# Static vacuum solutions from convergent null data expansions at space-like infinity

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## Abstract

We study formal expansions of asymptotically flat solutions to the static vacuum field equations which are determined by minimal sets of freely specifyable data referred to as 'null data'. These are given by sequences of symmetric trace free tensors at space-like infinity of increasing order. They are $1:1$ related to the sequences of Geroch multipoles. Necessary and sufficient growth estimates on the null data are obtained for the formal expansions to be absolutely convergent. This provides a complete characterization of all asymptotically flat solutions to the static vacuum field equations.

# 1   Introduction

In this article will be given a characterization of the asymptotically flat, static solutions to Einstein's vacuum field equations $Ric[\tilde{g}] = 0$. We thus consider Lorentz metrics which take in coordinates suitably adapted to a hypersurface orthogonal, time-like Killing field $K$ the form

$$\tilde{g} = v^2 \, dt^2 + \tilde{h}, \qquad v = v(x^c), \qquad \tilde{h} = \tilde{h}_{ab}(x^c) \, dx^a \, dx^b, \tag{1.1}$$

where $\tilde{h}$ denotes a negative definite metric on the time slices $\tilde{S}_c = \{t = c = const.\}$ and the Killing field is given by $K = \partial_t$. In this representation Einstein's vacuum field equations

reduce to the *static vacuum field equations*

$$R_{ab}[\tilde{h}] = \frac{1}{v}\, \tilde{D}_a\, \tilde{D}_b\, v, \qquad \Delta_{\tilde{h}}\, v = 0 \quad \text{on} \quad \tilde{S} \equiv \tilde{S}_0. \tag{1.2}$$

It will be assumed that $\tilde{S}$ is diffeomorphic to the complement of a closed ball $B_R(0)$ in $\mathbb{R}^3$ with a diffeomorphism whose components define coordinates $x^a$, $a = 1, 2, 3$, on $\tilde{S}$ in which the asymptotic flatness condition[1]

$$\tilde{h}_{ac} = \left(1 + \frac{2\,m}{|x|}\right)\delta_{ac} + O_k(|x|^{-(1+\epsilon)}), \qquad v = 1 - \frac{m}{|x|} + O_k(|x|^{-(1+\epsilon)}) \qquad \text{as} \quad |x| \to \infty, \tag{1.3}$$

is realized with some $\epsilon > 0$ and $k \geq 2$, where $|\,.\,|$ denotes the standard Euclidean norm.

Solutions to equations (1.2) satisfying the fall-off conditions (1.3) have been characterized by Reula ([23]) and Miao ([18]) in terms of boundary value problems for the static field equations where the data are prescribed on the sphere $\partial\tilde{S}$, which encompasses the asymptotic end.

Our interest in static solutions comes, however, from the observation that for vacuum solutions arising from asymptotically flat, time symmetric initial data asymptotic smoothness at null infinity appears to be related to asymptotic staticity of the data at space-like infinity ([14], [25]). To analyse this situtation we wish to control the static vacuum solutions in terms of quantities defined at space-like infinity.

Another reason for giving such a characterization results from the work by Corvino ([5], [6]), Corvino and Schoen ([7]), and Chruściel and Delay ([3], [4]). These authors deform given asymptotically flat vacuum data outside prescribed compact sets to vacuum data which are *exactly static or stationary near* or *asymptotically static or stationary at* space-like infinity and use such data to discuss the existence of null geodesically complete solutions which have a smooth asymptotic structure at null infinity. To assess the scope of these results it is desirable to have a complete description of the asymptotically flat static vacuum solutions in terms of asymptotic quantities.

A characterization of this type has been suggested by Geroch by giving a definition of multipole moments for static solutions ([16]). He assumes the metric $\tilde{h}$ to admit a smooth conformal extension in the following sense. With an additional point $i$, which is to represent space-like infinity, the set $S = \tilde{S} \cup \{i\}$ is assumed to acquire a smooth differential structure which induces on $\tilde{S}$ the given one, which makes $S$ diffeomorphic to an open ball in $\mathbb{R}^3$ with the center representing $i$, and which admits a function $\Omega \in C^2(S) \cap C^\infty(\tilde{S})$ with the properties

$$\Omega > 0 \quad \text{on} \quad \tilde{S}, \tag{1.4}$$

$$h_{ab} = \Omega^2\, \tilde{h}_{ab} \quad \text{extends to a smooth negative definite metric on } S, \tag{1.5}$$

$$\Omega = 0, \quad D_a\Omega = 0, \quad D_aD_b\Omega = -2\,h_{ab} \quad \text{at} \quad i, \tag{1.6}$$

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[1]The terms $O_k(|x|^{-(1+\epsilon)})$ behave like $O(|x|^{-(1+\epsilon+j)})$ under differentiations of order $j \leq k$.

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where $D$ denotes the covariant derivative operator defined by $h$. We note that these conditions are preserved under rescalings $h \to \vartheta^4 h$, $\Omega \to \vartheta^2 \Omega$ with smooth positive functions $\vartheta$ satisfying $\vartheta(i) = 1$.

With these assumptions Geroch defines a sequence of tensor fields $P$, $P_a$, $P_{a_2 a_1}$, ... near $i$ by setting[2]

$$P = \Omega^{-1/2} (1 - v), \quad P_a = D_a P, \quad P_{a_2 a_1} = \mathcal{C}(D_{a_2} P_{a_1} - \frac{1}{2} P R_{a_2 a_1}),$$

$$P_{a_{p+1}...a_1} = \mathcal{C}(D_{a_{p+1}} P_{a_p...a_1} - c_p P_{a_{p+1}...a_3} R_{a_2 a_1}), \quad \text{with} \quad c_p = \frac{p(2p-1)}{2}, \quad p = 2, 3, \ldots,$$

where $R_{ab}$ denotes the Ricci tensor of $h_{ab}$ and $\mathcal{C}$ the projector onto the symmetric, trace free part of the respective tensor fields. The multipole moments are then defined as the tensors

$$\nu = P(i), \qquad \nu_{a_p...a_1} = P_{a_p...a_1}(i), \quad p = 1, 2, 3, \ldots,$$

at $i$. Setting aside the monopole $\nu$, we will denote the remaining series of multipoles by

$$\mathcal{D}_{mp} = \{\nu_{a_1}, \nu_{a_2 a_1}, \nu_{a_3 a_2 a_1}, \ldots\}. \tag{1.7}$$

The problem of characterizing solutions to a quasi-linear, gauge-elliptic system of equations of the type (1.2) by a minimal set of data given at an ideal point representing space-like infinity is unusual and certainly quite different from a standard boundary value problem for (1.2). There are available some results which go into this direction but little has been done on the general question of existence.

Müller zum Hagen has shown that solutions $v$, $\tilde{h}_{ab}$ to (1.2) are real analytic in $\tilde{h}$-harmonic coordinates ([20]). The question to what extent the multipoles introduced above determine the metric $h_{ab}$ and the function $v$ raises the question whether this metric is real analytic even at $i$ in suitable coordinates and conformal scalings. Beig and Simon ([2]) have shown (under assumptions which have been relaxed later by Kennefick and O'Murchadha [17]) that the rescaled metric does indeed extend in a suitable gauge as a real analytic metric to $i$ if it is assumed that the ADM mass satisfies

$$m \neq 0. \tag{1.8}$$

We shall assume this result in the following and shall not go through the argument again, though its structural basis will be pointed out in passing. Beig and Simon also provide an argument which esssentially shows that given sets of multipoles determine unique formal expansions of 'formal solutions' to the static vacuum field equations.

For axisymmetric static vacuum solutions, which are special in admitting explicit descriptions ([26]), the question under which assumptions a sequences of multipoles does indeed determine a converging expansion of a static solution has been studied by Bäckdahl and Herberthson ([1]). For the general case, for which the freedom to prescribe data is much larger, this problem has never been analysed. For this reason the results referred to above remained essentially of heuristic value.

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[2]We depart from the convention of [16] by changing the sign of $P$.

It is the purpose of this article to derive, under the assumption (1.8), necessary and sufficient conditions for certain minimal sets of asymptotic data, denoted collectively by $\mathcal{D}_n$ and referred to as *null data*, to determine (unique) real analytic solutions and thus to provide a complete characterization of all possible asymptotically flat solutions to the static vacuum field equations. The behaviour of these solutions in the large will not be studied here. We shall only be interested in what could be called 'germs of static solutions at space-like infinity', for which $S$ may comprise only a neighbourhood of the point $i$ which is quite small in terms of $h$ (in terms of $\tilde{h}$ they cover still infinite domains extending to space-like infinity).

While the multipoles above are defined for any conformal gauge, it will be concenient for our analysis to remove the conformal gauge freedom. As shown below, the metric $h = \Omega^2\, \tilde{h}$ defined with the *preferred gauge*

$$\Omega = \left(\frac{1-v}{m}\right)^2,$$

on a suitable neighbourhood $\tilde{S}$ of space-like infinity, can be extended with (1.4) - (1.6) in suitable coordinates to a real analytic metric at $i$. The metric so obtained satisfies $R[h] = 0$ on $S$. In this gauge we get with the notation above

$$P = m, \quad P_a = 0 \quad P_{a_2 a_1} = -\frac{m}{2}\, s_{a_2 a_1}, \tag{1.9}$$

$$P_{a_{p+1}\ldots a_1} = \mathcal{C}(D_{a_{p+1}} P_{a_p \ldots a_1} - c_p\, P_{a_{p+1}\ldots a_3}\, s_{a_2 a_1}), \quad p = 2, 3, \ldots, \tag{1.10}$$

where $s_{ab}$ denotes the trace free part of the Ricci tensor of $h$. In the given gauge we consider now the set

$$\mathcal{D}_n = \{s_{a_2 a_1}(i),\ \mathcal{C}(D_{a_3} s_{a_2 a_1})(i),\ \mathcal{C}(D_{a_4} D_{a_3} s_{a_2 a_1})(i),\ \mathcal{C}(D_{a_5} D_{a_4} D_{a_3} s_{a_2 a_1})(i),\ \ldots\ldots\ \}.$$

Given $m \neq 0$ and the sequence $\mathcal{D}_n$ associated with $h$, one calculate the multipoles $\mathcal{D}_{mp}$ of $h$ and vice versa. The sets $\mathcal{D}_n$ and $\mathcal{D}_{mp}$ thus carry the same information, but $\mathcal{D}_n$ is easier to work with because the expressions are linear in the curvature.

Let now $c_{\mathbf{a}}$, $\mathbf{a} = 1, 2, 3$, be an $h$-orthonormal frame field near $i$ which is $h$-parallely propagate along the geodesics through $i$ and denote the covariant derivative in the direction of $c_{\mathbf{a}}$ by $D_{\mathbf{a}}$. We express the tensors in $\mathcal{D}_n$ in terms of this frame and write

$$\mathcal{D}_n^* = \{s_{\mathbf{a}_2 \mathbf{a}_1}(i),\ \mathcal{C}(D_{\mathbf{a}_3} s_{\mathbf{a}_2 \mathbf{a}_1})(i),\ \mathcal{C}(D_{\mathbf{a}_4} D_{\mathbf{a}_3} s_{\mathbf{a}_2 \mathbf{a}_1})(i),\ \mathcal{C}(D_{\mathbf{a}_5} D_{\mathbf{a}_4} D_{\mathbf{a}_3} s_{\mathbf{a}_2 \mathbf{a}_1})(i),\ \ldots\ \}. \tag{1.11}$$

We note that these tensors are defined uniquely up to a rigid rotation $c_{\mathbf{a}} \to s^{\mathbf{c}}{}_{\mathbf{a}}\, c_{\mathbf{c}}$ with $(s^{\mathbf{c}}{}_{\mathbf{a}}) \in O(3, \mathbb{R})$. These data will be referred to as the *null data of $h$ in the frame $c_{\mathbf{a}}$*.

It will be shown that if these data are derived from an real analytic metric $h$ near $i$ there exist constants $M, r > 0$ so that the components of these tensors satisfy the *Cauchy estimates*

$$|\mathcal{C}(D_{\mathbf{a}_p} \ldots D_{\mathbf{a}_1} s_{\mathbf{b}\, \mathbf{c}})(i)| \leq \frac{M\, p!}{r^p}, \qquad \mathbf{a}_p, \ldots, \mathbf{a}_1, \mathbf{b}, \mathbf{c} = 1, 2, 3, \qquad p = 0, 1, 2, \ldots\ .$$

Conversely, we get the following existence result.

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**Theorem 1.1** *Suppose $m \neq 0$ and*

$$\hat{\mathcal{D}}_n = \{\psi_{\mathbf{a}_2\mathbf{a}_1}, \ \psi_{\mathbf{a}_3\mathbf{a}_2\mathbf{a}_1}, \ \psi_{\mathbf{a}_4\mathbf{a}_3\mathbf{a}_2\mathbf{a}_1}, \ \dots \}, \tag{1.12}$$

*is a infinite sequence of symmetric, trace free tensors given in an orthonormal frame at the origin of a 3-dimensional Euclidean space. If there exist constants $M, r > 0$ such that the components of these tensors satisfy the estimates*

$$|\psi_{\mathbf{a}_p \ \dots \ \mathbf{a}_1 \mathbf{b} \mathbf{c}}| \leq \frac{M \, p!}{r^p}, \qquad \mathbf{a}_p, \dots, \mathbf{a}_1, \mathbf{b}, \mathbf{c} = 1, 2, 3, \qquad p = 0, 1, 2, \dots,$$

*then there exists a germ $(\tilde{h}, v)$ of an analytic, asymptotically flat, static vacuum solution at space-like infinity with ADM mass $m$, unique up to isometries, so that the null data implied by $h = \left( \frac{m}{1-v} \right)^4 \tilde{h}$ in a suitable frame $c_{\mathbf{a}}$ as described above satisfy*

$$\mathcal{C}(D_{\mathbf{a}_q} \ \dots \ D_{\mathbf{a}_3} s_{\mathbf{a}_2\mathbf{a}_1})(i) = \psi_{\mathbf{a}_q \ \dots \ \mathbf{a}_1}, \qquad q = 2, 3, 4, \dots \ .$$

A series of data of the form (1.12) (not necessarily satisfying any estimates) will in the following be referred to as *abstract null data*. The type of estimate imposed here on the abstract null data does not depend on the orthonormal frame in which they are given (cf. the discussion leading to (7.30)). Since these estimates are necessary as well as sufficient, all possible germs of asymptotically flat static vacuum solutions at space-like infinity are characterized by this result.

The proof of the result above will be given in terms of the conformal metric $h_{ab}$. For this purpose equations (1.2) are reexpressed in chapter 2 as 'conformal static vacuum field equations' for $h_{ab}$ and fields derived from $h_{ab}$ and $v$. In chapter 3 it is shown by a direct argument that in a certain setting a set of abstract null data defines the expansion coefficients of a formal expansion of a solution to these equations uniquely. Showing the convergence of the series so obtained appears difficult, however. Using the analyticity of the solutions to the conformal static vacuum field equations at the point $i$, we study in chapter 4 their analytic extensions into the complex domain. Denote by $\mathcal{N}_i$ the 'cone' with vertex at $i$ generated by the complex null geodesics through the point $i$. The null data are then represented by a function on $\mathcal{N}_i$, the component of the Ricci tensor obtained by contracting it with the null vector tangent to $\mathcal{N}_i$. In this setting the original problem assumes the form of a characteristic initial value problem with data prescribed on $\mathcal{N}_i$.

We wish to obtain the equations in a form which allows us to derive from prescribed estimates on the null data appropriate estimates on the expansion coefficients. This requires a choice of gauge which is suitably adapted to $\mathcal{N}_i$. Because of the vertex, any such gauge will necessarily be singular at a certain subset of the manifold. The manifold $\hat{S}$ considered in chapter 4 organizes the singularity in a geometric way. In chapter 5 the conformal static vaccum field equations are considered on $\hat{S}$, and it is shown how to determine a formal solution to the complete set of conformal field equations from a given set of abstract null data. The convergence of the series so obtained is shown in chapter 6. Making use of the Lemmas proven in the previous chapters, this result is translated in chapter 7 into a gauge which is regular near $i$ and allows us to prove Theorem 1.1. A translation of the estimates on the null data into equivalent estimates on the multipoles and a generalization of the present result to stationary solutions will be discussed elsewhere.

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# 2 The static field equations in the conformal setting

The existence problem will be analysed completely in terms of the conformally rescaled metric. We begin by describing the conformal gauge and then express the static field equations in terms of the conformal fields. This discussion follows essentially that of [12] and [14].

## 2.1 The choice of the conformal gauge

Consider a situation as described by conditions (1.4) - (1.6). If the metric $\tilde{h}$ is asymptotically flat and has vanishing Ricci scalar $R[\tilde{h}]$ on $\tilde{S}$ the function $\Omega$ satisfies (cf. [14])

$$(\Delta_h - \frac{1}{8}R[h])\,(\Omega^{-1/2}) = 0 \quad \text{on} \quad \tilde{S} \quad \text{and} \quad r\,\Omega^{-1/2} \to 1 \quad \text{as} \quad r \to 0,$$

where $r$ denotes the $h$-distance from $i$. Sufficiently close to $i$ one obtains the representation

$$\Omega^{-1/2} = \zeta^{-1/2} + W,$$

with smooth functions $\zeta$ and $W$ satisfying

$$(\Delta_h - \frac{1}{8}R[h])\,W = 0, \tag{2.1}$$

and

$$\zeta(i) = 0, \quad D_a\zeta(i) = 0, \quad D_aD_b\zeta(i) = -2\,h_{ab}. \tag{2.2}$$

The functions $\zeta$ and $W$ are real analytic if the metric $h$ is real analytic. In [2] Beig and Simon consider static vacuum metrics in the form

$$\tilde{g} = e^{2\,U}\,dt^2 + e^{-2\,U}\,\hat{h}_{ab}\,dx^a\,dx^b,$$

related to (1.1) by $v = e^U$ and $\hat{h}_{ab} = v^2\,\tilde{h}_{ab}$, and show that the function $\omega = (U/m)^2$ and the metric

$$h'_{ab} = \omega^2\,\hat{h}_{ab} = \Omega'^{\,2}\,\tilde{h}_{ab} \quad \text{with} \quad \Omega' = \omega\,e^U, \tag{2.3}$$

extend in $h'$-harmonic coordinates near $i$ to real analytic fields at $i$ so that $\Omega'$ satisfies requirements (1.4) - (1.6) with the $h'$-covariant derivative operator $D'$.

It follows that $\Omega'^{\,-1/2} = \zeta'^{\,-1/2} + W'$ with $\zeta' = \frac{\omega}{\cosh^2(U/2)}$ and $W' = \frac{m}{2}\frac{\sinh(U/2)}{U/2}$ ([12]). Assume $S$ to be chosen so that $U \neq 0$ on $\tilde{S}$. Rescaling with $\vartheta = W'/W'(i) > 0$ on $S$ gives

$$h = \vartheta^4\,h' = \Omega^2\,\tilde{h} \quad \text{with} \quad \Omega = \vartheta^2\,\Omega',$$

where the conformal factor can be written

$$\Omega = \left(\frac{1-v}{m}\right)^2 \quad \text{on} \quad S. \tag{2.4}$$

Because of equations (2.1) the metric $h$ has then vanishing Ricci scalar

$$R[h] = 0 \quad \text{on} \quad S, \tag{2.5}$$

and it follows that

$$\Omega^{-1/2} = \zeta^{-1/2} + W, \tag{2.6}$$

where

$$W = \frac{m}{2}, \qquad \zeta = \frac{1}{\mu}\left(\frac{1-v}{1+v}\right)^2 \quad \text{with} \quad \mu = \frac{m^2}{4}. \tag{2.7}$$

The fields $h$ and $\zeta$ are real analytic on $S$ and the functions $W$ and $\zeta$ satisfy (2.1), (2.2). In the following the gauge (2.4) and thus (2.5) - (2.7) will be assumed.

## 2.2   The conformal static vacuum field equations

The function $\zeta$ satisfies on $S$ the equation

$$\Delta_h\left(\zeta^{-1/2}\right) = 4\,\pi\,\delta_i, \tag{2.8}$$

where $\delta_i$ denotes the Dirac distribution with weight 1 at $i$. This equation implies

$$2\,\zeta\,s = D_a\zeta\,D^a\zeta \quad \text{on} \quad S \quad \text{with} \quad s = \frac{1}{3}\,\Delta_h\,\zeta, \tag{2.9}$$

which, together with (2.2), implies in turn the equation above. The function $\zeta^{-1/2}$ can be characterized as a fundamental solution of $\Delta_h$ with pole at $i$ so that $\zeta$ is real analytic on $S$ and satisfies (2.2). It is uniquely determined by $h$ because the expansion coefficients of $\zeta$ in $h$-normal coordinates centered at $i$ are recursively determined by (2.2), (2.9).

We derive now a representation of the static vacuum field equations (1.2) in terms of the conformal metric $h$ and fields derived from it. With (2.5) follows

$$R_{ab}[h] = s_{ab}, \tag{2.10}$$

where $s_{ab}$ is a trace free symmetric tensor field. The first of equations (1.2) implies in the gauge (2.4)

$$0 = \Sigma_{ab} \equiv D_a\,D_b\,\zeta - s\,h_{ab} + \zeta\,(1 - \mu\,\zeta)\,s_{ab}, \tag{2.11}$$

with $s$ as in (2.9). With the Bianchi identity $D^a s_{ab} = 0$ the integrability conditions

$$0 = \frac{1}{2}\,D^c\,\Sigma_{ca}, \qquad 0 = \frac{1}{\zeta}\left(D_{[c}\,\Sigma_{a]b} + \frac{1}{2}\,D^d\,\Sigma_{d[c}\,h_{a]b}\right)$$

for the overdetermined system (2.11) take the form

$$0 = S_a \equiv D_a\,s + (1 - \mu\,\zeta)\,s_{ab}\,D^b\,\zeta, \tag{2.12}$$

and

$$0 = H_{cab} \equiv (1 - \mu\,\zeta)\,D_{[c}s_{a]b} - \mu\,(2\,D_{[c}\zeta\,s_{a]b} + D^d\,\zeta\,s_{d[c}\,h_{a]b}). \tag{2.13}$$

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We note that this can be read as an expression of the Cotton tensor $B_{bca} = D_{[c}R_{a]b} - \frac{1}{4}D_{[c}R\,h_{a]b}$ in terms of the undifferentiated curvature. Its dualized version reads by (2.13)

$$B_{ab} = \frac{1}{2}\,B_{acd}\,\epsilon_b{}^{cd} = \frac{\mu}{1-\mu\,\zeta}(s_{da}\,\epsilon_b{}^{cd}D_c\zeta - \frac{1}{2}\,s_{de}\,\epsilon_{ba}{}^d D^e\zeta). \tag{2.14}$$

Equations (2.10), (2.11), (2.12), (2.13) together with conditions (2.2), which imply

$$s(i) = -2, \tag{2.15}$$

will be referred to as the *conformal static vacuum field equations* for the unknown fields

$$h, \ \zeta, \ s, \ s_{ab}. \tag{2.16}$$

The second of equations (1.2) implies that $R[\tilde{h}] = 0$ and can thus also be read as the conformally covariant Laplace equation for $v$. With the conformal covariance of the latter and (2.4), (2.5), (2.7), its conformal version reduces to (2.8). The identity

$$D_a(2\,\zeta\,s - D_c\,\zeta\,D^c\,\sigma) = 2\,\zeta\,S_a - 2\,\Sigma_{ac}\,D^c\zeta,$$

shows that (2.9), whence (2.8), is a consequence of equations (2.2) and (2.11). It follows that for given $m \neq 0$, which defines $W$ and $\mu$, a solution of the conformal static vacuum field equations provides a unique solution to the static vacuum field equations (1.2).

The system (2.10), (2.11), (2.12), (2.13) represents a quasi-linear, overdetermined, gauge-elliptic system of PDE's. The Ricci operator becomes elliptic in a suitable gauge and the elliptic character of the remaining equations can be seen by taking the trace of (2.11), by contracting (2.12) with $D^a$, and by contracting (2.13) with $D^c$ and using the Bianchi identity and (2.11) again so that in all three cases one obtains an equation with the Laplacian acting on the respective unknown. By deducing from the fall-off behaviour of the physical solution at space-like infinity a certain minimal smoothness of the conformal fields at $i$ and invoking a general theorem of Morrey ([19]) on elliptic systems of this type, Beig and Simon ([2]) concluded that the solutions are in fact real analytic at $i$. To avoid introducing additional constraints by taking derivatives, we shall deal with the system of first order above.

## 3  The exact sets of equations argument

Constructing solutions from minimal sets of data prescribed at $i$ poses quite an unusual problem for a system of the type of the static conformal field equations. To see how it might be done, we study expansions of the fields in normal coordinates.

For convenience assume in the following $S$ to coincide with a convex $h$-normal neighbourhood of $i$. Let $c_{\mathbf{a}}$, $\mathbf{a} = 1, 2, 3$, be an $h$-orthonormal frame field on $S$ which is parallely transported along the $h$-geodesics through $i$ and let $x^a$ denote normal coordinates centered at $i$ so that $c^b{}_{\mathbf{a}} \equiv\, < dx^b, c_{\mathbf{a}} >\, = \delta^b{}_{\mathbf{a}}$ at $i$. We refer to such a frame as *normal frame centered at $i$*. Its dual frame will be denoted by $\chi^{\mathbf{c}} = \chi^{\mathbf{c}}{}_b\,dx^b$.

At the point with coordinates $x^a$ the coefficients of the frame then satisfy

$$c^b{}_{\mathbf{a}}\,x^a = \delta^b{}_{\mathbf{a}}\,x^a, \qquad x_b\,c^b{}_{\mathbf{a}} = x_b\,\delta^b{}_{\mathbf{a}},$$

(where we set $x_a = x^b \delta_{ba}$ and assume, as in the following, that the summation rule does not distinguish between bold face and other indices). Equivalently, the coefficients of the dual frame satisfy

$$\chi^{\mathbf{a}}{}_b \, x^b = \delta^{\mathbf{a}}{}_b \, x^b, \qquad x_a \, \chi^{\mathbf{a}}{}_b = x_a \, \delta^{\mathbf{a}}{}_b, \tag{3.1}$$

which implies with the coordinate expression $h_{ab} = -\delta_{\mathbf{ac}} \, \chi^{\mathbf{a}}{}_b \, \chi^{\mathbf{c}}{}_d$ of the metric the well known characterization $x^a \, h_{ab} = -x^a \, \delta_{ab}$ of the $x^a$ as $h$-normal coordinates centered at $i$. In the following all tensor fields, except the frame field $c_{\mathbf{a}}$ and the coframe field $\chi^{\mathbf{c}}$, will be expressed in terms of this frame field, so that the metric is given by $h_{\mathbf{ab}} \equiv h(c_{\mathbf{a}}, c_{\mathbf{c}}) = -\delta_{\mathbf{ab}}$. With $D_{\mathbf{a}} \equiv D_{c_{\mathbf{a}}}$ the connection coefficients with respect to $c_{\mathbf{a}}$ are defined by $D_{\mathbf{a}} c_{\mathbf{c}} = \Gamma_{\mathbf{a}}{}^{\mathbf{b}}{}_{\mathbf{c}} \, c_{\mathbf{b}}$.

An analytic tensor field $T_{\mathbf{a}_1 \ldots \mathbf{a}_k}$ on $S$ has in the normal coordinates $x^a$ a *normal expansion* at $i$, which can be written (cf. [13])

$$T_{\mathbf{a}_1 \ldots \mathbf{a}_k}(x) = \sum_{p \geq 0} \frac{1}{p!} \, x^{c_p} \ldots x^{c_1} \, D_{\mathbf{c}_p} \ldots D_{\mathbf{c}_1} \, T_{\mathbf{a}_1 \ldots \mathbf{a}_k}(i). \tag{3.2}$$

(This is a convenient short version of the correct expression; more precisely, the $x^a$ should be replaced here by the components of the vector field $X$ which has in normal coordinates the expansion $X(x) = x^b \, \delta^{\mathbf{a}}{}_b \, c_{\mathbf{a}}$ and which is characterized by the conditions $D_V V = 0$, $V(i) = 0$.) In the following will be shown how normal expansions can be obtained for solutions

$$h_{\mathbf{ab}}, \; \zeta, \; s, \; s_{\mathbf{ab}}, \tag{3.3}$$

to the conformal static vacuum field equations. In 3 dimensions the curvature tensor satisfies

$$R_{abcd}[h] = 2\{h_{a[c} L_{d]b} + h_{b[d} L_{c]a}\} \quad \text{with} \quad L_{ab}[h] = R_{ab}[h] - \frac{1}{4} \, R[h] \, h_{ab},$$

and can be expressed because of (2.5) completely in terms of $s_{\mathbf{ab}}$. Once the latter is known, the connection coefficients $\Gamma_{\mathbf{a}}{}^{\mathbf{b}}{}_{\mathbf{c}}$ and the coefficients of the 1-forms $\chi^{\mathbf{a}}$ can thus be obtained, order by order, from the structural equations in polar coordinates (cf. [8]),

$$\frac{d}{ds} \left( s \, \chi^{\mathbf{a}}{}_b(s\,x^f) \right) = \delta^{\mathbf{a}}{}_b + \Gamma_{\mathbf{c}}{}^{\mathbf{a}}{}_{\mathbf{d}}(s\,x^f) \, s \, \chi^{\mathbf{c}}{}_b(s\,x^f) \, x^d,$$

$$\frac{d}{ds} \left( \Gamma_{\mathbf{a}}{}^{\mathbf{c}}{}_{\mathbf{e}}(s\,x^f) \, s \, \chi^{\mathbf{a}}{}_b(s\,x^f) \right) = R^{\mathbf{c}}{}_{\mathbf{eda}}(s\,x^f) \, x^d \, s \, \chi^{\mathbf{a}}{}_b(s\,x^f),$$

where $s$ denotes an affine parameter along the $h$-geodesics through $i$ with unit tangent vectors which vanishes at $i$, so that $s^2 = \delta_{ab} \, x^a \, x^b$.

By formally taking covariant derivatives, the expansion coefficients of $\zeta$ and $s$ up to order $m + 2$ resp. $m + 1$ can be obtained from equations (2.11) and (2.12) once $s_{\mathbf{ab}}$ is known up to order $m$. Calculating the expansion coefficients for $s_{\mathbf{ab}}$ by means of equation (2.13) leads, however, to some complicated algebra. It turns out that the latter simplifies considerably in the space spinor formalism.

