hep-th/0703285
ULB-TH/07-09
AEI-2007-014

An E_9 multiplet of BPS states

François Englert[a], Laurent Houart[b1], Axel Kleinschmidt[c], Hermann Nicolai[c] , Nassiba Tabti[b2]

[a]*Service de Physique Théorique, Université Libre de Bruxelles*
Campus Plaine C.P.225, Boulevard du Triomphe, B-1050 Bruxelles, Belgium
and
The International Solvay Institutes
Campus Plaine C.P.231, Boulevard du Triomphe, B-1050 Bruxelles, Belgium

[b]*Service de Physique Théorique et Mathématique, Université Libre de Bruxelles*
Campus Plaine C.P. 231, Boulevard du Triomphe, B-1050 Bruxelles, Belgium
and
The International Solvay Institutes
Campus Plaine C.P.231, Boulevard du Triomphe, B-1050 Bruxelles, Belgium

[c]*Max Planck Institute for Gravitational Physics, Albert Einstein Institute*
Am Mühlenberg 1, 14476 Potsdam, Germany

fenglert, lhouart, ntabti@ulb.ac.be
axel.kleinschmidt, hermann.nicolai@aei.mpg.de

Abstract

We construct an infinite E_9 multiplet of BPS states for 11D supergravity. For each positive real root of E_9 we obtain a BPS solution of 11D supergravity, or of its exotic counterparts, depending on two non-compact transverse space variables. All these solutions are related by U-dualities realised via E_9 Weyl transformations in the regular embedding $E_9 \subset E_{10} \subset E_{11}$. In this way we recover the basic BPS solutions, namely the KK-wave, the M2 brane, the M5 brane and the KK6-monopole, as well as other solutions admitting eight longitudinal space dimensions. A novel technique of combining Weyl reflexions with compensating transformations allows the construction of many new BPS solutions, each of which can be mapped to a solution of a dual effective action of gravity coupled to a certain higher rank tensor field. For real roots of E_{10} which are not roots of E_9, we obtain additional BPS solutions transcending 11D supergravity (as exemplified by the lowest level solution corresponding to the M9 brane). The relation between the dual formulation and the one in terms of the original 11D supergravity fields has significance beyond the realm of BPS solutions. We establish the link with the Geroch group of general relativity, and explain how the E_9 duality transformations generalize the standard Hodge dualities to an infinite set of 'non-closing dualities'.

[1]Research Associate F.N.R.S.
[2]F.R.I.A. Researcher

Contents

1 Introduction **3**

2 Basic BPS states in $E_{10} \subset E_{11}$ **7**

2.1 From E_{11} to E_{10} and the coset space E_{10}/K_{10}^{-} 7

2.2 The basic BPS solutions in 1 non-compact dimension 10

2.2.1 Generalities and Hodge duality 10

2.2.2 Levels 1 and 2: The 2-brane and the 5-brane 11

2.2.3 Levels 0 and 3: The Kaluza-Klein wave and the Kaluza-Klein monopole . 13

2.2.4 The exotic BPS solutions 16

3 E_9-branes and the infinite U-duality group **17**

3.1 The working hypothesis 17

3.2 The M2 - M5 system 19

3.2.1 The group-theoretical setting 19

3.2.2 The M2 sequence 22

3.2.3 The M5 sequence 29

3.3 The gravity tower 31

3.4 Analytic structure of BPS solutions and the Ernst potential 35

3.4.1 BPS solutions 37

3.4.2 Action of Geroch group 39

4 Dual formulation of the E_9 multiplet **42**

4.1 Effective actions 42

4.2 Charges and masses 45

4.2.1 From level 4 to level 6 47

4.2.2 The higher level solutions 49

5 Transcending 11D supergravity **49**

6 Summary and comments **52**

A Signature changes and compensations **54**

A.1 The gravity line 54

A.2 The brane towers . . . 55
A.2.1 Signatures of the M2 sequence . . . 55
A.2.2 Signatures of the M5 sequence . . . 56
A.3 The gravity towers . . . 58
A.3.1 Signatures of the wave sequence . . . 58
A.3.2 Signatures of the monopole sequence . . . 59

B Coset representatives of the gravity tower 60

C Masses and U-duality 61
C.1 The level 3 solution . . . 61
C.2 The level 4 solution . . . 62
C.3 The level 5 solution . . . 62
C.4 The level 6 solution . . . 63
C.5 Beyond level 6 . . . 63

D Level 4 by Buscher's duality 64

E Weyl transformations commute with compensations 66
E.1 case 1: $R_i \equiv R^{[6]}_{1-3n}$ and $R_f \equiv R^{[3]}_{1+3n}$ . . . 67
E.2 case 2: $R_i \equiv R^{[3]}_{-1-3n}$ and $R_f \equiv R^{[6]}_{-1+(3n+2)}$ . . . 67

