\operatorname{supp} \phi_0 \subset B_2, \quad \operatorname{supp} \phi_j \subset A_{2^{j-1}} \quad (j \geq 1), \quad \text{and} \quad \phi_j(x) := \phi_1(2^{1-j}x)(j \geq 1).$$

Scaling gives a one parmeter family of smooth partitions of unity

$$\phi_j^\epsilon(x) := \phi_j(\epsilon x) \quad (j \geq 0)$$

which satisfy

$$\operatorname{supp} \phi_0^\epsilon \subset B_{2/\epsilon},\ \operatorname{supp} \phi_j^\epsilon \subset A_{2^{j-1}/\epsilon} \quad (j \geq 1), \ \text{and} \ \phi_j^\epsilon(x) := \phi_1^\epsilon(2^{1-j}x)(j \geq 1). \tag{A.6}$$

Define a scaling operator by

$$S_j u(x) := u(2^{j-1}x). \tag{A.7}$$

This operator satisfies the following simple, but useful identities:

$$S_1 = 1\!\!1, \quad S_j \circ S_k = S_k \circ S_j = S_{k+j-1}, \tag{A.8}$$

$$S_j \phi_j^\epsilon = \phi_1^\epsilon \quad (j \geq 1), \tag{A.9}$$

$$\|S_j u\|_{L^p} = 2^{\frac{n(1-j)}{p}} \|u\|_{L^p}, \tag{A.10}$$

and

$$S_j \circ D^\alpha = 2^{(1-j)|\alpha|} D^\alpha \circ S_j. \tag{A.11}$$

Lemma A.1. *For $1 \leq p < \infty$, there exists a constant $C > 0$ independent of $\epsilon \geq 0$ such that*

$$\frac{1}{C}\|u\|^p_{L^p_{\delta,\epsilon}} \leq \|\phi_0^\epsilon u\|^p_{L^p} + \sum_{j=0}^{\infty}\|S_j(\phi_j^\epsilon u)\|^p_{L^p} \leq C\|u\|^p_{L^p_{\delta,\epsilon}}.$$

Proof. From the identity

$$\|u\|^p_{L^p} = \int_{B_{4/\epsilon}} |u|^p\, d^n x + \sum_{j=1}^{\infty}\int_{a_{2^{j+1}}/\epsilon} |u|^p\, d^n x$$

and a simple change of variables, it follows that

$$\|u\|^p_{L^p_{\delta,\epsilon}} = \|\sigma_\epsilon^{-\delta-n/p}u\|^p_{L^p(B_{4/\epsilon})} + \sum_{j=1}^{\infty} 2^{n(j-1)}\|S_j(\sigma_\epsilon^{-\delta-n/p}u)\|^p_{L^p(a_{4/\epsilon})}. \tag{A.12}$$

This identity and

$$\max_{x\in\overline{B}_{4/\epsilon}} \sigma_\epsilon(x)^{-\delta p-n} = \begin{cases} 2^{\frac{-\delta p-n}{2}} & \text{if } -\delta p - n \geq 0 \\ 1 & \text{if } -\delta p - n < 0 \end{cases},$$

$$\min_{x\in\overline{B}_{4/\epsilon}} \sigma_\epsilon(x)^{-\delta p-n} = \begin{cases} 1 & \text{if } -\delta p - n \geq 0 \\ 2^{\frac{-\delta p-n}{2}} & \text{if } -\delta p - n < 0 \end{cases},$$

$$\max_{x\in\overline{a}_{4/\epsilon}} (S_j\sigma_\epsilon)(x)^{-\delta p-n} = \begin{cases} (1+2^{2j})^{\frac{-\delta p-n}{2}} & \text{if } -\delta p - n \geq 0 \\ (1+2^{2(j-1)})^{\frac{-\delta p-n}{2}} & \text{if } -\delta p - n < 0 \end{cases},$$

$$\min_{x\in\overline{a}_{4/\epsilon}} (S_j\sigma_\epsilon)(x)^{-\delta p-n} = \begin{cases} (1+2^{2(j-1)})^{\frac{-\delta p-n}{2}} & \text{if } -\delta p - n \geq 0 \\ (1+2^{2j})^{\frac{-\delta p-n}{2}} & \text{if } -\delta p - n \leq 0 \end{cases},$$

show that

$$\frac{1}{C}\|u\|^p_{L^p(B_{4/\epsilon})} \leq \|\sigma_\epsilon^{-\delta-n/p}u\|^p_{L^p(B_{4/\epsilon})} \leq C\|u\|^p_{L^p(B_{4/\epsilon})} \tag{A.13}$$

and

$$\begin{aligned}\frac{1}{C}2^{-p\delta(j-1)}\|S_j u\|^p_{L^p(a_{4/\epsilon})} &\leq 2^{n(j-1)}\|S_j(\sigma_\epsilon^{-\delta-n/p}u)\|_{L^p(a_{4/\epsilon})} \\ &\leq C2^{-p\delta(j-1)}\|S_j u\|^p_{L^p(a_{4/\epsilon})}\end{aligned} \tag{A.14}$$

for some constant $C > 0$ which is independent of $\epsilon \geq 0$. Using a change of variable, the inequality (A.14) can be written as

$$\begin{aligned}&\frac{1}{C}2^{-p\delta(j-1)}2^{(1-j)n}\|u\|^p_{L^p(a_{2^{j+1}}/\epsilon)} \leq 2^{n(j-1)}\|S_j(\sigma_\epsilon^{-\delta-n/p}u)\|_{L^p(a_{4/\epsilon})} \\ &\quad\leq C2^{-p\delta(j-1)}2^{(1-j)n}\|u\|^p_{L^p(a_{2^{j+1}}/\epsilon)}.\end{aligned} \tag{A.15}$$

From

$$\sum_{k=0}^{2} \phi_k^\epsilon \Big|_{B_{4/\epsilon}} = \mathbb{1}_{B_{4/\epsilon}} \quad \text{and} \quad \sum_{k=0}^{2} \phi_{j+k}^\epsilon \Big|_{a_{2^{j+1}}/\epsilon} = \mathbb{1}_{a_{2^{j+1}}/\epsilon} \tag{A.16}$$

and (A.10), we obtain

$$\|u\|^p_{L^p(B_{4/\epsilon})} \leq C\left(\|\phi_0^\epsilon u\|^p_{L^p} + \sum_{k=1}^{3} \|S_k(\phi_k^\epsilon u)\|^p_{L^p}\right), \tag{A.17}$$

and

$$\|u\|^p_{L^p(a_{2^{j+1}}/\epsilon)} \leq C \sum_{k=0}^{2} 2^{n(j+k)} \|S_{j+k}(\phi^\epsilon_{j+k} u)\|^p_{L^p}. \tag{A.18}$$

Combining (A.12) with the inequalities (A.13), (A.15), (A.17) and (A.18) yields

$$\|u\|^p_{L^p_{\delta,\epsilon}} \leq C\left(\|\phi_0^\epsilon u\|^p_{L^p} + \sum_{j=1}^{\infty} 2^{-p\delta(j-1)} \|S_j(\phi_j^\epsilon u)\|^p_{L^p}\right) \tag{A.19}$$

for some constant $C > 0$ independent of $\epsilon \geq 0$.