To achieve the transition to the space-spinor formalism we introduce the constant van der Waerden symbols

$$\alpha^{AB}{}_a, \quad \alpha^a{}_{AB}, \quad a = 1, 2, 3, \quad A, B = 0, 1,$$

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which map one-index objects onto two-index objects which are symmetric in the two indices. If the latter are read as matrices, the symbols are given by

$$\xi^a \to \xi^{AB} = \alpha^{AB}{}_a \xi^a = \frac{1}{\sqrt{2}} \begin{pmatrix} -\xi^1 - i\xi^2 & \xi^3 \\ \xi^3 & \xi^1 - i\xi^2 \end{pmatrix},$$

$$\xi_a \to \xi_{AB} = \xi_a \alpha^a{}_{AB} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\xi_1 + i\xi_2 & \xi_3 \\ \xi_3 & \xi_1 + i\xi_2 \end{pmatrix}.$$

With the summation rule also applying to capital indices one gets

$$\delta^c{}_a = \alpha^c{}_{AB} \, \alpha^{AB}{}_a, \qquad - \delta_{ab} \alpha^a{}_{AB} \, \alpha^b{}_{CD} = -\epsilon_{A(C} \epsilon_{D)B} \equiv h_{ABCD},$$

$$a, b = 1, 2, 3, \quad A, B, C, D = 0, 1,$$

where the constant $\epsilon$-spinor is antisymmetric, $\epsilon_{AB} = -\epsilon_{BA}$, and satisfies $\epsilon_{01} = 1$. It is used to pull indices according to the rules $\iota_B = \iota^A \epsilon_{AB}$, $\iota^A = \epsilon^{AB} \iota_B$, so that $\epsilon_A{}^B$ corresponds to the Kronecker delta. We shall denote the 'scalar product' $\kappa_A \iota^A$ of two spinors $\kappa^A$ and $\iota^A$ occasionally also by $\epsilon(\kappa, \iota)$. It is important here to observe the order in which the spinors occur.

Given the van der Waerden symbols, we associate with an tensor field $T^{\mathbf{a}_1 \ldots \mathbf{a}_p}{}_{\mathbf{b}_1 \ldots \mathbf{b}_q}$ *given in the frame* $c_{\mathbf{a}}$ the space spinor field

$$T^{A_1 B_1 \ldots A_p B_p}{}_{C_1 D_1 \ldots C_q D_q} = T^{\mathbf{a}_1 \ldots \mathbf{a}_p}{}_{\mathbf{b}_1 \ldots \mathbf{b}_q} \alpha^{A_1 B_1}{}_{a_1} \ldots \ldots \alpha^{b_q}{}_{C_q D_q}$$

$$= T^{(A_1 B_1) \ldots (A_p B_p)}{}_{(C_1 D_1) \ldots (C_q D_q)}.$$

In the following we shall employ tensor or spinor notation as it appears convenient. With the spinor field

$$\tau^{AA'} = \epsilon_0{}^A \epsilon_{0'}{}^{A'} + \epsilon_1{}^A \epsilon_{1'}{}^{A'},$$

and the notation

$$\xi^+_{AB \ldots H} = \tau_A{}^{A'} \tau_B{}^{B'} \ldots \tau_H{}^{H'} \bar{\xi}_{A' B' \ldots H'},$$

where the bar denotes complex conjugation, one finds that a space spinor field

$$T_{A_1 B_1 \ldots A_p B_p} = T_{(A_1 B_1) \ldots (A_p B_p)},$$

arises from a real tensor field $T_{\mathbf{a}_1 \ldots \mathbf{a}_p}$ if and only if it satisfies the reality condition

$$T_{A_1 B_1 \ldots A_p B_p} = (-1)^p T^+_{A_1 B_1 \ldots A_p B_p}. \tag{3.4}$$

It follows in particular

$$\xi_{AB} \xi^{AB} = 2 \left( \xi_{00} \xi_{11} - \xi_{01} \xi_{01} \right) = 2 \det(\xi_{AB}) = -\delta_{ab} \xi^a \xi^b,$$

and we can have $\xi_{AB} \xi^{AB} = 0$ for vectors $\xi^{AB} \neq 0$ only if $\xi^a$ is complex. Since $\xi^{AB} = \xi^{(AB)}$, the relations $\xi_{AB} \xi^{AB} = 0$, $\xi^{AB} \neq 0$ imply by the equation above that $\xi^{AB} = \kappa^A \kappa^B$ for some $\kappa^A \neq 0$. This fact will allow us to interpret the data (1.11) as 'null data'.

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Any spinor field $T_{ABC...GH}$, symmetric or not, admits a decomposition into products of totally symmetric spinor fields and epsilon spinors which can be written schematically in the form (cf. [21])

$$T_{ABC...GH} = T_{(ABC...GH)} + \sum \epsilon's \times symmetrized \ contractions \ of \ T. \qquad (3.5)$$

Later on it will be important for us that spinor fields $T_{A_1 B_1 ... A_p B_p}$ arising from tensor fields $T_{\mathbf{a}_1 ... \mathbf{a}_p}$ satisfy

$$T_{(A_1 B_1 ... A_p B_p)} = \mathcal{C}(T_{\mathbf{a}_1 ... \mathbf{a}_p}) \alpha^{a_1}{}_{A_1 B_1} \cdots \alpha^{a_p}{}_{A_p B_p},$$

i.e. the projectors $\mathcal{C}$ onto the trace free symmetric part of tensors is represented in the space spinor notation simply by symmetrization. If convenient we shall denote the latter also by the symbol $sym$.

To discuss vector analysis in terms of spinors, a complex frame field and its dual 1-form field are defined by

$$c_{AB} = \alpha^a{}_{AB} \, c_{\mathbf{a}}, \qquad \chi^{AB} = \alpha^{AB}{}_a \, \chi^{\mathbf{a}},$$

so that $h(c_{AB}, c_{AB}) = h_{ABCD}$. If the derivative of a function $f$ in the direction of $c_{AB}$ is denoted by $c_{AB}(f) = f_{,a} \, c^a{}_{AB}$ and the spinor connection coefficients are defined by

$$\Gamma_{AB}{}^C{}_D = \frac{1}{2} \Gamma_{\mathbf{a}}{}^{\mathbf{b}}{}_{\mathbf{c}} \, \alpha^a{}_{AB} \, \alpha^{CH}{}_b \, \alpha^c{}_{DH}, \quad \text{so that} \quad \Gamma_{ABCD} = \Gamma_{(AB)(CD)},$$

the covariant derivative of a spinor field $\iota^A$ is given by

$$D_{AB} \, \iota^C = e^a{}_{AB}(\iota^C) + \Gamma_{AB}{}^C{}_B \, \iota^B.$$

If it is required to satisfies the Leibniz rule with respect to tensor products, it follows that covariant derivatives in the $c_{\mathbf{a}}$-frame formalism translate under contractions with the van der Waerden symbols into spinor covariant derivatives and vice versa.

The commutator of covariant spinor derivatives satisfies

$$(D_{CD} D_{EF} - D_{EF} D_{CD}) \, \iota^A = R^A{}_{BCDEF} \, \iota^B, \qquad (3.6)$$

with the curvature spinor

$$R_{ABCDEF} = \frac{1}{2} \left\{ (s_{ABCE} - \frac{R[h]}{6} h_{ABCE}) \epsilon_{DF} + (s_{ABDF} - \frac{R[h]}{6} h_{ABDF}) \epsilon_{CE} \right\},$$

where $R[h]$ is the Ricci scalar and $s_{ABCD} = s_{\mathbf{ab}} \alpha^a{}_{AB} \alpha^b{}_{CD}$ represents the trace free part of the Ricci tensor of $h$, which is completely symmetric, $s_{ABCD} = s_{(ABCD)}$. The gauge condition (2.5) implies

$$R_{ABCDEF} = \frac{1}{2} \left( s_{ABCE} \epsilon_{DF} + s_{ABDF} \epsilon_{CE} \right). \qquad (3.7)$$

In the space-spinor formalism equations (2.13) acquire the concise form

$$D_A{}^E s_{BCDE} = \frac{2 \, \mu}{1 - \mu \, \zeta} \, s_{E(BCD} D_{A)}{}^E \zeta. \qquad (3.8)$$

Applying to this equation and to the spinor versions of equations (2.11) and (2.12) the theory of 'exact sets of fields' discussed in [21], we get the following result.

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**Lemma 3.1** *Let there be given a sequence*

$$\hat{\mathcal{D}}_n = \{\psi_{A_2 B_2 A_1 B_1}, \ \psi_{A_3 B_3 A_2 B_2 A_1 B_1}, \ \psi_{A_4 B_4 A_3 B_3 A_2 B_2 A_1 B_1}, \ \ldots\},$$

*of totally symmetric spinors satisfying the reality condition 3.4. Assume that there exists a solution $h$, $\zeta$, $s$, $s_{ABCD}$ to the conformal static field equations (2.2), (2.10), (2.11), (2.12), (2.13) so that the spinors given by $\hat{\mathcal{D}}_n$ coincide with the null data $\mathcal{D}_n^*$ given by (1.11) of the metric $h$ in terms of an $h$-orthonormal normal frame centered at $i$, i.e.*

$$\psi_{A_p B_p \ldots A_3 B_3 A_2 B_2 A_1 B_1} = D_{(A_p B_p} \ldots D_{A_3 B_3} s_{A_2 B_2 A_1 B_1)}(i), \quad p \geq 2. \tag{3.9}$$

*Then the coefficients of the normal expansions (3.2) of the fields (2.16), in particular of*

$$s_{ABCD}(x) = \sum_{p \geq 0} \frac{1}{p!} x^{A_p B_p} \ldots x^{A_1 B_1} D_{A_p B_p} \ldots D_{A_1 B_1} s_{ABCD}(i), \tag{3.10}$$

*with $x^{AB} = \alpha^{AB}{}_a x^a$, are uniquely determined by the data $\hat{\mathcal{D}}_n$ and satisfy the reality conditions.*

PROOF. It holds $s_{ABCD}(i) = \psi_{ABCD}$ by assumption and the expansion coefficients for $\zeta$, $s$ of lowest order are given by (2.2), (2.15). The induction steps for $\zeta$ and $s$ being obvious by (2.11) and (2.12), we only need to consider $s_{ABCD}$ and (3.8). Assume $m \geq 0$. If spinors $D_{A_p B_p} \ldots D_{A_1 B_1} s_{CDEF}(i)$, $p \leq m$, have been obtained which satisfy (3.9) and, up to that order, equation (3.8), the totally symmetric part of

$$D_{A_{m+1} B_{m+1}} \ldots D_{A_1 B_1} s_{CDEF}(i),$$

is given by the prescribed data while its contractions, which define the remaining terms in the decomposition corresponding to (3.5), are determined as follows. Observing the symmetries involved, essentially two cases can occcur:

i) If one of the indices $B_j$ is contracted with $F$, say, the operator $D_{A_j B_j}$ can be commuted with other covariant derivatives, generating by (3.6), (3.7) only terms of lower order, until it applies directly to $s_{CDEF}$. Equation (3.8) then shows how to express the resulting term by quantities of lower order.

ii) If the index $B_j$ is contracted with $B_k$, $k \neq j$, the operators $D_{A_j B_j}$ and $D_{A_k B_k}$ can be commuted with other covariant derivatives, until the operator $D_{A_j H} D_{A_k}{}^H$ applies directly to $s_{CDEF}$. If the corresponding term is symmetrized in $A_j$ and $A_k$ the general identity

$$D_{H(A} D^H{}_{B)} s_{CDEF} = -2 \, s_{H(CDE} s_{F)AB}{}^H,$$

implied by (3.6), (3.7) shows that this term is in fact of lower order. If a contraction of $A_j$ and $A_k$ is involved, the general identity

$$D_{AB} D^{AB} s_{CDEF} = -2 \, D_F{}^G D_G{}^H s_{CDEH} + 3 \, s_{GH(CD} s_{E)F}{}^{GH},$$

shows together with (3.8) that the corresponding term can again be expressed in terms of quantities of lower order, showing that $D_{A_{m+1} B_{m+1}} \ldots D_{A_1 B_1} s_{CDEF}(i)$ is determined by our data and terms of order $\leq m$. That the expansion coefficients satisfy the reality

condition is a consequence of the formalism and the fact that they are satisfied by the data $\hat{\mathcal{D}}_n$. ∎

To achieve our goal, we have to show the convergence of the formal series determined in Lemma 3.1. This requires us to impose estimates on the free coefficients given by $\mathcal{D}_n$. We get the following result.

**Lemma 3.2** *A necessary condition for the formal series (3.10) determined in Lemma 3.1 to be absolutely convergent near the origin is that the data given by $\hat{\mathcal{D}}_n$ satisfy estimates of the type*

$$|\psi_{A_p B_p \ldots A_1 B_1 CDEF}| \leq \frac{p!\, M}{r^p}, \quad p = 0, 1, 2, \ldots, \tag{3.11}$$

*with some constants $M, r > 0$.*

PROOF. If $f$ is a real analytic function defined on some neighbourhood of the origin in $\mathbb{R}^n$, it can be analytically extended to a function which is defined, holomorphic, and bounded on a polydisc $P(0, r) = \{x \in \mathbb{C}^n |\, |x^j| < r, 1 \leq j \leq n\}$ with some $r > 0$. Its Taylor expansion $f = \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} \partial^\alpha f(0)\, x^\alpha$ is absolutely convergent on $P(0, r)$ with $\sup_{x \in P(0,r)} |f(x)| \leq M < \infty$ so that its derivatives satisfy the estimates

$$|\partial^\alpha f(0)| \leq \frac{\alpha!\, M}{r^{|\alpha|}} \leq \frac{|\alpha|!\, M}{r^{|\alpha|}}. \tag{3.12}$$

The first of these estimates are known as Cauchy inequalities. Here $\alpha \in \mathbb{N}^n$ denotes a multi-index and we use the notation $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\alpha! = \alpha_1! \cdot \ldots \cdot \alpha_n!$, $\partial^\alpha = \partial_1^{\alpha_1} \cdot \ldots \cdot \partial_n^{\alpha_n}$, and $x^\alpha = (x^1)^{\alpha_1} \cdot \ldots \cdot (x^n)^{\alpha_n}$.

If the series (3.10) and thus

$$s_{\mathbf{ab}}(x) = \sum_{p \geq 0} \frac{1}{p!}\, x^{c_p} \ldots x^{c_1}\, D_{\mathbf{c}_p} \ldots D_{\mathbf{c}_1}\, s_{\mathbf{ab}}(i), \tag{3.13}$$

is absolutely convergent near the origin, there exist therefore by the second of the estimates (3.12) constants $M_*, r_* > 0$ with

$$|D_{\mathbf{c}_p} \ldots D_{\mathbf{c}_1}\, s_{\mathbf{ab}}(i)| \leq \frac{p!\, M_*}{r_*^p}, \qquad \mathbf{c}_p, \ldots, \mathbf{c}_1, \mathbf{a}, \mathbf{b} = 1, 2, 3, \qquad p = 0, 1, 2, \ldots .$$

Observing the transition rule from tensor to spinor quantities, one gets from this the estimates

$$|D_{A_p B_p} \ldots D_{A_1 B_1}\, s_{CDEF}(i)| \leq \frac{p!\, M}{r^p}, \qquad A_p, B_p, \ldots E, F = 0, 1, \qquad p = 0, 1, 2, \ldots \tag{3.14}$$

with $M = 9\, c^2\, M_*$ and $r = r_*/3\, c$, where $c = \max_{a=1,2,3;\, A,B=0,1} |\alpha^a{}_{AB}|$. To derive from these estimates the estimates (3.11) we consider instead of (3.5) directly the symmetrization operator to get

$$|\psi_{A_p B_p \ldots A_1 B_1 CDEF}| = |D_{(A_p B_p} \ldots D_{A_1 B_1}\, s_{CDEF)}(i)| \leq$$

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$$\frac{1}{(2p+4)!} \sum_{\pi \in \mathcal{S}_{2p+4}} |D_{\pi(A_p B_p} \ldots D_{A_1 B_1} s_{CDEF)}(i)| \leq \frac{p!\, M}{r^p},$$

where $\mathcal{S}_m$ denotes the group of permutations of $m$ elements. ∎

We note for later use that if the derivatives of a smooth function $f$ satisfy estimates of the type (3.12) with some constants $M, r > 0$ then the function $f$ is real analytic near the origin because its Taylor series is majorized by

$$\sum_\alpha M\, r^{-|\alpha|}\, x^\alpha = \frac{M\, r^n}{(r - x^1) \cdot \ldots \cdot (r - x^n)}, \qquad |x^a| < 1, \tag{3.15}$$

and

$$\sum_\alpha \frac{|\alpha|!}{\alpha!} M\, r^{-|\alpha|}\, x^\alpha = \frac{M\, r}{(r - x^1 - \ldots - x^n)}, \qquad \sum_{j=1}^n |x^j| < 1. \tag{3.16}$$

## 3.1   Relations between null data and multipoles

We express the relation between the sequences $\mathcal{D}_n^*$ of null data and the sequences $\mathcal{D}_{mp}^*$ of multipoles of $h$ (in the same normal frame centered at $i$) in terms of space-spinor notation.

**Lemma 3.3** *The spinor fields $P_{A_p B_p \ldots A_1 B_1}$ near $i$, given by (1.9), (1.10), are of the form*

$$P_{A_p B_p \ldots A_1 B_1} = -\frac{m}{2} \left\{ D_{(A_p B_p} \ldots D_{A_3 B_3} s_{A_2 B_2 A_1 B_1)} + F_{A_p B_p \ldots A_1 B_1} \right\}, \tag{3.17}$$

*with symmetric spinor-valued functions*

$$F_p \equiv F_{A_p B_p \ldots A_1 B_1} = F_{A_p B_p \ldots A_1 B_1}[\{D_{(A_q B_q} \ldots D_{A_3 B_3} s_{A_2 B_2 A_1 B_1)}\}_{q \leq p-2}], \quad p \geq 2,$$

*which satisfy*

$$F_{A_2 B_2 A_1 B_1} = 0, \quad F_{A_3 B_3 A_2 B_2 A_1 B_1} = 0,$$

*and which are real linear combinations of symmetrized tensor products of*

$$s_{A_2 B_2 A_1 B_1}, \ D_{(A_3 B_3} s_{A_2 B_2 A_1 B_1)}, \ \ldots, D_{(A_{p-2} B_{p-2}} \ldots D_{A_3 B_3} s_{A_2 B_2 A_1 B_1)},$$

*for $p \geq 4$.*

PROOF. The first two results on $F$ follow by direct calculations from (1.9), (1.10). Inserting (3.17) into the recursion relation (1.10) gives for $p \geq 3$ the recursion relations

$$F_{A_{p+1} B_{p+1} \ldots A_1 B_1} = D_{(A_{p+1} B_{p+1}} F_{A_p B_p \ldots A_1 B_1)} \tag{3.18}$$

$$-c_p \left\{ s_{(A_{p+1} B_{p+1} A_p B_p} D_{A_{p-1} B_{p-1}} \ldots s_{A_2 B_2 A_1 B_1)} + s_{(A_{p+1} B_{p+1}} F_{A_{p-1} B_{p-1} \ldots A_1 B_1)} \right\}.$$

With the induction hypothesis which assumes the properties of the $F$'s stated above for $F_{A_q B_q \ldots A_1 B_1}$, $q \leq p$, the relations (3.18) imply these properties for $F_{A_{p+1} B_{p+1} \ldots A_1 B_1}$. ∎

A further calculation gives

$$F_4 = -c_3\, s_{(A_4 B_4 A_3 B_3}\, s_{A_2 B_2 A_1 B_1)}, \quad F_5 = -(2\, c_3 + c_4)\, s_{(A_5 B_5 A_4 B_4}\, D_{A_3 B_3} s_{A_2 B_2 A_1 B_1)},$$

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and by induction the recursion law above implies the general expressions

$$F_{2p} = \alpha_{2p}\, sym(s \otimes D^{2p-4}s) + \ldots + \omega_{2p}\, sym(\otimes^p s), \quad p \geq 3,$$

$$F_{2p+1} = \alpha_{2p+1}\, sym(s \otimes D^{2p-3}s) + \ldots + \omega_{2p+1}\, sym(\otimes^{p-1} s \otimes Ds), \quad p \geq 3,$$

with real coefficients $\alpha_{2p}, \alpha_{2p+1}, \ldots, \omega_{2p}, \omega_{2p+1}$. The first terms on the right hand sides denote the term with the highest power of $D$ occuring in the respective expression. The sum of the powers of $D$ occuring in each term is even in the case of $F_{2p}$ and odd in the case of $F_{2p+1}$. The sum of the powers of $D$ occuring in each of the terms indicated by dots lies between 2 and $2\,p - 4$ in the case of $F_{2p}$ and between 3 and $2\,p - 3$ in the case of $F_{2p+1}$. The coefficients indicated above are determined by

$$\alpha_6 = -(2\,c_3 + c_4 + c_5), \quad \alpha_7 = -(2\,c_3 + c_4 + c_5 + c_6), \quad \omega_5 = -(2\,c_3 + c_4), \quad \omega_6 = c_3\,c_5,$$

and, for $p \geq 3$, by

$$\alpha_{2p+1} = \alpha_{2p} - c_{2p}, \quad \alpha_{2p+2} = \alpha_{2p+1} - c_{2p+1},$$

$$\omega_{2p+1} = p\,\omega_{2p} - c_{2p}\,\omega_{2p-1}, \quad \omega_{2p+2} = -c_{2p+1}\,\omega_{2p},$$

which implies in particular

$$\omega_{2p} = (-1)^{p+1}\, \Pi_{l=1}^{p-1}\, c_{2l+1}, \quad p \geq 3. \tag{3.19}$$

Restricting the relation (3.17) to $i$ defines with the identification (3.9) a non-linear map which can be read as a map

$$\Psi : \{\, \hat{\mathcal{D}}_n \,\} \to \{\, \hat{\mathcal{D}}_{mp} \,\},$$

of the set of *abstract null data* into the set of *abstract multipoles* (i.e. sequences of symmetric spinors not necessarily derived from a metric) satisfying

$$\nu_{A_p B_p \ldots A_1 B_1} = -\frac{m}{2}\left(\psi_{A_p B_p \ldots A_1 B_1} + F_{A_p B_p \ldots A_1 B_1}[\{\psi_{A_q B_q \ldots A_1 B_1}\}_{q \leq p-2}]\right), \quad p \geq 2. \tag{3.20}$$

**Corollary 3.4** *For given $m$ the map $\Psi$ which maps sequences $\hat{\mathcal{D}}_n$ of abstract null data onto sequences $\hat{\mathcal{D}}_{mp}$ of abstract multipoles is bijective.*

PROOF. An inverse of $\Psi$ can be constructed because $F_2 = 0$, $F_3 = 0$, and the $F_p$ depend only on the $\psi_{A_q B_q \ldots A_1 B_1}$ with $q \leq p - 2$. The relations (3.20) therefore determine for a given sequence $\hat{\mathcal{D}}_{mp}$ recursively a unique sequence $\hat{\mathcal{D}}_n$. ∎

It follows that for a given metric $h$ the sequences of multipoles and the sequences of null data in a given standard frame carry the same information on $h$. The relation is not simple, however. It can happen that a sequence $\hat{\mathcal{D}}_n$ with only a finite number of non-vanishing members is mapped onto an sequence $\hat{\mathcal{D}}_{mp}$ with an infinite number of non-vanishing members and vice versa. For instance, the relations given above show that the sequence $\hat{\mathcal{D}}_n = \{\psi_2,\, 0,\, 0,\, 0, \ldots\}$ with $\psi_2 \equiv \psi_{A_2 B_2 A_1 B_1} \neq 0$ is mapped onto the sequence $\hat{\mathcal{D}}_{mp} = \{\nu_2,\, 0,\, \nu_4,\, 0,\, \nu_6,\, \ldots\}$ with $\nu_q = \nu_{A_q B_q \ldots A_1 B_1}$, where

$$\nu_2 = \psi_2, \qquad \nu_{2p} = (-1)^{p+1}\, (\Pi_{l=1}^{p-1}\, c_{2l+1})\, sym(\otimes^p \psi_2) \neq 0, \quad p \geq 2.$$

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# 4 The characteristic initial value problem

To complete the analysis one would have to show that the estimates (3.11) imply estimates of the type (3.14) for the coefficients of (3.10). The induction argument used in the proof of Lemma 3.1 leads, however, to complicated algebraic considerations. The commutation of covariant derivatives generates with the subsequent derivative operations more and more non-linear terms of lower order. Formalising this procedure to derive estimates does not look very attractive. To arrive at a formulation of our question which looks more similar to a boundary value problem to which Cauchy-Kowalevskaya type arguments apply, we make use of the inherent geometric nature of the problem and the geometric meaning of the null data.

The fields $h$, $\zeta$, $s$, $s_{ABCD}$ are necessarily real analytic in the normal coordinates $x^a$ and a standard frame $c_{AB}$ centered at $i$. They can thus be extended near $i$ by analyticity into the complex domain and considered as holomorphic fields on a complex analytic manifold $S_c$. Choosing $S_c$ to be a sufficiently small neighbourhood of $i$, we can assume the extended coordinates, again denoted by $x^a$, to define a holomorphic coordinate system on $S_c$ which identifies the latter with an open neighbourhood of the origin in $\mathbb{C}^3$. The original manifold $S$ is then a real, 3-dimensional, real analytic submanifold of the real, 6-dimensional, real analytic manifold underlying $S_c$. If $\alpha^a$, $\beta^a$, $a = 1, 2, 3$, define real local coordinates on the real 6-dimensional manifold underlying $S_c$ so that the holomorphic coordinates $x^a$ can be written $x^a = \alpha^a + i\,\beta^a$, we use the standard notation $\partial_{x^a} = \frac{1}{2}(\partial_{\alpha^a} - i\,\partial_{\beta^a})$ and $\partial_{\bar{x}^a} = \frac{1}{2}(\partial_{\alpha^a} + i\,\partial_{\beta^a})$. The assumption that the complex-valued function $f = f(x^a)$ be holomorphic is then equivalent to the requirement that $\partial_{\bar{x}^a} f = 0$ so that we will only have to deal with the operators $\partial_{x^a}$. Under the analytic extension the main differential geometric concepts and formulas remain valid. The coordinates $x^a$ and the extended frame, again denoted by $c_{AB}$, satisfy the same defining equations and the extended fields, denoted again by $h$, $\zeta$, $s$, $s_{ABCD}$, satisfy the conformal static vacuum field equations as before.

The analytic function $\Gamma = \delta_{ab}\, x^a\, x^b$ on $S$ extends to a holomorphic function on $S_c$ which satsifies again the eikonal equation $h^{ab}\, D_a\Gamma\, D_b\Gamma = -4\,\Gamma$. On $S$ it vanishes only at $i$, but the set

$$\mathcal{N}_i = \{p \in S_c |\ \Gamma(p) = 0\},$$

is an irreducible analytical set (cf. [22]) such that $\mathcal{N}_i \setminus \{i\}$ is 2-dimensional complex submanifold of $S_c$. It is the cone swept out by the complex null geodesics through $i$ and we will refer to it shortly as the *null cone at i*. While some of the following considerations may be reminiscent of considerations concerning cones swept out by real null geodesics through given points of 4-dimensional Lorentz spaces, there are basic differences. In the present case there do not exist splittings into future and past cones. The set $\mathcal{N}_i \setminus \{i\}$ is connected and its set of of complex null generators is diffeomorph to $P^1(\mathbb{C}) \sim S^2$. If $\mathcal{N}_i \setminus \{i\}$ is considered as a 4-dimensional submanifold of the 6-dimensional real manifold underlying $S_c$, the set of real null generators is not simply connected but diffeomorphic to $SO(3, \mathbb{R})$.

The set $\mathcal{N}_i$ will be important for geometrizing our problem. Let $u \to x^a(u)$ be a null geodesic through $i$ so that $x^a(0) = 0$. Its tangent vector is then of the form $\dot{x}^{AB} = \iota^A\,\iota^B$

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with a spinor field $\iota^A = \iota^A(u)$ satisfying $D_{\dot{x}}\iota^A = 0$ along the geodesic . Then

$$s_0(u) = \iota^A \, \iota^B \, \iota^C \, \iota^D \, s_{ABCD}(x(u)), \tag{4.1}$$

is an analytic function of $u$ with Taylor expansion

$$s_0 = \sum_{p=0}^{\infty} \frac{1}{p!} \, u^p \, \frac{d^p}{du^p} \, s_0(0),$$

where

$$\frac{d^p}{du^p} \, s_0(0) = \iota^{A_p} \, \iota^{B_p} \ldots \iota^C \, \iota^D \, D_{A_p B_p} \ldots D_{A_1 B_1} s_{ABCD}(i)$$

$$= \iota^{A_p} \, \iota^{B_p} \ldots \iota^C \, \iota^D \, D_{(A_p B_p} \ldots D_{A_1 B_1} s_{ABCD)}(i).$$

Knowing these expansion coefficients for initial null vectors $\iota^A \, \iota^B$ covering an open subset of the null directions at $i$ is equivalent to knowing the null data $\mathcal{D}_n^*$ of the metric $h$.

Our problem can thus be formulated as the boundary value problem for the conformal static vacuum equations with data given by the function (4.1) on $\mathcal{N}_i$, where the $\iota^A \iota^B$ are parallely propagated null vectors tangent to $\mathcal{N}_i$. The set $\mathcal{N}_i$ can be regarded as a (complex) characteristic of the (extended) operator $\Delta_h$ and also to the conformal static equations. Therefore we shall refer to this problem as the *characteristic initial value problem for the conformal static vacuum field equations with data on the null cone at space-like infinity.*

The conformal static vacuum field equations (2.10), (2.11), (2.12), (2.13) form a 3-dimensional analogue of the 4-dimensional conformal Einstein equations ([9]). Characteristic initial value problems for these two type of systems are therefore quite similar in character.

The existence of analytic solutions to characteristic initial value problems for the conformal Einstein equations has been shown in [10] by using Cauchy-Kowalevskaya type arguments. In the present case we shall employ somewhat different techniques for the following reason.

The remaining and in fact the main difficulty in our problem arises from fact that $\mathcal{N}_i$ is not a smooth hypersurface but an analytic set with a vertex at the point $i$. A characteristic initial value problem for the conformal Einstein equations with data on a cone has been studied in [11] and some of the techniques introduced there and further developed in [13] will be used in the following. The method we use to derive estimates on the expansion coefficients has apparently not been used before in the context of Einstein's field equations.

## 4.1 The geometric gauge

To obtain a setting in which the mechanism of calculating the expansion coefficients allows one to derive estimates on the coefficients from the conditions imposed on the data, a gauge needs to be chosen which is suitably adapted to the singular set $\mathcal{N}_i$. The coordinates and the frame field will then necessarily be singular and the frame will no longer define a smooth lift to the bundle of frames but a subset which becomes tangent to the fibres

over some points. The setting described in the following will organize this situation in a geometric way and provide control on the singularity and the smoothness of the fields.

Let $SU(2)$ be the group of complex $2 \times 2$ matrices $(s^A{}_B)_{A,B=0,1}$ satisfying

$$\epsilon_{AB}\, s^A{}_C\, s^B{}_D = \epsilon_{CD}, \quad \tau_{AB'}\, s^A{}_C\, \bar{s}^{B'}{}_{D'} = \tau_{CD'}, \tag{4.2}$$

where $s^B{}_D \to \bar{s}^{B'}{}_{D'}$ denotes complex conjugation. The map

$$SU(2) \ni s^A{}_B \to s^{(A}{}_{(C}\, s^{B)}{}_{D)} \to s^a{}_b = \alpha^a{}_{AB}\, s^A{}_C\, s^B{}_D\, \alpha^{CD}{}_b \in SO(3,R), \tag{4.3}$$

realizes the $2:1$ covering homomorphism of $SU(2)$ onto the group $SO(3,\mathbb{R})$. Under holomorphic extension the map above extends to a $2:1$ covering homomorphism of the group $SL(2,\mathbb{C})$ onto the group $SO(3,\mathbb{C})$, where $SL(2,\mathbb{C})$ denotes the group of complex $2 \times 2$ matrices satisfying only the first of conditions (4.2).