F Redundancy of the two gravity towers 68

G Structure of the A_1^+ U-duality group 70

References 72

1 Introduction

String theories, and particularly superstrings and their possible merging at the non-perturbative level in an elusive M-theory, are often viewed in the double perspective of a consistent quantum gravity theory and of fundamental interactions unification. It is of interest to inquire into the symmetries which would underlie the M-theory project, using as a guide symmetries rooted in its conjectured classical low energy limit, namely 11-dimensional supergravity whose bosonic action is

$$\begin{aligned}\mathcal{S}^{(11)} &= \frac{1}{16\pi G_{11}}\int d^{11}x\sqrt{-g^{(11)}}\Bigg(R^{(11)}-\frac{1}{2\cdot 4!}F_{\mu\nu\sigma\tau}F^{\mu\nu\sigma\tau}\\ &\qquad +\frac{1}{(144)^2}\epsilon^{\mu_1\ldots\mu_{11}}F_{\mu_1\ldots\mu_4}F_{\mu_5\ldots\mu_8}A_{\mu_9\mu_{10}\mu_{11}}\Bigg). \end{aligned} \tag{1.1}$$

Scalars in the dimensional reduction of the action Eq.(1.1) to three space-time dimensions realise non-linearly the maximal non-compact form of the Lie group E_8 as a coset $E_8/SO(16)$ where $SO(16)$ is its maximal compact subgroup. Here, the symmetry of the (2+1) dimensionally reduced action has been enlarged from the $GL(8)$ deformation group of the compact torus T^8 to the simple Lie group E_8. This symmetry enhancement stems from the detailed structure of the action Eq.(1.1).

Coset symmetries were first found in the dimensional reduction of 11-dimensional supergravity [1] to four space-time dimensions [2] but appeared also in other theories. They have been the subject of much study, and some classic examples are given in [3, 4, 5, 6, 7, 8]. In fact, all simple maximally non-compact Lie groups $\mathcal{G}$ can be generated from the reduction down to three dimensions of suitably chosen actions [9]. In particular the effective action of the 26-dimensional bosonic string without tachyonic term yields D_{24} and pure gravity in D space-time dimensions yields A_{D-3}.

It has been suggested that such actions, or possibly some unknown extensions of them, possess a much larger symmetry than the one revealed by their dimensional reduction to three space-time dimensions in which all fields, except $(2+1)$-dimensional gravity itself, are scalars. Such hidden symmetries would be, for each simple Lie group $\mathcal{G}$, the Lorentzian 'overextended' $\mathcal{G}^{++}$ [10] or the 'very-extended' $\mathcal{G}^{+++}$ [11, 12, 13] Kac–Moody algebras generated respectively by adding 2 or 3 nodes to the Dynkin diagram defining $\mathcal{G}$. One first adds the affine node, then a second node connected to it by a single line to get the $\mathcal{G}^{++}$ Dynkin diagram and then similarly a third one connected to the second to generate $\mathcal{G}^{+++}$. In particular, the E_8 invariance of the dimensional reduction to three dimensions of 11-dimensional supergravity would be enlarged to $E_8^{++}\equiv E_{10}$ [5, 14] or to $E_8^{+++}\equiv E_{11}$ [15]. In our quest for the symmetries of M-theory we shall restrict here our considerations to E_{10} and E_{11} and their gravity subalgebras A_8^{++} and A_8^{+++}. The extension of the Dynkin diagram of E_8 to E_{11} is depicted in Fig.1.

To explore the possible fundamental significance of these huge symmetries a Lagrangian formulation [14] *explicitly* invariant under E_{10} has been proposed. It was constructed as a reparametrisation invariant σ-model of fields depending on one parameter t, identified as a time parameter, living on the coset space E_{10}/K_{10}^+. Here K_{10}^+ is the subalgebra of E_{10} invariant under the Chevalley involution. The σ-model contains an infinite number of fields and is built in a recursive way by a level expansion of E_{10} with respect to its subalgebra A_9 [14, 16] whose Dynkin diagram is the 'gravity line' defined in Fig.1, with the node 1 deleted[1]. The level of an irreducible representation of A_9 occurring in the decomposition of the adjoint representation of E_{10} counts the number of times the simple root α_{11} not pertaining to the gravity line appears in the decomposition. The σ-model, limited to the roots up to level 3 and height 29, reveals a perfect match with the bosonic equations of motion of 11-dimensional supergravity in the vicinity of the spacelike singularity of the cosmological billiards [19, 20, 21], where fields depend only on time. It was conjectured that space derivatives are hidden in some higher level fields of the σ-model [14]. We shall label this one-dimensional σ-model S^{cosmo}.

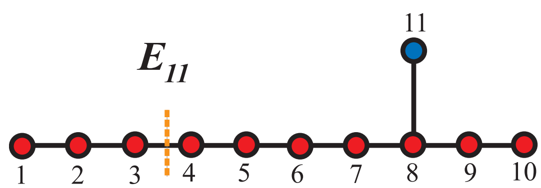


Figure 1: *The Dynkin diagram of E_{11} and its gravity line. The roots 3 and 2 extend E_8 to E_{10} and the additional root 1 to E_{11}. The Dynkin diagram of the A_{10} subalgebra of E_{11}, represented in the figure by the horizontal line is its 'gravity line'.*

An alternate E_{10} σ-model parametrised by a space variable x^1 can be formulated on a coset space E_{10}/K_{10}^-, where K_{10}^- is invariant under a 'temporal' involution ensuring the Lorentz invariance $SO(1,9)$ (rather than $SO(10)$ in S^{cosmo}) at each level in the A_9 decomposition of E_{10}. This σ-model provides a natural framework for studying static solutions. It yields all the basic BPS solutions of 11D supergravity [22], namely the KK-wave, the M2 brane, the M5 brane and the KK6-monopole, smeared in all space dimensions but one, as well as their exotic counterparts. We shall label the action of this σ-model S^{brane} [23]. The algebras K_{10}^+ and K_{10}^- are both subalgebras of the algebra K_{11}^- invariant under the temporal involution defined on E_{11}, which selects the Lorentz group $SO(1,10) = K_{11}^- \cap A_{10}$ in the A_{10} decomposition of E_{11} [22, 23].