Since $\operatorname{supp} \phi_0^\epsilon \subset B_{4/\epsilon}$ and $\|\phi_0^\epsilon\|_{L^\infty} = \|\phi_0\|_{L^\infty}$, we get from (A.13) that

$$\|\phi_0^\epsilon u\|^p_{L^p} \leq \|\phi_0^\epsilon\|^p_{L^\infty} \|u\|^p_{L^p(B_{4/\epsilon})} \leq C\|\sigma_\epsilon^{-\delta-n/p} u\|^p_{L^p(B_{4/\epsilon})} \tag{A.20}$$

for some constant $C > 0$ independent of $\epsilon \geq 0$. Next,

$$\begin{aligned} &2^{-p\delta(j-1)} \|S_j(\phi_j^\epsilon u)\|^p_{L^p} \\ &\quad \leq 2^{-p\delta(j-1)} 2^{n(1-j)} \|\phi_j^\epsilon u\|_{L^p(A_{2^{j-1}}/\epsilon)} && \text{by (A.10) and (A.6),} \\ &\quad \leq 2^{-p\delta(j-1)} 2^{n(1-j)} \|\phi_1\|_{L^\infty} \|\phi_j^\epsilon u\|_{L^p(\cup_{k=-1}^{1} a_{2^{j-k}}/\epsilon)} && \text{since } \|\phi_1\|_{L^\infty} = \|\phi_j^\epsilon\|_{L^\infty}. \end{aligned}$$

So there exists a constant $C > 0$ independent of $\epsilon \geq 0$ such that

$$2^{-p\delta j-1} \|S_j(\phi_j^\epsilon u)\|^p_{L^p} \leq \begin{cases} C\left(\|u\|^p_{L^p(B_{4/\epsilon})} + \|u\|^p_{L^p(a_{8/\epsilon})}\right) & \text{if } j = 1 \\ C 2^{-p\delta(j-1)} 2^{n(1-j)} \sum_{k=0}^{2} \|u\|^p_{L^p(a_{2^{j-1+k}}/\epsilon)} & \text{if } j \geq 2 \end{cases}. \tag{A.21}$$

Therefore

$$\begin{aligned} &\|\phi_0^\epsilon u\|^p_{L^p} + \sum_{j=1}^{\infty} 2^{-p\delta(j-1)} \|S_j(\phi_j^\epsilon u)\|^p_{L^p} \\ &\quad \leq C\left(\|\sigma_\epsilon^{-\delta-n/p} u\|^p_{L^p(B_{4/\epsilon})} + \sum_{j=1}^{\infty} 2^{n(j-1)} \|S_j(\sigma_\epsilon^{-\delta-n/p} u)\|^p_{L^p(a_{4/\epsilon})}\right) \\ &\qquad\qquad \text{by (A.15), (A.20), and (A.21)} \\ &\quad \leq C\|u\|^p_{L^p_{\delta,\epsilon}} \qquad \text{by (A.12),} \end{aligned}$$

where $C > 0$ is a constant independent of $\epsilon \geq 0$. The proof then follows from this inequality and (A.19). □

The above lemma shows that the norm

$$|||u|||^p_{L^p_{\delta,\epsilon}} := \|\phi_0^\epsilon u\|^p_{L^p} + \sum_{j=1}^{\infty} 2^{-p\delta(j-1)} \|S_j(\phi_j^\epsilon u)\|^p_{L^p}$$

is equivalent for $1 \leq p < \infty$, independent of $\epsilon \geq 0$, to the weighted norm $\|u\|_{L^p_{\delta,\epsilon}}$. For $p = \infty$, the appropriate norm is

$$|||u|||_{L^\infty_{\delta,\epsilon}} := \sup\left\{\|\phi_0^\epsilon u\|_{L^\infty}, 2^{-\delta(j-1)}\|\phi_j^\epsilon u\|_{L^\infty} \quad (j \geq 1)\right\}$$

and it is easy to see that there exists a constant $C > 0$ independent of $\epsilon \geq 0$ such that

$$\frac{1}{C}\|u\|_{L^\infty_{\delta,\epsilon}} \leq |||u|||_{L^\infty_{\delta,\epsilon}} \leq C\|u\|_{L^\infty_{\delta,\epsilon}}.$$

The same arguments used in proving the previous lemma can be used to establish the following generalization.

Lemma A.2. *For* $1 \leq p < \infty$, *let*

$$|||u|||^p_{W^{k,p}_{\delta,\epsilon}} := \|\phi_0^\epsilon u\|^p_{W^{k,p}} + \sum_{j=1}^{\infty} 2^{-p\delta(j-1)} \|S_j(\phi_j^\epsilon u)\|^p_{W^{k,p}}, \tag{A.22}$$

and for $p = \infty$ *let*

$$|||u|||_{W^{k,\infty}_{\delta,\epsilon}} := \sup\{\|\phi_0^\epsilon u\|_{W^{k,\infty}}, 2^{-\delta(j-1)}\|S_j(\phi_j^\epsilon u)\|_{W^{k,\infty}} \ (j \geq 1)\}. \tag{A.23}$$

Then there exists a constant $C > 0$ *independent of* $\epsilon \geq 0$ *such that*

$$\frac{1}{C}\|u\|^p_{W^{k,p}_{\delta,\epsilon}} \leq |||u|||^p_{W^{k,p}_{\delta,\epsilon}} \leq C\|u\|^p_{W^{k,p}_{\delta,\epsilon}}.$$

For the remainder of this section, we will use the two equivalent norms $\|\cdot\|_{W^{k,p}_{\delta,\epsilon}}$ and $|||\cdot|||_{W^{k,p}_{\delta,\epsilon}}$ interchangeably and refer to both using the notation $\|\cdot\|_{W^{k,p}_{\delta,\epsilon}}$. From (A.22), it follows that there exist a constant $C > 0$ independent of $\epsilon \geq 0$ such that

$$\|u\|_{W^{k_2,p}_{\delta_2,\epsilon}} \leq C\|u\|_{W^{k_1,p}_{\delta_1,\epsilon}} \quad \text{whenever } k_2 \leq k_1 \text{ and } \delta_1 \leq \delta_2. \tag{A.24}$$

Thus we have the inclusion $W^{k_1,p}_{\delta_1,\epsilon} \subset W^{k_2,p}_{\delta_2,\epsilon}$ for $k_2 \leq k_1$ and $\delta_1 \leq \delta_2$. The representation (A.22) is particularly useful for extending estimates from the usual Sobolev spaces $W^{k,p}_\delta$ to the weighted ones $W^{k,p}_{\delta,\epsilon}$ ($\epsilon > 0$) as the next lemma shows. It also makes clear the philosophy behind deriving weighted Sobolev inequalities which is to derive global estimates from scaling and local Sobolev inequalities [1].

We remark that the norm $|||\cdot|||_{W^{k,p}_{\delta,1}}$, as an alternate representation for the standard weighted norms $\|\cdot\|_{W^{k,p}_{\delta,1}}$, was introduced by Maxwell in [25]. There he used the norm to define the weighted Sobolev spaces for non-integral k (see also [4]). Here we will only be interested in integral k.

Lemma A.3. *Suppose* $\epsilon_0 > 0$ *and for all* $u \in C^\infty(\mathbb{R}^n, V)$, $u \mapsto F_1(u)$ *is a map that satisfies*

$$\phi_0^\epsilon F_1(u) = \phi_0^\epsilon F_1((\phi_0^\epsilon + \phi_1^\epsilon)u),$$
$$\phi_j^\epsilon F_1(u) = \phi_j^\epsilon F_1\left(\sum_{k=-1}^{1} \phi_{j+k}^\epsilon u\right) \quad (j \geq 1),$$
$$S_j F_1(u) = 2^{-(j-1)\lambda} F_1(S_j u) \quad (j \geq 1),$$

and F_α $(\alpha = 2, 3, 4, 5)$ *are linear operators on* V.

(*i*) *If there is an estimate of the form*

$$\|F_1(u)\|_{W^{k_1,p_1}} \leq C_1 \|F_2(u)\|_{W^{k_2,p_2}},$$

where $p_1 \geq p_2$, *then*

$$\|F_1(u)\|_{W^{k_1,p_1}_{\delta_1,\epsilon}} \leq C \|F_2(u)\|_{W^{k_2,p_2}_{\delta_2,\epsilon}}$$

for some constant $C > 0$ *independent of* $\epsilon \in [0, \epsilon_0]$ *provided* $\delta_1 + \lambda \geq \delta_2$.

(*ii*) *If there exists an estimate of the form*

$$\|F_1(u)\|_{W^{k_1,p_1}} \leq C_1 \|F_2(u)\|_{W^{k_2,p_2}} \|F_3(u)\|_{W^{k_3,p_3}} + C_2 \|F_2(u)\|_{W^{k_4,p_4}} \|F_1(u)\|_{W^{k_5,p_5}},$$

where $\dfrac{1}{p_1} = \dfrac{1}{p_2} + \dfrac{1}{p_3} = \dfrac{1}{p_4} + \dfrac{1}{p_5}$ $(1 \leq p_1 \leq p_\alpha \leq \infty \quad \alpha = 2, 3, 4, 5)$, *then*

$$\|F_1(u)\|_{W^{k_1,p_1}_{\delta_1,\epsilon}} \leq C\left(C_1 \|F_2(u)\|_{W^{k_2,p_2}_{\delta_2,\epsilon}} \|F_3(u)\|_{W^{k_3,p_3}_{\delta_3,\epsilon}} + C_2 \|F_4(u)\|_{W^{k_4,p_4}_{\delta_4,\epsilon}} \|F_5(u)\|_{W^{k_5,p_5}_{\delta_5,\epsilon}}\right)$$

for some constant $C > 0$ *independent of* $\epsilon \in [0, \epsilon_0]$ *provided* $\delta_1 + \lambda \geq \max\{\delta_2 + \delta_3, \delta_4 + \delta_5\}$.