We will make use of the principal bundle of normalized spin frames $SU(S) \xrightarrow{\pi} S$ with structure group $SU(2)$. A point $\delta \in SU(S)$ is given by a pair of spinors $\delta = (\delta_0^A, \delta_1^A)$ at a given point of $S$ which satisfies

$$\epsilon(\delta_A, \delta_B) = \epsilon_{AB}, \quad \epsilon(\delta_A, \delta^+{}_{B'}) = \tau_{AB'}, \tag{4.4}$$

where the lower index, which labels the members of the spin frame, is assumed to acquire a prime under the "+"-operation. The action of the structure group is given for $s \in SU(2)$ by

$$\delta \to \delta \cdot s \quad \text{where} \quad (\delta \cdot s)_A = s^B{}_A\, \delta_B.$$

The projection $\pi$ maps a frame $\delta$ onto its base point in $S$. The bundle of spin frames is mapped by a $2:1$ bundle morphism $SU(S) \xrightarrow{p} SO(S)$ onto the bundle $SO(S) \xrightarrow{\pi'} S$ of oriented, orthonormal frames on $S$ so that $\pi' \circ p = \pi$. For any spin frame $\delta$ we can identify by (4.4) the matrix $(\delta_B^A)_{A,A,B=0,1}$ with an element of the group $SU(2)$. With this reading the map $p$ will be assumed to be realized by

$$SU(S) \ni \delta \to p(\delta)_{AB} = \delta_A^E\, \delta_B^F\, c_{EF} \in SO(S),$$

where $c_{AB}$ denotes the normal frame field on $S$ introduced before. We refer to $p(\delta)$ as the frame associated with the spin frame $\delta$.

Under holomorphic extension the bundle $SU(S) \to S$ is extended to the principal bundle $SL(S_c) \xrightarrow{\pi} S_c$ of spin frames $\delta = (\delta_0^A, \delta_1^A)$ at given points of $S_c$ which satisfy only the first of conditions (4.4). Its structure group is $SL(2,\mathbb{C})$. The bundle $SU(S) \xrightarrow{\pi} S$ is embedded into $SL(S_c) \xrightarrow{\pi} S_c$ as a real analytic subbundle. The bundle morphism $p$ extends to a $2:1$ bundle morphism, again denoted by $p$, of $SL(S_c) \xrightarrow{\pi} S_c$ onto the bundle $S0(S_c) \xrightarrow{\pi'} S_c$ of oriented, normalized frames of $S_c$ with structure group $SO(3,\mathbb{C})$. We shall make use of several structures on $SM(S_c)$.

With each $\alpha \in sl(2,\mathbb{C})$, i.e. $\alpha = (\alpha^A{}_B)$ with $\alpha_{AB} = \alpha_{BA}$, is associated a *vertical vector field* $Z_\alpha$ tangent to the fibres, which is given at $\delta \in SL(S_c)$ by $Z_\alpha(\delta) = \frac{d}{dv}(\delta \cdot exp(v\,\alpha))|_{v=0}$, where $v \in \mathbb{C}$ and $exp$ denotes the exponential map $sl(2,\mathbb{C}) \to SL(2,\mathbb{C})$.

The $\mathbb{C}^3$-valued *solder form* $\sigma^{AB} = \sigma^{(AB)}$ maps a tangent vector $X \in T_\delta SL(S_c)$ onto the components of its projection $T_\delta(\pi)X \in T_{\pi(\delta)}S_c$ in the frame $p(\delta)$ associated with $\delta$

so that $T_\delta(\pi)X = <\sigma^{AB}, X> p(\delta)_{AB}$. It follows that $<\sigma^{AB}, Z_\alpha> = 0$ for any vertical vector field $Z_\alpha$.

The $sl(2,\mathbb{C})$-valued *connection form* $\omega^A{}_B$ on $SL(S_c)$ transforms with the adjoint transformation under the action of $SL(2,\mathbb{C})$ and maps any vertical vector field $Z_\alpha$ onto its generator so that $<\omega^A{}_B, Z_\alpha> = \alpha^A{}_B$.

With $x^{AB} = x^{(AB)} \in \mathbb{C}^3$ is associated the *horizontal vector field* $H_x$ on $SL(S_c)$ which is horizontal in the sense that $<\omega^A{}_B, H_x> = 0$ and which satisfies $<\sigma^{AB}, H_x> = x^{AB}$. Denoting by $H_{AB}$, $A, B = 0, 1$, the horizontal vector fields satisfying $<\sigma^{AB}, H_{CD}> = h^{AB}{}_{CD}$, it follows that $H_x = x^{AB} H_{AB}$. An integral curve of a horizontal vector field projects onto an $h$-geodesic and represents a spin frame field which is parallely transported along this geodesic.

A holomorphic spinor field $\psi$ on $S_c$ is represented on $SL(S_c)$ by a holomorphic spinor-valued function $\psi_{A_1 \ldots A_j}(\delta)$ on $SL(S_c)$, given by the components of $\psi$ in the frame $\delta$. We shall use the notation $\psi_k = \psi_{(A_1 \ldots A_j)_k}$, $k = 0, \ldots, j$, where $(\ldots \ldots)_k$ denotes the operation *'symmetrize and set $k$ indices equal to $1$ the rest equal to $0$'*. If $\psi$ is symmetric, these functions completely specify $\psi$. They are referred to as the *esssential components of $\psi$*.

## 4.2 The submanifold $\hat{S}$ of $SL(S_c)$

We combine the construction of a coordinate system and a frame field with the definition of an analytic submanifold $M$ of $SL(S_c)$ which is obtained as follows. We choose a spin frame $\delta^*$ in the fibre of $SL(S_c)$ over $i$ which is projected by $\pi'$ onto the frame $c_{AB}$ at considered $i$ before. The curve

$$C \ni v \to \delta(v) = \delta^* \cdot s(v) \in SL(S_c),$$

with

$$s(v) = exp(v\,\alpha) = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in sl(2, C), \tag{4.5}$$

in the fibre of $SL(S_c)$ over $i$ defines a vertical, 1-dimensional, holomorphic submanifold $I$ through $\delta^*$ on which $v$ defines a coordinate. The associated family of frames $e_{AB} = e_{AB}(v)$ at $i$ is given explicitly by

$$e_{00}(v) = c_{00} + 2\,v\,c_{01} + v^2\,c_{11}, \quad e_{01}(v) = c_{01} + v\,c_{11}, \quad e_{11}(v) = c_{11}.$$

The following construction is carried out in some neighbourhood of $I$. If the latter is chosen small enough all the following statements will be correct.

The set $I$ is moved with the flow of $H_{11}$ to obtain a holomorphic 2-manifold $U_0$ of $SL(S_c)$ containing $I$. The parameter on the integral curves of $H_{11}$ which vanishes on $I$ will be denoted by $w$ and $v$ is extended to $U_0$ by assuming it to be constant on the integral curves of $H_{11}$. All these integral curves are mapped by $\pi$ onto the null geodesics $\gamma(w)$ with affine parameter $w$ and tangent vector $\gamma'(0) = c_{11}$ at $\gamma(0) = i$. The parameter $v$ specifies frame fields which are parallely propagted along $\gamma$.

The set $U_0$ is moved with the flow of $H_{00}$ to obtain a holomorphic 3-submanifold $\hat{S}$ of $SL(S_c)$ containing $U_0$. We denote by $u$ the parameter on the integral curves of $H_{00}$ which vanishes on $U_0$, extend $v$ and $w$ to $\hat{S}$ by assuming them to be constant on the integral

curves of $H_{00}$. The functions $z^1 = u$, $z^2 = v$, $z^3 = w$ define holomorphic coordinates on $\hat{S}$. The restriction the projection to $\hat{S}$ will be again denoted by $\pi$.

The projections of the integral curves of $H_{00}$ with a fixed value of $w$ sweep out, together with $\gamma$, the cone $\mathcal{N}_{\gamma(w)}$ near $\gamma(w)$ which is generated by the null geodesics through the point $\gamma(w)$. On the null geodesics $u$ is an affine parameter which vanishes at $\gamma(w)$ while $v$ parametrizes the different generators. In terms of the base space $S_c$ our gauge is based on the nested family of cones $\mathcal{N}_{\gamma(w)}$ which share the generator $\gamma$. The set $W_0 = \{w = 0\}$, which projects onto $\mathcal{N}_i \setminus \gamma$, will define the intial data set for our problem. The map $\pi$ induces a biholomorphic diffeomorphism of $\hat{S}' \equiv \hat{S} \setminus U_0$ onto $\pi(\hat{S}')$. The singularity of the gauge at points of $U_0$ (resp. over $\gamma$) consists in $\pi$ dropping rank on $U_0$ because the curves $w = const.$ on $U_0$ are tangent to the fibres over $\gamma(w)$ where $\partial_v = Z_\alpha$. The null curve $\gamma(w)$ will be referred to as the singular generator of $\mathcal{N}_i$ in the gauge determined by the spin frame $\delta^*$ resp. the corresponding frame $c_{AB}$ at $i$.

The solder and the connection form pull back to holomorphic 1-forms on $\hat{S}$, which will be denoted again by $\sigma^{AB}$ and $\omega^A{}_B$. Corresponding to the behaviour of $\pi$ the 1-forms $\sigma^{00}$, $\sigma^{01}$, $\sigma^{11}$ are linearly independent on $\hat{S}'$ while the rank of this system drops to 2 on $U_0$ because $< \sigma^{AB}, \partial_v > = < \sigma^{AB}, Z_\alpha > = 0$. If the pull back of the curvature form $\Omega^A{}_B = \frac{1}{2} r^A{}_{BCDEF} \sigma^{CD} \wedge \sigma^{EF}$ to $\hat{S}$ is denoted again by $\Omega^A{}_B$, the solder and the connection form satisfy the structural equations

$$d\sigma^{AB} = -\omega^A{}_C \wedge \sigma^{CB} - \omega^B{}_C \wedge \sigma^{AC}, \qquad d\omega^A{}_B = -\omega^A{}_C \wedge \omega^C{}_B + \Omega^A{}_B.$$

By construction of $\hat{S}$ we have

$$< \sigma^{AB}, \partial_v > = 0, \quad < \sigma^{AB}, \partial_w > = \epsilon_1{}^A \epsilon_1{}^B \quad \text{on} \quad U_0,$$

$$< \omega^A{}_B, \partial_w > = 0, \quad < \omega^A{}_B, \partial_v > = < \omega^A{}_B, Z_\alpha > = \epsilon_1{}^A \epsilon_B{}^0 \quad \text{on} \quad U_0,$$

$$< \sigma^{AB}, \partial_u > = \epsilon_0{}^A \epsilon_0{}^B \text{ and } < \omega^A{}_B, \partial_u > = 0 \text{ on } \hat{S} \text{ while } < \sigma^{AB}, \partial_v > \neq 0 \text{ on } \hat{S}'.$$

To obtain more precise information on $\sigma^{AB}$ and $\omega^A{}_B$ we note the following general properties (cf. [11] and [13] for more details). If, for given $x^{AB} \in C^3$, the Lie derivative with respective $H_x$ is denoted by $\mathcal{L}_x$, then

$$\mathcal{L}_x \sigma^{AB} = 2 x^{C(A} \omega^{B)}{}_C, \qquad < \mathcal{L}_x \omega^A{}_B, . > = < \Omega^A{}_B, H_x \wedge . > .$$

Since $0 = [\partial_u, \partial_v] = [H_{00}, \partial_v]$ on $\hat{S}$ and $\Omega^A{}_B$ is horizontal, it follows that

$$\partial_u < \sigma^{AB}, \partial_v > = 2 \epsilon_0{}^{(A} < \omega^{B)}{}_0, \partial_v >, \ \ \partial_u < \omega^A{}_B, \partial_v > |_{u=0} = < \Omega^A{}_B, H_x \wedge Z_\alpha > |_{u=0} = 0,$$

which gives with the relations above

$$< \omega^A{}_B, \partial_v > = \epsilon_1{}^A \epsilon_B{}^0 + O(u^2) \text{ whence } < \omega^A{}_B, \partial_v > = 2 u \epsilon_0{}^{(A} \epsilon_1{}^{B)} + O(u^3) \quad \text{as} \quad u \to 0.$$

Similarly we obtain with $0 = [\partial_u, \partial_w] = [H_{00}, \partial_w]$ on $\hat{S}$

$$\partial_u < \sigma^{AB}, \partial_w > = 2 \epsilon_0{}^{(A} < \omega^{B)}{}_0, \partial_w >, \qquad \partial_u < \omega^A{}_B, \partial_w > |_{u=0} = \frac{1}{2} r^A{}_{B0011}.$$

In terms of the coordinates $z^a$ we thus get $\sigma^{AB} = \sigma^{AB}{}_a\, dz^a$ on $\tilde{S}'$ with a co-frame matrix

$$(\sigma^{AB}{}_a) = \begin{pmatrix} 1 & \sigma^{00}{}_2 & \sigma^{00}{}_3 \\ 0 & \sigma^{01}{}_2 & \sigma^{01}{}_3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & O(u^3) & O(u^2) \\ 0 & u + O(u^3) & O(u^2) \\ 0 & 0 & 1 \end{pmatrix} \quad \text{as} \quad u \to 0. \quad (4.6)$$

On $\hat{S}'$ there exist unique, holomorphic vector fields $e_{AB}$ which satisfy

$$< \sigma^{AB}, e_{EF} > = h^{AB}{}_{EF}.$$

If we write $e_{AB} = e^a{}_{AB}\, \partial_{z^a}$, the properties noted above imply for the frame coefficients

$$(e^a{}_{AB}) = \begin{pmatrix} 1 & e^1{}_{01} & e^1{}_{11} \\ 0 & e^2{}_{01} & e^2{}_{11} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & O(u^2) & O(u^2) \\ 0 & \frac{1}{2u} + O(u) & O(u) \\ 0 & 0 & 1 \end{pmatrix} \quad \text{as} \quad u \to 0. \quad (4.7)$$

In the following we shall write

$$e^a{}_{AB} = e^{*a}{}_{AB} + \hat{e}^a{}_{AB}, \tag{4.8}$$

with singular part

$$e^{*a}{}_{AB} = \delta^a_1\, \epsilon_A{}^0\, \epsilon_B{}^0 + \delta^a_2\, \frac{1}{u}\, \epsilon_{(A}{}^0\, \epsilon_{B)}{}^1 + \delta^a_3\, \epsilon_A{}^1\, \epsilon_B{}^1, \tag{4.9}$$

and holomorphic functions $\hat{e}^a{}_{AB}$ on $\hat{S}$ which satisfy

$$\hat{e}^a{}_{AB} = O(u) \quad \text{as} \quad u \to 0. \tag{4.10}$$

We define connections coefficients on $\hat{S}'$ by writing $\omega^A{}_B = \Gamma_{CD}{}^A{}_B\, \sigma^{CD}$ with

$$\Gamma_{CD\,AB} \equiv\, < \omega_{AB}, e_{CD} >,$$

so that $\Gamma_{CD\,AB} = \Gamma_{(CD)\,(AB)}$. The definition of the frame then implies

$$\Gamma_{00\,AB} = 0 \quad \text{on} \quad \hat{S} \quad \text{and} \quad \Gamma_{11\,AB} = 0 \quad \text{on} \quad U_0,$$

and it follows from the discussion above that

$$\Gamma_{ABCD} = \Gamma^*_{ABCD} + \hat{\Gamma}_{ABCD}, \tag{4.11}$$

with singular part

$$\Gamma^*_{AB\,CD} = -\frac{1}{u}\, \epsilon_{(A}{}^0\, \epsilon_{B)}{}^1\, \epsilon_C{}^0\, \epsilon_D{}^0, \tag{4.12}$$

and holomorphic functions $\hat{\Gamma}_{ABCD}$ on $\hat{S}$ which satisfy

$$\hat{\Gamma}_{ABCD} = O(u) \quad \text{as} \quad u \to 0. \tag{4.13}$$

The singular parts are 'universal' in the sense that their expressions only depend on the construction of $\hat{S}$ and not on properties of the metric. If the latter is flat the functions $\hat{e}^a{}_{AB}$ and $\hat{\Gamma}_{ABCD}$ vanish on $\hat{S}$. With the frame and the connection coefficients so defined we have the spin frame calculus in its standard form. The expressions above imply for any holomorphic spinor valued function $\psi_{A...C}$ that $D_{00}\,\psi_{A...C}$ and $D_{11}\,\psi_{A...C}$ extend to $\hat{S}$ as holomorphic functions so that

$$D_{00}\,\psi_{A...C} = \partial_u\,\psi_{A...C} \quad \text{on} \quad \hat{S} \quad \text{and} \quad D_{11}\,\psi_{A...C} = \partial_w\,\psi_{A...C} \quad \text{on} \quad U_0.$$

## 4.3  Tensoriality and expansion type

A holomorphic function on $SL(S_c)$ induces a holomorphic function on $\hat{S}$ which can be considered as a holomorphic function of the coordinates $z^a$. While these coordinates are holomorphic on the submanifold $\hat{S}$ of $SL(S_c)$, the induced map $\pi$ of $\hat{S}$ into $S_c$ is singular on $U_0$. As a consequence, not every holomorphic function of the $z^a$ can arise as a pull-back to $\hat{S}$ of a holomorphic function on $SL(S_c)$. The latter must have a special type of expansion in terms of the $z^a$ which reflects the particular relation between the 'angular' coordinate $v$ the 'radial' coordinate $u$. The following notion will be important for our discussion.

**Definition**: *A holomorphic function $f$ on $\hat{S}$ will be said to be of v-finite expansion type $k_f$, with $k_f$ an integer, if it has in terms of the coordinates $u$, $v$, and $w$ a Taylor expansion at the origin of the form*

$$f = \sum_{p=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{2\,m+k_f} f_{m,n,p}\, u^m\, v^n\, w^p,$$

*where it is assumed that $f_{m,n,p} = 0$ if $2\,m + k_f < 0$.*

We note that the construction of $\hat{S}$ does not distinguish the set $I = \pi^{-1}(i)$ from the sets $\pi^{-1}(\gamma(w))$. Correspondingly, the Taylor expansions of the function $f$ above at points $(0, 0, w_0)$ with $w_0$ close to 0 have the same structure with respect to $u$ and $v$.

**Lemma 4.1**  *Let $\phi_{A_1 \ldots A_j}$ be a holomorphic, symmetric, spinor-valued function on $SL(S_c)$. Then the restrictions of its essential components $\phi_k = \phi_{(A_1 \ldots A_j)_k}$, $0 \le k \le j$, to $\hat{S}$ satisfy*

$$\partial_v\, \phi_k = (j - k)\, \phi_{k+1}, \quad k = 0, \ldots, j, \quad on \quad U_0, \tag{4.14}$$

*(where we set $\phi_{j+1} = 0$) and $\phi_k$ is of expansion type $j - k$.*

PROOF. : In the following we consider $\hat{S}$ as a submanifold of $SL(S_c)$. The tensorial transformation law of $\phi$ under the action of the 1-parameter subgroup (4.5) with generator $\alpha^A{}_B = \epsilon_1{}^A \epsilon_B{}^0$ implies

$$Z_\alpha\, \phi_k = (j - k)\, \phi_{k+1} \quad \text{for} \quad 0 \le k \le j \quad \text{on} \quad SL(S_c),$$

and thus (4.14) because $Z_\alpha = \partial_v$ on $U_0$. From the relations above follows in particular that

$$Z_\alpha^{j-k+1}\, \phi_k = 0 \quad \text{on} \quad SL(S_c). \tag{4.15}$$

A general horizontal vector field $H_x$ has with $Z_\alpha$ the commutator

$$[Z_\alpha, H_x] = H_{\alpha \cdot x},$$

where $\alpha$ acts on $x^{AB} = x^{(AB)}$ according to the induced action by

$$x^{AB} \rightarrow (\alpha \cdot x)^{AB} = \alpha^A{}_C\, x^{CB} + \alpha^B{}_C\, x^{AC} = 2\,\epsilon_1^{(A}\, x^{B)0}.$$

With $x^{AB} = \epsilon_0{}^A \epsilon_0{}^B$, so that $H_x = H_{00}$, it follows

$$[Z_\alpha, H_{00}] = 2\, H_{01}, \quad [Z_\alpha, H_{01}] = H_{11}, \quad [Z_\alpha, H_{11}] = 0.$$

By induction this gives the operator equations

$$Z_\alpha^n H_{00} = n(n-1) H_{11} Z_\alpha^{n-2} + 2n H_{01} Z_\alpha^{n-1} + H_{00} Z_\alpha^n, \quad n \geq 1,$$

and, more generally,

$$Z_\alpha^n H_{00}^m = a_{n,m} H_{11}^m Z_\alpha^{n-2m} + \sum_{l=0}^{2m-1} A_{n,m,l} Z_\alpha^{n-l} + H_{00}^m Z_\alpha^n, \quad m, n \geq 1,$$

where the $a_{n,m}$ are real coefficients, the $A_{n,m,l}$ denote operators which are sums of products of horizontal vector fields, and the terms in which $Z_\alpha$ formally appears with negative exponent are assumed to vanish. With (4.15) this implies

$$Z_\alpha^n H_{00}^m \phi_k = 0 \quad \text{for} \quad n > 2m + j - k \quad \text{on} \quad SL(S_c).$$

The results follows because $Z_\alpha^n H_{00}^m \phi_k = \partial_v^n \partial_u^m \phi_k$ at points of $U_0$. ∎

## 4.4   The null data on $W_0$

We shall derive an expansion of the restriction of the essential component $s_0$ of the Ricci spinor to the hypersurface $W_0$, i.e.

$$s_0(u,v) = s_{(ABCD)_0}|_{W_0},$$

in terms of quantities on the base space $S_c$. Consider the normal frame $c_{AB}$ on $S_c$ near $i$ which agrees at $i$ with the frame associated with $\delta^*$ and denote by

$$\mathcal{D}_n^* \equiv \{ D^*_{(A_1 B_1} \ldots D^*_{A_p B_p} s^*_{ABCD)}(i), \ p = 0, 1, 2, \ldots \},$$

the corresponding null data of $h$ in the frame $c_{AB}$. Choose a fixed value of $v$ and consider $s = s(v)$ as in (4.5). The vector $H_{00}(\delta^* \cdot s)$ then projects onto the null vector $s^A{}_0 s^B{}_0 c_{AB}$ at $i$. Since $c_{AB}$ is a normal frame near $i$, the null vector field $s^A{}_0 s^B{}_0 c_{AB}$ is tangent to a null geodesic $\eta = \eta(u,v)$ on $\mathcal{N}_i$ with affine parameter $u$ with $u = 0$ at $i$ and the integral curve of $H_{00}$ through $\delta^* \cdot s$ projects onto this null geodesic. It follows from this with the explicit expression for $s = s(v)$ that

$$s_0(u,v) = s^A{}_0(v)\, s^B{}_0(v)\, s^C{}_0(v)\, s^D{}_0(v)\, s^*_{ABCD}|_{\eta(u,v)} \tag{4.16}$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} u^m s^{A_1}{}_0(v)\, s^{B_1}{}_0(v) \ldots s^D{}_0(v)\, D^*_{(A_1 B_1} \ldots D^*_{A_m B_m} s^*_{ABCD)}(i)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{2m+4} \psi_{m,n}\, u^m\, v^n,$$

with

$$\psi_{m,n} = \frac{1}{m!} \binom{2m+4}{n} D^*_{(A_1 B_1} \ldots D^*_{A_m B_m} s^*_{ABCD)_n}(i), \quad 0 \leq n \leq 2m+4.$$

This formula shows us how to determine the function $s_0(u,v)$ from the null data $\mathcal{D}_n^*$ and vice versa. We note that the expansion above is consistent with $s_0$ being of $v$-finite expansion type 4. We shall refer to (4.16) as the *null data on $W_0$* in our gauge.

# 5 The conformal static vacuum field equations on $\hat{S}$

With the frame $e_{AB}$ and the connection coefficients $\gamma_{ABCD}$ on $\hat{S}$ we have the standard frame calculus available. Given the fields $\zeta$, $s$, $s_{ABCD}$, we define on $\hat{S}$ the quantities

$$t_{AB}{}^{EF}{}_{CD}\, e^a{}_{EF} \equiv (\Gamma_{AB}{}^{EF}{}_{CD} - \Gamma_{CD}{}^{EF}{}_{AB})\, e^a{}_{EF} - e^a{}_{CD,b}\, e^b{}_{AB} + e^a{}_{AB,b}\, e^b{}_{CD},$$

$$R_{ABCDEF} \equiv r_{ABCDEF} - \frac{1}{2}\left\{ s_{ABCE}\, \epsilon_{DF} + s_{ABDF}\, \epsilon_{CE} \right\},$$

with

$$r_{ABCDEF} \equiv e_{CD}(\Gamma_{EFAB}) - e_{EF}(\Gamma_{CDAB})$$
$$+\Gamma_{EF}{}^{K}{}_{C}\,\Gamma_{KDAB} + \Gamma_{EF}{}^{K}{}_{D}\,\Gamma_{CKAB} - \Gamma_{CD}{}^{K}{}_{E}\,\Gamma_{KFAB}$$
$$-\Gamma_{CD}{}^{K}{}_{F}\,\Gamma_{EKAB} + \Gamma_{EF}{}^{K}{}_{B}\,\Gamma_{CDAK} - \Gamma_{CD}{}^{K}{}_{B}\,\Gamma_{EFAK}$$
$$-t_{CD}{}^{GH}{}_{EF}\,\Gamma_{GHAB},$$

$$\Sigma_{AB} \equiv D_{AB}\,\zeta - \zeta_{AB},$$

$$\Sigma_{ABCD} \equiv D_{AB}\,\zeta_{CD} - s\,h_{ABCD} + \zeta\,(1 - \mu\,\zeta)\,s_{ABCD},$$

$$S_{AB} \equiv D_{AB}\,s + (1 - \mu\,\zeta)\,s_{ABCD}\,\zeta^{CD},$$

$$H_{ABCD} \equiv D_{A}{}^{E}s_{BCDE} - \frac{2\,\mu}{1 - \mu\,\zeta}\,s_{E(BCD}\,\zeta_{A)}{}^{E}.$$

In terms of the fields on the left hand side, which have been introduced as labels for the equations as well as for the discussion of their interdependencies, the conformal static vacuum equations read

$$t_{AB}{}^{EF}{}_{CD}\, e^a{}_{EF} = 0, \qquad R_{ABCDEF} = 0,$$

$$\Sigma_{AB} = 0,$$

$$\Sigma_{ABCD} = 0, \qquad S_{AB} = 0, \qquad H_{ABCD} = 0.$$

The first equation is Cartan's first structural equation with the requirement that the (metric) connection be torsion free ($t_{AB}{}^{EF}{}_{CD}$ being the torsion tensor). The second equation is Cartan's second structural equation with the requirement that the Ricci tensor coincides with the trace free tensor $s_{ab}$. The third equation defines $\zeta_{AB}$, the remaining equations have been considered before.

To discuss these equations in detail we need to write them out in our gauge, observing in particular the nature of the singularities in (4.8) and (4.11).

The equations $t_{AB}{}^{EF}{}_{00}\,e^a{}_{EF}=0$:

$$\partial_u \hat{e}^1{}_{01} + \frac{1}{u}\,\hat{e}^1{}_{01} = -2\,\hat{\Gamma}_{0101} + 2\,\hat{\Gamma}_{0100}\,\hat{e}^1{}_{01},$$

$$\partial_u \hat{e}^2{}_{01} + \frac{1}{u}\,\hat{e}^2{}_{01} = \frac{1}{u}\,\hat{\Gamma}_{0100} + 2\,\hat{\Gamma}_{0100}\,\hat{e}^2{}_{01},$$

$$\partial_u \hat{e}^1{}_{11} = -2\,\hat{\Gamma}_{1101} + 2\,\hat{\Gamma}_{1100}\,\hat{e}^1{}_{01},$$

$$\partial_u \hat{e}^2{}_{11} = \frac{1}{u}\hat{\Gamma}_{1100} + 2\,\hat{\Gamma}_{1100}\,\hat{e}^2{}_{01}.$$

The equations $R_{AB00EF}=0$:

$$\partial_u \hat{\Gamma}_{0100} + \frac{2}{u}\,\hat{\Gamma}_{0100} - 2\,\hat{\Gamma}^2_{0100} = \frac{1}{2}\,s_0,$$

$$\partial_u \hat{\Gamma}_{0101} + \frac{1}{u}\,\hat{\Gamma}_{0101} - 2\,\hat{\Gamma}_{0100}\,\hat{\Gamma}_{0101} = \frac{1}{2}\,s_1,$$

$$\partial_u \hat{\Gamma}_{0111} + \frac{1}{u}\,\hat{\Gamma}_{0111} - 2\,\hat{\Gamma}_{0100}\,\hat{\Gamma}_{0111} = \frac{1}{2}\,s_2,$$

$$\partial_u \hat{\Gamma}_{1100} + \frac{1}{u}\,\hat{\Gamma}_{1100} - 2\,\hat{\Gamma}_{1100}\,\hat{\Gamma}_{0100} = s_1,$$

$$\partial_u \hat{\Gamma}_{1101} - 2\,\hat{\Gamma}_{1100}\,\hat{\Gamma}_{0101} = s_2,$$

$$\partial_u \hat{\Gamma}_{1111} - 2\,\hat{\Gamma}_{1100}\,\hat{\Gamma}_{0111} = s_3.$$

The equations $\Sigma_{00}=0$, $\Sigma_{00CD}=0$, $S_{00}=0$:

$$0 = \partial_u \zeta - \zeta_{00},$$

$$0 = \partial_u\,\zeta_{00} + \zeta\,(1-\mu\,\zeta)\,s_0,$$

$$0 = \partial_u\,\zeta_{01} + \zeta\,(1-\mu\,\zeta)\,s_1,$$

$$0 = \partial_u\,\zeta_{11} - s + \zeta\,(1-\mu\,\zeta)\,s_2,$$

$$0 = \partial_u s + (1-\mu\,\zeta)\,(s_0\,\zeta_{11} - 2\,s_1\,\zeta_{01} + s_2\,\zeta_{00}).$$

The equations $-H_{0(BCD)_k}=0$ in the order $k=0,1,2,3$:

$$\partial_u\,s_1 - \frac{1}{2\,u}(\partial_v\,s_0 - 4\,s_1) - \hat{e}^1{}_{01}\partial_u\,s_0 - \hat{e}^2{}_{01}\partial_v\,s_0$$

$$= -4\,\hat{\Gamma}_{0101}\,s_0 + 4\,\hat{\Gamma}_{0100}\,s_1 - \frac{2\,\mu}{(1-\mu\,\zeta)}\,\{s_0\,\zeta_{01} - s_1\,\zeta_{00}\},$$

$$\partial_u\,s_2 - \frac{1}{2\,u}(\partial_v\,s_1 - 3\,s_2) - \hat{e}^1{}_{01}\partial_u\,s_1 - \hat{e}^2{}_{01}\partial_v\,s_1$$

$$= -\hat{\Gamma}_{0111}\,s_0 - 2\,\hat{\Gamma}_{0101}\,s_1 + 3\,\hat{\Gamma}_{0100}\,s_2 - \frac{\mu}{2\,(1-\mu\,\zeta)}\,\{s_0\,\zeta_{11} + 2\,s_1\,\zeta_{01} + 3\,s_2\,\zeta_{00}\},$$

$$\partial_u\,s_3 - \frac{1}{2\,u}(\partial_v\,s_2 - 2\,s_3) - \hat{e}^1{}_{01}\partial_u\,s_2 - \hat{e}^2{}_{01}\partial_v\,s_2$$

$$= -2\hat{\Gamma}_{0111}\, s_1 + 2\,\hat{\Gamma}_{0100}\, s_3 - \frac{\mu}{(1-\mu\,\zeta)}\left\{s_1\,\zeta_{11} + s_3\,\zeta_{00}\right\},$$

$$\partial_u\, s_4 - \frac{1}{2\,u}(\partial_v\, s_3 - s_4) - \hat{e}^1{}_{01}\partial_u\, s_3 - \hat{e}^2{}_{01}\partial_v\, s_3$$

$$= -3\,\hat{\Gamma}_{0111}\, s_2 + 2\,\hat{\Gamma}_{0101}\, s_3 + \hat{\Gamma}_{0100}\, s_4 - \frac{\mu}{2\,(1-\mu\,\zeta)}\left\{3\, s_2\,\zeta_{11} - 2\, s_3\,\zeta_{01} - s_4\,\zeta_{00}\right\}.$$

These equations, referred to as the $\partial_u$-*equations*, will be read as a system of PDE's for the set of functions

$$\hat{e}^1{}_{01},\ \ \hat{e}^2{}_{01},\ \ \hat{e}^1{}_{11},\ \ \hat{e}^2{}_{11},\ \ \hat{\Gamma}_{01AB},\ \ \hat{\Gamma}_{11AB},\ \ \zeta,\ \ \zeta_{AB},\ \ s,\ \ s_1,\ \ s_2,\ \ s_3,\ \ s_4,$$

which comprises all the unknowns with the exception of $s_0$. The following features of them will be important.