Elucidating the role of the huge number of E_{10} and E_{11} generators is an important problem in the Kac–Moody approach to M-theory. In this paper we focus on the real roots of E_{10}, and in particular to those belonging to its regular affine subalgebra E_9. We consider fields parametrising the Borel representatives of the coset space E_{10}/K_{10}^-, that is the Cartan and the positive generators of E_{10}. E_{10} is taken to be embedded regularly in E_{11}. The Dynkin diagrams of E_{10} and E_9 are obtained from the Dynkin diagram of Fig.1 by deleting successively the nodes 1 and 2. We find that each positive real root determines a BPS state in space-time[2], where a BPS state is defined by the no-force condition allowing superposition of configurations centred at different space-time points. We obtain explicitly an infinite E_9 multiplet of BPS

[1]Level expansion of $\mathcal{G}^{+++}$ algebras in terms of a subalgebra A_{D-1} have been considered in [17, 18].

[2]This is in line with the analysis of [24] where BPS states are associated with E_{11} roots.

static solutions of 11D supergravity depending on two non-compact space variables. They are all related by U-dualities realised by the E_9 Weyl transformations.

An obvious question at this point concerns the relation between our results and the conjecture of [25], according to which the BPS solitons of the toroidally compactified theory should transform under the arithmetic group $E_9(\mathbb{Z})$. Like [26], we here consider only the Weyl group of E_9, realized as a subgroup of $E_9(\mathbb{Z})$. More specifically, we consider the Weyl groups of various affine $A_1^+ \equiv A_1^{(1)}$ subgroups of E_9. Each one of these is then found to act via inversions and integer shifts on a certain analytic function characterising the given BPS solution, cf. Eq.(3.52) — much like the modular group $SL(2,\mathbb{Z})$ (see Appendix G for details on the embedding of the Weyl group into the Kac–Moody group itself). There is also an action on the conformal factor, which is, however, more complicated and cannot be interpreted in this simple way (and which can be locally undone by a conformal coordinate transformation). The remaining Weyl reflections 'outside' the given A_1^+, and associated to the simple roots of the E_9, are realized as permutations. However, we should like to stress that the full $E_9(\mathbb{Z})$ contains many more transformations than those considered here.

We show that the BPS states we find admit an equivalent description as solutions of effective actions, where the infinite set of E_9 fields in the Borel representative of E_{10}/K_{10}^- play the role of matter fields in a 'dual' metric. We label the description in terms of 11D supergravity fields the 'direct' one and the description in terms of Borel fields the 'dual' one. Comparing the direct with the dual description sheds some light on the significance of the E_9 real roots. We see indeed that the Borel fields corresponding to these roots are related to the supergravity fields by an infinite set of 'non-closing dualities' generalizing the Hodge duality. This is a feature which is well-known in the context of the Geroch symmetry of standard $D = 4$ gravity reduced to two dimensions [27, 28, 29, 30]. There one also defines an infinite set of dual potentials from the standard fields and uses the infinite Geroch group to generate new solutions. The Geroch group is the affine A_1^+ extension of $SL(2)$ and we will make use of various subgroups of $E_9 \subset E_{10}$ isomorphic to this basic affine group.

One of the original motivations for the present work was to get a better understanding of the significance of the higher level fields in E_{10} and maybe expand the known dictionary beyond level three (rather height 29). Since the role of some of these higher levels is understood in the two-dimensional E_9 context in terms of the generalised dual potentials one can anticipate that they will play a similar role in the one-dimensional σ-models S^{brane} and S^{cosmo}. Indeed this is what we find in the BPS case but we have not obtained an analytic form for these non-closing dualitites in the form of an extended dictionary. Still it is clear that their definition is not restricted to BPS states.

Borel fields attached to roots of E_{10} that are not roots of E_9 define BPS states depending on one non-compact space variable. These may not admit a direct description in terms of 11D supergravity fields but their dual description is well defined. The explicit BPS solution attached to a level 4 root of E_{10} is obtained, in agreement with reference [24], and describes the M9 solution in 11 dimensions which is the 'uplifting' of the D8-brane of Type IIA string theory.

The paper is organised as follows.

In Section 2 we review the construction of the σ-model S^{brane} and its relation to the basic BPS solutions of 11D supergravity and of its exotic counterparts.