Proof. We only proof part (ii) for $1 \leq p_\alpha < \infty$. Part (i) can be proved in a similar manner using the inequality

$$\left(\sum_j a_j^p\right)^{1/p} \leq \left(\sum_j a_j^q\right)^{1/q} \quad \text{for } a_j \geq 0 \text{ and } 0 < q \leq p \tag{A.25}$$

instead of Hölder's and Minkowski's inequalities. See also the proof of Theorem 1.2 in [1].

Recall Hölder's and Minkowski's inequalities which state that for $1 \leq p \leq q \leq r < \infty$, $1/p = 1/q + 1/r$ and any two sequences $a_j, b_j \geq 0$ that the following holds

$$\left(\sum_j a_j^p b_j^p\right)^{1/p} \leq \left(\sum_j a_j^q\right)^{1/q} \left(\sum_j b_j^r\right)^{1/r} \tag{A.26}$$

and

$$\left(\sum_j (a_j + b_j)^p\right)^{1/p} \leq \left(\sum_j a_j^p\right)^{1/p} + \left(\sum_j a_j^p\right)^{1/p}. \tag{A.27}$$

Next, suppose $j \geq 2$. Then

$$
\begin{aligned}
&\|S_j(\phi_j^\epsilon F_1(u))\|^{p_1}_{W^{k_1,p_1}} \\
&= \left\|\phi_1^\epsilon S_j F_1\left(\sum_{k=-1}^{1} \phi_{j+k}^\epsilon u)\right)\right\|^{p_1}_{W^{k_1,p_1}} \leq C2^{-(1-j)p_1\lambda}\left\|F_1\left(\sum_{k=-1}^{1} S_j\phi_{j+k}^\epsilon u\right)\right\|^{p_1}_{W^{k_1,p_1}} \\
&\leq C2^{-(j-1)p_1\lambda}\left(C_1\left\|F_2\left(\sum_{k=-1}^{1} S_j\phi_{j+k}^\epsilon u\right)\right\|_{W^{k_2,p_2}}\left\|F_3\left(\sum_{k=-1}^{1} S_j\phi_{j+k}^\epsilon u\right)\right\|_{W^{k_3,p_3}}\right. \\
&\quad \left.+C_2\left\|F_4\left(\sum_{k=-1}^{1} S_j\phi_{j+k}^\epsilon u\right)\right\|_{W^{k_4,p_4}}\left\|F_5\left(\sum_{k=-1}^{1} S_j\phi_{j+k}^\epsilon u\right)\right\|_{W^{k_5,p_5}}\right)^{p_1},
\end{aligned}
$$

where $C > 0$ is a constant independent of $\epsilon \geq 0$. Note that in deriving this, we have used the fact that $\|\phi_1^\epsilon\|_{W^{k_1,\infty}}$ is bounded for $\epsilon \in [0, \epsilon_0]$. From the above inequality, we see that

$$
\begin{aligned}
&2^{-\delta_1 p_1(j-1)}\|S_j(\phi_j^\epsilon F_1(u))\|^{p_1}_{W^{k_1,p_1}} \\
&\leq C\left(C_1 2^{-\delta_2(j-1)}\left(\sum_{k=-1}^{1}\|F_2(S_{j+k}(\phi_{j+k}^\epsilon u))\|_{W^{k_2,p_2}}\right)2^{-\delta_3(j-1)}\right. \\
&\quad \times\left(\sum_{k=-1}^{1}\|F_3(S_{j+k}(\phi_{j+k}^\epsilon u))\|_{W^{k_3,p_3}}\right) \\
&\quad +C_2 2^{-\delta_4(j-1)}\left(\sum_{k=-1}^{1}\|F_4(S_{j+k}(\phi_{j+k}^\epsilon u))\|_{W^{k_4,p_4}}\right)2^{-\delta_5(j-1)} \\
&\quad \left.\times\left(\sum_{k=-1}^{1}\|F_5(S_{j+k}(\phi_{j+k}^\epsilon u))\|_{W^{k_5,p_5}}\right)\right)^{p_1},
\end{aligned}
$$

where we have used $\delta_1 + \lambda \geq \max\{\delta_2 + \delta_3, \delta_4 + \delta_5\}$. The above inequality along with (A.26) and (A.27) imply

$$
\begin{aligned}
&\left(\sum_{j=1}^{\infty} 2^{-\delta_1 p_1(j-1)}\|S_j(\phi_j^\epsilon F_1(u))\|^{p_1}_{W^{k_1,p_1}}\right)^{1/p_1} \\
&\leq C\left(C_1\left(\sum_{j=1}^{\infty} 2^{-\delta_2 p_2(j-1)}\|F_2(S_j(\phi_j^\epsilon u))\|^{p_2}_{W^{k_2,p_2}}\right)^{1/p_2}\right. \\
&\quad \times\left(\sum_{j=1}^{\infty} 2^{-\delta_3 p_3(j-1)}\|F_3(S_j(\phi_j^\epsilon u))\|^{p_3}_{W^{k_3,p_3}}\right)^{1/p_3}
\end{aligned}
$$

$$+C_2\left(\sum_{j=1}^{\infty}2^{-\delta_4 p_4(j-1)}\|F_4(S_j(\phi_j^\epsilon u))\|^{p_4}_{W^{k_4,p_4}}\right)^{1/p_4}$$
$$\times\left(\sum_{j=1}^{\infty}2^{-\delta_5 p_5(j-1)}\|F_5(S_j(\phi_j^\epsilon u))\|^{p_5}_{W^{k_5,p_5}}\right)^{1/p_5}\Bigg),$$

and hence

$$\left(\sum_{j=1}^{\infty}2^{-\delta_1 p_1(j-1)}\|S_j(\phi_j^\epsilon F_1(u))\|^{p_1}_{W^{k_1,p_1}}\right)^{1/p_1}\leq C\Bigg(C_1\|F_2(u)\|_{W^{p_2,k_2}_{\delta_2,\epsilon}}\|F_3(u)\|_{W^{p_3,k_3}_{\delta_3,\epsilon}}$$
$$+C_2\|F_4(u)\|_{W^{p_4,k_4}_{\delta_4,\epsilon}}\|F_5(u)\|_{W^{p_5,k_5}_{\delta_5,\epsilon}}\Bigg). \tag{A.28}$$

Similar arguments show that

$$\left(\|\phi^\epsilon F_1(u)\|^{p_1}_{W^{k_1,p_1}_{\delta_1,\epsilon}}+\|S_1(\phi_1^\epsilon F_1(u))\|^{p_1}_{W^{k_1,p_1}_{\delta_1,\epsilon}}\right)^{1/p_1}\leq C\Bigg(C_1\|F_2(u)\|_{W^{p_2,k_2}_{\delta_2,\epsilon}}\|F_3(u)\|_{W^{p_3,k_3}_{\delta_3,\epsilon}}$$
$$+C_2\|F_4(u)\|_{W^{p_4,k_4}_{\delta_4,\epsilon}}\|F_5(u)\|_{W^{p_5,k_5}_{\delta_5,\epsilon}}\Bigg), \tag{A.29}$$

for some constant $C > 0$ independent of $\epsilon \in [0,\epsilon_0]$. The proof now follows from the two inequalities (A.28) and (A.29). □

The next lemma is a variation of the previous one and can be proved in the same fashion.

Lemma A.4. *Suppose* $\epsilon_0 > 0$ *and for all* $u \in C^\infty(\mathbb{R}^n, V)$, $u \mapsto F_1(u)$ *is a map that satisfies*

$$\phi_0^\epsilon F_1(u) = \phi_0^\epsilon F_1((\phi_0^\epsilon+\phi_1^\epsilon)u),$$
$$\phi_j^\epsilon F_1(u) = \phi_j^\epsilon F_1\left(\sum_{k=-1}^{1}\phi_{j+k}^\epsilon u\right)\quad (j\geq 1),$$
$$S_jF_1(u) = 2^{-(j-1)\lambda}F_1(S_ju)\quad (j\geq 1),$$

and

$$F_2 = DP_2,\quad F_3 = P_3,\quad F_4 = DP_4,\quad \textit{and}\quad F_5 = P_5,$$

where P_α $(\alpha = 2,3,4,5)$ *are linear operators on* V.