All $\partial_u$-equations are *interior equations on the hypersurfaces* $\{w = w_0\}$ in the sense that only derivatives in the directions of $u$ and $v$ are involved.

The equations are singular with terms $u^{-1}$ occuring in various places. It will be seen later that these terms come with the 'right' signs to possess (unique) solutions which are holomorphic in $u$, $v$ and $w$. Remarkably, the equations for the $s_k$ ensure regular solution to have the correct tensorial behaviour by the occurrence of terms $u^{-1}$ with factors $\partial_v\, s_k - (4-k)\, s_{k+1}$. By Lemma 4.1 we know that they have to vanish $U_0$.

The system splits into a hierarchy of subsystems, with

$$t_{01}{}^{EF}{}_{00}\, e^2{}_{EF} = 0, \quad R_{000001} = 0,$$

being the first subsystem,

$$t_{01}{}^{EF}{}_{00}\, e^1{}_{EF} = 0, \quad R_{010001} = 0, \quad \Sigma_{00} = 0, \quad \Sigma_{0000} = 0, \quad \Sigma_{0001} = 0, \quad H_{0000} = 0,$$

being the second subsystem, and so on. The hierarchy has the following property. If $s_0$ is given on $\{w = w_0\}$, the first subsystem reduces to singular system of ODE's. Given its solution, the second subsystem also reduces to a system of ODE's (with coefficients which are calculated from the functions known so far by operation interior to $\{w = w_0\}$), and so on. Thus, given $s_0$ and the appropriate initial data on $U_0 \cap \{w = w_0\}$, all unknowns can be determined on $\{w = w_0\}$ by solving a sequence of systems of ODE's in the independent variable $u$.

The functions $\hat{e}^a{}_{AB}$ and $\hat{\Gamma}_{ABCD}$ vanish on $U_0$ by our gauge conditions. Therefore only initial data for $\zeta$, $\zeta_{AB}$, $s$, and $s_k$ need to be determined on $U_0$ and the function $s_0$ needs to be provided on $\{w = w_0\}$. Since $s_0$ will be prescribed on $W_0$ as our initial datum, an equation is needed to determine its evolution off $W_0$. For this purpose we will consider the following equations.

The equations $H_{1(BCD)_k} = 0$ in the order $k = 0, 1, 2, 3$:

$$\partial_w\, s_0 - \frac{1}{2\,u}(\partial_v\, s_1 - 3\, s_2) + \hat{e}^1{}_{11}\partial_u\, s_0 + \hat{e}^2{}_{11}\partial_v\, s_0 - \hat{e}^1{}_{01}\partial_u\, s_1 - \hat{e}^2{}_{01}\partial_v\, s_1 \qquad (5.1)$$

$$= -(\hat{\Gamma}_{0111} - 4\,\hat{\Gamma}_{1101})\,s_0 - (2\,\hat{\Gamma}_{0101} + 4\,\hat{\Gamma}_{1100})\,s_1 + 3\,\hat{\Gamma}_{0100}\,s_2 + \frac{2\,\mu}{(1 - \mu\,\zeta)}\,\frac{1}{4}\,\{s_0\,\zeta_{11} + 2\,s_1\,\zeta_{01} - 3\,s_2\,\zeta_{00}\}\,,$$

$$\partial_w\,s_1 - \frac{1}{2\,u}(\partial_v\,s_2 - 2\,s_3) + \hat{e}^1{}_{11}\partial_u\,s_1 + \hat{e}^2{}_{11}\partial_v\,s_1 - \hat{e}^1{}_{01}\partial_u\,s_2 - \hat{e}^2{}_{01}\partial_v\,s_2$$

$$= \hat{\Gamma}_{1111}\,s_0 - (2\,\hat{\Gamma}_{0111} - 2\,\hat{\Gamma}_{1101})\,s_1 - 3\,\hat{\Gamma}_{1100}\,s_2 + 2\,\hat{\Gamma}_{0100}\,s_3 + \frac{2\,\mu}{(1 - \mu\,\zeta)}\,\frac{1}{2}\,\{s_1\,\zeta_{11} - s_3\,\zeta_{00}\}\,,$$

$$\partial_w\,s_2 - \frac{1}{2\,u}(\partial_v\,s_3 - 2\,s_4) + \hat{e}^1{}_{11}\partial_u\,s_2 + \hat{e}^2{}_{11}\partial_v\,s_2 - \hat{e}^1{}_{01}\partial_u\,s_3 - \hat{e}^2{}_{01}\partial_v\,s_3$$

$$= 2\,\hat{\Gamma}_{1111}\,s_1 - 3\,\hat{\Gamma}_{0111}\,s_2 - (2\,\hat{\Gamma}_{1100} - 2\,\hat{\Gamma}_{0101})\,s_3 + \hat{\Gamma}_{0100}\,s_4 + \frac{2\,\mu}{(1 - \mu\,\zeta)}\,\frac{1}{4}\,\{3\,s_2\,\zeta_{11} - 2\,s_3\,\zeta_{01} - s_4\,\zeta_{00}\}\,,$$

$$\partial_w\,s_3 - \frac{1}{2\,u}\partial_v\,s_4 + \hat{e}^1{}_{11}\partial_u\,s_3 + \hat{e}^2{}_{11}\partial_v\,s_3 - \hat{e}^1{}_{01}\partial_u\,s_4 - \hat{e}^2{}_{01}\partial_v\,s_4$$

$$= 3\,\hat{\Gamma}_{1111}\,s_2 - (4\,\hat{\Gamma}_{0111} + 2\,\hat{\Gamma}_{1101})\,s_3 - (\hat{\Gamma}_{1100} - 4\,\hat{\Gamma}_{0101})\,s_4 + \frac{2\,\mu}{(1 - \mu\,\zeta)}\,\{s_3\,\zeta_{11} - s_4\,\zeta_{01}\}\,.$$

All singular terms cancel in the equations $0 = H_{0(BCD)_{k+1}} + H_{1(BCD)_k}$, which are given in the order $k = 0, 1, 2$ by

$$\partial_w\,s_0 - \partial_u\,s_2 + \hat{e}^1{}_{11}\partial_u\,s_0 + \hat{e}^2{}_{11}\partial_v\,s_0 \qquad\qquad (5.2)$$

$$= 4\,\hat{\Gamma}_{1101}\,s_0 - 4\,\hat{\Gamma}_{1100}\,s_1 + \frac{\mu}{(1 - \mu\,\zeta)}\,\{s_0\,\zeta_{11} + 2\,s_1\,\zeta_{01} - 3\,s_2\,\zeta_{00}\}\,,$$

$$\partial_w\,s_1 - \partial_u\,s_3 + \hat{e}^1{}_{11}\partial_u\,s_1 + \hat{e}^2{}_{11}\partial_v\,s_1$$

$$= \hat{\Gamma}_{1111}\,s_0 + 2\,\hat{\Gamma}_{1101}\,s_1 - 3\,\hat{\Gamma}_{1100}\,s_2 - \frac{2\,\mu}{(1 - \mu\,\zeta)}\,\{s_1\,\zeta_{11} - s_3\,\zeta_{00}\}\,,$$

$$\partial_w\,s_2 - \partial_u\,s_4 + \hat{e}^1{}_{11}\partial_u\,s_2 + \hat{e}^2{}_{11}\partial_v\,s_2$$

$$= 2\,\hat{\Gamma}_{1111}\,s_1 - 2\,\hat{\Gamma}_{1100}\,s_3 + \frac{\mu}{(1 - \mu\,\zeta)}\,\{3\,s_2\,\zeta_{11} - 2\,s_3\,\zeta_{01} - s_4\,\zeta_{00}\}\,.$$

We can consider (5.1) or (5.2) as equation prescribing the propagation of $s_0$ transverse to the hypersurfaces $\{w = const.\}$.

Because $\Gamma_{11CD} = 0$ on $U_0$, the equations $\Sigma_{11} = 0$, $\Sigma_{11CD} = 0$, $S_{11} = 0$ reduce on $U_0$ to the ODE's

$$\partial_w\,\zeta = \zeta_{11}, \qquad \partial_w\,\zeta_{CD} = s\,h_{11CD} - \zeta\,(1 - \mu\,\zeta)\,s_{11CD}, \qquad \partial_w\,s = -(1 - \mu\,\zeta)\,s_{11CD}\,\zeta^{CD}.$$

By (2.2), (2.15) we must impose

$$\zeta = 0, \quad \zeta_{AB} = 0, \quad s(i) = -2 \quad \text{on} \quad I = \{u = 0, w = 0\}.$$

This implies with the equations above

$$\zeta = 0, \quad \zeta_{01} = 0, \quad \zeta_{11} = 0 \quad \text{on} \quad U_0 = \{u = 0\}. \qquad\qquad (5.3)$$

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To determine $\zeta$, $\zeta_{AB}$, and $s$ on $U_0$ it remains to solve on $U_0$ the equations

$$\partial_w \zeta_{00} = s, \quad \partial_w s = -s_4 \zeta_{00}. \tag{5.4}$$

The tensorial properties of $\zeta_{AB}$ and $s$ imply with (5.3) that

$$\partial_v^n \zeta_{00} = 0, \quad \partial_v^n \hat{s} = 0 \quad \text{on} \quad U_0 \quad \text{for} \quad n \geq 1. \tag{5.5}$$

Later it will be important for that these relations can in fact be deduced from (5.3), (5.4), (5.6), and the initial conditions on $I$.

To ensure the tensor relations for the $s_k$ and thus the existence of regular solutions to the equation for the $s_k$, we determine the initial data for $s_1, \ldots, s_4$ on $U_0$ by imposing the conditions

$$\partial_v s_k = (4 - k) s_{k+1}, \quad k = 0, \ldots, 3, \quad \text{on} \quad U_0. \tag{5.6}$$

They imply recursively the expressions

$$\partial_v^n \partial_w^p s_k = \frac{(4-k)!}{4!} \partial_v^{k+n} \partial_w^p s_0, \quad k = 0, \ldots 4, \quad n, p \geq 0 \quad \text{at} \quad \{u = 0, v = 0, w = 0\}.$$

## 5.1 Calculating the formal expansion

Since the equations are overdetermined there are various ways to determine a formal expansion of the solution. It will follow from Lemma 5.5 that the expansion obtained by the following procedure will lead to a formal solution of the full system of equations. A solution obtained by any other procedure with this property will thus have to coincide with the present one. It will be convenient to replace $s$ by the unknown

$$\hat{s} = 2 + s.$$

For certain discussions it is useful to write

$$s_k = s_k^* + \hat{s}_k \quad \text{with} \quad \partial_u s_k^* = 0 \quad \text{and} \quad s_k^*|_{u=0} = s_k|_{u=0} \quad \text{so that} \quad \hat{s}_k = O(u) \quad \text{as} \quad u \to 0.$$

By (5.6) we can then assume that

$$\partial_v s_k^* = (4 - k) s_{k+1}^*,$$

and the $\partial_u$-equations for the $\hat{s}_k$ can be written in the form

$$0 = -H_{0\,(BCD)_k} = \partial_u \hat{s}_{k+1} + \frac{4-k}{2\,u} \hat{s}_{k+1} - \frac{1}{2\,u} \partial_v \hat{s}_k + \hat{e}^a{}_{01} \partial_a (s_k^* + \hat{s}_k)$$

$$+ terms\ of\ zeroth\ order,$$

so that the coefficient $(4 - k)/2$ of the singular term $u^{-1} \hat{s}_{k+1}$ is positive and the term $u^{-1} \partial_v \hat{s}_k$, which involves the unknown $\hat{s}_k$ determined in an earlier step of the integration procedure, creates no problem because $\hat{s}_k = 0$ on $U_0$. Writing

$$x = (\hat{e}^a{}_{AB}, \hat{\Gamma}_{ABCD}, \zeta, \zeta_{AB}, \hat{s}, s_1, s_2, s_3, s_4),$$

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so that the full set of unknowns are given by $x$ and $s_0$, we proceed as follows.

On $W_0$ we prescribe $s_0$ as given in (4.16) with the null data $\mathcal{D}_n^*$ satisfying the reality conditions and the estimates (3.11). By (5.6) all components of $x$ can be determined on $I$.

We successivly integrate the subsystems in the hierachy of $\partial_u$-equations to determine all components of $x$ on $W_0$. These will be holomorphic in $u$ and $v$ and unique, because the relevant operators in the singular equations are of the form $\partial_u f + c\, u^{-1} f$ with non-negative constants $c$ (a proof of this statement follows from the derivation of the estimates discussed below).

The equation $H_{0100} + H_{1000} = 0$ is used to determine $\partial_w s_0$ from the fields $x$ and $s_0$ on $W_0$ as a holomorphic function of $u$ and $v$.

Applying the operator $\partial_w$ formally to the $\partial_u$-equations, one obtains equations for $\partial_w x$ on $W_0$ which can be solved with the initial data on $\{w = 0, u = 0\}$ which are obtained by using (5.4) and by applying $\partial_w$ to (5.6). Applying $\partial_w$ to the equation $H_{0100} + H_{1000} = 0$, one obtains $\partial^2 s_0$ on $W_0$.

Repeating these steps by applying successively the operator $\partial_w^p$, $p = 2, 3, \ldots$, one gets an sequence of functions $\partial_w^p x$, $\partial_w^p s_0$ on $W_0$, which are holomorphic in $u$ and $v$.

Expanding the functions so obtained at $u = 0$, $v = 0$ we get the following result.

**Lemma 5.1** *The procedure described above determines at the point $O = (u = 0, v = 0, w = 0)$ from the data $s_0$, given on $W_0$ according to (4.16), a unique sequence of expansion coefficients*
$$\partial_u^m \partial_v^n \partial_w^p f(O), \qquad m, n, p = 0, 1, 2, \ldots,$$
*where $f$ stands for any of the functions*
$$\hat{e}^a{}_{AB},\ \hat{\Gamma}_{ABCD},\ \zeta,\ \zeta_{AB},\ \hat{s},\ s_j.$$
*If the corresponding Taylor series are absolutely convergent in some neighbourhood $P$ of $O$, they define a solution to the $\partial_u$-equations and to the equation $H_{1000} = 0$ on $P$ which satisfies on $P \cap U_0$ equations (5.6) and $\Sigma_{11} = 0$, $\Sigma_{11CD} = 0$, $S_{11} = 0$.*

By Lemma 4.1 all spinor-valued functions should have a specific $v$-finite expansion type. The following result will be important for our convergence proof.

**Lemma 5.2** *If the data $s_0$ are given on $W_0$ as in (4.16), the formal expansions of the fields obtained in Lemma 5.1 correspond to ones of functions of $v$-finite expansion types given by*
$$k_{s_j} = 4 - j, \quad k_{\zeta_i} = 2 - i, \quad k_\zeta = 0, \quad k_s = k_{\hat{s}} \leq 2,$$
$$k_{\hat{e}^1_{AB}} = -A - B, \quad k_{\hat{e}^2_{AB}} = 3 - A - B \quad for \quad AB = 01,\, 10\ or\ 11.$$
$$k_{\hat{\Gamma}_{01AB}} = 2 - A - B, \quad k_{\hat{\Gamma}_{11AB}} = 1 - A - B \quad for \quad A, B = 0\ or\ 1.$$

**Remark 5.3** *The scalar functions $s$, $\hat{s}$ should have expansion type $k_s = k_{\hat{s}} = 0$. As pointed out below, this does not follow with the simple arguments used here. Since it will not be important for the following discussions, we shall make no effort to retrieve this information from the equations.*

PROOF. We note the following properties of $v$-finite expansion types:

For given integer $k$ the functions of expansion type $k$ form a complex vector space which comprises the functions of expansion type $\leq k$.

If the functions $f$ and $g$ have expansion type $k_f$ and $k_g$ respectively, their product $f g$ has expansion type $k_{fg} = k_f + k_g$.

If $f$ has expansion type $k_f$, the function $\partial_u f$ has expansion type $k_f + 2$. Conversely, if $\partial_u f$ has expansion type $k_f + 2$ and if the function independent of $u$ which agrees on $U_0$ with $f$ has expansion type $k_f$ (for instance if $f|_{u=0} = 0$), then $f$ has expansion type $k_f$.

If $f$ has expansion type $k_f$ and $f|_{u=0} = 0$ then $\frac{1}{u} f$ has expansion type $k_f + 2$.

If $f$ has expansion type $k_f$, the function $\partial_v f$ has expansion type $k_f - 1$.

If $f$ has an expansion type, the function $\partial_w f$ has the same expansion type.

Applying these rules one can check that the expansion types listed above are consistent with the $\partial_u$-equations, the equation $H_{1000} + H_{0100} = 0$ and the equations $S_{11} = 0$, $\Sigma_{1100} = 0$ used on $U_0$ in the sense that all terms in the equations have the same expansion types.

Assuming the given expansion types for the $s_k$, the $\partial_u$-equations for the $\hat{\Gamma}_{ABCD}$ imply at lowest order in $u$ that in general the $k_{\hat{\Gamma}_{ABCD}}$ must take the values given above. It follows then from the $\partial_u$-equations for the $\hat{e}^a_{AB}$ at lowest order in $u$ that the $k_{\hat{e}^a_{AB}}$ must take in general the values above. The remaining $\partial_u$-equations then imply at lowest order the other expansion types.

With these observations the Lemma follows from our procedure by a straightforward though lengthy induction argument. We do not write out the details. ∎

The equation

$$0 = S_{00} = \partial_u s + (1 - \mu\,\zeta)\, s_{00CD}\, \zeta^{CD},$$

should imply more precisely $k_s = 0$, because the expansion type of the tensorial component $s_{00CD}\, \zeta^{CD}$ should be 2. The contraction of the spinor fields on the right hand side implies cancellations which lower the expansion types of the contracted quantities on the right hand side. These cancellations cannot be controlled in the explicit expression

$$0 = \partial_u s + (1 - \mu\,\zeta)\,(s_0\,\zeta_{11} - 2\,s_1\,\zeta_{01} + s_2\,\zeta_{00}),$$

by the simple rules given above, they only suggest an expansion type $k_s \leq 2$. Fortunately, this does not prevent us from determining the other expansion types. In the equation

$$0 = \Sigma_{0011} = \partial_u \zeta_{11} - s + \zeta\,(1 - \mu\,\zeta)\, s_{0011},$$

$s$ is added to a field of expansion type 2 and the equation

$$0 = S_{11} = \partial_w s + s_{11CD}\, \zeta^{CD} = \partial_w s + s_{1111}\, \zeta_{00} \quad \text{on} \quad U_0,$$

is consistent with $k_s \leq 2$. No further equation involving $s$ is needed in the convergence proof.

## 5.2 The complete set of equations on $\hat{S}$

Because only a certain subset of the system of equations has been used to determine the formal expansions of the fields, it remains to be shown that the latter define in fact a

formal solution to the complete system of conformal static vacuum field equations. To simplify stating the following result it will be assumed in this subsection that the formal expansions for

$$\hat{e}^a{}_{AB}, \ \hat{\Gamma}_{ABCD}, \ \zeta, \ \zeta_{AB}, \ \hat{s}, \ s_j,$$

determined in Lemma 5.1 define in fact absolutely convergent series on an open neighbourhood of the point $O$, which we assume to coincide with $\hat{S}$. There will arise no problem from this assumption because the following two Lemmas will not be used in the derivation of the estimates in the next section.

**Lemma 5.4** *With the assumptions above the corresponding functions*

$$e^a{}_{AB}, \ \Gamma_{ABCD}, \ \zeta, \ \zeta_{AB}, \ s, \ s_j,$$

*satisfy the complete set of the conformal vacuum field equations on the set $U_0$ in the sense that the fields*

$$t_{AB}{}^{EF}{}_{CD}, \ R_{ABCDEF}, \ \Sigma_{AB}, \ \Sigma_{ABCD}, \ S_{AB}, \ H_{ABCD},$$

*calculated from these functions on $\hat{S} \setminus U_0$ have vanishing limit as $u \to 0$.*

PROOF. Because of the equations solved already and the symmetries involved, we only need to examine the behaviour of the fields

$$t_{11}{}^{EF}{}_{01}, \ R_{AB0111}, \ \Sigma_{01}, \ \Sigma_{01CD}, \ S_{01}, \ H_{1(BCD)_k}, \quad k = 1, 2, 3,$$

near $U_0$.

With (4.8), (4.9), (4.11), (4.12) the $\partial_u$-equations imply for the frame and the dual frame coefficients the slightly stronger results (4.6), (4.7). A direct calculation gives then

$$t_{01}{}^{EF}{}_{11} = 2\,\Gamma_{01}{}^{(E}{}_1\epsilon_1{}^{F)} - \Gamma_{11}{}^{(E}{}_0\epsilon_1{}^{F)} - \Gamma_{11}{}^{(E}{}_1\epsilon_0{}^{F)} - \sigma^{EF}{}_a\,(e^a{}_{11,c}\,e^c{}_{01} - e^a{}_{01,c}\,e^c{}_{11}) = O(u),$$

as $u \to 0$.

With the particular result

$$t_{01}{}^{01}{}_{11} = \Gamma_{0111} - \frac{1}{2}\,e^2{}_{11,2} - \frac{1}{2\,u}\,e^1{}_{11} + O(u^2) = O(u),$$

follows

$$R_{000111} = \Gamma_{1100,1}\,e^1{}_{01} + \Gamma_{1100,2}\,e^2{}_{01} - \Gamma_{0100,1}\,e^1{}_{11} - \Gamma_{0100,2}\,e^2{}_{11} - \Gamma_{0100,3}$$

$$-\Gamma_{1100}\,\Gamma_{1100} + 2\,\Gamma_{0100}\,(\Gamma_{1101} - \Gamma_{0111}) - t_{01}{}^{01}{}_{11}\,\Gamma_{0100} - t_{01}{}^{11}{}_{11}\,\Gamma_{1100} - \frac{1}{2}\,s_{0011}$$

$$= \frac{1}{2\,u}\,(\Gamma_{1100,2} - 2\,\Gamma_{1101} + 3\,\Gamma_{0111} - \frac{1}{2}\,e^2{}_{11,2} - \frac{3}{2\,u}\,e^1{}_{11}) - \frac{1}{2}\,s_{0011} + O(u) \to$$

$$\frac{1}{2}\left(\partial_v\,\partial_u\,\Gamma_{1100} - 2\,\partial_u\,\Gamma_{1101} + 3\,\partial_u\,\Gamma_{0111} - \frac{1}{2}\,\partial_v\,\partial_u\,e^2{}_{11} - \frac{3}{4}\,\partial_u^2\,e^1{}_{11} - s_{0011}\right) = 0 \text{ as } u \to 0,$$

where the $\partial_u$-equations and the relation $\partial_v s_1 = 3 s_2$ on $U_0$ are used to calculate the limit. Similarly,

$$R_{010111} = \frac{1}{2\,u}\,\Gamma_{1101,2} - \frac{1}{2\,u}\,\Gamma_{1111} - \frac{1}{2}\,s_{0111} + O(u)$$

$$\rightarrow \frac{1}{2}\,(\partial_v\,\partial_u\,\Gamma_{1101} - \partial_u\,\Gamma_{1111} - s_{0111}) = 0 \quad \text{as} \quad u \rightarrow 0,$$

where the $\partial_u$-equations and the relation $\partial_v s_2 = 2 s_3$ on $U_0$ are used,

$$R_{110111} = \frac{1}{2\,u}\,\Gamma_{1111,2} - \frac{1}{2}\,s_{1111} + O(u) \rightarrow \frac{1}{2}\,(\partial_v\,\partial_u\,\Gamma_{1111} - s_{1111}) = 0 \quad \text{as} \quad u \rightarrow 0,$$

where the $\partial_u$-equations and the relation $\partial_v s_3 = s_4$ on $U_0$ are used.

By (5.3) and the remark following (5.5) we know that $\zeta = 0$, $\zeta_{01} = 0$, $\zeta_{11} = 0$, $\partial_v \zeta_{00} = 0$, $\partial_v s = 0$ on $U_0$. The $\partial_u$-equations and (5.6) imply

$$\Sigma_{01} = \frac{1}{2\,u}\partial_v\,\zeta - \zeta_{01} + O(u) \rightarrow \frac{1}{2}\partial_v\,\zeta_{00} - \zeta_{01} = 0,$$

$$\Sigma_{0100} = \frac{1}{2\,u}\,(\partial_v\,\zeta_{00} - 2\,\zeta_{01}) + O(u) \rightarrow \frac{1}{2}\,(\partial_v\,\partial_u\,\zeta_{00} - 2\,\partial_u\,\zeta_{01}) = 0,$$

$$\Sigma_{0101} = \frac{1}{2\,u}\,(\partial_v\,\zeta_{01} - \zeta_{11}) + \frac{1}{2}\,s + O(u) \rightarrow \frac{1}{2}\,(\partial_v\,\partial_u\,\zeta_{01} - \partial_u\,\zeta_{11} + s) = 0,$$

$$\Sigma_{0111} = \frac{1}{2\,u}\,\partial_v\,\zeta_{11} + O(u) \rightarrow \frac{1}{2}\,\partial_v\,\partial_u\,\zeta_{11} = 0.$$

$$S_{01} = \frac{1}{2\,u}\,\partial_v\,s + s_{0111}\,\zeta_{00} + O(u) \rightarrow \frac{1}{2}\,(\partial_v\,\partial_u\,s + 2\,s_{0111}\,\zeta_{00}) = \frac{1}{2}\,\partial_v\,(\partial_u\,s + s_{0011}\,\zeta_{00}) = 0,$$

as $u \rightarrow 0$.

With our assumptions (and formally setting $s_5 = 0$) we get for $k = 0, \ldots, 3$

$$\gamma_k \equiv \lim_{u\to 0}\,(-2\,H_{0(ABC)_k}) = (6 - k)\,\partial_u\,s_{k+1} - \partial_v\,\partial_u\,s_k - (4 - k)\,\mu\,s_{k+1}\,\zeta_{00},$$

$$\beta_k \equiv \lim_{u\to 0}\,(-2\,H_{1(ABC)_k}) = 2\,\partial_w\,s_k - \partial_v\,\partial_u\,s_{k+1} + (3 - k)\,\partial_u\,s_{k+2} - (3 - k)\,\mu\,s_{k+2}\,\zeta_{00}.$$

The expected tensorial nature of $s_{ABCD}$ and $H_{ABCD}$ (cf. Lemma 4.1) would imply

$$4\,\beta_1 = \partial_v\,\beta_0 - \partial_v\,\gamma_1 + 2\,\gamma_2, \qquad 12\,\beta_2 = \partial_v^2\,\beta_0 - \partial_v^2\,\gamma_1 - 2\,\partial_v\,\gamma_2 + 4\,\gamma_3,$$

$$24\,\beta_3 = \partial_v^3\,\beta_0 - \partial_v^3\,\gamma_1 - 2\,\partial_v^2\,\gamma_2 - 8\,\partial_v\,\gamma_3 \quad \text{on} \quad U_0.$$

It turns out that these relations can in fact be verified by a direct calculation with the expressions for $\gamma_k$, $\beta_k$ obtained above. Because the equations used to establish Lemma 5.1 imply $\gamma_k = 0$, $\beta_0 = 0$, it follows that $\beta_1 = \beta_2 = \beta_3 = 0$ so that in fact $H_{ABCD} \rightarrow 0$ as $u \rightarrow 0$. $\blacksquare$

We can now prove the desired result.

**Lemma 5.5** *The functions*

$$e^a{}_{AB}, \, \Gamma_{ABCD}, \, \zeta, \, \zeta_{AB}, \, s, \, s_j,$$

*corresponding to the expansions determined in Lemma 5.1 satisfy the complete set of conformal vacuum field equations on the set $\hat{S}$.*

PROOF. It needs to be shown that the *zero quantities*

$$t_{01}{}^{EF}{}_{11}, \quad R_{AB0111}, \quad \Sigma_{01}, \quad \Sigma_{11}, \quad \Sigma_{01CD}, \quad \Sigma_{11CD}, \quad S_{01}, \quad S_{11}, \quad H_{1ABCD},$$

vanish on $\hat{S}$. For this purpose we shall derive a system of *subsidiary equations* for these fields.

Given the fields

$$e^a{}_{AB}, \, \Gamma_{ABCD}, \, \zeta, \, \zeta_{AB}, \, s, \, s_{ABCD},$$

we have the 1-forms $\sigma^{AB}$ dual to $e_{AB}$ and the connection form $\omega^A{}_B = \Gamma_{CD}{}^A{}_B \, \sigma^{CD}$.