In Section 3, we classify *all* E_9 generators in A_1^+ subgroups with central charge in E_{10}. We select particular A_1^+ subgroups containing two infinite 'brane' towers of generators, or one infinite 'gravity' tower. The tower generators are recurrences of the generators defining the basic M2 and M5, or the KK-wave and the KK6-monopole. The other A_1^+ subgroups needed to span all E_9 generators are obtained from the chosen ones by Weyl transformations in the $A_8 \subset E_9$ gravity line. The fields characterising the basic BPS solutions smeared to two space dimensions are encoded as parameters in Borel representatives of E_{10}/K_{10}^-: each basic solution is fully determined by a specific positive generator associated to a specific positive real root. All $E_9 \subset E_{10}$ real roots are related by Weyl transformations. We use sequences of Weyl reflexions to reach any positive real root from roots corresponding to basic BPS solutions. We then express through dualities and compensations the fields defined by a given root in terms of the 11-dimensional metric and the 3-form potential. We verify that these fields yield a new solution of 11D supergravity or of its exotic counterparts. In this way we generate an infinite multiplet of E_9 BPS solutions depending on two space variables. In the string theory context this constitutes an infinite sequence of U-dualities realised as Weyl transformations of E_9. It is shown that the full BPS multiplet of states is characterised by group transformations preserving the analyticity of the Ernst potential originally introduced in the context of the Geroch symmetry of 4-dimensional gravity with one time and one space Killing vectors.

Section 4 discusses the nature of the different BPS states. One introduces the dual formalism which proves a convenient tool to analyse the charge and mass content of the E_9 BPS states. The masses are defined and computed in the string theory context. We show that the E_9 multiplet can be split into three different classes according to the A_9 level l. For $0 \leq l \leq 3$ one gets the basic BPS states smeared to two non-compact space dimensions. For levels 4, 5 and 6 the BPS states depending on two non-compact space variables can not be 'unsmeared' in higher space dimensions. We qualify the eight remaining space dimensions, and the time dimension, as longitudinal ones. For $l > 6$ all BPS states admit nine longitudinal dimensions, including time, and we argue that they are all compact.

Section 5 shows that E_{10} fields associated to real roots which are not in E_9 are BPS solutions of S^{brane} and admit thus a space-time description with one non-compact transverse space dimension. They may not admit a direct description but the dual description is still well defined. These facts are exemplified by a level 4 field which yields the M9 brane.

We summarize the results in Section 6 and discuss the emergence of non-closing dualities.

Several appendices complement arguments in the main part of the paper.

2 Basic BPS states in $E_{10} \subset E_{11}$

2.1 From E_{11} to E_{10} and the coset space E_{10}/K_{10}^-

We recall that the Kac-Moody algebra E_{11} is entirely defined by the commutation relations of its Chevalley generators and by the Serre relations [31]. The Chevalley presentation consists of the generators e_m, f_m and $h_m,\ m = 1, 2, \ldots 11$ with commutation relations

$$\begin{aligned} [h_m, h_n] = 0 \quad , \quad [h_m, e_n] &= a_{mn} e_n\,, \\ [h_m, f_n] &= -a_{mn} f_n \quad , \quad [e_m, f_n] = \delta_{mn} h_m\,, \end{aligned} \tag{2.1}$$

where a_{mn} is the Cartan matrix which can be expressed in terms of scalar products of the simple roots α_m as

$$a_{mn} = 2\frac{\langle \alpha_m, \alpha_n \rangle}{\langle \alpha_m, \alpha_m \rangle}\,. \tag{2.2}$$

The Cartan subalgebra is generated by h_m, while the positive (negative) step operators are the e_m (f_n) and their multi-commutators, subject to the Serre relations

$$[e_m, [e_m, \ldots, [e_m, e_n] \ldots]] = 0, \quad [f_m, [f_m, \ldots, [f_m, f_n] \ldots]] = 0\,, \tag{2.3}$$

where the number of e_m (f_m) acting on e_n (f_n) is given by $1 - a_{mn}$. The Cartan matrix of E_{11} is encoded in its Dynkin diagram depicted in Fig.1. Erasing the node 1 defines the regular embedding of its E_{10} hyperbolic subalgebra and erasing the nodes 1 and 2 yields the regular embedding of the affine E_9.[3]

E_{11} contains a subgroup $GL(11)$ such that $SL(11) \equiv A_{10} \subset GL(11) \subset E_{11}$. The generators of the $GL(11)$ subalgebra are taken to be $K^a{}_b$ $(a, b = 1, 2, \ldots, 11)$ with commutation relations

$$[K^a{}_b, K^c{}_d] = \delta^c_b K^a{}_d - \delta^a_d K^c{}_b\,. \tag{2.4}$$

The relation between the commuting generators $K^a{}_a$ of $GL(11)$ and the Cartan generators h_m of E_{11} in the Chevalley basis follows from comparing the commutation relations Eqs.(2.1) and (2.4) and from the identification of the simple roots of E_{11}. These are $e_m = \delta^a_m K^a{}_{a+1},\ m = 1, \ldots, 10$ and $e_{11} = R^{9\,10\,11}$ where R^{abc} is a generator in E_{11} that is a third rank anti-symmetric tensor under A_{10}. One gets

$$h_m = \delta^a_m (K^a{}_a - K^{a+1}{}_{a+1}) \qquad m = 1, \ldots, 10 \tag{2.5}$$

$$h_{11} = -\frac{1}{3}(K^1{}_1 + \ldots + K^8{}_8) + \frac{2}{3}(K^9{}_9 + K^{10}{}_{10} + K^{11}{}_{11})\,. \tag{2.6}$$

The positive (negative) step operators in the A_{10} subalgebra are the $K^a{}_b$ with $b > a$ ($b < a$). The adjoint representation of E_{11} can be written as an infinite direct sum of representations of the $GL(11)$ generated by the $K^a{}_b$. This is known as the A_{10} level decomposition of E_{11} [14, 17, 16].