(*i*) *If there exists an estimate of the form*

$$\|F_1(u)\|_{W^{k_1,p_1}} \le C_1\|F_2(u)\|_{W^{k_2,p_2}},$$

where $p_1 \ge p_2$*, then there exists a constant* $C > 0$ *independent of* $\epsilon \in [0, \epsilon_0]$ *such that*

$$\|F_1(u)\|_{W^{k_1,p_1}_{\delta_1,\epsilon}} \le C\left(\|F_2(u)\|_{W^{k_2,p_2}_{\delta_2-1,\epsilon}} + \epsilon\|P_2u\|_{W^{k_2,p_2}_{\delta_2,\epsilon}}\right)$$

provided $\delta_1 + \lambda \ge \delta_2$.

(*ii*) *If there exists an estimate of the form*

$$\|F_1(u)\|_{W^{k_1,p_1}} \le C_1\|F_2(u)\|_{W^{k_2,p_2}}\|F_3(u)\|_{W^{k_3,p_3}} + C_2\|F_2(u)\|_{W^{k_4,p_4}}\|F_1(u)\|_{W^{k_5,p_5}},$$

where $\frac{1}{p_1} = \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{p_4} + \frac{1}{p_5}$ $(1 \le p_1 \le p_\alpha \le \infty \quad \alpha = 2, 3, 4, 5)$*, then*

$$\begin{aligned}\|F_1(u)\|_{W^{k_1,p_1}_{\delta_1,\epsilon}} \le C\Bigg(&C_1\left(\|F_2(u)\|_{W^{k_2,p_2}_{\delta_2-1,\epsilon}} + \epsilon\|P_2u\|_{W^{k_2,p_2}_{\delta_2,\epsilon}}\right)\|F_3(u)\|_{W^{k_3,p_3}_{\delta_3,\epsilon}}\\ &+C_2\left(\|F_4(u)\|_{W^{k_4,p_4}_{\delta_4-1,\epsilon}} + \epsilon\|P_4u\|_{W^{k_4,p_4}_{\delta_4,\epsilon}}\right)\|F_5(u)\|_{W^{k_5,p_5}_{\delta_5,\epsilon}}\Bigg)\end{aligned}$$

for some constant $C > 0$ *independent of* $\epsilon \in [0, \epsilon_0]$ *provided* $\delta_1 + \lambda \ge \max\{\delta_2 + \delta_3, \delta_4 + \delta_5\}$.

Remark A.5. By using the generalized Hölder's inequality, part (ii) of Lemmas A.3 and A.4 can be extended in the obvious fashion if there exist estimates of the form

$$\|F_1(u)\|_{W^{k_1,p_1}} \le C\|F_2(u)\|_{W^{k_2,p_2}}\|F_3(u)\|_{W^{k_3,p_3}}\cdots\|F_N(u)\|_{W^{k_N,p_N}},$$

where $\frac{1}{p_1} = \sum_{i=2}^{N}\frac{1}{p_i}$ $(1 \le p_1 \le p_i \le \infty)$, F_1 is as in Lemma A.3, and F_i $(i \ge 2)$ are of the form $F_i = P_i$ or $F_i = DP_i$ with P_i a linear operator on V.

We will now use these two lemmas to extend various inequalities from the standard Sobolev spaces to the weighted ones. All of these inequalities have been derived before by various authors, see for example [1,4,7,8,25,30]. The new aspect here is that we show that the constants in the inequalities are independent of $\epsilon \ge 0$ and hence we find inequalities that interpolate between the weighted ($\epsilon > 0$) and the standard ones ($\epsilon = 0$). We begin with a weighted Hölder inequality.

Lemma A.6. *Suppose* $\epsilon_0 > 0$, $\delta_1 = \delta_1 + \delta_2$ *and* $\frac{1}{p_1} = \frac{1}{p_2} + \frac{1}{p_3}$*. Then there is a constant* $C > 0$ *independent of* $\epsilon \in [0, \epsilon_0]$ *such that*

$$\|uv\|_{L^{p_1}_{\delta_1,\epsilon}} \le C\|u\|_{L^{p_2}_{\delta_2,\epsilon}}\|v\|_{L^{p_3}_{\delta_3,\epsilon}}$$

for all $u \in L^{p_2}_{\delta_2,\epsilon}$ *and* $v \in L^{p_3}_{\delta_3,\epsilon}$.

Proof. Follows directly from Hölder's inequality and Lemma A.3. □

Next, we consider weighted versions of the Sobolev inequalities.

Lemma A.7.

(i) *For* $\epsilon_0 > 0$ *and* $k > n/p$ *there exists a constant* $C > 0$ *independent of* $\epsilon \in [0, \epsilon_0]$ *such that*

$$\|u\|_{L^\infty_{\delta,\epsilon}} \leq C\|u\|_{W^{k,p}_{\delta,\epsilon}}$$

for all $u \in W^{k,p}_{\delta,\epsilon}$. *Moreover* $u \in C^0_{\delta,\epsilon}$ *and for* $\epsilon > 0$, $u(x) = o(|x|^\delta)$ *as* $|x| \to \infty$.

(ii) *For* $\epsilon_0 > 0$ *and* $1 \leq p < n$ *there exists a constant* $C > 0$ *independent of* $\epsilon \in [0, \epsilon_0]$ *such that*

$$\|u\|_{L^{np/(n-p)}_{\delta,\epsilon}} \leq C\left(\|Du\|_{L^p_{\delta-1,\epsilon}} + \epsilon\|u\|_{L^p_{\delta,\epsilon}}\right)$$

for all $u \in W^{1,p}_{\delta,\epsilon}$.

Proof.

(i) The estimate $\|u\|_{L^\infty_{\delta,\epsilon}} \leq C\|u\|_{W^{k,p}_{\delta,\epsilon}}$ for some constant $C > 0$ independent of $\epsilon \geq 0$ follows from the usual Sobolev inequality $\|u\|_{L^\infty} \leq C\|u\|_{W^{k,p}}$ $(k > n/p)$ and Lemma A.3. Since $\|\cdot\|_{W^{k,p}_{\delta,\epsilon}}$ for $\epsilon > 0$ is equivalent to $\|\cdot\|_{W^{k,p}_{\delta,1}}$, the statement $u(x) = \text{o}(|x|^\delta)$ as $|x| \to \infty$ for $\epsilon > 0$ follows from Theorem 1.2 in [1].

(ii) Follows from Lemma A.4 and the Sobolev inequality $\|u\|_{L^{np/(n-p)}} \leq C\|Du\|_{L^p}$ which holds for all $u \in W^{1,p}$ where $1 \leq p < n$. □

In addition to the Sobolev inequalities, we will also require weighted versions of the multiplication and Moser inequalities. We first consider the multiplication inequalities.

Lemma A.8. *Suppose* $\epsilon_0 > 0$, $1 \leq p < \infty$, $k_1, k_2 \geq k_3$, $k_3 < k_1 + k_2 - n/p$, $\delta_1 + \delta_2 \leq \delta_3$, *and* $V_1 \times V_2 \to V_3 \;: (u, v) \mapsto uv$ *is a multiplication. Then there exists a constant* $C > 0$ *independent of* $\epsilon \in [0, \epsilon_0]$ *such that*

$$\|uv\|_{W^{k_3,p}_{\delta_3,\epsilon}} \leq C\|u\|_{W^{k_1,p}_{\delta_1,\epsilon}}\|v\|_{W^{k_2,p}_{\delta_2,\epsilon}}$$

for all $u \in W^{k_1,p}_{\delta_1,\epsilon}$ *and* $v \in W^{k_2,p}_{\delta_2,\epsilon}$.