To derive the subsidiary system we consider the torsion form

$$\Theta^{AB} = \frac{1}{2} \, t_{CD}{}^{AB}{}_{EF} \, \sigma^{CD} \wedge \sigma^{EF},$$

and the form

$$\Omega^{*A}{}_B \equiv \Omega^A{}_B - \hat{\Omega}^A{}_B = \frac{1}{2} \, R^A{}_{BCDEF} \, \sigma^{CD} \wedge \sigma^{EF},$$

obtained as difference of the curvature form

$$\Omega^A{}_B = \frac{1}{2} \, r^A{}_{BCDEF} \, \sigma^{CD} \wedge \sigma^{EF},$$

and the form

$$\hat{\Omega}^A{}_B = \frac{1}{2} \, s^A{}_{BCE} \, \sigma^C{}_F \wedge \sigma^{EF}.$$

The following general relations will be used: The identity $\sigma^a \wedge \sigma^b \wedge \sigma^c = \epsilon^{abc} \nu$ with $\nu = \frac{1}{3!} \epsilon_{def} \, \sigma^d \wedge \sigma^e \wedge \sigma^f$, which holds in 3-dimensional spaces. In space spinor form it takes the form

$$\sigma^{AB} \wedge \sigma^{CD} \wedge \sigma^{EF} = \epsilon^{AB\,CD\,EF} \nu \quad \text{with} \quad \epsilon^{AB\,CD\,EF} = \frac{i}{\sqrt{2}} \left( \epsilon^{AC} \epsilon^{BF} \epsilon^{DE} - \epsilon^{AE} \epsilon^{BD} \epsilon^{FC} \right),$$

which implies

$$\sigma^{AB} \wedge \sigma^C{}_D \wedge \sigma^{ED} = -i \sqrt{2} \, \epsilon^{A(C} \epsilon^{E)B} \nu = i \sqrt{2} \, h^{AB\,CE} \nu,$$

and thus

$$\hat{\Omega}^A{}_B \wedge \sigma^{BD} = \frac{1}{2} \, s^A{}_{BCE} \, \sigma^{BD} \wedge \sigma^C{}_F \wedge \sigma^{EF} = 0.$$

The equations

$$i_H (\alpha \wedge \beta) = i_H \alpha \wedge \beta + (-1)^k \alpha \wedge i_H \beta, \qquad \mathcal{L}_H \alpha = (d \circ i_H + i_H \circ d) \alpha,$$

which holds for arbitrary vector field $H$, $k$-form $\alpha$, and $j$-form $\beta$. Finally, we note that in the presence of torsion the Ricci identity for a spinor field $\iota_{E...H}$ of degree $m$ reads

$$(D_{AB} D_{CD} - D_{CD} D_{AB}) \iota_{EF...H} = - \iota_{LF...H} \, r^L{}_{E\,ABCD} - \iota_{EL...H} \, r^L{}_{F\,ABCD} - \ldots$$

$$- \iota_{EF...L} \, r^L{}_{H\,ABCD} - t_{AB}{}^{KL}{}_{CD} D_{KL} \iota_{EF...H}.$$

We shall derive now the subsidiary equations. The fields $\Theta^{AB}$ and $\Omega^A{}_B$ satisfy the first structural equation

$$d\sigma^{AB} = -\omega^A{}_C \wedge \sigma^{CB} - \omega^B{}_C \wedge \sigma^{AC} + \Theta^{AB},$$

and the second structural equation

$$d\omega^A{}_B = -\omega^A{}_C \wedge \omega^C{}_B + \Omega^A{}_B,$$

respectively. These equations imply

$$d\Theta^{AB} = 2\,\Omega^{(A}{}_C \wedge \sigma^{B)C} - 2\,\omega^{(A}{}_C \wedge \Theta^{B)C} = 2\,\Omega^{*(A}{}_C \wedge \sigma^{B)C} - 2\,\omega^{(A}{}_C \wedge \Theta^{B)C}.$$

We set $H = e_{00}$ and observe that the gauge conditions and the $\partial_u$-equations imply

$$i_H \sigma^{AB} = \epsilon_0{}^A \epsilon_0{}^B = h_{00}{}^{AB}, \qquad i_H \omega^A{}_B = 0, \qquad i_H \Theta^{AB} = 0, \qquad i_H \Omega^{*A}{}_B = 0.$$

It follows that
$$\mathcal{L}_H \Theta^{AB} = (d \circ i_H + i_H \circ d)\,\Theta^{AB} = 2\,\Omega^{*(A}{}_0\, \epsilon_0{}^{B)},$$

and thus
$$\mathcal{L}_H < \Theta^{AB}, e_{01} \wedge e_{11} > =$$

$$2 < \Omega^{*(A}{}_0, e_{01} \wedge e_{11} > \epsilon_0{}^{B)} + < \Theta^{AB}, [H, e_{01}] \wedge e_{11} > + < \Theta^{AB}, e_{01} \wedge [H, e_{11}] > .$$

The first structural equation, the gauge conditions, and the $\partial_u$-equations imply

$$0 = < \Theta^{EF}, H \wedge e_{CD} > e_{EF} = -\Gamma_{CD}{}^{EF}{}_{00}\, e_{EF} - [H, e_{CD}],$$

whence
$$[H, e_{CD}] = -2\,\Gamma_{CD01}\, e_{00} + 2\,\Gamma_{CD00}\, e_{01}.$$

This implies

$$\mathcal{L}_H < \Theta^{AB}, e_{01} \wedge e_{11} > = 2\,\Gamma_{0100} < \Theta^{AB}, e_{01} \wedge e_{11} > + 2 < \Omega^{*(A}{}_0, e_{01} \wedge e_{11} > \epsilon_0{}^{B)},$$

i.e.
$$(\partial_u + \frac{1}{u})\, t_{01}{}^{AB}{}_{11} = 2\,\hat{\Gamma}_{0100}\, t_{01}{}^{AB}{}_{11} + 2\,R^{(A}{}_{0\,01\,11}\epsilon_0{}^{B)}. \tag{5.7}$$

With the first structural equation we obtain

$$d\hat{\Omega}_{AB} - \omega^H{}_A \wedge \hat{\Omega}_{HB} - \omega^H{}_B \wedge \hat{\Omega}_{AH} = \frac{1}{2}\, D_{GH}\, s_{ABCD}\, \sigma^{GH} \wedge \sigma^C{}_F \wedge \sigma^{DF} = \frac{i}{\sqrt{2}}\, H^E{}_{ABE}\, \nu,$$

and from the second structural equation we get

$$d\,\Omega_{AB} - \omega^H{}_A \wedge \Omega_{HB} - \omega^H{}_B \wedge \Omega_{AH} = 0,$$

which give together

$$d\,\Omega^*_{AB} - \omega^H{}_A \wedge \Omega^*_{HB} - \omega^H{}_B \wedge \Omega^*_{AH} = -\frac{i}{\sqrt{2}}\,H^E{}_{ABE}\,\nu,$$

and thus, with the equations above,

$$(\partial_u + \frac{1}{u})\,R_{AB\,01\,11} = 2\,\hat{\Gamma}_{0100}\,R_{AB\,01\,11} + \frac{1}{2}\,H_{1AB0}. \tag{5.8}$$

The identity

$$D_{AB}\,\Sigma_{CD} - D_{CD}\,\Sigma_{AB} = t_{AB}{}^{EF}{}_{CD}\,D_{EF}\zeta + \Sigma_{CDAB} - \Sigma_{ABCD},$$

gives with the gauge conditions and the $\partial_u$-equations

$$\partial_u\,\Sigma_{CD} + \frac{2}{u}\,\epsilon_{(C}{}^0\,\epsilon_{D)}{}^1\,\Sigma_{01} = 2\,\hat{\Gamma}_{CD00}\,\Sigma_{01} + \Sigma_{CD00}. \tag{5.9}$$

The identity

$$D_{AB}\,\Sigma_{CDEF} - D_{CD}\,\Sigma_{ABEF} = -2\,\zeta_{K(E}\,R^K{}_{F)ABCD} + t_{AB}{}^{GH}{}_{CD}\,D_{GH}\zeta_{EF}$$

$$+S_{CD}\,h_{ABEF} - S_{AB}\,h_{CDEF}$$

$$+(1-2\,\mu\,\zeta)\,(\Sigma_{AB}\,s_{CDEF} - \Sigma_{CD}\,s_{ABEF}) + \zeta\,(1-\mu\,\zeta)\,(\epsilon_{CA}\,H_{BDEF} + \epsilon_{DB}\,H_{CAEF}),$$

implies with the gauge conditions and the $\partial_u$-equations

$$\partial_u\,\Sigma_{CDEF} + \frac{2}{u}\,\epsilon_{(C}{}^0\,\epsilon_{D)}{}^1\,\Sigma_{01EF} = \tag{5.10}$$

$$2\,\hat{\Gamma}_{CD00}\,\Sigma_{01EF} + S_{CD}\,h_{00EF} - (1-2\,\mu\,\zeta)\,\Sigma_{CD}\,s_{00EF} + \zeta\,(1-\mu\,\zeta)\,\epsilon_{D0}\,H_{C0EF}.$$

The identity

$$D_{AB}\,S_{CD} - D_{CD}\,S_{AB} = t_{AB}{}^{EF}{}_{CD}\,D_{EF}s - \mu\,\{\Sigma_{AB}\,s_{CDEF} - \Sigma_{CD}\,s_{ABEF}\}\,\zeta^{EF}$$

$$(1-\mu\,\zeta)\,\{\Sigma_{AB}{}^{EF}\,s_{CDEF} - \Sigma_{CD}{}^{EF}\,s_{ABEF} + (\epsilon_{CA}\,H_{BDEF} + \epsilon_{DB}\,H_{CAEF})\,\zeta^{EF}\},$$

implies with the gauge conditions and the $\partial_u$-equations

$$\partial_u\,S_{CD} + \frac{2}{u}\,\epsilon_{(C}{}^0\,\epsilon_{D)}{}^1\,S_{01} = \tag{5.11}$$

$$2\,\hat{\Gamma}_{CD00}\,S_{01} + \mu\,\Sigma_{CD}\,s_{00EF}\,\zeta^{EF} - (1-\mu\,\zeta)\,\{\Sigma_{CD}{}^{EF}\,s_{00EF} - \epsilon_{D0}\,H_{C0EF}\,\zeta^{EF}\}.$$

Finally we have the identity

$$2\,D^{EF}\,H_{EFAB} = -4\,s_{K(BGH}\,R^K{}_{A)}{}^{EG}{}_E{}^H + t^E{}_F{}^{KL}{}_{EH}\,D_{KL}\,s_{AB}{}^{FH} \tag{5.12}$$

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$$-\frac{4\,\mu}{1-\mu\,\zeta}\,s_{H(ABF}\,\Sigma^{EF}{}_{G)}{}^{H}-\frac{2\,\mu^2}{(1-2\,\mu)^2}\,\Sigma^{EF}\,s_{H(ABF}\,\zeta_{E)}{}^{H}$$

$$+\frac{\mu}{1-\mu\,\zeta}\,\left\{2\,H_{EHAB}\,\zeta^{EH}-2\,H^{E}{}_{EH(A}\,\zeta_{B)}{}^{H}\right\},$$

where the right hand side is a linear function of the zero quantities. The gauge conditions and the equations $H_{0ABC}=0$, $H_{1000}=0$ imply for the left hand side

$$D^{EF}\,H_{EFAB}=\partial_u\,H_{11AB}+\frac{1}{u}\,\left\{H_{11AB}+H_{110(A}\,\epsilon_{B)}{}^{0}\right\}-(\frac{1}{2\,u}\,\partial_v+\hat{e}^a{}_{01}\partial_{z^a})\,H_{10AB} \quad (5.13)$$

$$-2\,\hat{\Gamma}_{0100}\,H_{11AB}-\hat{\Gamma}_{010A}\,H_{110B}-\hat{\Gamma}_{010B}\,H_{110A}+\hat{\Gamma}_{011A}\,H_{100B}+\hat{\Gamma}_{011B}\,H_{100A}+\hat{\Gamma}_{1100}\,H_{10AB}.$$

Equations (5.7), (5.8), (5.9), (5.10), (5.11), and equation (5.12) with (5.13) observed on the left hand side provide the system of subsidiary equations. Note that the right hand side of this system is a linear function of the zero quantities. It implies with Lemma 5.4 that all zero quantities vanish on $\hat{S}$. ∎

If the series considered in Lemma 5.1 are absolutely convergent it thus follows from Lemma 5.5 that they define in fact a solution to the complete set of static conformal vacuum field equations on $\hat{S}$.

# 6 Convergence of the formal expansion

Let there be given a given sequence

$$\hat{\mathcal{D}}_n=\{\psi_{A_2B_2A_1B_1},\ \psi_{A_3B_3A_2B_2A_1B_1},\ \psi_{A_4B_4A_3B_3A_2B_2A_1B_1},\ \ldots\},$$

of totally symmetric spinors as in Lemma 3.1 and set in the expansion (4.16) of $s_0(u,v)$

$$D^*_{(A_1B_1}\,\cdots\,D^*_{A_mB_m}\,s^*_{ABCD)}(i)=\psi_{A_1B_1\ldots A_mB_mABCD},\quad m\geq 0.$$

Observing the estimates (3.11), one finds as a necessary condition for the function $s_0$ on $W_0$ to determine an analytic solution to the conformal static vacuum field equations that its non-vanishing Taylor coefficients at the point $O$ satisfy estimates of the form

$$|\partial_u^m\,\partial_v^n\,s_0(O)|=m!\,n!\,|\psi_{m,n}|\leq\binom{2m+4}{n}\,m!\,n!\,M\,r_1^{-m},\quad m\geq 0,\quad 0\leq n\leq 2\,m+4.$$

$$(6.1)$$

A slightly different type of estimate will be more convenient for us.

**Lemma 6.1** *For given $\rho_0\in\mathbb{R}$, $0<\rho_0\leq e^2$, there exist positive constants $r_0$, $c_0$ so that (6.1) implies estimates of the form*

$$|\partial_u^m\,\partial_v^n\,s_0(O)|\leq c_0\,\frac{m!\,n!\,r_0^m\,\rho_0^n}{(1+m)^2\,(1+n)^2},\quad m\geq 0,\quad 0\leq n\leq 2\,m+4. \qquad (6.2)$$

PROOF. With $r_0 = 4\,e^6\,r_1^{-1}\,\rho_0^{-2}$ and $c_0 = 16\,M\,e^8\,\rho_0^{-4}$, the estimate $1 \leq \binom{2\,m+4}{n} \leq 2^{2m+4}$, which follows from the binomial law $(1+x)^{2m+4} = \sum_{n=0}^{2m+4} \binom{2\,m+4}{n} x^n$, and the estimate $e^x \geq 1 + x$, which holds for $x \geq 0$, we get

$$\binom{2m+4}{n} m!\,n!\,M\,r_1^{-m} \leq 16\,M\,m!\,n!\,(4\,r_1^{-1})^m = c_0\,m!\,n!\,\frac{r_0^m}{(e^m)^2}\,\frac{\rho_0^n}{(e^n)^2}\,\left(\frac{\rho_0}{e^2}\right)^{2\,m+4-n}$$

$$\leq c_0\,m!\,n!\,\frac{r_0^m}{(1+m)^2}\,\frac{\rho_0^n}{(1+n)^2}, \quad m \geq 0, \quad 0 \leq n \leq 2\,m+4.$$

∎

The following Lemma provides our main estimates.

**Lemma 6.2** *Suppose $s_0 = s_0(u,v)$ is a holomorphic function defined on some open neighbourhood $U$ of $O = \{u = 0, v = 0, w = 0\}$ in $W_0 = \{w = 0\}$ which has an expansion of the form*

$$s_0(u,v) = \sum_{m=0}^{\infty} \sum_{n=0}^{2m+4} \psi_{m,n}\,u^m\,v^n,$$

*so that its Taylor coefficients at the point $O$ satisfy estimates of the type (6.2) with some positive constants $c_0^*$, $r_0$, and $\rho_0 < 1/2$. Then there exist positive constants $r \geq r_0$, $\rho$, $c_{\hat{e}_{AB}^a}$, $c_{\hat{\Gamma}_{ABCD}}$, $c_\zeta$, $c_{\zeta_i}$, $c_{\hat{s}}$, $c_k$ so that the expansion coefficients determined from $s_0$ in Lemma 5.1 satisfy for $m, n, p = 0, 1, 2, \ldots$*

$$|\partial_u^m\,\partial_v^n\,\partial_w^p\,s_k(O)| \leq c_k\,\frac{r^{m+p}\,(m+p)!\,\rho^n\,n!}{(m+1)^2\,(n+1)^2\,(p+1)^2}, \tag{6.3}$$

*and*

$$|\partial_u^m\,\partial_v^n\,\partial_w^p\,f(O)| \leq c_f\,\frac{r^{m+p-1}\,(m+p)!\,\rho^n\,n!}{(m+1)^2\,(n+1)^2\,(p+1)^2}, \tag{6.4}$$

*where $f$ stands for any of the functions $\hat{e}_{AB}^a$, $\hat{\Gamma}_{ABCD}$, $\zeta$, $\zeta_i$, $\hat{s}$.*

**Remark 6.3** *Observing the $v$-finite expansion types discussed in Lemma 5.2, we can replace the right hand sides in the estimates above by zero if $n$ is large enough relative to $m$. This will not be pointed out at each step and for convenience the estimates will be written as above. The expansion types obtained in Lemma 5.2 will become important and will be observed, however, when we derive the estimates.*

We shall make use of arguments discussed in [24]. The following four Lemmas are essentially given in that article.

**Lemma 6.4** *For any non-negative integer $n$ there is a positive constant $C$ independent of $n$ so that*

$$\sum_{k=0}^{n} \frac{1}{(k+1)^2(n-k+1)^2} \leq C\,\frac{1}{(n+1)^2}.$$

PROOF. Denoting by $[n/2]$ the largest integer $\leq n/2$, we get with $C = \sum_{k=0}^{\infty} \frac{8}{(k+1)^2}$

$$\sum_{k=0}^{n} \frac{1}{(k+1)^2(n-k+1)^2} \leq \sum_{k=0}^{[n/2]} \frac{2}{(k+1)^2(n-k+1)^2}$$

$$\leq \sum_{k=0}^{[n/2]} \frac{2}{(k+1)^2([n/2]+1)^2} \leq C \frac{1}{(n+1)^2}.$$

■

In the following $C$ will always denote the constant above.

**Lemma 6.5** *For any integers $m$, $n$, $k$, $j$ with $0 \leq k \leq m$, and $0 \leq j \leq n$ resp. $0 \leq j \leq n-1$ holds*

$$\binom{m}{k}\binom{n}{j} \leq \binom{m+n}{k+j} \qquad resp. \qquad \binom{m}{k}\binom{n-1}{j} \leq \binom{m+n}{k+j}.$$

PROOF. This follows by induction, using the general formula $\binom{n+1}{j} = \binom{n}{j} + \binom{n}{j-1}$, or by expanding $(x+y)^{m+n} = (x+y)^m (x+y)^n$, using the binomial law $(x+y)^p = \sum_{j=0}^{p} \binom{p}{j} x^j y^{p-j}$. ■

If $f$ is holomorphic on the polydisk $P = \{(u,v,w,) \in \mathbb{C}^3 | \; |u| \leq 1/r_1, |v| \leq 1/r_2, |w| \leq 1/r_3\}$, with some $r_1, r_2, r_3 > 0$, one has the Cauchy estimates

$$|\partial_u^m \partial_v^n \partial_w^p f(O)| \leq r_1^m \, r_2^n \, r_3^p \, m! \, n! \, p! \, \sup_P |f|, \qquad m,n,p = 0,1,2,\ldots \qquad (6.5)$$

where $O$ denotes the origin $u = 0$, $v = 0$, $w = 0$. We need a slight modification of this.

**Lemma 6.6** *If $f$ is holomorphic near $O$, there exist positive constants $c$, $r_0$, $\rho_0$ so that*

$$|\partial_u^m \partial_v^n \partial_w^p f(O)| \leq c \frac{r^{m+p} (m+p)! \, \rho^n \, n!}{(m+1)^2 \, (n+1)^2 \, (p+1)^2}, \qquad m,n,p = 0,1,2,\ldots$$

*for any $r \geq r_0$, $\rho \geq \rho_0$. If in addition $f(0,v,0) = 0$, the constants can be chosen such that*

$$|\partial_u^m \partial_v^n \partial_w^p f(O)| \leq c \frac{r^{m+p-1} (m+p)! \, \rho^n \, n!}{(m+1)^2 \, (n+1)^2 \, (p+1)^2}, \qquad m,n,p = 0,1,2,\ldots$$

*for any $r \geq r_0$, $\rho \geq \rho_0$.*

PROOF. Choosing an estimate of the type (6.5) with $r_1 = r_3$ and setting $c = \alpha \sup_P |f|$, $r_0 = e^2 r_1 = e^2 r_3$, $\rho_0 = e^2 r_2$ with some $\alpha > 0$, one gets from (6.5)

$$|\partial_u^m \partial_v^n \partial_w^p f(O)| \leq c\,\alpha^{-1} r_0^{m+p} (m+p)! \, \rho_0^n \, n! \, e^{-2(m+n+p)} \leq c\,\alpha^{-1} \frac{r_0^{m+p} (m+p)! \, \rho_0^n \, n!}{(m+1)^2 \, (n+1)^2 \, (p+1)^2}.$$

With $\alpha = 1$ the monotonicity of $x \to x^q$, $q \geq 0$, $x > 0$ implies the first estimate. With $\alpha = r_0$ the estimate above implies

$$|\partial_u^m \, \partial_v^n \, \partial_w^p \, f(O)| \leq c \, \frac{r_0^{m+p-1} \, (m+p)! \, \rho_0^n \, n!}{(m+1)^2 \, (n+1)^2 \, (p+1)^2}.$$

If $f(0, v, 0) = 0$, then $\partial_u^0 \, \partial_v^n \, \partial_w^0 \, f(O) = 0$ for $n \in N_0$ and the last relation remains true for $m + p = 0$, i.e. $m = 0$ and $p = 0$, if $r_0$ and $\rho_0$ are replaced by $r \geq r_0$ and $\rho \geq \rho_0$. If $m + p > 0$ the result follows as above. ∎

**Lemma 6.7** *Let $m$, $n$, $p$ be non-negative integers and $f_i$, $i = 1, \ldots, N$, be smooth complex valued functions of $u$, $v$, $w$ on some neighbourhood $U$ of $O$ whose derivatives satisfy on $U$ (resp. at a given point $p \in U$) estimates of the form*

$$|\partial_u^j \, \partial_v^k \, \partial_w^l \, f_i| \leq c_i \, \frac{r^{j+l+q_i} \, (j+l)! \, \rho^k \, k!}{(j+1)^2 \, (k+1)^2 \, (l+1)^2} \quad for \quad 0 \leq j \leq m, \ 0 \leq k \leq n, \ 0 \leq l \leq p,$$

*with some positive constants $c_i$, $r$, $\rho$ and some fixed integers $q_i$ (independent of $j, k, l$). Then one has on $U$ (resp. at $p$) the estimates*

$$|\partial_u^m \, \partial_v^n \, \partial_w^p \, (f_1 \cdot \ldots \cdot f_N)| \leq C^{3 \, (N-1)} \, c_1 \cdot \ldots \cdot c_N \, \frac{r^{m+p+q_1+\ldots+q_N} \, (m+p)! \, \rho^n \, n!}{(m+1)^2 \, (n+1)^2 \, (p+1)^2}. \quad (6.6)$$

**Remark 6.8** *(i) Lemma 6.7 remains obviously true if $m$, $n$, $p$ are replaced in (6.6) by integers $m'$, $n'$, $p'$ with $0 \leq m' \leq m$, $0 \leq n' \leq n$, $0 \leq p' \leq p$.*
*(ii) By the argument given below the factor $C^{3 \, (N-1)}$ in (6.6) can be replaced by $C^{(3-r) \, (N-1)}$ if $r$ of the integers $m$, $n$, $p$ vanish.*

PROOF. We prove the case $N = 2$. The general case then follows with the first of Remarks 6.8 by an induction argument. With the estimates above and Lemmas (6.4) and (6.5) we get on $U$ (resp. at $p$)

$$|\partial_u^m \, \partial_v^n \, \partial_w^p \, (f_1 \, f_2)| \leq \sum_{j=0}^{m} \sum_{k=0}^{n} \sum_{l=0}^{p} \binom{m}{j} \binom{n}{k} \binom{p}{l} |\partial_u^j \, \partial_v^k \, \partial_w^l \, f_1| |\partial_u^{m-j} \, \partial_v^{n-k} \, \partial_w^{p-l} \, f_2|$$

$$\leq \sum_{j=0}^{m} \sum_{k=0}^{n} \sum_{l=0}^{p} \binom{m}{j} \binom{n}{k} \binom{p}{l} \frac{c_1 \, r^{j+l+q_1} \, (j+l)! \, \rho^k \, k!}{(j+1)^2 \, (k+1)^2 \, (l+1)^2} \frac{c_2 \, r^{m-j+p-l+q_2} \, (m-j+p-l)! \, \rho^{n-k} \, (n-k)!}{(m-j+1)^2 \, (n-k+1)^2 \, (p-l+1)^2}$$

$$\leq \sum_{j=0}^{m} \sum_{k=0}^{n} \sum_{l=0}^{p} \frac{\binom{m}{j} \binom{p}{l}}{\binom{m+p}{j+l}} \frac{c_1 \, c_2 \, r^{m+p+q_1+q_2} \, (m+p)! \, \rho^n \, n!}{(j+1)^2 \, (k+1)^2 \, (l+1)^2 \, (m-j+1)^2 \, (n-k+1)^2 \, (p-l+1)^2}$$

$$\leq C^3 \, c_1 \, c_2 \, \frac{r^{m+p+q_1+q_2} \, (m+p)! \, \rho^n \, n!}{(m+1)^2 \, (n+1)^2 \, (p+1)^2}.$$

∎

We are now able to proof our main estimates.

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**Proof of Lemma** 6.2. The proof will essentially be given by induction with respect to $m$ and $p$, following the procedure which led to Lemma 5.1. It is easy to see that the constants can be chosen to satisfy the estimates at lowest order. Leaving the choice of the constants open, we will derive from the induction hypothesis for the derivatives of the next order estimates of the form

$$|\partial_u^m \partial_v^n \partial_w^p s_k(O)| \leq c_k \frac{r^{m+p} (m+p)! \, \rho^n \, n!}{(m+1)^2 (n+1)^2 (p+1)^2} A_{s_k},$$

$$|\partial_u^m \partial_v^n \partial_w^p f(O)| \leq c_f \frac{r^{m+p-1} (m+p)! \, \rho^n \, n!}{(m+1)^2 (n+1)^2 (p+1)^2} A_f,$$

with certain constants $A_{s_k}$, $A_f$ which depend on $m$, $n$, $p$ and the constants $c_k$, $c_f$, $r$, and $\rho$. Sometimes superscripts will indicate to which order of differentiability particular constants $A_{s_k}$, $A_f$ refer. Occasionally we will have to make assumptions on $r$ to proceed with the induction step. We shall collect these conditions and the constants $A_{s_k}$, $A_f$, or estimates for them, and at the end it will be shown that the constants $c_k$, $c_f$, $r$, and $\rho$ can be adjusted so that all conditions are satisfied and $A_{s_k} \leq 1$, $A_f \leq 1$. This will complete the induction proof.

In the following it is understood that as above a function in a modulus sign is evaluated at the origin $O$. The symbol $x$ will stand for any of the fields

$$\hat{e}^a{}_{AB}, \; \hat{\Gamma}_{ABCD}, \; \zeta, \; \zeta_0, \; \zeta_1, \; \zeta_2, \; \hat{s}, \; s_1, \; s_2, \; s_3, \; s_4.$$

For the quantities which are known to vanish at $I$ the estimates are correct for $m = 0$, $p = 0$. Since we consider $\hat{s}$ as an unknown and $s(0) = -2$ as part of the equations, we thus only need to discuss the $s_k$. They are given on $I$ by

$$s_k = \frac{(4-k)!}{4!} \, \partial_v^k s_0.$$

It thus follows by our assumptions

$$|\partial_u^0 \partial_v^n \partial_w^0 s_k| = |\frac{(4-k)!}{4!} \partial_v^{k+n} s_0|$$

$$\leq \left\{ \begin{array}{cc} \frac{(4-k)!}{4!} c_0 \frac{\rho^{n+k} (n+k)!}{(n+k+1)^2} & \text{for} \quad n \leq 4-k \\ 0 \quad \text{for} \quad n > 4-k \end{array} \right\} = c_k \frac{\rho^n \, n!}{(n+1)^2} A_{s_k}^{m=0,p=0},$$

with

$$A_{s_k}^{m=0,p=0} = \frac{c_0}{c_k} \rho^k \, h_{k,n} \leq \frac{c_0}{c_k} \rho^k,$$

because

$$h_{k,n} \equiv \left\{ \begin{array}{cc} \frac{(4-k)!}{4!} \frac{(n+k)!}{n!} \frac{(n+1)^2}{(n+k+1)^2} & \text{for} \quad n \leq 4-k \\ 0 \quad \text{for} \quad n > 4-k \end{array} \right\} \leq 1.$$

We should study now under which conditions on the constants it can be shown by induction with respect to $m$ that the quantities $|\partial_u^m \partial_v^n \partial_w^0 x|$, $n \in N_0$, satisfy the estimates

given in the Lemma. We shall skip the details of this step, because the arguments used here are similar to those used to discuss the quantities $|\partial_u^m \, \partial_v^n \, \partial_w^p \, x|$ for general $p$ and the requirements obtained in that case are in fact stronger that those obtained for $p = 0$.

It will be assumed now that $p \geq 1$, that the estimates for $|\partial_u^m \, \partial_v^n \, \partial_w^l \, x|$ given in the Lemma hold true for $m, n \in N_0$, $0 \leq l \leq p - 1$, and try to determine conditions so that the induction step $p - 1 \to p$ can be performed.

By taking formal derivatives of the equation

$$0 = H_{0100} + H_{1000},$$

we get with our assumptions

$$|\partial_u^m \, \partial_v^n \, \partial_w^p \, s_0| \leq |\partial_u^{m+1} \, \partial_v^n \, \partial_w^{p-1} \, s_2| + |\partial_u^m \, \partial_v^n \, \partial_w^{p-1} \, (\hat{e}^1 \,_{11} \, \partial_u \, s_0)|$$

$$+|\partial_u^m \, \partial_v^n \, \partial_w^{p-1} \, (\hat{e}^2 \,_{11} \, \partial_v \, s_0)| + 4 \, |\partial_u^m \, \partial_v^n \, \partial_w^{p-1} \, (\hat{\Gamma}_{1101} \, s_0 + \hat{\Gamma}_{1100} \, s_1)|$$

$$+\mu \, |\partial_u^m \, \partial_v^n \, \partial_w^{p-1} \, (\frac{1}{1 - \mu \, \zeta} \, \{s_0 \, \zeta_2 + 2 \, s_1 \, \zeta_1 - 3 \, s_2 \, \zeta_0\})|.$$

For the first term on the right hand side follows immediately

$$|\partial_u^{m+1} \, \partial_v^n \, \partial_w^{p-1} \, s_2| \leq c_2 \, \frac{r^{m+p} \, (m + p)! \, \rho^n \, n!}{(m + 2)^2(n + 1)^2 p^2}.$$

A slight variation of the calculations in the proof Lemma 6.7 gives

$$|\partial_u^m \, \partial_v^n \, \partial_w^{p-1} \, (\hat{e}^1 \,_{11} \, \partial_u \, s_0)| \leq$$

$$\sum_{j=0}^m \sum_{k=0}^n \sum_{l=0}^{p-1} \binom{m}{j} \binom{n}{k} \binom{p-1}{l} |\partial_u^j \, \partial_v^k \, \partial_w^l \, \hat{e}^1 \,_{11}||\partial_u^{m-j+1} \, \partial_v^{n-k} \, \partial_w^{p-l-1} \, s_0| \leq$$

$$\sum_{j=0}^m \sum_{k=0}^n \sum_{l=0}^{p-1} \frac{\binom{m}{j} \binom{p-1}{l}}{\binom{m+p}{j+l}} \frac{c_{\hat{e}^1 \,_{11}} \, c_0 \, r^{m+p-1} \, (m + p)! \, \rho^n \, n!}{(j + 1)^2 \, (k + 1)^2 \, (l + 1)^2 \, (m - j + 2)^2 \, (n - k + 1)^2 \, (p - l)^2}$$

$$\leq C^3 \, c_{\hat{e}^1 \,_{11}} \, c_0 \, \frac{r^{m+p-1} \, (m + p)! \, \rho^n \, n!}{(m + 2)^2 \, (n + 1)^2 \, p^2},$$

where the sum over $j$ has been extended in the last step to $m + 1$.