[3] We will usually not distinguish in notation between group and algebra since it should be clear from the context which is meant.

The $K^a{}_b$ define the level zero positive (negative) step operators. The positive (negative) level l step operators are defined by the number of times the root α_{11} appears in the decomposition of the adjoint representation of E_{11} into irreducible representations of A_{10} . At level 1, one has the single representation spanned by the anti-symmetric tensor R^{abc}. At each level the number of irreducible representations of A_{10} is finite and the symmetry properties of the irreducible tensors $R^{c_1 \ldots c_r}$ ($R_{c_1 \ldots c_r}$) are fixed by the Young tableaux of the representations. In what follows, positive level step operators will always be denoted with upper indices and negative level ones with lower ones. For positive level l the number of indices on a generator $R^{c_1 \cdots c_r}$ is $r = 3l$.

The Borel group formed by the Cartan generators and the positive level 0 generators can be taken as representative of the coset space $GL(11)/SO(1, 10)$ and hence the parameters of the Borel group can be used to define in a particular gauge the 11-dimensional metric $g_{\mu\nu}$ which spans this coset space at a given space-time point. We note that the subgroup $SO(1, 10)$ of $GL(11)$ is the subgroup invariant under a temporal involution Ω^0 which generalizes the Chevalley involution by allowing the identification of the tensor index 1 to be the time index. Namely we define Ω^0 by the map

$$K^a{}_b \overset{\Omega^0}{\mapsto} -\epsilon_a \epsilon_b K^b{}_a \,, \tag{2.7}$$

with $\epsilon_a = -1$ if $a = 1$ and $\epsilon_a = +1$ otherwise. This suggests to define in general all E_{11} fields as parameters of the coset space E_{11}/K^-_{11} where K^-_{11} is invariant under the more general temporal involution Ω [22] with map

$$K^a{}_b \overset{\Omega}{\mapsto} -\epsilon_a \epsilon_b K^b{}_a \quad , \quad R^{c_1 \ldots c_r} \overset{\Omega}{\mapsto} -\epsilon_{c_1} \ldots \epsilon_{c_r} R_{c_1 \ldots c_r} \,. \tag{2.8}$$

Here $R_{c_1 \ldots c_r}$ is the negative step operator corresponding to the positive one $R^{c_1 \cdots c_r}$. One sees that $K^-_{11} \cap A_{10} = SO(1, 10)$. We shall henceforth label the A_{10} Dynkin subdiagram of E_{11}, the 'gravity line' depicted in Fig.1.

For the regular embedding of E_{10} in E_{11} obtained by deleting the node 1 in Fig.1, the description in terms of $GL(10)$ follows from the description of E_{11} in terms of $GL(11)$. Omitting the generators $K^1{}_2, K^2{}_1$ and $K^1{}_1$, the relations Eqs.(2.4), (2.5) remain valid and Eq.(2.6) becomes

$$h_{11} \to -\frac{1}{3}(K^2{}_2 + \ldots + K^8{}_8) + \frac{2}{3}(K^9{}_9 + K^{10}{}_{10} + K^{11}{}_{11}) \,. \tag{2.9}$$

The temporal involution Ω, $\varepsilon_a = +1$ for $a = 2, 3 \ldots, 11$ reduces to the Chevalley involution acting on the E_{10} generators. It leaves invariant a subalgebra K^+_{10} of E_{10}. The Borel representative of the coset space E_{10}/K^+_{10} is now parametrized by A_9 tensor fields in the Euclidean metric $GL(10)/SO(10)$. Taking these fields to be functions of the remaining time coordinate 1, one can built a σ-model on this coset space. This model has been used mainly to study cosmological solutions [21, 32] and we shall label the action of this σ-model [14] as S^{cosmo}.

We could of course have chosen 2 instead of 1 as time coordinate in $GL(11) \subset E_{11}$. This change of time coordinate can be obtained by performing the E_{11} Weyl reflexion W_{α_1} sending $\alpha_1 \to -\alpha_1$ and $\alpha_2 \to \alpha_1 + \alpha_2$. Choosing the gravity line of Fig.1 to be the reflected one, one finds that its time coordinate has switched from 1 to 2. This results from the fact that the temporal involution does not in general commute with Weyl reflexions [33, 34, 23], a property that has far reaching consequences, as reviewed in Appendix A. Deleting the node 1 in Fig.1 we obtain the Dynkin diagram of E_{10} endowed with the temporal involution $\Omega_{(\lambda)}$, $\lambda = 2$, with $\varepsilon_a = -1$ for $a = 2$ and $+1$ for $a = 3, 4 \ldots 11$ in Eq.(2.8). This involution leaves invariant a subalgebra K_{10}^- of E_{10}. The coset space E_{10}/K_{10}^- accommodates the Lorentzian metric $GL(10)/SO(1,9)$. Performing products of E_{10} Weyl reflexions on the gravity line W_{α_i}, $i = 2, \ldots, 10$, one obtains ten possible different identifications of the time coordinate from $\Omega_{(\lambda)}$, $\lambda = 2, 3 \ldots 11$. The σ-model build upon the coset E_{10}/K_{10}^- can be constructed for any choice of λ in $\Omega_{(\lambda)}$. These formulations of the σ-model are all equivalent up to the field redefinitions by E_{10} Weyl transformations and we shall label them by the generic notation S^{brane}, leaving implicit the choice of the time coordinate λ.