Proof. This proof does not follow directly from Lemma A.3, but can be proved in a simlar fashion. To see this first recall the Sobolev mlutiplication inequality

$$\|uv\|_{W^{k_3,p}} \leq C\|u\|_{W^{k_1,p}}\|v\|_{W^{k_2,p}} \tag{A.30}$$

which holds for $1 \leq p < \infty$, $k_1, k_2 \geq k_3$, and $k_3 < k_1 + k_2 - n/p$. So

$$
\begin{aligned}
\|uv\|_{W^{k_3,p}_{\delta_3}} &= \left(\|\phi_0^\epsilon uv\|^p_{W^{k_3,p}} + \sum_{j=1}^{\infty} 2^{-p\delta_3(j-1)} \|S_j(\phi_j^\epsilon uv)\|^p_{W^{k_3,p}} \right)^{1/p} \\
&\leq C \Bigg(\|\phi_0^\epsilon u(\phi_0^\epsilon + \phi_1^\epsilon)v\|^p_{W^{k_1,p}} \\
&\quad + \sum_{j=1}^{\infty} 2^{-p\delta_3(j-1)} \|S_j(\phi_j^\epsilon u) S_j\left(\sum_{k=-1}^{1} \phi_{j+k}^\epsilon v \right)\|^p_{W^{k_1,p}} \Bigg)^{1/p} \\
&\leq C \Bigg(\|\phi_0^\epsilon u(\phi_0^\epsilon + \phi_1^\epsilon)v\|^{p/2}_{W^{k_3,p}} \\
&\quad + \sum_{j=1}^{\infty} 2^{-(p/2)\delta_3(j-1)} \|S_j(\phi_j^\epsilon u) S_j\left(\sum_{k=-1}^{1} \phi_{j+k}^\epsilon v \right)\|^{p/2}_{W^{k_1,p}} \Bigg)^{2/p} \\
&\leq C \Bigg(\|\phi_0^\epsilon u\|^{p/2}_{W^{k_1,p}} \|\phi_0^\epsilon v\|^{p/2}_{W^{k_2,p}} \\
&\quad + \sum_{j=1}^{\infty} 2^{-(p/2)\delta_1(j-1)} \|S_j(\phi_j^\epsilon u)\|^{p/2}_{W^{k_1,p}} 2^{-(p/2)\delta_2(j-1)} \|S_j(\phi_j^\epsilon v)\|_{W^{k_2,p}} \Bigg)^{2/p} \\
&\leq C \left(\|\phi_0^\epsilon u\|^p_{W^{k_1,p}} + \sum_{j=1}^{\infty} 2^{-p\delta_1(j-1)} \|S_j(\phi_j^\epsilon u)\|^p_{W^{k_1,p}} \right)^{1/p} \\
&\quad \times \left(\|\phi_0^\epsilon v\|^p_{W^{k_2,p}} + \sum_{j=1}^{\infty} 2^{-p\delta_2(j-1)} \|S_j(\phi_j^\epsilon v)\|^p_{W^{k_2,p}} \right)^{1/p} \\
&\leq C \|u\|_{W^{k_1,p}_{\delta_1}} \|v\|_{W^{k_2,p}_{\delta_2}},
\end{aligned}
$$

where in deriving the third, fourth, and fifth lines we used (A.25), (A.30), and (A.26), respectively. □

Lemma A.9.

(i) *If* $\epsilon_0 > 0$ *and* $\delta_1 \geq \max\{\delta_2 + \delta_3, \delta_4 + \delta_5\}$, *then there exists a constant* $C > 0$ *independent of* $\epsilon \in [0, \epsilon_0]$ *such that*

$$
\|uv\|_{H^k_{\delta_1,\epsilon}} \leq C \left(\|u\|_{H^k_{\delta_2,\epsilon}} \|v\|_{L^\infty_{\delta_3,\epsilon}} + \|v\|_{H^k_{\delta_4,\epsilon}} \|u\|_{L^\infty_{\delta_5,\epsilon}} \right)
$$

for all $u \in H^k_{\delta_2,\epsilon} \cap L^\infty_{\delta_5,\epsilon}$ *and* $v \in H^k_{\delta_4,\epsilon} \cap L^\infty_{\delta_3,\epsilon}$.

(ii) *If* $\epsilon_0 > 0$ *and* $\delta_1 \geq \max\{\delta_2 + \delta_3, \delta_4 + \delta_5\}$, *then there exists a constant* $C > 0$ *independent of* $\epsilon \in [0, \epsilon_0]$ *such that*

$$
\begin{aligned}
\|[D^\alpha, u]v\|_{L^2_{\delta_1-|I|,\epsilon}} \leq C \Bigg(&\left(\|Du\|_{H^{k-1}_{\delta_2-1,\epsilon}} + \epsilon \|u\|_{L^2_{\delta_2,\epsilon}} \right) \|v\|_{L^\infty_{\delta_3,\epsilon}} \\
&+ \left(\|Du\|_{L^\infty_{\delta_4-1,\epsilon}} + \epsilon \|u\|_{L^\infty_{\delta_4,\epsilon}} \right) \|v\|_{H^{k-1}_{\delta_5,\epsilon}} \Bigg)
\end{aligned}
$$

for all $|\alpha| \leq k$, $u \in H^k_{\delta_2,\epsilon} \cap W^{1,\infty}_{\delta_4,\epsilon}$ *and* $v \in H^{k-1}_{\delta_5,\epsilon} \cap L^\infty_{\delta_3,\epsilon}$.

(iii) *Suppose* $\epsilon_0 > 0$, $F \in C^\ell(V, \mathbb{R}^m)$ *is a map that satisfies* $DF \in C^{k-1}_b(V, \mathbb{R}^m)$, *and* $1 \leq |\alpha| \leq k$. *Then there exists a* $C > 0$ *independent of* $\epsilon \in [0, \epsilon_0]$ *such that*

$$\|D^\alpha F(u)\|_{L^2_{\delta-|\alpha|,\epsilon}} \leq C\|DF\|_{C^{k-1}_b}\|u\|^{k-1}_{L^\infty}\left(\|Du\|_{H^{k-1}_{\delta-1,\epsilon}} + \epsilon\|u\|_{L^2_{\delta,\epsilon}}\right)$$

for all $u \in H^k_{\delta,\epsilon} \cap L^\infty$.

(iv) *Suppose* $\epsilon_0 > 0$ *and* $F \in C^k_b(V, \mathbb{R}^m)$. *Then there exists a* $C > 0$ *independent of* $\epsilon \in [0, \epsilon_0]$ *such that*

$$\|F(u)\|_{H^k_{\delta,\epsilon}} \leq C\|F\|_{C^k_b}(1 + \|u\|^{k-1}_{L^\infty})\|u\|_{H^k_{\delta,\epsilon}}$$

for all $u \in H^k_{\delta,\epsilon} \cap L^\infty$.

Proof. Inequalities (i)–(iv) follow directly from (A.24), Lemmas A.3 and A.4, and the following standard Sobolev inequalities:

(i) $\|uv\|_{H^k} \leq C\left(\|u\|_{H^k}\|v\|_{L^\infty} + \|v\|_{H^k}\|u\|_{L^\infty}\right)$ for all $u \in H^k \cap L^\infty$ and $v \in H^k \cap L^\infty$.

(ii) $\|[D^\alpha, u]v\|_{L^2} \leq C\left(\|Du\|_{H^{k-1}}\|v\|_{L^\infty} + \|Du\|_{L^\infty}\|v\|_{H^{k-1}}\right)$ for all $|\alpha| \leq k$, $u \in H^k \cap W^{1,\infty}$ and $v \in H^{k-1} \cap L^\infty$.

(iii) Suppose $F \in C^\ell(V, \mathbb{R}^m)$ is a map that satisfies $DF \in C^{k-1}_b(V, \mathbb{R}^m)$ and $1 \leq |\alpha| \leq k$. Then $\|\partial^\alpha F(u)\|_{L^2} \leq C\|DF\|_{C^{k-1}_b}\|u\|^{k-1}_{L^\infty}\|Du\|_{H^{k-1}}$ for all $u \in H^k \cap L^\infty$.

(iv) Suppose $F \in C^k_b(V, \mathbb{R}^m)$. Then $\|F(u)\|_{H^k} \leq C\|F\|_{C^k_b}(1 + \|u\|^{k-1}_{L^\infty})\|u\|_{H^k}$ for all $u \in H^k \cap L^\infty$.

Note that we have used $\|\cdot\|_{L^\infty_{0,\epsilon}} = \|\cdot\|_{L^\infty}$. □

In addition to the Moser inequalities, we also need to know when the map $u \mapsto F(u)$ is locally Lipschitz on H^k_δ.