Similarly one gets

$$|\partial_u^m \, \partial_v^n \, \partial_w^{p-1} \, (\hat{e}^2 \,_{11} \, \partial_v \, s_0)| \leq$$

$$\sum_{j=0}^m \sum_{k=0}^n \sum_{l=0}^{p-1} \binom{m}{j} \binom{n}{k} \binom{p-1}{l} |\partial_u^j \, \partial_v^k \, \partial_w^l \, \hat{e}^2 \,_{11}||\partial_u^{m-j} \, \partial_v^{n-k+1} \, \partial_w^{p-l-1} \, s_0| \leq$$

$$\sum_{j=0}^m \sum_{k=0}^n \sum_{l=0}^{p-1} \frac{\binom{m}{j} \binom{n}{k} \binom{p-1}{l}}{\binom{m+p-1}{j+l} \binom{n+1}{k}} \frac{c_{\hat{e}^2 \,_{11}} \, c_0 \, r^{m+p-2} \, (m + p - 1)! \, \rho^{n+1} \, (n + 1)!}{(j + 1)^2 \, (k + 1)^2 \, (l + 1)^2 \, (m - j + 1)^2 \, (n - k + 2)^2 \, (p - l)^2} \leq$$

$$C^3 \, c_{\hat{e}^2 \,_{11}} \, c_0 \, \frac{r^{m+p-2} \, (m + p - 1)! \, \rho^{n+1} \, (n + 1)!}{(m + 1)^2 \, (n + 2)^2 \, p^2},$$

where the sum over $k$ has been extended in the last step to $n + 1$.

We emphasis here again an observation which is important for us. By Lemma 5.2 the terms $\partial_u^j \, \partial_v^k \, \partial_w^l \, \hat{e}^2{}_{11}$ and $\partial_u^{m-j} \, \partial_v^{n-k+1} \, \partial_w^{p-l-1} \, s_0$ in the second line vanish if $k > 2\,j+1$ and $n - k + 1 > 2\,(m - j) + 4$ respectively. This implies, consistent with Lemma 5.2, that the term on the left hand side vanishes if $n > 2\,m + 4$. When we estimate the expression in the last line above we can thus assume without error that $n \le 2\,m + 4$.

Lemma 6.7 implies immediately

$$4 \, |\partial_u^m \, \partial_v^n \, \partial_w^{p-1} \, (\hat{\Gamma}_{1101} \, s_0 + \hat{\Gamma}_{1100} \, s_1)| \le 4\,C^3 \, (c_0 \, c_{\hat{\Gamma}_{1101}} + c_1 \, c_{\hat{\Gamma}_{1100}}) \, \frac{r^{m+p-2} \, (m + p - 1)! \, \rho^n \, n!}{(m + 1)^2 \, (n + 1)^2 \, p^2},$$

and, observing that $\zeta(O) = 0$,

$$\mu \, |\partial_u^m \, \partial_v^n \, \partial_w^{p-1} \, (\frac{1}{1 - \mu \, \zeta} \, \{s_0 \, \zeta_2 + 2 \, s_1 \, \zeta_1 - 3 \, s_2 \, \zeta_0\})|$$

$$\le \mu \, \sum_{l=0}^{\infty} |\partial_u^m \, \partial_v^n \, \partial_w^{p-1} \, ((\mu \, \zeta)^l \, \{s_0 \, \zeta_2 + 2 \, s_1 \, \zeta_1 - 3 \, s_2 \, \zeta_0\})|$$

$$\le \mu \, \sum_{l=0}^{\infty} \mu^l \, c_\zeta^l \, C^{3\,(l+1)} \, (c_0 \, c_{\zeta_2} + 2 \, c_1 \, c_{\zeta_1} + 3 \, c_2 \, c_{\zeta_0}) \frac{r^{m+p-l-2} \, (m + p - 1)! \, \rho^n \, n!}{(m + 1)^2 \, (n + 1)^2 \, p^2}$$

$$= \frac{\mu}{1 - \frac{\mu \, c_\zeta \, C^3}{r}} \, C^3 \, (c_0 \, c_{\zeta_2} + 2 \, c_1 \, c_{\zeta_1} + 3 \, c_2 \, c_{\zeta_0}) \frac{r^{m+p-2} \, (m + p - 1)! \, \rho^n \, n!}{(m + 1)^2 \, (n + 1)^2 \, p^2},$$

where it is assumed that

$$r > \mu \, c_\zeta \, C^3.$$

Together this gives

$$|\partial_u^m \, \partial_v^n \, \partial_w^p \, s_0| \le c_2 \, \frac{r^{m+p} \, (m + p)! \, \rho^n \, n!}{(m + 2)^2 (n + 1)^2 p^2}$$

$$+ C^3 \, c_{\hat{e}^1{}_{11}} \, c_0 \, \frac{r^{m+p-1} \, (m + p)! \, \rho^n \, n!}{(m + 2)^2 \, (n + 1)^2 \, p^2}$$

$$+ C^3 \, c_{\hat{e}^2{}_{11}} \, c_0 \, \frac{r^{m+p-2} \, (m + p - 1)! \, \rho^{n+1} \, (n + 1)!}{(m + 1)^2 \, (n + 2)^2 \, p^2}$$

$$+ 4\,C^3 \, (c_0 \, c_{\hat{\Gamma}_{1101}} + c_1 \, c_{\hat{\Gamma}_{1100}}) \, \frac{r^{m+p-2} \, (m + p - 1)! \, \rho^n \, n!}{(m + 1)^2 \, (n + 1)^2 \, p^2}$$

$$+ \frac{\mu}{1 - \frac{\mu \, c_\zeta \, C^3}{r}} \, C^3 \, (c_0 \, c_{\zeta_2} + 2 \, c_1 \, c_{\zeta_1} + 3 \, c_2 \, c_{\zeta_0}) \frac{r^{m+p-2} \, (m + p - 1)! \, \rho^n \, n!}{(m + 1)^2 \, (n + 1)^2 \, p^2}$$

$$\le c_0 \, \frac{r^{m+p} \, (m + p)! \, \rho^n \, n!}{(m + 1)^2 (n + 1)^2 (p + 1)^2} \, A_{s_0}^*,$$

with a factor

$$A_{s_0}^* = \frac{c_2}{c_0} \, \frac{(m + 1)^2 \, (p + 1)^2}{(m + 2)^2 \, p^2} + \frac{1}{r} \, C^3 \, c_{\hat{e}^1{}_{11}} \, \frac{(m + 1)^2 \, (p + 1)^2}{(m + 2)^2 \, p^2}$$

$$+\frac{1}{r^2}\,C^3\,c_{\hat{e}^2\,11}\,\frac{\rho\,(n+1)^3\,(p+1)^2}{(n+2)^2\,p^2\,(m+p)}+\frac{4}{r^2}\,C^3\,(c_{\hat{\Gamma}_{1101}}+\frac{c_1}{c_0}\,c_{\hat{\Gamma}_{1100}})\,\frac{(p+1)^2}{p^2\,(m+p)}$$

$$+\frac{1}{r^2}\,\frac{\mu}{1-\frac{\mu\,c_\zeta\,C^3}{r}}\,C^3\,(c_{\zeta_2}+2\,\frac{c_1}{c_0}\,c_{\zeta_1}+3\,\frac{c_2}{c_0}\,c_{\zeta_0})\frac{(p+1)^2}{p^2\,(m+p)}.$$

Recalling that we can assume $n \le 2\,m+4$ in the third term on the right hand side, this finally gives

$$A^*_{s_0} \le 4\,\frac{c_2}{c_0}+\frac{4}{r}\,C^3\,c_{\hat{e}^1\,11}+\frac{20\,\rho}{r^2}\,C^3\,c_{\hat{e}^2\,11}+\frac{16}{r^2}\,C^3\,(c_{\hat{\Gamma}_{1101}}+\frac{c_1}{c_0}\,c_{\hat{\Gamma}_{1100}})$$

$$+\frac{1}{r^2}\,\frac{4\,\mu}{1-\frac{\mu\,c_\zeta\,C^3}{r}}\,C^3\,(c_{\zeta_2}+2\,\frac{c_1}{c_0}\,c_{\zeta_1}+3\,\frac{c_2}{c_0}\,c_{\zeta_0}).$$

We have the relations

$$s_k=\frac{(4-k)!}{4\,!}\,\partial_v^k\,s_0 \quad \text{on} \quad U_0,$$

the equation $0=H_{0100}+H_{1000}$ reduces to

$$\partial_w\,s_0=\partial_u\,s_2+3\,\mu\,s_2\,\zeta_0 \quad \text{on} \quad U_0,$$

and we have seen that

$$\partial_v\,\zeta_0=0 \quad \text{on} \quad U_0.$$

This implies for $p \ge 1$ the estimates

$$|\partial_u^0\,\partial_v^n\,\partial_w^p\,s_k| \le \frac{(4-k)!}{4\,!}\,(|\partial_u^1\,\partial_v^{n+k}\,\partial_w^{p-1}\,s_2|+3\,\mu\,|\partial_u^0\,\partial_v^{n+k}\,\partial_w^{p-1}\,(s_2\,\zeta_0)|)$$

$$\le \left\{ \begin{array}{cc} \frac{(4-k)!}{4!}\,c_2\,\frac{r^p\,p!\,\rho^{n+k}\,(n+k)!}{4\,p^2\,(n+k+1)^2} & \text{for} \quad n \le 4-k \\ 0 \quad \text{for} \quad n > 4-k \end{array} \right\}$$

$$+\left\{ \begin{array}{cc} 3\,\mu\,\frac{(4-k)!}{4!}\,\sum_{l=0}^{p-1}\binom{p-1}{l}|\partial_v^{n+k}\,\partial_w^l\,s_2|\,|\partial_w^{p-1-l}\,\zeta_0| & \text{for} \quad n \le 2-k \\ 0 \quad \text{for} \quad n > 2-k \end{array} \right\}$$

$$\le c_k\,\frac{r^p\,p!\,\rho^n\,n!}{(n+1)^2(p+1)^2}\,A^{m=0,p\ge1}_{s_k},$$

with

$$A^{m=0,p\ge1}_{s_k}=\frac{c_2}{c_k}\,\rho^k\,f_{k,n}+\frac{3}{r}\,\mu\,C\,\frac{c_2\,c_{\zeta_0}}{c_k}\,\rho^k\,g_{k,n} \le \frac{c_2}{c_k}\,\rho^k+\frac{12}{r}\,\mu\,C\,\frac{c_2\,c_{\zeta_0}}{c_k}\,\rho^k,$$

because

$$f_{k,n} \equiv \left\{ \begin{array}{cc} \frac{(4-k)!}{4!}\,\frac{(n+k)!\,(n+1)^2\,(p+1)^2}{n!\,(n+k+1)^2\,4\,p^2} & \text{for} \quad n \le 4-k \\ 0 \quad \text{for} \quad n > 4-k \end{array} \right\} \le 1,$$

$$g_{k,n} \equiv \left\{ \begin{array}{cc} \frac{(4-k)!}{4!}\,\frac{(n+k)!\,(n+1)^2\,(p+1)^2}{n!\,(n+k+1)^2\,p^3} & \text{for} \quad n \le 2-k \\ 0 \quad \text{for} \quad n > 2-k \end{array} \right\} \le 4.$$

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From the equation $\Sigma_{1100} = 0$, which reads

$$\partial_w \zeta_0 = -2 + \hat{s} \quad \text{on} \quad U_0,$$

follows

$$|\partial_u^0 \partial_v^n \partial_w \zeta_0| = |\partial_v^n (-2 + \hat{s})| = 2\,\delta_0^n \leq c_{\zeta_0} \frac{\rho^n\, n!}{(n+1)^2}\, A_{\zeta_0}^{m=0,p=1},$$

with

$$A_{\zeta_0}^{m=0,p=1} = \frac{2}{c_{\zeta_0}}.$$

Furthermore, for $p \geq 2$,

$$|\partial_u^0 \partial_v^n \partial_w^p \zeta_0| = |\partial_v^n \partial_w^{p-1} \hat{s}| = c_{\hat{s}} \frac{r^{p-1}\,(p-1)!\,\rho^n\, n!}{(n+1)^2\, p^2} \leq c_{\zeta_0} \frac{r^p\, p!\,\rho^n\, n!}{(n+1)^2\,(p+1)^2}\, A_{\zeta_0}^{m=0,p\geq 2},$$

with

$$A_{\zeta_0}^{m=0,p\geq 2} = \frac{1}{r}\, \frac{c_{\hat{s}}}{c_{\zeta_0}}\, \frac{(p+1)^2}{p^3} \leq \frac{2}{r}\, \frac{c_{\hat{s}}}{c_{\zeta_0}}.$$

The equation $S_{11} = 0$, which reads

$$\partial_w \hat{s} = -s_4\, \zeta_0 \quad \text{on} \quad U_0,$$

implies

$$|\partial_u^0 \partial_v^n \partial_w \hat{s}| = 0 \leq c_{\hat{s}} \frac{\rho^n\, n!}{(n+1)^2},$$

and for $p \geq 2$

$$|\partial_u^0 \partial_v^n \partial_w^p \hat{s}| = |\partial_u^0 \partial_v^n \partial_w^{p-1} (s_4\, \zeta_0)| \leq C^2\, c_4\, c_{\zeta_0} \frac{r^{p-2}\,(p-1)!\,\rho^n\, n!}{(n+1)^2\, p^2}$$

$$\leq c_{\hat{s}} \frac{r^{p-1}\, p!\,\rho^n\, n!}{(n+1)^2\,(p+1)^2}\, A_{\hat{s}}^{m=0,p\geq 2},$$

with

$$A_{\hat{s}}^{m=0,p\geq 2} = \frac{1}{r}\, C^2\, \frac{c_4\, c_{\zeta_0}}{c_{\hat{s}}}\, \frac{(p+1)^2}{p^3} \leq \frac{2}{r}\, C^2\, \frac{c_4\, c_{\zeta_0}}{c_{\hat{s}}}.$$

Having studied the quantities $|\partial_u^m \partial_v^n \partial_w^p x|$ for $m = 0$, we shall now derive the conditions which arise from the requirement that we can obtain the desired estimates for these quantities inductively for all positive integers $m$. We shall provide detailed arguments only for some representative $\partial_u$-equations and just state the analogues results for the remaining equations.

Multiplication of the equation

$$\partial_u \hat{e}^2{}_{01} + \frac{1}{u}\, \hat{e}^2{}_{01} = \frac{1}{u}\, \hat{\Gamma}_{0100} + 2\,\hat{\Gamma}_{0100}\, \hat{e}^2{}_{01},$$

with $u$ and formal differentiation gives with Lemma 6.6 for $m \geq 1$

$$|\partial_u^m \, \partial_v^n \, \partial_w^p \, \hat{e}^2{}_{01}| \leq \frac{1}{m+1} \left( |\partial_u^m \, \partial_v^n \, \partial_w^p \, \hat{\Gamma}_{0100}| + 2\, m \, |\partial_u^{m-1} \, \partial_v^n \, \partial_w^p \, (\hat{\Gamma}_{0100} \, \hat{e}^2{}_{01})| \right)$$

$$\leq \frac{1}{m+1} \left( c_{\hat{\Gamma}_{0100}} \frac{r^{m+p-1}\,(m+p)!\,\rho^n\,n!}{(m+1)^2\,(n+1)^2\,(p+1)^2} + 2\, m \, C^3 \, c_{\hat{e}^2{}_{01}} \, c_{\hat{\Gamma}_{0100}} \frac{r^{m+p-3}\,(m+p-1)!\,\rho^n\,n!}{m^2\,(n+1)^2\,(p+1)^2} \right)$$

$$= c_{\hat{e}^2{}_{01}} \frac{r^{m+p-1}\,(m+p)!\,\rho^n\,n!}{(m+1)^2\,(n+1)^2\,(p+1)^2} \, A_{\hat{e}^2{}_{01}}^{m \geq 1},$$

with

$$A_{\hat{e}^2{}_{01}}^{m \geq 1} = \frac{c_{\hat{\Gamma}_{0100}}}{c_{\hat{e}^2{}_{01}}} \frac{1}{m+1} + \frac{1}{r^2} \, C^3 \, c_{\hat{\Gamma}_{0100}} \frac{2\,(m+1)}{m\,(m+p)}.$$

Proceeding in a similar way with the equations for the other frame coefficients one gets for the factors which need to be controlled the estimates

$$A_{\hat{e}_{01}^2}^{m \geq 1} \leq \frac{c_{\hat{\Gamma}_{0100}}}{2\,c_{\hat{e}_{01}^2}} + \frac{4}{r^2} \, C^3 \, c_{\hat{\Gamma}_{0100}}, \qquad A_{\hat{e}_{11}^2}^{m \geq 1} \leq \frac{c_{\hat{\Gamma}_{1100}}}{c_{\hat{e}_{11}^2}} + \frac{8}{r^2} \, C^3 \, \frac{c_{\hat{\Gamma}_{1100}} \, c_{\hat{e}_{01}^2}}{c_{\hat{e}_{11}^2}},$$

$$A_{\hat{e}_{01}^1}^{m \geq 1} \leq \frac{4}{r} \, \frac{c_{\hat{\Gamma}_{0101}}}{c_{\hat{e}_{01}^1}} + \frac{4}{r^2} \, C^3 \, c_{\hat{\Gamma}_{0100}}, \qquad A_{\hat{e}_{11}^1}^{m \geq 1} \leq \frac{8}{r} \, \frac{c_{\hat{\Gamma}_{1101}}}{c_{\hat{e}_{11}^1}} + \frac{8}{r^2} \, C^3 \, \frac{c_{\hat{\Gamma}_{1100}} \, c_{\hat{e}_{01}^1}}{c_{\hat{e}_{11}^1}}.$$

The same inequalities, with $C^3$ replaced by $C^2$, are obtained in the case $p = 0$. In the last two inequalities the occurence of $1/r$ in both terms reflects the fact that $\hat{e}_{01}^1$ and $\hat{e}_{11}^1$ are both of the order $O(u^2)$ near $O$.

Multiplication of the equation

$$\partial_u \, \hat{\Gamma}_{0100} + \frac{2}{u} \, \hat{\Gamma}_{0100} = 2\, \hat{\Gamma}_{0100}^2 + \frac{1}{2} \, s_0,$$

with $u$ and formal differentiation gives for $m \geq 1$

$$|\partial_u^m \, \partial_v^n \, \partial_w^p \, \hat{\Gamma}_{0100}| \leq \frac{m}{m+2} (2\, |\partial_u^{m-1} \, \partial_v^n \, \partial_w^p \, \hat{\Gamma}_{0100}| + \frac{1}{2} \, |\partial_u^{m-1} \, \partial_v^n \, \partial_w^p \, s_0|)$$

$$\leq \frac{2\, m}{m+2} \, C^3 \, c_{\hat{\Gamma}_{0100}}^2 \, \frac{r^{m+p-3}\,(m+p-1)!\,\rho^n\,n!}{m^2\,(n+1)^2\,(p+1)^2} + \frac{m}{2\,(m+2)} \, c_0 \, \frac{r^{m+p-1}\,(m+p-1)!\,\rho^n\,n!}{m^2\,(n+1)^2\,(p+1)^2}$$

$$\leq c_{\hat{\Gamma}_{0100}} \frac{r^{m+p-1}\,(m+p)!\,\rho^n\,n!}{(m+1)^2\,(n+1)^2\,(p+1)^2} \, A_{\hat{\Gamma}_{0100}}^{m \geq 1},$$

with

$$A_{\hat{\Gamma}_{0100}}^{m \geq 1} = \frac{1}{r^2} \, C^3 \, c_{\hat{\Gamma}_{0100}} \frac{2\,(m+1)^2}{m\,(m+2)\,(m+p)} + \frac{c_0}{c_{\hat{\Gamma}_{0100}}} \frac{(m+1)^2}{2\,m\,(m+2)\,(m+p)}.$$

Proceeding in a similar way with the equations for the other connection coefficients one gets for the factors which need to be controlled the estimates

$$A_{\hat{\Gamma}_{0100}}^{m \geq 1} \leq \frac{c_0}{c_{\hat{\Gamma}_{0100}}} + \frac{4}{r^2} \, C^3 \, c_{\hat{\Gamma}_{0100}}, \qquad A_{\hat{\Gamma}_{0101}}^{m \geq 1} \leq \frac{c_1}{c_{\hat{\Gamma}_{0101}}} + \frac{4}{r^2} \, C^3 \, c_{\hat{\Gamma}_{0100}},$$

$$A^{m\geq 1}_{\hat{\Gamma}_{0111}} \leq \frac{c_2}{c_{\hat{\Gamma}_{0111}}} + \frac{4}{r^2}\, C^3\, c_{\hat{\Gamma}_{0100}}, \qquad A^{m\geq 1}_{\hat{\Gamma}_{1100}} \leq \frac{2\,c_1}{c_{\hat{\Gamma}_{1100}}} + \frac{4}{r^2}\, C^3\, c_{\hat{\Gamma}_{0100}},$$

$$A^{m\geq 1}_{\hat{\Gamma}_{1101}} \leq \frac{4\,c_2}{c_{\hat{\Gamma}_{1101}}} + \frac{8}{r^2}\, C^3\, \frac{c_{\hat{\Gamma}_{1100}} c_{\hat{\Gamma}_{0101}}}{c_{\hat{\Gamma}_{1101}}}, \qquad A^{m\geq 1}_{\hat{\Gamma}_{1111}} \leq \frac{4\,c_3}{c_{\hat{\Gamma}_{1111}}} + \frac{8}{r^2}\, C^3\, \frac{c_{\hat{\Gamma}_{1100}} c_{\hat{\Gamma}_{0111}}}{c_{\hat{\Gamma}_{1111}}},$$

The same inequalities, with $C^3$ replaced by $C^2$, are obtained in the case $p = 0$. Being slightly more generous, one gets inequalities which can be written in the concise form

$$A^{m\geq 1}_{\hat{\Gamma}_{01AB}} \leq \frac{c_{A+B}}{c_{\hat{\Gamma}_{01AB}}} + \frac{4}{r^2}\, C^3\, c_{\hat{\Gamma}_{0100}}, \quad A^{m\geq 1}_{\hat{\Gamma}_{11AB}} \leq \frac{4\,c_{A+B+1}}{c_{\hat{\Gamma}_{11AB}}} + \frac{8}{r^2}\, C^3\, \frac{c_{\hat{\Gamma}_{1100}} c_{\hat{\Gamma}_{01AB}}}{c_{\hat{\Gamma}_{11AB}}}, \quad A, B = 0, 1,$$

where the $c_{A+B}$, $c_{A+B+1}$ denote for suitable numerical values of the indices $A$, $B$ the constants $c_0, \ldots, c_4$.

The analogous discussion of the equations

$$\partial_u \zeta = \zeta_0,$$

$$\partial_u \zeta_0 = -\zeta\,(1 - \mu\,\zeta)\,s_0,$$

$$\partial_u \zeta_1 = -\zeta\,(1 - \mu\,\zeta)\,s_1,$$

$$\partial_u \zeta_2 = -2 + \hat{s} - \zeta\,(1 - \mu\,\zeta)\,s_2,$$

$$\partial_u \hat{s} - (1 - \mu\,\zeta)\,(s_0\,\zeta_{11} - 2\,s_1\,\zeta_{01} + s_2\,\zeta_{00}),$$

does not require new considerations. For the factors which need to be controlled we get the estimates

$$A^{m\geq 1, p\geq 0}_{\zeta} \leq \frac{4}{r}\,\frac{c_{\zeta_0}}{c_{\zeta}},$$

$$A^{m\geq 1, p\geq 0}_{\zeta_0} \leq \frac{4}{r}\, C^3\, \frac{c_0\, c_{\zeta}}{c_{\zeta_0}} + \frac{4}{r^2}\, \mu\, C^6\, \frac{c_0\, c_{\zeta}^2}{c_{\zeta_0}}, \qquad A^{m\geq 1, p\geq 0}_{\zeta_1} \leq \frac{4}{r}\, C^3\, \frac{c_1\, c_{\zeta}}{c_{\zeta_1}} + \frac{4}{r^2}\, \mu\, C^6\, \frac{c_0\, c_{\zeta}^2}{c_{\zeta_1}},$$

$$A^{m\geq 1, p\geq 0}_{\zeta_2} \leq \left\{ \begin{array}{ll} \frac{8}{c_{\zeta_2}} + \frac{4}{r}\left(\frac{c_{\hat{s}}}{c_{\zeta_2}} + C^3\, \frac{c_2\, c_{\zeta}}{c_{\zeta_2}}\right) + \frac{4}{r^2}\, \mu\, C^6\, \frac{c_2\, c_{\zeta}^2}{c_{\zeta_2}} & \text{for} \quad m = 1, n = 0, p = 0, \\ \frac{4}{r}\left(\frac{c_{\hat{s}}}{c_{\zeta_2}} + C^3\, \frac{c_2\, c_{\zeta}}{c_{\zeta_2}}\right) + \frac{4}{r^2}\, \mu\, C^6\, \frac{c_2\, c_{\zeta}^2}{c_{\zeta_2}} & \text{otherwise} \end{array} \right\},$$

$$A^{m\geq 1}_{\hat{s}} \leq \left(\frac{4}{r}\, C^3 + \frac{4}{r^2}\, \mu\, C^6\, c_{\zeta}\right)\left(\frac{c_0\, c_{\zeta_2}}{c_{\hat{s}}} + 2\,\frac{c_1\, c_{\zeta_1}}{c_{\hat{s}}} + \frac{c_2\, c_{\zeta_0}}{c_{\hat{s}}}\right).$$

We consider the $\partial_u$-equations for the curvature component $s_1$. Multiplication with $2\,u$ gives

$$2\,u\,\partial_u s_1 + 4\,s_1 = \partial_v s_0 + 2\,u\,\hat{e}^1_{\ 01}\partial_u s_0 + 2\,u\,\hat{e}^2_{\ 01}\partial_v s_0$$

$$-8\,u\,(\hat{\Gamma}_{0101}\,s_0 - \hat{\Gamma}_{0100}\,s_1) - u\,\frac{4\,\mu}{(1 - \mu\,\zeta)}\,\{s_0\,\zeta_1 - s_1\,\zeta_0\},$$

which implies for $m \geq 1$

$$|\partial_u^m\, \partial_v^n\, \partial_w^p\, s_1| \leq \frac{1}{2\,m + 4}\,|\partial_u^m\, \partial_v^{n+1}\, \partial_w^p\, s_0|$$

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$$+\frac{2\,m}{2\,m+4}\,\left(|\partial_u^{m-1}\,\partial_v^n\,\partial_w^p\,(\hat{e}^1{}_{01}\,\partial_u\,s_0)| + |\partial_u^{m-1}\,\partial_v^n\,\partial_w^p\,(\hat{e}^2{}_{01}\,\partial_v\,s_0)|\right)$$

$$+\frac{4\,m}{2\,m+4}\,\left(2\,|\partial_u^{m-1}\,\partial_v^n\,\partial_w^p\,(\hat{\Gamma}_{0101}\,s_0 - \hat{\Gamma}_{0100}\,s_1)| + \mu\,|\partial_u^{m-1}\,\partial_v^n\,\partial_w^p\,\{\frac{1}{1-\mu\,\zeta}\,(s_0\,\zeta_1 - s_1\,\zeta_0)\}|\right).$$

The terms arising here are estimated in a similar way as the terms in the curvature equation above. Again the expansion types allows one assumed that $0 \le n \le 2\,m+4-k$. Again $r$ is restricted to values with

$$r > \mu\,c_\zeta\,C^3.$$

Proceeding similarly with the other $\partial_u$-equations for the curvature, the following estimates are obtained for the factors which need to be controlled.

$$A_{s_1}^{m\ge 1} \le \frac{c_0}{c_1}\,\rho + \frac{1}{r}\,C^3\,\frac{c_0}{c_1}\,c_{\hat{e}_{01}^1} + \frac{8\,\rho}{r^2}\,C^3\,\frac{c_0}{c_1}\,c_{\hat{e}_{01}^2} + \frac{8}{r^2}\,C^3\,(\frac{c_0}{c_1}\,c_{\hat{\Gamma}_{0101}} + c_{\hat{\Gamma}_{0100}})$$

$$+\frac{1}{r^2}\,C^3\,\frac{4\,\mu}{1-\frac{\mu\,c_\zeta\,C^3}{r}}\,(\frac{c_0}{c_1}\,c_{\zeta_1} + c_{\zeta_0})$$

$$A_{s_2}^{m\ge 1} \le \frac{c_1}{c_2}\,\rho + \frac{1}{r}\,C^3\,\frac{c_1}{c_2}\,c_{\hat{e}_{01}^1} + \frac{8\,\rho}{r^2}\,C^3\,\frac{c_1}{c_2}\,c_{\hat{e}_{01}^2} + \frac{4}{r^2}\,C^3\,(\frac{c_0}{c_2}\,c_{\hat{\Gamma}_{0111}} + 2\,\frac{c_1}{c_2}\,c_{\hat{\Gamma}_{0101}} + 3\,c_{\hat{\Gamma}_{0100}})$$

$$+\frac{1}{r^2}\,C^3\,\frac{2\,\mu}{1-\frac{\mu\,c_\zeta\,C^3}{r}}\,(\frac{c_0}{c_2}\,c_{\zeta_2} + 2\,\frac{c_1}{c_2}\,c_{\zeta_1} + 3\,c_{\zeta_0})$$

$$A_{s_3}^{m\ge 1} \le \frac{c_2}{c_3}\,\rho + \frac{1}{r}\,C^3\,\frac{c_2}{c_3}\,c_{\hat{e}_{01}^1} + \frac{8\,\rho}{r^2}\,C^3\,\frac{c_2}{c_3}\,c_{\hat{e}_{01}^2} + \frac{8}{r^2}\,C^3\,(\frac{c_1}{c_3}\,c_{\hat{\Gamma}_{0111}} + c_{\hat{\Gamma}_{0100}})$$

$$+\frac{1}{r^2}\,C^3\,\frac{4\,\mu}{1-\frac{\mu\,c_\zeta\,C^3}{r}}\,(\frac{c_1}{c_3}\,c_{\zeta_2} + c_{\zeta_0})$$

$$A_{s_3}^{m\ge 1} \le \frac{c_3}{c_4}\,\rho + \frac{1}{r}\,C^3\,\frac{c_3}{c_4}\,c_{\hat{e}_{01}^1} + \frac{8\,\rho}{r^2}\,C^3\,\frac{c_3}{c_4}\,c_{\hat{e}_{01}^2} + \frac{4}{r^2}\,C^3\,(3\,\frac{c_2}{c_4}\,c_{\hat{\Gamma}_{0111}} + 2\,\frac{c_3}{c_4}\,c_{\hat{\Gamma}_{0101}} + c_{\hat{\Gamma}_{0100}})$$

$$+\frac{1}{r^2}\,C^3\,\frac{2\,\mu}{1-\frac{\mu\,c_\zeta\,C^3}{r}}\,(3\,\frac{c_2}{c_4}\,c_{\zeta_2} + 2\,\frac{c_3}{c_4}\,c_{\zeta_1} + c_{\zeta_0}).$$

This gives all the needed information.