For sake of completeness, we recall the construction of S^{brane} [22, 23]. We take as representatives of E_{10}/K_{10}^- the elements of the Borel group of E_{10} which we write as[4]

$$\mathcal{V}(x^1) = \exp\left[\sum_{a \geq b} h_b{}^a(x^1) K^b{}_a\right] \exp\left[\sum \frac{1}{r!} A_{a_1 \ldots a_r}(x^1) R^{a_1 \ldots a_r} + \cdots\right], \tag{2.10}$$

where from now on all indices run from 2 to 11. The first exponential contains only level zero operators and the second one the positive step operators of E_{10} of levels strictly greater than zero. Define

$$v(x^1) = \frac{d\mathcal{V}}{dx^1}\mathcal{V}^{-1} \quad \widetilde{v}(x^1) = -\Omega_{(\lambda)}\, v(x^1) \qquad v_{sym} = \frac{1}{2}(v + \widetilde{v}), \tag{2.11}$$

with λ equal to the chosen time coordinate. Using the invariant scalar product $\langle\cdot|\cdot\rangle$ for E_{10} one obtains a σ-model constructed on the coset E_{10}/K_{10}^-

$$S^{brane} = \int dx^1 \frac{1}{n(x^1)} \langle v_{sym}(x^1) | v_{sym}(x^1)\rangle, \tag{2.12}$$

where $n(x^1)$ is an arbitrary lapse function ensuring reparametrisation invariance on the world-line. Explicitly, defining

$$e_\mu{}^m = (e^{-h})_\mu{}^m \qquad g_{\mu\nu} = e_\mu{}^m e_\nu{}^n \eta_{mn}, \tag{2.13}$$

where η_{mn} is the Lorentz metric with $\eta_{\lambda\lambda} = -1$, one writes Eq.(2.12) as

$$S^{brane} = S_0 + \sum_A S_A, \tag{2.14}$$

[4]As a warning to the reader we note that this type of Borel gauge for the coset space is not always accessible since the denominator K_{10}^- is not the compact subgroup of the split E_{10} and therefore the Iwasawa decomposition theorem fails. A simple finite-dimensional example is the coset space $SL(2)/SO(1,1)$ which will also play a role below. In this space one cannot find upper triangular representatives for matrices of the form

$$\mathcal{V} = \begin{bmatrix} a & b \\ a & c \end{bmatrix} \qquad \text{with} \quad \big[a(c-b) = 1\big],$$

since the lightlike first column vector cannot be Lorentz boosted to a spacelike one.

with

$$S_0 = \frac{1}{4}\int dx^1 \frac{1}{n(x^1)}(g^{\mu\nu}g^{\sigma\tau} - g^{\mu\sigma}g^{\nu\tau})\frac{dg_{\mu\sigma}}{dx^1}\frac{dg_{\nu\tau}}{dx^1}, \tag{2.15}$$

$$S_A = \frac{1}{2r!}\int dx^1 \frac{1}{n(x^1)}\left[\frac{DA_{\mu_1\ldots\mu_r}}{dx^1} g^{\mu_1\mu'_1}\ldots g^{\mu_r\mu'_r}\frac{DA_{\mu'_1\ldots\mu'_r}}{dx^1}\right]. \tag{2.16}$$

Here D/dx^1 is a group covariant derivative and all indices run from 2 to 11. Note that in Eq.(2.13) one may extend the range of indices to include $\mu, \nu, m, n = 1$, using the embedding relation E_{10} in E_{11} which reads [13]

$$h_1{}^1 = \sum_{a=2}^{11} h_a{}^a \qquad , \qquad h_1{}^a = 0 \qquad a = 2, 3, \ldots 11\,. \tag{2.17}$$

Up to level 3 and height 29, the fields in Eqs.(2.15) and (2.16) can be identified with fields of 11D supergravity [14]. They can be used, as shall now be recalled, to characterise its basic BPS solutions depending on one non-compact space variable.