Lemma A.10. *Suppose* $\epsilon_0 > 0$, $F \in C^\ell_b(V, \mathbb{R})$, $F(0) = 0$, $\delta \leq 0$, *and* $k \leq \ell$, *and* $k > n/2$. *Then for each* $R > 0$ *there exists a* $C > 0$ *independent of* $\epsilon \in [0, \epsilon_0]$ *such that*

$$\|F(u_1) - F(u_2)\|_{H^k_{\delta,\epsilon}} \leq C\|u_1 - u_2\|_{H^k_{\delta,\epsilon}} \quad \textit{for all } u_1, u_2 \in B_R(H^k_{\delta,\epsilon}).$$

Proof. See the proof of Lemma B.6 in [30]. □

We conclude this section with a lemma comparing the norms $\|\cdot\|_{L^p_\delta}$ and $\|\cdot\|_{L^p_{\delta,\epsilon}}$.

Lemma A.11.

(i) *If* $\delta \leq -n/p$, $1 \leq p \leq \infty$, *and* $0 \leq \epsilon \leq 1$, *then*

$$\epsilon^{-\delta-n/p}\|u\|_{L^p_\delta} \leq \|u\|_{L^p_{\delta,\epsilon}} \leq \|u\|_{L^p_\delta}$$

for all $u \in L^p_\delta$.

(ii) If $-n/p < \delta$, $1 \le p < \infty$, *and* $0 < \epsilon \le 1$, *then*

$$\|u\|_{L^p_\delta} \le \|u\|_{L^p_{\delta,\epsilon}} \le \epsilon^{-\delta-n/p}\|u\|_{L^p_\delta}$$

for all $u \in L^p_\delta$.

Proof. (i) By assumption $0 \le \epsilon \le 1$, and so we have $\epsilon\sigma_1(x) \le \sigma_\epsilon(x) \le \sigma_1(x)$ for all $x \in \mathbb{R}^n$. By assumption $-\delta - n/p > 0$ and so we get $\epsilon^{-\delta-n/p}\sigma_1^{-\delta-n/p} \le \sigma_\epsilon^{-\delta-np} \le \sigma_1^{-\delta-np}$. Therefore, directly from the definition of the weighted norm, we find $\epsilon^{-\delta-n/p}\|u\|_{L^p_\delta} \le \|u\|_{L^p_{\delta,\epsilon}} \le \|u\|_{L^p_\delta}$. Part (ii) is proved in a similar fashion. □

B. Quasilinear Symmetric Hyperbolic Systems

In this section we establish a local existence and uniqueness theorem for a particular form of the quasilinear symmetric hyperbolic system on the weighted Sobolev spaces H^k_δ. In [30], we proved a local existence and uniqueness theorem for quasilinear parabolic systems on the H^k_δ spaces by adapting the approach of Taylor [39] (see Theorem 7.2, p. 330, and Proposition 7.7, p. 334) which is based on using mollifiers to construct a sequence of approximate solutions and then showing that the sequence converges to a true solution. Here, we will again follow the same approach for quasilinear symmetric hyperbolic systems and adapt the local existence and uniqueness theorems of Taylor (see Proposition 2.1, p. 370) to work on the weighted Sobolev spaces. We will only provide a brief sketch of the proof since the proof is very similar to the one in [30] and the details can easily be filled in by the reader. Related existence results have been derived independently in [4] using a different method.

The hyperbolic equations that we will consider are of the form

$$b^0(u, v)\partial_t v = b^j(u, v)\partial_i v + f(u, v)v + h, \tag{B.1}$$

$$v|_{t=0} = v_0, \tag{B.2}$$

where

(i) the map $u = u(t, x)$ is $\mathbb{R}^r$-valued while the maps $v = v(t, x)$ and $h = h(t, x)$ are $\mathbb{R}^m$-valued,
(ii) $b^0, b^j, f \in C^k_b(\mathbb{R}^r \times \mathbb{R}^m, \mathbb{M}_{m\times m})$ $(j = 1, \ldots, n)$,
(iii) b^0 and b^j $(j = 1, \ldots, n)$ are symmetric, and
(iv) there exists a constant $\omega > 0$ such that

$$b^0(\xi_1, \xi_2) \ge \omega \mathbb{I}_{m\times m} \quad \text{for all } (\xi_1, \xi_2) \in \mathbb{R}^r \times \mathbb{R}^m. \tag{B.3}$$

B.1. Galerkin method. Let $j \in C^\infty_0(\mathbb{R}^n)$ be any function that satisfies $j \ge 0$, $j(x) = 0$ for $|x| \ge 1$, and $\int_{\mathbb{R}^n} j(x)\, d^n x = 1$. Following the standard prescription, we construct from j the mollifier $j_\eta(x) := \eta^{-n} j(x/\eta)$ $(\eta > 0)$ and the smoothing operator

$$J_\eta(u)(x) := j_\epsilon * u(x) = \int_{\mathbb{R}^n} j_\eta(x - y)u(y)\, d^n y.$$

Following Taylor ([39], Ch. 16, Sects. 1 & 2), we first solve the approximating equation

$$b^0(u, J_\eta v_\eta)\partial_t v_\eta = J_\eta b^j(u, J_\eta v_\eta)\partial_i J_\eta v_\eta + J_\eta f(u, J_\eta v_\eta)J_\eta v_\eta + J_\eta h, \tag{B.4}$$
$$v_\eta|_{t=0} = v_0, \tag{B.5}$$

and later show that the solutions v_ϵ converge to a solution of (B.1)–(B.2) as $\eta \to 0$.

Proposition B.1. *Suppose* $T_1, T_2 > 0$, $\eta > 0$, $\delta \le \gamma \le 0$, $k > n/2$, $v_0 \in H^k_\delta$, $u \in C^0([-T_1, T_2], H^k_\gamma)$, *and* $h \in C^0([-T_1, T_2], H^k_\delta)$ *for some* $T > 0$. *Then there exists a* $T_* > 0$ ($T_* < T_1, T_2$) *and a unique* $v_\eta \in C^1((-T_*, T_*), H^k_\delta)$ *that solves the initial value problem* (B.4)–(B.5). *Moreover if* $\sup_{0\le t<T_*} \|v_\eta(t)\|_{H^k_\delta} < \infty$ *then there exists a* $T^* \in (T_*, T_2]$ *such that* v_η *extends to a unique solution on* $(-T_*, T^*)$.

Proof. Fix $\eta > 0$ and define

$$F(t, v) := (b^0(u, J_\eta v))^{-1}(J_\eta b^j(u, J_\eta v)\partial_i J_\eta v + J_\eta f(u, J_\eta v)J_\eta v + J_\eta h).$$

Then the approximating equations (B.4)–(B.5) can be written as the first order differential equation $\dot{v} = F(v)$; $v(0) = v_0$ on H^k_δ. If we can show that F is continuous and is Lipshitz in a neighborhood of v_0 in H^k_δ, then the proof follows immediately from standard existence, uniqueness, and continuation theorems for ODEs on Banach spaces.

To prove that F is locally Lipshitz, we first prove the following lemma.

Lemma B.2. *Suppose* $\delta \le \gamma \le 0$, $\ell > n/2$ *and that* $f \in C^\ell_b(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{M}_{m\times m})$. *Then for each* $u \in H^\ell_\gamma$ *and* $R > 0$ *there exists a constant* $C > 0$ *such that*

$$\|f(u, v_1)v_1 - f(u, v_2)v_2\|_{H^\ell_\delta} \le C\|v_1 - v_2\|_{H^\ell_\delta}$$

for all $v_1, v_2 \in B_R(H^\ell_\delta)$.