To arrange now the constants so that the induction argument can successfully carried out, we proceed as follows. The estimates for the decisive factors which have been obtained above are of the general form

$$A \le \alpha + \frac{1}{r}\,\beta + \frac{1}{r^2}\,\gamma,$$

with $\alpha$, $\beta$, and $\gamma$ depending on all the constants except $r$. If $\beta = 0$ and $\gamma = 0$ it suffices to ensure $\alpha \le 1$. In the other cases we require $\alpha \le a$ where $a$ is a given constant, $a < 1$, and then choose $r$ large enough so that $A \le 1$. A first set of conditions arising this way reads

$$\frac{c_k}{c_{k+1}} \rho \le a, \quad \frac{c_0}{c_k} \rho^k \le 1, \quad \frac{c_2}{c_k} \rho^k \le a, \quad 4\frac{c_2}{c_0} \le a.$$

These conditions can be satisfied simultaneously. The first equation implies $c_k \ge (\rho/a)^k c_0$. With

$$c_k = (\frac{\rho}{a})^k c_0^*,$$

where $0 < \rho, a < 1$, the first two relations hold true, the forth relation implies $\rho^2 \le a^3/4$ and with this restriction the third relation holds as well. We choose

$$\rho = \rho_0, \quad a = (4\rho_0^2)^{1/3}.$$

The conditions

$$\frac{2}{c_{\zeta_0}} \le 1, \quad \frac{8}{c_{\zeta_2}} \le a,$$

are met by setting

$$c_{\zeta_0} \equiv 2, \quad c_{\zeta_2} \equiv \frac{8}{a}.$$

The conditions

$$\frac{c_{A+B}}{c_{\hat{\Gamma}_{01AB}}} \le a, \quad \frac{4c_{1+A+B}}{c_{\hat{\Gamma}_{11AB}}} \le a, \quad A, B = 0, 1,$$

are then dealt with by setting

$$c_{\hat{\Gamma}_{01AB}} \equiv \frac{1}{a} c_{A+B}, \quad c_{\hat{\Gamma}_{11AB}} \equiv \frac{1}{a} c_{1+A+B}.$$

The conditions

$$\frac{c_{\hat{\Gamma}_{0100}}}{c_{\hat{e}_{01}^2}} \le a, \quad \frac{c_{\hat{\Gamma}_{1100}}}{c_{\hat{e}_{11}^2}} \le a,$$

are satisfied by setting

$$c_{\hat{e}_{01}^2} \equiv \frac{1}{a} c_{\hat{\Gamma}_{0100}}, \quad c_{\hat{e}_{11}^2} \equiv \frac{1}{a} c_{\hat{\Gamma}_{1100}}.$$

After this we choose some positive constants

$$\hat{e}_{01}^1, \quad \hat{e}_{11}^1, \quad c_\zeta, \quad c_{\zeta_1}, \quad c_{\hat{s}}.$$

That these constants are not further restricted by the procedure reflects the fact that the corresponding functions vanish to higher order at $O$. Their choice affects, however, the value of the constant $r$. After all constants except $r$ have been fixed we can choose $r$ so large that

$$r > \max\{r_0, \ \mu c_\zeta C^3\},$$

and that all the $A$'s are $\le 1$. This finishes the induction proof. $\square$

The following statement of the convergence result, obtained by using the $v$-finite expansion types of the various functions, emphasizes the role of $v$ as an angular coordinate.

**Lemma 6.9** *The estimates (6.3) and (6.4) for the derivatives of the functions $s_k$ and $f$ and the expansion types given in Lemma 5.2 imply that the associated Taylor series are absolutely convergent in the domain $|v| < \frac{1}{\alpha \rho}$, $|u| + |w| < \frac{\alpha^2}{r}$, for any real number $\alpha$, $0 < \alpha \leq 1$. It follows that the formal expansion determined in Lemma 5.1 defines indeed a (unique) holomorphic solution to the conformal static vacuum field equations which induces the datum $s_0$ on $W_0$.*

PROOF. The estimates (6.3) and (6.4) imply

$$|\partial_u^m \, \partial_v^n \, \partial_w^p \, s_k(O)| \leq \frac{c_k}{\alpha^{4-k}} \, \frac{(r/\alpha^2)^{m+p} \, (m+p)! \, (\alpha \, \rho)^n \, n!}{(m+1)^2 \, (n+1)^2 \, (p+1)^2} \, \alpha^{4-k+2m+2p-n}$$

$$\leq \frac{c_k}{\alpha^{4-k}} \, \frac{(r/\alpha^2)^{m+p} \, (m+p)! \, (\alpha \, \rho)^n \, n!}{(m+1)^2 \, (n+1)^2 \, (p+1)^2} \quad \text{for} \quad n \leq 2\,m + 4 - k, \quad m, p = 0, 1, 2, \ldots$$

$$|\partial_u^m \, \partial_v^n \, \partial_w^p \, f(O)| \leq \frac{c_f}{\alpha^{k_f - 2}} \, \frac{(r/\alpha^2)^{m+p-1} \, (m+p)! \, (\alpha \, \rho)^n \, n!}{(m+1)^2 \, (n+1)^2 \, (p+1)^2} \, \alpha^{k_f + 2m + 2p - n}$$

$$\leq \frac{c_f}{\alpha^{k_f - 2}} \, \frac{(r/\alpha^2)^{m+p-1} \, (m+p)! \, (\alpha \, \rho)^n \, n!}{(m+1)^2 \, (n+1)^2 \, (p+1)^2} \quad \text{for} \quad n \leq 2\,m + k_f, \quad m, p = 0, 1, 2, \ldots \,.$$

Since the other derivatives vanish because of the respective expansion types, the first assertion is an immediate consequence of the majorizations (3.15), (3.16). The second assertion then follows with Lemma 5.5. ∎

# 7    Analyticity at space-like infinity

Due to our singular gauge the holomorphic solution of the conformal static field equations obtained in Lemma 6.9 does not cover a full neighbourhood of the point $i$. To analyse the situation we study the part of the solution which we have obtained by the convergence proof in terms of a normal frame based on the frame $c_{AB}$ at $i$ and associated normal coordinates. We write the geodesic equation $D_{\dot{z}} \dot{z} = 0$ for $z^a(s) = (u(s), v, (s), w(s))$ in the form

$$\dot{z}^a = m^{AB} \, e_{AB}^a = m^{AB} \, (e_{AB}^{*a} + \hat{e}_{AB}^a),$$

$$\dot{m}^{AB} = -2 \, m^{CD} \Gamma_{CD} \,^{(A} \,_B \, m^{B)E} = -2 \, m^{CD} \Gamma_{CD}^* \,^{(A} \,_B \, m^{B)E} - 2 \, m^{CD} \hat{\Gamma}_{CD} \,^{(A} \,_B \, m^{B)E},$$

With the explicit expressions for the singular parts, the system takes the form

$$\dot{u} = \quad m^{00} + m^{AB} \, \hat{e}_{AB}^1, \qquad \dot{m}^{00} = -2 \, m^{CD} \hat{\Gamma}_{CD} \,^0 \,_B \, m^{0B},$$

$$\dot{v} = \frac{1}{u} \, m^{01} + m^{AB} \, \hat{e}_{AB}^2, \qquad \dot{m}^{01} = -\frac{1}{u} \, m^{01} \, m^{00} - 2 \, m^{CD} \hat{\Gamma}_{CD} \,^{(0} \,_B \, m^{1)B},$$

$$\dot{w} = m^{11}, \qquad\qquad\qquad \dot{m}^{11} = -\frac{2}{u} \, m^{01} \, m^{01} - 2 \, m^{CD} \hat{\Gamma}_{CD} \,^1 \,_B \, m^{1B}.$$

These equations have to be solved with the initial conditions

$$u|_{s=0} = 0, \quad w|_{s=0} = 0, \tag{7.1}$$

for the curves to start at $i$. An arbitrary value

$$v_0 = v|_{s=0}, \qquad (7.2)$$

can be prescribed to determine the $\partial_u$-$\partial_w$-plane over $i$ in which the tangent vector is lying, and an arbitrary choice of

$$m^{AB}|_{s=0} = m_0^{AB} = m_0^{AB}\,\epsilon_0{}^A\,\epsilon_0{}^B + m_0^{AB}\,\epsilon_1{}^A\,\epsilon_1{}^B, \qquad \dot{u}_0 \neq 0,$$

can be prescribed to specify the tangent vector in the $\partial_u$-$\partial_w$-plane. Regularity and the equations require

$$m_0^{00} = \dot{u}|_{s=0} \equiv \dot{u}_0, \quad m_0^{01} = 0, \quad m_0^{11} = \dot{w}|_{s=0} \equiv \dot{w}_0. \qquad (7.3)$$

If the frame $e_{AB}$ at a point of $I$ is identified with its projection into $T_i S_c$, then

$$m_0^{AB}\,e_{AB} = m_0^{AB}\,s^C{}_A(v_0)\,s^D{}_B(v_0)\,c_{CD} = m^{*AB}\,c_{AB},$$

holds at $i$ with

$$m^{*00} = \dot{u}_0, \quad m^{*01} = \dot{u}_0\,v_0, \quad m^{*11} = \dot{u}_0\,v_0^2 + \dot{w}_0, \qquad \dot{u}_0 \neq 0.$$

For arbitrarily given $m^{*AB} \in C^3$ with $m^{*00} \neq 0$ this relation determines $\dot{u}_0$, $v_0$, $\dot{w}_0$ uniquely. Using $c_{AB} = \alpha^a{}_{AB}\,c_\mathbf{a}$, the tangent vectors can be written $m^{*AB}\,c_{AB} = x^a\,c_\mathbf{a}$ with

$$x^1 = \frac{1}{\sqrt{2}}\,(\dot{w}_0 + (v_0^2-1)\,\dot{u}_0), \quad x^2 = \frac{i}{\sqrt{2}}\,(\dot{w}_0 + (v_0^2+1)\,\dot{u}_0), \quad x^3 = \sqrt{2}\,v_0\,\dot{u}_0 \qquad \dot{u}_0 \neq 0, \; (7.4)$$

or, equivalently,

$$\dot{u}_0(x^a) = -\frac{x^1 + i\,x^2}{\sqrt{2}}, \quad v_0(x^a) = -\frac{x^3}{x^1 + i\,x^2}, \quad \dot{w}_0(x^a) = \frac{\delta_{ab}\,x^a\,x^b}{\sqrt{2}\,(x^1 + i\,x^2)}, \quad x^1 + i\,x^2 \neq 0.$$
$$(7.5)$$

The vectors $x^a c_\mathbf{a}$ cover all directions at $i$ except those tangent to the complex null hyperplane $(c_\mathbf{1} + i\,c_\mathbf{2})^\perp = \{a(c_\mathbf{1} + i\,c_\mathbf{2}) + b\,c_\mathbf{3}|\; a, b \in \mathbb{C}\}$.

To determine the normal frame centered at $i$ and based on the frame $c_{AB}$ at $i$, we write the equation $D_{\dot{x}} c_{AB} = 0$ for the normal frame as an equation for the transformation $t^A{}_B \in SL(2, \mathbb{C})$, which relates the frame $e_{AB}$ to the normal frame $c_{AB} = t^C{}_A\,t^D{}_B\,e_{CD}$. The resulting equation

$$0 = \frac{d}{ds}(t^C{}_A\,t^D{}_B) + m^{GH}\,\Gamma_{GH}{}^{CD}{}_{EF}\,t^E{}_A\,t^F{}_B,$$

can be written in the form $\dot{t}^A{}_B = -m^{DE}\Gamma_{DE}{}^A{}_C\,t^C{}_B$. Taking into account the structure of the connection coefficients, this gives

$$\dot{t}^A{}_B = -\frac{1}{u}\,m^{01}\,\epsilon_1{}^A\,t^0{}_B - m^{DE}\,\hat{\Gamma}_{DE}{}^A{}_C\,t^C{}_B. \qquad (7.6)$$

This equation has to be solved along $z(s)$ with the initial condition

$$t^A{}_B|_{s=0} = s^A{}_B(-v_0). \tag{7.7}$$

The initial value problems above make sense because the functions $\hat{e}^a{}_{AB}$ and $\hat{\Gamma}_{ABCD}$ are, by Lemma 6.9, holomorphic near the point $u = 0$, $v = v_0$, $w = 0$ for any prescribed value of $v_0$. The singularity of the system at that particular point requires, however, some attention.

We prepare the statement and the proof of the existence result, to be given in Lemma 7.2, by casting the system of ODE's into a suitable form. It will be convenient to make use of the *replacements resp. change of notation*

$$v \to v_0 + v, \quad m^{AB} \to m_0^{AB} + m^{AB}, \tag{7.8}$$

so that all unknowns vanish at $s = 0$. Furthermore, by setting

$$\tilde{e}^a_{AB}(u, v, w) = \hat{e}^a_{AB}(u, v_0 + v, w), \qquad \tilde{\Gamma}_{ABCD}(u, v, w) = \hat{\Gamma}_{ABCD}(u, v_0 + v, w),$$

we define functions $\tilde{e}^a_{AB}$, $\tilde{\Gamma}_{ABCD}$ of the new unknowns which are holomorphic near $u = v = w = 0$. The regular equations read with this notation

$$\dot{u} = \dot{u}_0 + m^{00} + \dot{w}_0\,\tilde{e}^1_{11} + 2\,\tilde{e}^1_{01}\,m^{01} + \tilde{e}^1_{11}\,m^{11},$$

$$\dot{w} = \dot{w}_0 + m^{11},$$

$$\dot{m}^{00} = -2\left\{\dot{u}_0\,\dot{w}_0\,\tilde{\Gamma}_{1101} + \dot{u}_0\,(2\,\tilde{\Gamma}_{0101}\,m^{01} + \tilde{\Gamma}_{1101}\,m^{11}) + \dot{w}_0\,(\tilde{\Gamma}_{1101}\,m^{00} + \tilde{\Gamma}_{1111}\,m^{01})\right.$$

$$\left. + 2\tilde{\Gamma}_{0101}\,m^{00}\,m^{01} + 2\tilde{\Gamma}_{0111}\,m^{01}\,m^{01} + \tilde{\Gamma}_{1101}\,m^{00}\,m^{11} + \tilde{\Gamma}_{1111}\,m^{01}\,m^{11}\right\}$$

The singular equations take the form

$$u\,\dot{v} = m^{01} + u\,(\dot{w}_0\,\tilde{e}^2_{AB} + 2\,\tilde{e}^2_{01}\,m^{01} + \tilde{e}^2_{11}\,m^{11})$$

$$u\,\dot{m}^{01} = -\dot{u}_0\,m^{01} - m^{00}\,m^{01} + u\left\{\dot{u}_0\,\dot{w}_0\,\tilde{\Gamma}_{1100} - \dot{w}_0^2\,\tilde{\Gamma}_{1111}\right.$$

$$+ \dot{u}_0\,(2\,\tilde{\Gamma}_{0100}\,m^{01} + \tilde{\Gamma}_{1100}\,m^{11}) + \dot{w}_0\,(\tilde{\Gamma}_{1100}\,m^{00} - 2\,\tilde{\Gamma}_{0111}\,m^{01} - 2\,\tilde{\Gamma}_{1111}\,m^{11})$$

$$\left. + 2\,\tilde{\Gamma}_{0100}\,m^{00}\,m^{01} - 2\,\tilde{\Gamma}_{0111}\,m^{01}\,m^{11} + \tilde{\Gamma}_{1100}\,m^{00}\,m^{11} - \tilde{\Gamma}_{1111}\,m^{11}\,m^{11}\right\},$$

$$u\,\dot{m}^{11} = -2\,m^{01}\,m^{01} + 2\,u\left\{\dot{w}_0^2\,\tilde{\Gamma}_{1101} + \dot{w}_0\,(2\,\tilde{\Gamma}_{0101}\,m^{01} + \tilde{\Gamma}_{1100}\,m^{01} + 2\,\tilde{\Gamma}_{1101}\,m^{11})\right.$$

$$\left. + 2\,\tilde{\Gamma}_{0100}\,m^{01}\,m^{01} + 2\,\tilde{\Gamma}_{0101}\,m^{01}\,m^{11} + \tilde{\Gamma}_{0100}\,m^{01}\,m^{11} + \tilde{\Gamma}_{1101}\,m^{11}\,m^{11}\right\}.$$

Finally, equation (7.6) reads

$$\dot{t}^A{}_B = -\frac{1}{u}\, m^{01}\, \epsilon_1{}^A\, t^0{}_B - (2\, m^{01}\, \hat{\Gamma}_{01}{}^A{}_C + \dot{w}_0\, \hat{\Gamma}_{11}{}^A{}_C + m^{11}\, \hat{\Gamma}_{11}{}^A{}_C)\, t^C{}_B. \tag{7.9}$$

After applying $\partial_s$ resp. $\partial_s^2$ to the geodesic equations and restricting all equations to $s = 0$ one obtains with the initial conditions (7.1), (7.2), (7.3) the relations

$$\dot{v}|_{s=0} = 0, \qquad \dot{m}^{AB}|_{s=0} = 0, \qquad \ddot{u}|_{s=0} = 0, \tag{7.10}$$

and, by taking a further derivative,

$$\partial_s^3 u(0) = \dot{u}_0^2\, \dot{w}_0\, \left\{\partial_u^2\, \hat{e}^1{}_{11} - 2\, \partial_u\, \hat{\Gamma}_{1101}\right\}_{u=0,v=v_0,w=0}.$$

This gives with the $\partial_u$-equations

$$\partial_s^3 u(0) = -4\, \dot{u}_0^2\, \dot{w}_0\, (s_2)_{u=0,v=v_0,w=0} = -\frac{1}{3}\dot{u}_0^2\, \dot{w}_0\, (\partial_v^2 s_0)_{u=0,v=v_0,w=0}, \tag{7.11}$$

which can be determined from the null data.

Because of Lemma 6.9 and the behaviour (4.7), (4.13) of the metric and the connection coefficients, which follows from the $\partial_u$-equations, there exist functions $f$, $g$, $h$, $k$, $l$ which are holomorphic on a polycylinder $P_{\epsilon'} = \{x \in \mathbb{C}^6|\ |x_j| < \epsilon'\}$ with some $\epsilon' > 0$ so that the equations above can be written

$$\dot{u} = \dot{u}_0 + m^{00} + u^2\, f, \tag{7.12}$$
$$u\, \dot{v} = m^{01} + u^2\, g, \tag{7.13}$$
$$\dot{w} = \dot{w}_0 + m^{11}, \tag{7.14}$$
$$\dot{m}^{00} = u\, h, \tag{7.15}$$
$$u\, \dot{m}^{01} = -\dot{u}_0\, m^{01} - m^{00}\, m^{01} + u^2\, k, \tag{7.16}$$
$$u\, \dot{m}^{11} = -2\, m^{01}\, m^{01} + u^2\, l, \tag{7.17}$$

with $f$, $g$, $h$, $k$, $l$ depending on the $\mathbb{C}^6$-valued function $z(s)$ comprising our unknowns in the form

$$z(s) = (z^j(s))_{j=1,\dots,6} = (u(s),\, v(s),\, w(s),\, m^{00}(s),\, m^{01}(s),\, m^{11}(s)),$$

(which agrees after the replacement $v \to v - v_0$ in the first 3 components with the notation introduced earlier).

If $F$ stands for any of the functions $f$, $g$, $h$, $k$, $l$, then it has on $P_{\epsilon'}$ an absolutely convergent expansion

$$F = \sum_{\alpha \in N^6} F_\alpha\, z^\alpha,$$

at $z^j = 0$, where again the multi-index notation is used. If $0 < \epsilon < \epsilon'$, there exists thus an $M > 0$ so that

$$\sup_{x \in P_\epsilon} \sum_\alpha |F_\alpha|\, |z^\alpha| \le M.$$

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**Lemma 7.1** *Let $p \geq 0$ be an integer and $c$ and $t$ real numbers which satisfy with the constant $C$ of Lemma 6.4*

$$c \geq \frac{M}{C}, \qquad t \geq \max\{1, \frac{cC}{\epsilon}\}. \tag{7.18}$$

*If the derivatives of the functions $z^j(s)$ at $s = 0$ exist and statisfy the estimates*

$$|\partial_s^k z^j| \leq c \frac{t^{k-1} k!}{(k+1)^2}, \quad k = 1, \ldots, 6, \qquad k \leq p,$$

*then*

$$|\partial_s^p F(z(s))|_{s=0} \leq c \frac{t^p p!}{(p+1)^2}.$$

*If, in addition, $u$ satisfies $u(0) = 0$, $\dot{u}(0) = \dot{u}_0$ and*

$$|\partial_s^k u(s)|_{s=0} \leq c \frac{t^{k-2} k!}{(k+1)^2}, \qquad 2 \leq k \leq p,$$

*then*

$$|\partial_s^p(u \, F(z(s)))|_{s=0} \leq |\dot{u}_0| \, c \frac{t^{p-1} p!}{p^2} + c^2 \, C \frac{t^{p-2} p!}{(p+1)^2},$$

*for $p \geq 1$, where the second term on the right hand side is to be dropped if $p < 2$, and*

$$|\partial_s^p(u^2 \, F(z(s)))|_{s=0} \leq 2 \, |\dot{u}_0|^2 \, c \frac{t^{p-2} p!}{(p-1)^2} + 4 \, |\dot{u}_0| \, c^2 \, C \frac{t^{p-3} p!}{(p+1)^2} + c^3 \, C^2 \frac{t^{p-4} p!}{(p+1)^2},$$

*for $p \geq 2$, where the second term on the right hand side is to be dropped if $p < 3$ and the third term is to be dropped if $p < 4$.*

In the following a function in the modulus sign has argument $s = 0$.

PROOF. Observing Lemma 6.7 and the subsequent remark, one gets

$$|\partial_s^p F(z)| \leq \sum_{|\alpha| \leq p} |F_\alpha| |\partial_s^p z^\alpha| \leq \sum_{|\alpha| \leq p} |F_\alpha| \, C^{|\alpha|-1} \, c^{|\alpha|} \frac{t^{p-|\alpha|} p!}{(p+1)^2}$$

$$\leq \frac{1}{cC} \sum_{|\alpha| \leq p} |F_\alpha| \left(\frac{cC}{t}\right)^{|\alpha|} c \frac{t^p p!}{(p+1)^2} \leq \frac{M}{cC} \, c \frac{t^p p!}{(p+1)^2} \leq c \frac{t^p p!}{(p+1)^2},$$

by the choice of $c$ and $t$. With Lemma 6.4 this gives

$$|\partial_s^p(u \, F(z))| \leq p \, |\dot{u}_0| \, |\partial_s^{p-1} F(z)| + \sum_{j=2}^{p} \binom{p}{j} |\partial_s^j u| \, |\partial_s^{p-j} F(z)|$$

$$\leq p \, |\dot{u}_0| \, c \frac{t^{p-1} (p-1)!}{p^2} + \sum_{j=2}^{p} \binom{p}{j} c \frac{t^{j-2} (j)!}{(j+1)^2} \, c \frac{t^{p-j} (p-j)!}{(p-j+1)^2} \leq |\dot{u}_0| \, c \frac{t^{p-1} p!}{p^2} + c^2 \, C \frac{t^{p-2} p!}{(p+1)^2},$$

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and similarly

$$|\partial_s^p(u^2\, F(z))| \leq \sum_{l=0}^{p} \binom{p}{l} \sum_{j=0}^{l} \binom{l}{j} |\partial_s^j u|\, |\partial_s^{l-j} u|\, |\partial_s^{p-l} F(z)|$$

$$= 4 \binom{p}{2} |\dot{u}_0|^2\, |\partial_s^{p-2} F(z)| + \sum_{l=3}^{p} \binom{p}{l} 2\, l\, |\dot{u}_0|\, |\partial_s^{l-1} u|\, |\partial_s^{p-l} F(z)|$$

$$+ \sum_{l=2}^{p} \binom{p}{l} \sum_{j=2}^{l-2} \binom{l}{j} |\partial_s^j u|\, |\partial_s^{l-j} u|\, |\partial_s^{p-l} F(z)|$$

$$\leq 2\, |\dot{u}_0|^2\, c\, \frac{t^{p-2}\, p\,!}{(p-1)^2} + 4\, |\dot{u}_0|\, c^2\, C\, \frac{t^{p-3}\, p\,!}{(p+1)^2} + c^3\, C^2\, \frac{t^{p-4}\, p\,!}{(p+1)^2},$$

∎

**Lemma 7.2**  *The requirement that $z(s)$ be a holomorphic solution of equations (7.12) -
(7.17) near $s = 0$ satisfying $x(0) = 0$ and $\partial_s u(0) = \dot{u}_0 \neq 0$ determines a unique formal
expansion of $z(s)$ at $s = 0$. There exist real constants $c$ and $t$ satisfying*

$$c \geq \max\left\{4\,|\dot{u}_0|, 4\,|\dot{w}_0|, |\dot{u}_0|^2\,|\,\dot{w}_0|\,|(\partial_v^2 s_0)_{u=0,v=v_0,w=0}|, \frac{M}{C}\right\}, \qquad t \geq \max\left\{1, \frac{c\,C}{\epsilon}\right\}, \tag{7.19}$$

*with $C$ the constant of Lemma 6.4, so that the Taylor coefficients of $z(s)$ at $s = 0$ satisfy
the estimates*

$$|\partial_s^q z^j| \leq c\,\frac{t^{q-1}\, q\,!}{(q+1)^2}, \qquad q = 0, 1, 2, \dots, \tag{7.20}$$

*and the Taylor coefficients of $u(s)$ at $s = 0$ satisfy in addition the estimates*

$$|\partial_s^{q+2} u| \leq c\,\frac{t^q\, (q+2)!}{(q+3)^2}, \qquad q = 0, 1, 2, \dots. \tag{7.21}$$

*It follows that for any given initial data $\dot{u}_0$, $v_0$, $\dot{w}_0$ with $\dot{u}_0 \neq 0$ there exists a number
$t = t(\dot{u}_0, v_0, \dot{w}_0)$ and a unique holomorphic solutions $z^j(s) = z^j(s, \dot{u}_0, v_0, \dot{w}_0)$ of the initial
value problem for the geodesic equations with initial data as described above which is defined
for $|s| \leq 1/t$. The functions $z^j(s, \dot{u}_0, v_0, \dot{w}_0)$ are in fact holomorphic functions of all four
variables in a certain domain.*

PROOF. The existence of a unique formal expansion follows immediately by applying
$\partial_s^p$ for $p = 1, 2, 3, \dots$ formally to equations (7.12) - (7.17), restricting to $s = 0$, and
observing $\dot{u}_0 \neq 0$ and the initial data.

That the estimates (7.20) hold for $q = 0, 1$ follows from the initial condition $x(0) = 0$,
the equations at $s = 0$ and our conditions on $c$ and $t$. That the estimates (7.21) hold for
$q = 0, 1$ follows from (7.10), (7.11), and our conditions on $c$ and $t$.

Let $p \geq 1$ be an integer. We show that $c$ and $t$ can be chosen such that the estimates
(7.20), (7.21) for $q \leq p$ imply with the equations the corresponding estimates for $p + 1$.