2.2 The basic BPS solutions in 1 non-compact dimension

2.2.1 Generalities and Hodge duality

The basic BPS solutions of 11D supergravity are the 2-brane (M2) and its magnetic counterpart the 5-brane (M5), and in the pure gravity sector the Kaluza–Klein wave (KK-wave) whose magnetic counterpart is Kaluza-Klein monopole (KK6-monopole). These are static solutions which, wrapped on tori, leave respectively 8, 5, 9 and 3 non-compact space dimensions. It is convenient to express the magnetic solutions in terms of an 'electric' potential with a time index. This is done by trading *on the equations of motion* the field strength for its Hodge dual. For the M5 the field $F_{\mu_1\mu_2\mu_3\mu_4} = 4\,\partial_{[\mu_1}A_{\mu_2\mu_3\mu_4]}$ has as dual the field $\widetilde{F}_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7} = 7\,\partial_{[\nu_1}A_{\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7]}$ defined by

$$\sqrt{-g}\widetilde{F}^{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7} = \frac{1}{4!}\epsilon^{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7\mu_1\mu_2\mu_3\mu_4}F_{\mu_1\mu_2\mu_3\mu_4}\,, \tag{2.18}$$

For the KK6-monopole the KK-potential $A_\mu^{(\nu)}$ in terms of the vielbein $e_\mu{}^n$ is given by $A_\mu^{(\nu)} = -e_\mu{}^n(e^{-1})_n{}^\nu$ where μ labels the non-compact directions, ν is the Taub-NUT direction in coordinate indices and n is the Taub-NUT direction in flat frame indices and there is no summation on n. The field strength is $F^{(\nu)}_{\mu_1\mu_2} = \partial_{\mu_1}A^{(\nu)}_{\mu_2} - \partial_{\mu_2}A^{(\nu)}_{\mu_1}$. Its dual is $\widetilde{F}_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7\nu_8\nu_9|\nu_9} = 9\,\partial_{[\nu_1}A_{\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7\nu_8\nu_9]|\nu_9}$ where

$$\sqrt{-g}\widetilde{F}^{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7\nu_8\nu_9|\nu_9} = \frac{1}{2}\epsilon^{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7\nu_8\nu_9\mu_1\mu_2}F^{(\nu_9)}_{\mu_1\mu_2}\,. \tag{2.19}$$

These BPS solutions depend on space variables in the non-compact dimensions only. One may further compactify on tori some of these 'transverse' directions. This increases accordingly the number of Killing vectors and one obtains in this way new 'smeared' solutions depending only on the space variables in the remaining non-compact dimensions[5]. As we shall see the smearing process is straightforward, except for the smearing of magnetic solutions to one non-compact dimensions, which can only be performed in the electric language, hinting on the fundamental significance of the dual formulation as will indeed be later confirmed.

As recalled below, all BPS solutions, smeared up to one non-compact dimensions are solutions of the σ-model S^{brane} given by Eq.(2.14) where x^1 is identified with the non-compact space dimension [22, 23].

2.2.2 Levels 1 and 2: The 2-brane and the 5-brane

For the 2-brane (M2) solution of 11D supergravity wrapped in the directions 10 and 11, we choose 9 as the time coordinate, so that the only non vanishing component of the 3-form potential is $A_{9\,10\,11}$. For the 5-brane (M5) wrapped in the directions 4,5,6,7,8, we choose 3 as time coordinate so that the 3-form potential is still $A_{9\,10\,11}$ and the Hodge dual Eq.(2.18) is $A_{3\,4\,5\,6\,7\,8}$. One gets for these BPS solutions the following metric and fields

$$\begin{aligned} \text{M2} \quad : \quad & g_{11} = g_{22} = H^{1/3}\,, \quad g_{33} = g_{44} = \cdots = g_{88} = H^{1/3}\,, \quad -g_{99} = g_{10\,10} = g_{11\,11} = H^{-2/3}\,, \\ & A_{9\,10\,11} = \frac{1}{H}\,, \end{aligned} \tag{2.20}$$

$$\begin{aligned} \text{M5} \quad : \quad & g_{11} = g_{22} = H^{2/3}\,, \quad -g_{33} = g_{44} = \cdots = g_{88} = H^{-1/3}\,, \quad g_{99} = g_{10\,10} = g_{11\,11} = H^{2/3}\,, \\ & A_{3\,4\,5\,6\,7\,8} = \frac{1}{H}\,. \end{aligned} \tag{2.21}$$

Here H is a harmonic function of the non-compact space dimensions [(1,2,3,4,5,6,7,8) for M2 and (1,2,9,10,11) for M5] with δ-function singularities at the location of the branes. Smearing simply reduces the number of variables in the harmonic functions to those labelling the remaining non-compact dimensions. For instance a single M2 (M5) brane smeared to two non-compact dimensions, located at the origin of the coordinates (x^1, x^2), is described by $H = (q/2\pi)\ln r = (q/2\pi)\ln\sqrt{(x^1)^2 + (x^2)^2}$ where q is an electric (magnetic) charge density. When smeared to one non-compact dimension one gets $H = (q/2)\,|x^1|$. We see that in the one-dimensional case only the electric dual description of the magnetic M5 brane is available, as the duality relating the 6-form Eq.(2.18) to the 3-form supergravity potential requires at least two transverse dimensions. Note that this one-dimensional solution is still a solution of 11D supergravity because the replacement in the equations of motion of the 4-form field strength by the 7-form dual is valid as long as the Chern-Simons term contributions vanish, as is indeed the case for the above brane