Proof. Let f(0,0) = c and $g(x, y) = f(x, y) - c$ so that $g(0, 0) = 0$. Then

$$f(u, v_1)v_1 - f(u, v_2)v_2 = c(v_1 - v_2) + (g(u, v_1) - g(u, v_2))v_1 + g(u, v_2)(v_1 - v_2).$$

Since $\gamma \le 0$ and $\ell > n/2$, we get from Lemma A.8 that

$$\begin{aligned}\|f(u, v_1)v_1 - f(u, v_2)v_2\|_{H^\ell_\delta} \le C &\left(1 + \|g(u, v_2)\|_{H^\ell_\gamma}\right) \|v_1 - v_2\|_{H^\ell_\delta} \\ &+\|v_1\|_{H^\ell_\delta}\|g(u, v_1) - g(u, v_2)\|_{H^\ell_\gamma}.\end{aligned}$$

By Lemma A.10, Lemmas A.7 and A.9, and (A.24), we get from the above inequality that

$$\|f(u, v_1)v_1 - f(u, v_2)v_2\|_{H^\ell_\delta} \le C(\|u\|_{H^\ell_\gamma}, \|v_1\|_{H^\ell_\delta}, \|v_2\|_{H^\ell_\delta})\|v_1 - v_2\|_{H^\ell_\delta},$$

where $P(y_1, y_2, y_3)$ is a polynomial. This proves the lemma. □

Using Lemma A.7 of [30], it is not difficult to prove the following variation of the above lemma.

Lemma B.3. *Suppose* $\delta \le \gamma \le 0$, $\eta > 0$, $\ell > n/2$ *and that* $f \in C_b^\ell(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{M}_{m\times m})$. *Then for each* $u \in H_\gamma^\ell$ *and* $R > 0$ *there exist a constant* $C > 0$ *such that*

$$\|f(u, J_\eta v_1)DJ_\eta v_1 - f(u, J_\eta v_2)DJ_\eta v_2\|_{H_\delta^\ell} \le C\|v_1 - v_2\|_{H_\delta^\ell}$$

for all $v_1, v_2 \in B_R(H_\delta^\ell)$.

The proof now follows easily from the above lemmas, Lemma A.7 of [30], and the estimates of Appendix A, which show that for any $R > 0$ the map $F : ([-T_1, T_2] \times B_R(H_{\delta-1}^k) \to H_{\delta-1}^k$ is continuous and moreover there exists a constant $C > 0$ such that $\|F(t, v_1) - F(t, v_2)\|_{H_\delta^k} \le C\|v_1 - v_2\|_{H_\delta^k}$ for all $v_1, v_2 \in B_R(H_\delta^k)$. □

B.2. Energy estimates. Fix $k > n/2 + 1$. By Proposition B.1, we have a sequence of solutions $v_\eta \in C^1([-T(\eta), T(\eta)], H_\delta^k)$ $(0 < T(\eta) \le T_1, T_2)$ to the approximating Eqs. (B.4)–(B.5). The goal is to derive bounds on v_η in the H_δ^k spaces independent of η. To do this, we use energy estimates which we now describe.

Lemma B.4. *Suppose* $a^0 \in C^1([0,\tau], W^{1,\infty})$, $a^j \in C^0([0,\tau], W^{1,\infty})$, $f \in C^0([0,\tau], L_\lambda^2)$ *and that* $w \in C^1([0,\tau], L_\lambda^2)$ *satisfies the equation*

$$a^0 \partial_t w = J_\eta a^j \partial_j J_\eta w + g.$$

Then there exists a constant $C > 0$ *independent of* $\eta > 0$ *such that*

$$\frac{d}{dt}\left\langle w|a^0 w\right\rangle_{L_\lambda^2} \le C\left[(1 + \|\mathrm{div}\, a\|_{L^\infty} + \|\vec{a}\|_{L^\infty})\,\|w\|_{L_\lambda^2}^2 + \|g\|_{L_\lambda^2}\|w\|_{L_\lambda^2}\right],$$

where $\mathrm{div}\, a = \partial_t a^0 + \partial_j a^j$ *and* $\vec{a} = (a^1, \ldots, a^n)$.

Proof. First, we have

$$\begin{aligned}\frac{d}{dt}\left\langle w|a^0 w\right\rangle_{L_\lambda^2} &= 2\left\langle w|a^0\partial_t w\right\rangle_{L_\lambda^2} + \left\langle w|\partial_t a^0 w\right\rangle_{L_\lambda^2}\\ &= 2\left\langle w|J_\eta a^j \partial_j J_\eta w\right\rangle_{L_\lambda^2} + 2\left\langle w|g\right\rangle_{L_\lambda^2} + \left\langle w|\partial_t a^0 w\right\rangle_{L_\lambda^2}.\end{aligned}$$

Letting $J_\eta^\dagger$ denote the adjoint of J_η with respect to the inner-product (A.4), we can write the above expression as

$$\frac{d}{dt}\left\langle w|a^0 w\right\rangle_{L_\lambda^2} = 2\left\langle J_\eta^\dagger w|a^j \partial_j J_\eta w\right\rangle_{L_\lambda^2} + 2\left\langle w|g\right\rangle_{L_\lambda^2} + \left\langle w|\partial_t a^0 w\right\rangle_{L_\lambda^2}. \tag{B.6}$$

Integration by parts shows that

$$\left\langle J_\eta^\dagger w|a^j \partial_j J_\eta w\right\rangle_{L_\lambda^2} = -\left\langle \partial_j J_\eta^\dagger w|a^j J_\eta w\right\rangle_{L_\lambda^2} - \left\langle J_\eta^\dagger w|(\partial_j a^j + a^j \rho^{-1}\partial_j \rho) J_\eta w\right\rangle_{L_\lambda^2}, \tag{B.7}$$

where $\rho = \sigma_1^{-2\lambda-n}$. Since $\|\rho^{-1}\partial_j\rho\|_{L^\infty} < \infty$, together Lemmas B.7 and B.8 of [30] and (B.7) imply that

$$\left\langle J_\eta^\dagger w|a^j \partial_j J_\eta w\right\rangle_{L_\lambda^2} \le -\left\langle \partial_j J_\eta w|a^j J_\eta w\right\rangle_{L_\lambda^2} + C(1 + \|\partial_i a^i\|_{L^\infty} + \|\vec{a}\|_{L^\infty})\|w\|_{L_\lambda^2}^2. \tag{B.8}$$

Again integrating by parts and using Lemma B.8 of [30], we find that

$$-\left\langle \partial_j J_\eta w | a^j J_\eta w \right\rangle_{L^2_\lambda} \le C(1 + \|\partial_i a^i\|_{L^\infty} + \|\vec{a}\|_{L^\infty})\|w\|^2_{L^2_\lambda}. \tag{B.9}$$

The proof now follows from the Cauchy-Schwartz inequality and Eqs. (B.6), (B.8), and (B.9). □

Let $v^\alpha_\eta = D^\alpha v_\eta$, $b^0_\eta = b^0(u, J_\eta v_\eta)$, $b^j_\eta = b^j(u, J_\eta v_\eta)$ and $f_\eta = f(u, J_\eta v_\eta)J_\eta v_\eta$. The evolution equation (B.4) implies that

$$\partial_t v_\eta = (b^0_\eta)^{-1} J_\eta b^j_\eta \partial_j J_\eta v_\eta + (b^0_\eta)^{-1} f_\eta + (b^0_\eta)^{-1} h. \tag{B.10}$$

Differentiating this equation yields

$$b^0_\eta \partial_t v^\alpha_\eta = J_\eta b^j_\eta \partial_j J_\eta v^\alpha_\eta + g^\alpha, \tag{B.11}$$

where

$$g^\alpha = b^0_\eta [D^\alpha, (b^0_\eta)^{-1} J_\eta b^j_\eta]\partial_j J_\eta v_\eta + b^0_\eta D^\alpha \left((b^0_\eta)^{-1} J_\eta f_\eta\right) + b^0_\eta D^\alpha \left((b^0_\eta)^{-1} J_\eta h\right). \tag{B.12}$$