From equation (7.15) and Lemma 7.1 (with the provisos given there not repeated here) follows

$$|\partial_s^{p+1} m^{00}| = |\partial_s^p (u\, h)| \leq |\dot{u}_0|\, c\, \frac{t^{p-1}\, p\,!}{p^2} + c^2\, C\, \frac{t^{p-2}\, p\,!}{(p+1)^2} \leq A_{00}\, c\, \frac{t^p\,(p+1)\,!}{(p+2)^2},$$

with

$$A_{00} = \frac{1}{t}\, |\dot{u}_0|\, \frac{p\,!}{p^2}\, \frac{(p+2)^2}{(p+1)\,!} + \frac{1}{t^2}\, c\, C\, \frac{p\,!}{(p+1)^2}\, \frac{(p+2)^2}{(p+1)\,!} \leq \frac{5}{t}\, |\dot{u}_0| + \frac{2}{t^2}\, c\, C.$$

Similarly one gets from (7.12)

$$|\partial_s^{p+2} u| \leq |\partial_s^{p+1} m^{00}| + |\partial_s^{p+1} (u^2\, f)| \leq A_{m^{00}}\, c\, \frac{t^p\,(p+1)\,!}{(p+2)^2}$$

$$+2\, |\dot{u}_0|^2\, c\, \frac{t^{p-1}\,(p+1)\,!}{p^2} + 4\, |\dot{u}_0|\, c^2\, C\, \frac{t^{p-2}\,(p+1)\,!}{(p+2)^2} + c^3\, C^2\, \frac{t^{p-3}\,(p+1)\,!}{(p+2)^2} \leq A_u\, c\, \frac{t^p\,(p+2)\,!}{(p+3)^2},$$

with

$$A_u = A_{00}\, \frac{(p+1)\,!}{(p+2)^2}\, \frac{(p+3)^2}{(p+2)\,!} + \frac{2}{t}\, |\dot{u}_0|^2\, \frac{(p+3)^2}{p^2(p+2)} + \frac{4}{t^2}\, |\dot{u}_0|\, c\, C\, \frac{(p+3)^2}{(p+2)^3} + \frac{1}{t^3}\, c^2\, C^2\, \frac{(p+3)^2}{(p+2)^3}$$

$$\leq \frac{3}{t}\, |\dot{u}_0|(1 + 4\, |\dot{u}_0|) + \frac{1}{t^2}\, c\, C\, (1 + 4\, |\dot{u}_0|) + \frac{1}{t^3}\, c^2\, C^2,$$

and from (7.14)

$$|\partial_s^{p+1} w| = |\partial_s^p m^{11}| \leq c\, \frac{t^{p-1}\, p\,!}{(p+1)^2} \leq A_w\, c\, \frac{t^p\,(p+1)\,!}{(p+2)^2},$$

with

$$A_w = \frac{1}{t}\, \frac{(p+2)^2}{(p+1)^3} \leq \frac{2}{t}.$$

Applying $\partial_s^{p+1}$ to equation (7.16) and observing the initial conditions, gives at $s = 0$ for $p \geq 1$

$$(p+2)\, \dot{u}_0\, \partial_s^{p+1} m^{01} = -\sum_{j=2}^{p+1} \binom{p+1}{j}\, \partial_s^j u\, \partial_s^{p+2-j} m^{01}$$

$$-\sum_{j=1}^{p} \binom{p+1}{j}\, \partial_s^j m^{00}\, \partial_s^{p+1-j} m^{01} + \partial_s^{p+1}(u^2\, k),$$

whence

$$|\partial_s^{p+1} m^{01}| \leq \frac{1}{(p+2)\, |\dot{u}_0|} \left\{ \sum_{j=2}^{p+1} \binom{p+1}{j}\, c^2\, \frac{t^{j-2}\, j\,!}{(j+1)^2}\, \frac{t^{p+1-j}\,(p+2-j)\,!}{(p+3-j)^2} \right.$$

$$\left. +\sum_{j=1}^{p} \binom{p+1}{j}\, c^2\, \frac{t^{j-1}\, j\,!}{(j+1)^2}\, \frac{t^{p-j}\,(p+1-j)\,!}{(p+2-j)^2} + |\partial_s^{p+1}(u^2\, k)| \right\}$$

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$$\leq \frac{1}{|\dot{u}_0|} c^2\, C\, t^{p-1}\, (p+1)! \left\{ \frac{1}{(p+3)^2} + \frac{1}{(p+2)^2} \right\} + 2\,|\dot{u}_0|\, c\, \frac{t^{p-1}\,(p+1)!}{p^2\,(p+2)}$$

$$+ 4\,c^2\, C\, \frac{t^{p-2}\,(p+1)!}{(p+2)^3} + \frac{1}{|\dot{u}_0|}\, c^3\, C^2\, \frac{t^{p-3}\,(p+1)!}{(p+2)^3} = A_{01}\, c\, \frac{t^p\,(p+1)!}{(p+2)^2},$$

with

$$A_{01} = \frac{1}{t}\left\{ \frac{c\,C}{|\dot{u}_0|}(1 + \frac{(p+2)^2}{(p+3)^2}) + 2\,|\dot{u}_0|\frac{(p+2)}{p^2} \right\} + \frac{4\,c\,C}{t^2}\frac{1}{p+2} + \frac{c^2\,C^2}{t^3\,|\dot{u}_0|}\frac{1}{p+2}$$

$$\leq \frac{1}{t}\left\{ \frac{2\,c\,C}{|\dot{u}_0|} + 4\,|\dot{u}_0| \right\} + \frac{2\,c\,C}{t^2} + \frac{c^2\,C^2}{t^3\,|\dot{u}_0|}.$$

Similarly we get from (7.13)

$$|\partial_s^{p+1} v| \leq \frac{1}{(p+1)\,|\dot{u}_0|}\left\{ \sum_{j=2}^{p+1} \binom{p+1}{j}\, |\partial_s^j u|\,|\partial_s^{p+2-j} v| + |\partial_s^{p+1} m^{01}| + |\partial_s^{p+1}(u^2\,h)| \right\}$$

$$\leq \frac{1}{(p+1)\,|\dot{u}_0|}\left\{ \sum_{j=2}^{p+1} \binom{p+1}{j}\, c^2\, \frac{t^{j-2}\,j!}{(j+1)^2}\, \frac{t^{p+1-j}\,(p+2-j)!}{(p+3-j)^2} + |\partial_s^{p+1} m^{01}| + |\partial_s^{p+1}(u^2\,h)| \right\}$$

$$\leq A_v\, c\, \frac{t^p\,(p+1)!}{(p+2)^2},$$

with

$$A_v = \frac{A_{01}}{(p+1)\,|\dot{u}_0|} + \frac{1}{t}\frac{2\,c\,C}{|\dot{u}_0|}\frac{(p+2)^2}{(p+3)^2} + \frac{2\,|\dot{u}_0|}{t}\frac{(p+2)^2}{p\,(p+1)} + \frac{4\,c\,C}{t^2}\frac{1}{p+1} + \frac{c^2\,C^2}{t^3\,|\dot{u}_0|}\frac{1}{p+1}$$

$$\leq \frac{1}{t}\left\{ 9\,|\dot{u}_0| + 2 + \frac{2\,c\,C}{|\dot{u}_0|} + \frac{c\,C}{|\dot{u}_0|^2} \right\} + \frac{c\,C}{t^2}\left\{ 2 + \frac{1}{|\dot{u}_0|} \right\} + \frac{c^2\,C^2}{t^3}\left\{ \frac{1}{|\dot{u}_0|} + \frac{1}{|\dot{u}_0|^2} \right\},$$

and finally from (7.17)

$$|\partial_s^{p+1} m^{11}| \leq \frac{1}{(p+1)\,|\dot{u}_0|}\left\{ \sum_{j=2}^{p+1} \binom{p+1}{j}\, c^2\, \frac{t^{j-2}\,j!}{(j+1)^2}\, \frac{t^{p+1-j}\,(p+2-j)!}{(p+3-j)^2} \right.$$

$$\left. + \sum_{j=1}^{p} \binom{p+1}{j}\, c^2\, \frac{t^{j-1}\,j!}{(j+1)^2}\, \frac{t^{p-j}\,(p+1-j)!}{(p+2-j)^2} + |\partial_s^{p+1}(u^2\,l)| \right\} \leq A_{11}\, c\, \frac{t^p\,(p+1)!}{(p+2)^2},$$

with

$$A_{11} \leq \frac{1}{t}\left\{ 18\,|\dot{u}_0| + \frac{2\,c\,C}{|\dot{u}_0|} \right\} + \frac{2\,c\,C}{t^2} + \frac{c^2\,C^2}{t^3\,|\dot{u}_0|}.$$

From the estimates for the $A$'s it follows that given a choice of $c$ which satisfies the first of the estimates (7.19), we can determine $t$ large enough such that the second of the estimates (7.19) and the conditions

$$A_u, A_v, A_w, A_{00}, A_{01}, A_{11} \leq 1,$$

are satisfied. With this choice the induction step can be carried out.

It follows immediately from estimates (7.20) that the Taylor expansions of the functions $z^j$ at $s = 0$, $z^j(s) = \sum_{p=0}^{\infty} z_p^j s^p$ with $z_p^j = \frac{1}{p!} \partial_s^p z^j(0)$, are absolutely convergent for $|s| < 1/t$.

The coefficients $z_p^j = z_p^j(\dot{u}_0, v_0, \dot{w}_0)$ depend on $v_0$ via the expansion coefficients of the functions $\tilde{e}_{AB}^a$, $\tilde{\Gamma}_{ABCD}$. This implies a polynomial dependence of the $z_p^j$ on $v_0$ due to the $v$-finite expansion types of the functions $\hat{e}_{AB}^a$, $\hat{\Gamma}_{ABCD}$. The explicit dependence of the right hand sides of equations (7.12) - (7.17) on $\dot{u}_0$ and $\dot{w}_0$ alone would lead to a polynomial dependence of the $z_p^j$ on $\dot{u}_0$ and $\dot{w}_0$. The occurence of the factors $u$ on the left hand sides of equations (7.15) - (7.17) implies, however, that the $z_p^j$ are polynomials in $\dot{u}_0$, $v_0$, $\dot{w}_0$ divided by certain powers of $\dot{u}_0$.

The number $t$ which restricts the domain of convergence ensured by our argument depends via $\epsilon$ and $M$ on $v_0$, and via $c$ and the $A$'s on $\dot{u}_0$, $1/\dot{u}_0$ and $\dot{w}_0$ with the effect that $t \to \infty$ as $\dot{u}_0 \to 0$. It follows, however, from the form of the estimates (7.20) and the way they have been obtained that for $(\dot{u}_0, v_0, \dot{w}_0)$ in a compactly embedded subset $U$ of $(\mathbb{C} \setminus \{0\}) \times \mathbb{C} \times \mathbb{C}$ a common number $t$ can be determined so that the Taylor series will be absolutely convergent for $(s, \dot{u}_0, v_0, \dot{w}_0) \in P_{1/t}(0) \times U$.

If $K$ is compact in $P_{1/t}(0) \times U$, there exists $t' > t$ with $K \subset P_{1/t'}(0) \times U$ and it follow from (7.20) that the sequence of holomorphic functions $f_n^j = \sum_{p=0}^{n} z_p^j s^p$ on $P_{1/t}(0) \times U$ satisfies

$$\sup_K |f_n^j - z^j| \leq \sum_{p=n+1}^{\infty} c \frac{t^{p-1}}{(p+1)^2} \left(\frac{1}{t'}\right)^p \leq \frac{c}{t'} \frac{(t/t')^n}{1 - t/t'} \to 0 \quad \text{as} \quad n \to \infty,$$

so that the $f_n^j$ converge uniformly to $z^j$ on $K$. Standard results on compactly converging sequences of holomorphic functions ([22]) then imply that the $z^j = z^j(s, \dot{u}_0, v_0, \dot{w}_0)$ are holomorphic function of all four variables on $P_{1/t}(0) \times U$. ∎

**Lemma 7.3** *Along the geodesic corresponding to $s \to z^j(s, \dot{u}_0, v_0, \dot{w}_0)$ equations (7.9) have a unique holomorphic solution $t^A{}_B(s) = t^A{}_B(s, \dot{u}_0, v_0, \dot{w}_0)$ satisfying the initial conditions (7.7). The functions $t^A{}_B(s, \dot{u}_0, v_0, \dot{w}_0)$ are holomorphic in all four variables in the domain where the $z^j(s, \dot{u}_0, v_0, \dot{w}_0)$ are holomorphic.*

PROOF. By the previous discussion we have $m^{01} = O(s^2)$, $u = O(s)$ with $\dot{u}_0 \neq 0$ so that $m^{01}/u = O(s)$ as $s \to 0$. It follows that equation (7.9) is in fact a linear ODE with holomorphic coefficients and the Lemma follows from standard ODE theory. ∎

For later use we note that (7.7), (7.9) imply as an immediate consequence that

$$t^{-1A}{}_B(s) = s^A{}_B(v_0) + O(|s|^2) \quad \text{as} \quad s \to 0. \tag{7.22}$$

To discuss the transformation to normal coordinates the notation employed before the transition (7.8) will be used again, so that

$$s \to z^a(exp(s\, x^a c_a)) = z^a(s, \dot{u}_0, v_0, \dot{w}_0),$$

denotes in the coordinates $z^1 = u$, $z^2 = v$, $z^3 = w$ the geodesic which has at $s = 0$ the tangent vector $x^a c_{\mathbf{a}}$ with $x^a = x^a(\dot{u}_0, v_0, \dot{w}_0)$ at $i$. We note that by the discussion above

$$u(s, \dot{u}_0, v_0, \dot{w}_0) = \dot{u}_0\, s + O(|s|^3), \quad v(s, \dot{u}_0, v_0, \dot{w}_0) = v_0 + O(|s|^2), \quad w(s, \dot{u}_0, v_0, \dot{w}_0) = \dot{w}_0\, s + O(|s|^3). \tag{7.23}$$

In terms of the map (7.5) the transformation of the normal coordinates $x^c$ centered at $i$ and based on the frame $c_{\mathbf{a}}$ at $i$ into the coordinates $z^a$ is the given by

$$x^a \to z^a(x^c) = z^a(1, \dot{u}_0(x^c), v_0(x^c), \dot{w}_0(x^c)), \tag{7.24}$$

for small enough $|x^a|$ with $x^1 + i\, x^2 \neq 0$. The geodesics being given in normal coordinates by the curves $s \to s\, x^a$, this implies

$$s\, x^a \to z^a(1, \dot{u}_0(s\, x^c), v_0(s\, x^c), \dot{w}_0(s\, x^c)) = z^a(s, \dot{u}_0(x^c), v_0(x^c), \dot{w}_0(x^c)).$$

We use the relation on the right hand side to derive a convenient expression for the map (7.24). Observing that

$$\dot{u}_0(s\, x^c) = s\, \dot{u}_0(x^c), \quad v_0(s\, x^c) = v_0(x^c), \quad \dot{w}_0(s\, x^c) = s\, \dot{w}_0(x^c), \quad s \in C,$$

by (7.5), we write $x^a = s\, x_*^a$ with $s$ chosen so that $\dot{u}_0(x_*^c) = 1$, whence $\dot{u}_0(x^c) = s$, and get with the relation above the map (7.24) in the form

$$z^a(x^c) = z^a(1, \dot{u}_0(x^c), v_0(x^c), \dot{w}_0(x^c)) = z^a(s, \dot{u}_0(x_*^c), v_0(x_*^c), \dot{w}_0(x_*^c))$$

$$= z^a\left(\dot{u}_0(x^c), 1, v_0(x^c), \frac{\dot{w}_0(x^c)}{\dot{u}_0(x^c)}\right).$$

With (7.23) this gives, as $|x| \equiv \sqrt{\delta_{ab}\, \bar{x}^a\, x^b} \to 0$, $x^1 + i\, x^2 \neq 0$,

$$u(x^c) = -\frac{x^1 + i\, x^2}{\sqrt{2}} + O(|x|^3), \quad v(x^c) = -\frac{x^3}{x^1 + i\, x^2} + O(|x|^2), \tag{7.25}$$

$$w(x^c) = \frac{1}{\sqrt{2}}\left(x^1 - i\, x^2 + \frac{(x^3)^2}{x^1 + i\, x^2}\right) + O(|x|^3) = \frac{\delta_{ab}\, x^a\, x^b}{\sqrt{2}\,(x^1 + i\, x^2)} + O(|x|^3). \tag{7.26}$$

In the flat case the order symbols must be omitted in these expressions.

With (4.6), (7.22) and

$$du = -\frac{1}{\sqrt{2}}(dx^1 + i\, dx^2) + O(|x|^2), \quad dv = \frac{dx^3}{\sqrt{2}\, u} + \frac{v}{\sqrt{2}\, u}(dx^1 + i\, dx^2) + O(|x|),$$

$$dw = \frac{1}{\sqrt{2}}\left(dx^1 - i\, dx^2 - 2\, v\, dx^3 - v^2\,(dx^1 + i\, dx^3)\right) + O(|x|^2),$$

one gets for the forms $\chi^{AB} = \chi^{AB}{}_c\, dx^c$ dual to the normal frame $c_{AB}$ indeed

$$\chi^{AB}(x^c) = t^{-1A}{}_C\, t^{-1B}{}_D\,(\sigma^{CD}{}_1\, du + \sigma^{CD}{}_2\, dv + \sigma^{CD}{}_3\, dw) = (\alpha^{AB}{}_a + \hat{\chi}^{AB}{}_a)\, dx^a,$$

with some functions $\hat{\chi}^{AB}{}_a(x^c)$ which satisfies $\hat{\chi}^{AB}{}_a = O(|x|^2)$ as $|x| \to 0$. Correspondingly, the coefficients $c^a_{AB} = \langle dx^a, c_{AB} \rangle$ of the normal frame in the normal coordinates satisfy

$$c^a{}_{AB}(x^c) = \alpha^a{}_{AB} + \hat{c}^a{}_{AB}(x^c),$$

with holomorphic functions $\hat{c}^a{}_{AB}(x^c)$ which satisfy $\hat{c}^a{}_{AB}(x^c) = O(|x|^2)$ as $|x| \to 0$

Since the three 1-forms $\alpha^{AB}{}_a\, dx^a$ are linearly independent this shows that for small $|x^c|$ the coordinate transformation $x^a \to z^a(x^c)$, where defined, is non-degenerate and the forms $\chi^{AB}$ behave as required by normal forms in normal coordinates. The relations (3.1), which characterize coefficients of normal forms in normal coordinates, are a consequence of the equations satisfied by $z^a(s)$ and $t^A{}_B(s)$. All the tensor fields which enter the conformal static vacuum field equations can now be expressed in term of the coordinates $x^c$ and the frame field $c_{AB}$.

All ingredients are now available to derive our main result.

**Proof of Theorem 1.1** The coordinates $x^c$ cover a certain (connected) domain $U$ in $\mathbb{C}^3$ on which the frame vector fields $c^a{}_{AB}\, \partial/\partial_{x^c}$ exist, are linearly independent and holomorphic and where the other tensor fields expressed in terms of the $x^a$ and $c_{AB}$ are holomorphic. It follows from Lemmas 6.9, 7.2, and 7.3 that given any initial data $\dot{u}_0$, $v_0$, $\dot{w}_0$ with $\dot{u}_0 \neq 0$, there exists a solution $z^a(s, \dot{u}_0, v_0, \dot{w}_0)$ of the geodesics equations on the solution provided by Lemma 6.9 which is defined for $|s| \leq 1/t$ with some $t > 0$. The dicussion above shows, however, that $t$ will become large if $|v_0|$ becomes large or $|\dot{u}_0|$ becomes very small. This implies that the domain $U$ will not contain the hypersurface $x^1 + i\,x^2 = 0$ but its boundary will become tangent to this hypersurface at $x^a = 0$. From the estimates obtained so far it cannot be concluded that the coordinates extend holomorphically to a domain containing an open neighbourhood of the origin.

To analyse this question, we make use of the remaining gauge freedom to perform with some $t^A{}_B \in SU(2)$ a rotation $\delta^* \to \delta^* \cdot t$ of the spin frame and the associated rotation

$$c_{AB} \to c^t_{AB} = t^C{}_A\, t^D{}_B\, c_{CD}$$

of the frame $c_{AB}$ at $i$ on which the construction of the submanifold $\hat{S}$ and the related gauge is based. Starting with these frames at $i$ all the previous constructions and derivations can be repeated.

Let $u'$, $v'$, $w'$ and $e^t_{AB}$ denote the analogues in the new gauge of the coordinates $u$, $v$, $w$ and the frame $e_{AB}$. The sets $\{w = 0\}$ and $\{w' = 0\}$ are then both to be thought of as lift of the set $\mathcal{N}_i$ to the bundle of spin frames, the coordinates $u$ and $u'$ can both be interpreted as affine parameters on the null generators of $\mathcal{N}_i$ which vanish at $i$, the coordinates $v$, $v'$ both label these null generators, and the frame vectors $e_{00}$ and $e^t_{00}$ can be identified with auto-parallel vector fields tangent to the null generators.

If $v$ and $v'$ then label the same generator $\eta$ of $\mathcal{N}_i$, a relation

$$s^C{}_0(v')\, s^D{}_0(v')\, t^E{}_C\, t^F{}_D\, c_{EF} = e^t_{00} = f^2\, e_{00} = f^2\, s^C{}_0(v)\, s^D{}_0(v)\, c_{CD},$$

must hold at $i$ with some $f \neq 0$ and $e^t_{00} = f^2\, e_{00}$ must hold in fact along $\eta$, with $f$ constant along $\eta$ because $e^t_{00}$ and $e_{00}$ are auto-parallel. Absorbing the undetermined sign in $f$, this leads to

$$t^E{}_C\, s^C{}_0(v') = f\, s^E{}_0(v).$$

With

$$(t^A{}_B) = \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} \quad \text{where} \quad a, c \in C, \quad |a|^2 = |c|^2 = 1, \tag{7.27}$$

this gives

$$v' = \frac{-c + a\,v}{\bar{a} + \bar{c}\,v}, \quad f = \frac{1}{\bar{a} + \bar{c}\,v}, \quad \text{resp.} \quad v = \frac{c + \bar{a}\,v'}{a - \bar{c}\,v'}, \quad f = a - \bar{c}\,v'.$$

Moreover, the relations

$$< du, e_{00} > = 1 = < du', e_{00}^t > = < du', f^2\,e_{00} >,$$

imply for the affine parameters satisfy along $\eta$

$$u = f^2\,u',$$

so that $\eta(u', v') = \eta(u, v)$ holds with these relations. We note that choices of $t^A{}_B$ with $c \neq 0$ can supply new information, because then $v \to \infty$ as $v' \to a/\bar{c}$ so that the singular generator of the $c_{AB}$-gauge, about whose neighbourhood we need information, is then contained in the regular domain of the $c^t_{AB}$-gauge.

For the null datum in the new gauge one gets with (4.16)

$$s_0^t(u', v') = s^A{}_0(v') \ldots s^C{}_0(v')\,t^E{}_A \ldots t^H{}_D\,s^*_{E \ldots H}|_{\eta(u', v')} = f^4\,s_0(u, v)$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!}\,u'^m\,f^{2m+4}\,s^{A_1}{}_0(v)\,s^{B_1}{}_0(v) \ldots s^D{}_0(v)\,D^*_{(A_1 B_1} \ldots D^*_{A_m B_m}\,s^*_{ABCD)}(i)$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!}\,u'^m\,f^{2m+4}\,s^{A_1}{}_0(v')\,s^{B_1}{}_0(v') \ldots s^D{}_0(v')\,D^t_{(A_1 B_1} \ldots D^t_{A_m B_m}\,s^t_{ABCD)}(i),$$

and thus

$$s_0^t(u', v') = \sum_{m=0}^{\infty} \sum_{n=0}^{2m+4} \psi^t_{m,n}\,u'^m\,v'^n, \tag{7.28}$$

with

$$D^t_{(A_1 B_1} \ldots D^t_{A_m B_m}\,s^t_{ABCD)}(i) \equiv t^{G_1}{}_{A_1}\,t^{H_1}{}_{B_1} \ldots t^N{}_D\,D^*_{(G_1 H_1} \ldots D^*_{G_m H_m}\,s^*_{LKMN)}(i),$$

and

$$\psi^t_{m,n} = \frac{1}{m!} \binom{2m+4}{n}\,D^t_{(A_1 B_1} \ldots D^t_{A_m B_m}\,s^t_{ABCD)_n}(i)$$

$$= \frac{1}{m!} \binom{2m+4}{n} \sum_{j=0}^{2m+4} \binom{2m+4}{j}\,t^{(G_1}{}_{(A_1}\,t^{H_1}{}_{B_1} \ldots t^{N)_j}{}_{D)_n}\,D^*_{(G_1 H_1} \ldots D^*_{G_m H_m}\,s^*_{LKMN)j}(i)$$

$$= \binom{2m+4}{n} \sum_{j=0}^{2m+4} t^{(G_1}{}_{(A_1}\,t^{H_1}{}_{B_1} \ldots t^{N)_j}{}_{D)_n}\,\psi_{m,j}.$$

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It is convenient to write this in the form

$$\psi_{m,n}^t = \sum_{j=0}^{2\,m+4} \binom{2\,m+4}{n}^{1/2} \binom{2\,m+4}{j}^{-1/2} T_{2\,m+4}{}^j{}_n(t)\,\psi_{m,j} \qquad (7.29)$$

where the numbers

$$T_{2\,m+4}{}^j{}_n(t) = \binom{2\,m+4}{n}^{1/2} \binom{2\,m+4}{j}^{1/2} t^{(G_1}{}_{(A_1}\, t^{H_1}{}_{B_1} \dots t^{N)_j}{}_{D)_n},$$

are so defined ([11]) that they represent the matrix elements of a unitary representation of $SU(2)$ and thus satisfy

$$|T_{2\,m+4}{}^j{}_n(t)| \le 1, \quad m = 0, 1, 2, \dots, \quad 0 \le j,\, n \le 2\,m + 4.$$

With the expressions above it is easy to see that the type of the estimate (3.11) and the type of the resulting estimate (6.1) are preserved under the gauge transformation. With (7.28) and (7.29) follows from (6.1) at the point $O' = (u' = 0, v' = 0)$

$$|\partial_{u'}^m \partial_{v'}^n s_0^t(O')| = m!\,n!\,|\psi_{m,n}^t| \le m!\,n!\, \sum_{j=0}^{2\,m+4} \binom{2\,m+4}{n}^{1/2} \binom{2\,m+4}{j}^{-1/2} |T_{2\,m+4}{}^j{}_n(t)|\,|\psi_{m,j}|$$

$$(7.30)$$

$$\le m!\,n!\, \sum_{j=0}^{2\,m+4} \binom{2\,m+4}{n}^{1/2} \binom{2\,m+4}{j}^{1/2} M\,r_1^{-m} \le m!\,n!\, \binom{2\,m+4}{n} \sum_{j=0}^{2\,m+4} \binom{2\,m+4}{j} M\,r_1^{-m}$$

$$= m!\,n!\, \binom{2\,m+4}{n} M'\,r_t^{-m},$$

with $M' = 16\,M$ and $r_t = r_1/4$.

Assuming now that $c \ne 0$ in (7.27), the resulting $c_{AB}^t$-*gauge* can be studied from two different points of view:

i) The singular generator of $\mathcal{N}_i$ in the $c_{AB}^t$-gauge will coincide with the regular generator of $\mathcal{N}_i$ on which $v = -\bar{a}/\bar{c}$ in the $c_{AB}$-gauge. By starting from the solution in the $c_{AB}$-gauge, we are thus able to directly determine near that generator the transformation into the $c_{AB}^t$-gauge and to determine the expansion of the solution in the $c_{AB}$-gauge in terms of the coordinates $u'$, $v'$, $w'$ and the frame field $e_{AB}^t$.

ii) Alternatively, with the null data $s_0^t(u', v')$ at hand, one can go through the discussions of the previous sections to show the existence of a solution to the conformal static vacuum equations in the coordinates $u'$, $v'$, $w'$ pertaining to the $c_{AB}^t$-gauge. All the observations made above, in particular statements about domains of convergence, apply to this solution as well. Important for us is that this solution covers the generator $v' = a/\bar{c}$ near $u' = 0$ and $w' = 0$, which corresponds to the singular generator in the $c_{AB}$-gauge.

Because the formal expansions of the fields in terms of $u'$, $v'$, $w'$ are uniquely determined by the data $s_0^t(u', v')$, the solutions obtained by the two methods are holomorphically related to each other on certain domains by the gauge transformation obtained in

(i). The solution obtained in (ii) can be expressed in terms of the normal coordinates $x_t^a$ and the normal frame field $c_{AB}^t$ so that the $x_t^a$ cover an certain domain $U_t \subset \mathbb{C}^3$ and the frame field $c_{AB}^t$ is non-degenerate and all our tensor fields expressed in terms of $x_t^a$ and $c_{AB}^t$ are holomorphic on $U_t$ as discussed above. It follows then that the solution in the $c_{AB}$-gauge and the solution in the $c_{AB}^t$-gauge are related on certain domains by the simple transformation (cf. (4.3))

$$x_t^a = t^{-1\,a}{}_b\,x^b, \quad c_{AB}^t = t^C{}_A\,t^D{}_B\,c_{CD}.$$

Extending this as a coordinate and frame transformation to the solution obtained in (ii) to express all field in terms $x^a$ and $c_{AB}$ so that they are defined and holomorphic on $t^{-1}\,U_t$, one finds that the solution obtained in (ii) and our original solution define in fact genuine holomorphic extensions of each other because each one covers the singular generator of the other one away from the origin in a regular way.

By letting $t^A{}_B$ go through $SU(2)$ and observing the corresponding extensions, one obtains in fact a holomorphic solution to the conformal static vacuum field equations in the normal coordinates $x^a$ centered at $i$ associated with the frame $\delta^*$ resp. $c_{AB}$ at $i$ on a domain which covers a full neighbourhood of space-like infinity. Consider again the solution we obtained in the $c_{AB}$-gauge. From the discussion above it follows that the domain $U$ in $\mathbb{C}^3$ on which the solution is holomorphic in the coordinates $x^a$ covers a (connected) domain $U'$ of the hypersurface $\{x^3 = 0\}$ of $\mathbb{C}^3$ which has empty intersection with the line $\{x^1 + i\,x^3 = 0, \, x^3 = 0\}$ (corresponding to the singular generator of the $c_{AB}$-gauge) and whose boundary becomes tangent to this line at the origin $x^a = 0$. Under the transition

$$\dot{u}_0 \to \dot{u}_0, \quad v_0 \to e^{i\,\theta/2}\,v_0, \quad \dot{w}_0 \to e^{i\,\theta}\,\dot{w}_0, \quad \theta \in R,$$

which leaves the quantities $|\dot{u}_0|$, $|v_0|$, $|\dot{w}_0|$ entering the estimates above invariant, one gets by (7.4)

$$x^1 + i\,x^2 \to x^1 + i\,x^2, \quad x^1 - i\,x^2 \to e^{i\,\theta}\,(x^1 - i\,x^2), \quad x^3 \to e^{i\,\theta}\,x^3.$$

Thus the set $U'$ can be assumed to be invariant under this transformation.

Consider now the $c_{AB}^{t^*}$-gauge where the special transformation $t^{*\,A}{}_B$ is given by (7.27) with $a = 0$, $c = 1$. Let $U'_{t^*}$ denote the subset of the hypersurface $\{x_{t^*}^3 = 0\}$ in $\mathbb{C}^3$ analogous to $U'$. It has empty insection with the line $\{x_{t^*}^1 + i\,x_{t^*}^2 = 0, \, x_{t^*}^3 = 0\}$ but its boundary becomes tangent to it at $x_{t^*}^a = 0$. It holds

$$c_{00}^{t^*} = c_{11}, \quad c_{01}^{t^*} = -c_{01}, \quad c_{11}^{t^*} = c_{00} \quad \text{at} \quad i,$$

and the corresponding normal coordinates are related by

$$x_{t^*}^1 = -x^1, \quad x_{t^*}^2 = x^2, \quad x_{t^*}^3 = -x^3.$$

The holomorphic transformation $\{x_{t^*}^3 = 0\} \ni (x_{t^*}^1, x_{t^*}^2) \to (-x^1, x^2) \in \{x^3 = 0\}$ maps $U'_{t^*}$ onto a subset of $\mathbb{C}^2 \sim \mathbb{C}^2 \times \{0\}$, denoted by $t^{*\,-1}\,U'_{t^*}$, which has non-empty intersection with $U'$. After the transformation above the two solutions coincide on $t^{*\,-1}\,U'_{t^*} \cap U'$.

On the other hand, the image of the $c_{AB}^{t^*}$-regular line $\{x_{t^*}^1 - i\,x_{t^*}^2 = 0, \, x_{t^*}^3 = 0\} \cap U'_{t^*}$ under this transformation contains the intersection of a neighbourhood of the origin with

the singular line $\{x^1 - i\,x^2 = 0,\, x^3 = 0,\, x^a \neq 0\}$ of the $c_{AB}$-gauge. In fact, the set $t^{*\,-1}\,U'_{t*} \cup U'$, which admits a holomorphic extension of our solution in the coordinates $x^a$ and the frame $c_{AB}$, contains a punctured neighbourhood of the origin. As we have seen above, the field $c_{AB}$ on this neighbourhood extends continously to the origin.

Let now $x^a_* \neq 0$ be an arbitrary point in $\mathbb{C}^3$. We want to show that the solution extends in the coordinates $x^a$ to a domain which covers the set $s\,x^a_*$ for $0 < |s| < \epsilon$ for some $\epsilon > 0$. Since $x^a_* = y^a + i\,z^a$ with $y^a,\, z^a \in \mathbb{R}^3$ there is a vector $u^a \in \mathbb{R}^3$ of unit length and orthogonal to $x^a$ with respect to the standard product $u \cdot x = \delta_{ab}\,u^a\,x^b$. Consider the $c^t_{AB}$-gauges with $t^A{}_B \in SU(2)$ so that $u^a{}_t = t^{-1\,a}{}_b\,u^b = \delta^a{}_3$. It follows then that $x^a_{*t} = t^{-1\,a}{}_b\,x^b_* \in \{x^3_t = 0\}$ and by the preceeding observation $t^A{}_B$ can in fact be chosen such that there exist an $\epsilon > 0$ so that the points $s\,x^a_{*t}$ with $0 < |s| < \epsilon$ are covered by $U'_t$. Transforming back we find that the set $U \in \mathbb{C}^3$ covered by the coordinates $x^a$ can be extended so that the points $s\,x^a_*$ with $0 < |s| < \epsilon$ are covered by $U$ and all field are holomorphic on $U$ in the coordinates $x^a$. It follows that we can assume $U$ to contain a punctured neighbourhood of the origin in which the solution is holomorphic in the normal coordinates $x^a$ and the normal frame $c_{AB}$. Since holomorphic functions in more than one dimensions cannot have isolated singularities ([15]) the solution is then in fact holomorphic on a full neighbourhood of the origin $x^a = 0$, which represents the point $i$.

By Lemma 3.1 the exact sets of equations argument determines from null data satisfying the reality conditions a formal expansion of the solution with expansion coefficients satisfying the reality conditions. By the various uniqueness statements obtained in the Lemmas this expansion must coincide with the expansion in normal coordinates of the solution obtained above. This implies the existence of a 3-dimensional real slice on which the tensor fields satisfy the reality conditions. It is obtained by requiring the coordinates $x^a$ to assume values in $\mathbb{R}^3$. $\square$

# 8    Acknowledgements

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