To simplify the following estimates, we will assume that $b^j(0,0) = 0$. It is not difficult to treat the case where $b^j(0,0) \neq 0$. Recalling that $\delta \le \gamma \le 0$ and $k > n/2+1$, we get from the calculus inequalities of Appendix A and Lemma A.7 of [30] the following estimate

$$\begin{aligned}
\|b^0_\eta[D^\alpha, (b^0_\eta)^{-1} J_\eta b^j_\eta]\partial_j J_\eta v_\eta\|_{L^2_{\delta-|\alpha|}} &\le \|b^0_\eta\|_{L^\infty}\|[D^\alpha, (b^0_\eta)^{-1} J_\eta b^j_\eta]\partial_j J_\eta v_\eta\|_{L^2_{\delta-|\alpha|}} \\
&\le C\left(\|(b^0_\eta)^{-1} J_\eta b^j_\eta\|_{H^k_0}\|\partial_j J_\eta v_\eta\|_{L^\infty_{\delta-1}} + \|(b^0_\eta)^{-1} J_\eta b^j_\eta\|_{W^{1,\infty}}\|\partial_j J_\eta v_\eta\|_{H^{k-1}_{\delta-1}}\right) \\
&\le C\Big[(1 + (\|u\|_{L^\infty} + \|v_\eta\|_{L^\infty})^{k-1})\left(\|u\|_{H^k_0} + \|v_\eta\|_{H^k_0}\right)\|v_\eta\|_{W^{1,\infty}_\delta} \\
&\quad + (1 + \|u\|_{W^{1,\infty}} + \|v_\eta\|_{W^{1,\infty}})\|v_\eta\|_{H^k_\delta}\Big],
\end{aligned}$$

where C is independent of η. By the Sobolev inequality (Lemma A.7) we have

$$\|u\|_{W^{1,\infty}} \le C\|u\|_{H^k_\eta}, \quad \|v\|_{W^{1,\infty}} + \|v\|_{W^{1,\infty}_\delta} \le C\|v_\eta\|_{H^k_\delta},$$

and hence

$$\|b^0_\eta[D^\alpha, (b^0_\eta)^{-1} J_\eta b^j_\eta]\partial_j J_\eta v_\eta\|_{L^2_{\delta-|\alpha|}} \le P(\|u\|_{H^k_\eta}, \|v_\eta\|_{H^k_\delta})$$

for a η independent polynomial $P(y_1, y_2)$. The other terms in g^α can be estimated in a similar fashion to get

$$\|g^\alpha\|_{L^2_{\delta-|\alpha|}} \le P\left(\|u\|_{H^k_\gamma}, \|v_\eta\|_{H^k_\delta}, \|h\|_{H^k_\delta}\right), \tag{B.13}$$

where as above $P(y_1, y_2, y_3)$ is an η independent polynomial . It can also be shown using the calculus inequalities and (B.10) that

$$\|\mathrm{div}\, b\|_{L^\infty} \le P(\|u\|_{H^k_\gamma}, \|v_\eta\|_{H^k_\delta}, \|h\|_{H^k_\delta}). \tag{B.14}$$

Finally, we note that

$$\|\vec{b}\|_{L^\infty} \leq C. \tag{B.15}$$

Next, if we define

$$|||v_\eta|||^2_{k,\delta} := \sum_{|\alpha|\leq k} \left\langle D^\alpha v_\eta | b^0_\eta D^\alpha v_\eta \right\rangle_{L^2_{\delta-|\alpha|}},$$

then by (B.3) and (B.15) there exists a constant $C > 0$ independent of η such that

$$C^{-1}\|v_\eta\|_{H^k_\delta} \leq |||v_\eta|||_{k,\delta} \leq C\|v_\eta\|_{H^k_\delta}. \tag{B.16}$$

Since $\sup_{0\leq t\leq T}\|u(t)\|_{H^k_\gamma} < \infty$ and $\sup_{0\leq t\leq T]}\|h(t)\|_{H^k_\delta} < \infty$, Lemma (B.4) and (B.13), (B.14), (B.15), and (B.16) imply that

$$\frac{d}{dt}|||v_\eta|||^2_{k,\delta} \leq C(|||v_\eta|||_{k,\delta})|||v_\eta|||_{k,\delta} \tag{B.17}$$

or equivalently

$$\frac{d}{dt}|||v_\eta|||_{k,\delta} \leq P(|||v_\eta|||_{k,\delta})$$

for a polynomial $P(y)$ with positive coefficients that are independent of $\eta > 0$. By Gronwall's inequality, (B.17), and Proposition B.1, this implies that there exists constants $T_*, K > 0$, both independent of $\eta > 0$, such that $T(\eta) \geq T_*$ and

$$\sup_{0\leq t\leq T_*}\|v_\eta(t)\|_{H^k_\delta} \leq K. \tag{B.18}$$

Using the time reversed version of the equation (i.e. sending $t \mapsto -t$) we also get, shrinking T_* if necessary, that

$$\sup_{-T_*\leq t\leq 0}\|v_\eta(t)\|_{H^k_\delta} \leq K. \tag{B.19}$$

Finally, from (B.10), (B.18), (B.19), Lemma A.7 of [30], Lemmas A.7 and A.9, and (A.24), we see, increasing K if necessary, that

$$\sup_{-T_*\leq t\leq T_*}\|\partial_t v_\eta(t)\|_{H^{k-1}_\delta} \leq K. \tag{B.20}$$

B.3. Local existence and uniqueness. To get local existence following the approach of Taylor (see Theorem 1.2, p. 362 in [39]), we let $\eta \searrow 0$ and use the bounds (B.18)–(B.20) obtained from the energy estimates to extract a weakly convergent subsequence of v_η which has a limit that solves the initial value problem (B.1)–(B.2). Since the proof is very similar to that of Theorem B.2 in [30], we omit the details.

Proposition B.5. *Suppose $T_1, T_2 > 0$, $\delta \leq \gamma \leq 0$, $k > n/2+1$, $v_0 \in H^k_\delta$, $u \in C^0([-T_1, T_2], H^k_\gamma)$ and $h \in C^0([-T_1, T_2], H^k_\delta)$. Then there exists a $T_* > 0$ $T_* < min\{T_1, T_2\}$ and a $v \in L^\infty((-T_*, T_*), H^k_\delta) \cap \mathrm{Lip}((-T_*, T_*), H^{k-1}_\delta)$ that solves the initial value problem* (B.1)–(B.2).

Using the estimates of Appendix B of [30] and of Appendix A and B.2 of this paper, it is not difficult to adapt the proofs of Propositions 1.3–1.5, pp. 364–365, in [39] to get the following theorem.

Theorem B.6. *The solution v from Proposition B.5 is unique in $L^\infty((-T_1, T_2), H^k_{\mathrm{loc}}) \cap \mathrm{Lip}((-T_1, T_2), H^{k-1}_{\mathrm{loc}})$ and satisfies the additional regularity*

$$v \in C^0((-T_*, T_*), H^k_\delta) \cap C^1((-T_*, T_*), H^{k-1}_\delta).$$

Moreover, if $T_* < T_2$ and $\sup_{0 \le T < T_*} \|v(t)\|_{W^{1,\infty}} < \infty$, then there exists a $T^* \in (T_*, T_2]$ such that the solution can be extended to a solution of (B.1)–(B.2) *on $(-T_*, T^*)$.*

For linear systems, the energy estimate (see Lemma 7.1 with $\epsilon = 1$) ensures, via the continuation principle of the above theorem, that the solutions can be continued as long as the functions $u(t)$ and $h(t)$ are defined.

Proposition B.7. *Suppose $T_1, T_2 > 0$, $\delta \le \gamma \le 0$, $k > n/2 + 1$, $v_0 \in H^k_\delta$, $u \in C^0([-T_1, T_2], H^k_\gamma)$ and $h \in C^0([-T_1, T_2], H^k_\delta)$. Then the initial value problem*

$$b^0(u)\partial_t v = b^j(u)\partial_i v + f(u)v + h, \tag{B.21}$$

$$v|_{t=0} = v_0 \tag{B.22}$$

has a solution

$$v \in C^0([-T_1, T_2], H^k_\delta) \cap C^1([-T_1, T_2], H^{k-1}_\delta))$$

that is unique in $L^\infty((-T_1, T_2), H^k_{\mathrm{loc}}) \cap \mathrm{Lip}((-T_1, T_2), H^{k-1}_{\mathrm{loc}})$.

Let $[n/2]$ denote the largest integer with $[n/2] \le n/2$ and $k_0 = [n/2] + 2$. Then differentiating the solution from Theorem B.6, with respect to t, and using Proposition B.7 yields the following result.

Corollary B.8. *Suppose $k = k_0 + s$, $u \in \bigcap_{\ell=0}^s C^\ell([-T_1, T_2], H^{k-\ell}_\gamma)$ and $h \in \bigcap_{\ell=0}^s C^\ell([-T_1, T_2], H^{k-\ell}_\delta)$. Then the solution from B.5 satisfies the additional regularity*

$$v \in \bigcap_{\ell=0}^{s+1} C^\ell((-T_*, T_*), H^{k-\ell}_\delta).$$

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