[5] In the string language, the smearing process amounts to introducing image branes in the compact dimensions and averaging them over the torus radii (or equivalently considering in the non-compact dimensions distances large compared to these radii). Compact dimensions which cannot be 'unsmeared' are labelled 'longitudinal'. It will be convenient in what follows to take the time dimension as compact (and longitudinal). Decompactifying longitudinal space-time dimensions does not affect the field dependence of the solutions. However longitudinal dimensions cannot always be decompactified, as exemplified by the Taub-NUT direction of the KK6-monopole. This feature will be studied in detail in Section 4.2.

solutions. Actually the electric description of the M5, Eq.(2.21), is a solution of the following effective action (in any number of transverse non-compact dimensions)

$$\mathcal{S}^{(11)}_{M5} = \frac{1}{16\pi G_{11}} \int d^{11}x \sqrt{-g^{(11)}} \left[R^{(11)} - \frac{1}{2 \cdot 7!} \widetilde{F}_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7} \widetilde{F}^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7} \right] . \tag{2.22}$$

We recall [22] how the M2 solution of 11D supergravity smeared over all dimensions but one can be obtained as a solution of the the σ-model by putting in S^{brane} (Eq.(2.14)) all non-Cartan fields to zero except the level 1 3-form component $A_{9\,10\,11}$ with time in 9. Similarly the M5 solution smeared in all directions but one solves the equations of motion of this σ-model by retaining for the non-Cartan fields only the component $A_{3\,4\,5\,6\,7\,8}$ of the level 2 6-form with time in 3. These are respectively parameters of the Borel generators

$$R_1^{[3]} \stackrel{def}{=} R^{9\,10\,11} \tag{2.23}$$

$$R_2^{[6]} \stackrel{def}{=} R^{345678} , \tag{2.24}$$

where the subscripts denote the A_9 level in the decomposition of the adjoint representation of E_{10}. As we will see in more detail below, the roots corresponding to the elements $R_1^{[3]}$ and $R_2^{[6]}$ have scalar product -2 and thus give rise to an infinite-dimensional affine A_1^+ subalgebra.

These two solutions of the σ-model are characterised by Cartan fields ${h_a}^a, a = 2, 3, \ldots, 11$. One has[6] [22]

$$\begin{aligned}
\text{M2} \ : \ & {h_a}^a = \left\{ \frac{-1}{6} \,\Big|\, \frac{-1}{6}, \frac{-1}{6}, \frac{-1}{6}, \frac{-1}{6}, \frac{-1}{6}, \frac{-1}{6}, \frac{-1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\} \ln H , \qquad {h_a}^b = 0 \text{ for } a \neq b \\
& {h_a}^a {K^a}_a = \frac{1}{2} \ln H \ \cdot \ h_{11} , \\
& A_{9\,10\,11} = \frac{1}{H} ,
\end{aligned} \tag{2.25}$$

$$\begin{aligned}
\text{M5} \ : \ & {h_a}^a = \left\{ \frac{-1}{3} \,\Big|\, \frac{-1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{-1}{3}, \frac{-1}{3}, \frac{-1}{3} \right\} \ln H , \qquad {h_a}^b = 0 \text{ for } a \neq b \\
& {h_a}^a {K^a}_a = \frac{1}{2} \ln H \ \cdot \ (-h_{11} - {K^2}_2) , \\
& A_{3\,4\,5\,6\,7\,8} = \frac{1}{H} ,
\end{aligned} \tag{2.26}$$

where we used Eq.(2.9) and $H(x^1)$ is a harmonic function. To the left of the | symbol in the first line of Eqs.(2.25) and (2.26), we have added the field ${h_1}^1$, evaluated from the embedding of E_{10} in E_{11} Eq.(2.17). All other quantities in these equations are defined in E_{10}. From Eq.(2.13), the σ-model results Eqs.(2.25) and (2.26) are equivalent to the supergravity results Eqs.(2.20) and (2.21) for branes smeared to one space dimension.

[6]For simplicity we have chosen zero for the integration constants in the solutions of the equations of motion of the fields $A_{9\,10\,11}$ and $A_{3\,4\,5\,6\,7\,8}$.

2.2.3 Levels 0 and 3: The Kaluza-Klein wave and the Kaluza-Klein monopole

We now examine the BPS solutions involving only gravity.

First consider the KK-wave solution. The supergravity solution with time in 3 and torus compactification in the 11 direction, is

$$ds^2 = -H^{-1}(dx^3)^2 + (dx^1)^2 + (dx^2)^2 + (dx^4)^2 + \cdots + (dx^{10})^2 + H[dx^{11} - A_3^{(11)}dx^3]^2\,, \quad (2.27)$$

where the electric potential $A_3^{(11)}$ is related to the harmonic function H in nine space dimensions with suitable source δ-function singularities by

$$A_3^{(11)} = (1/H) - 1\,. \quad (2.28)$$

Smearing over any number of space dimensions results in taking H as a harmonic function of only the remaining non-compact space variables. For non-compact space dimension $d > 2$, the constant -1 in Eq.(2.28) makes the potential vanish in the asymptotic Minkowskian space-time if the limit of H at spatial infinity is chosen to be one. For $d = 2$ or 1 space is not asymptotically flat and we keep for convenience the non vanishing constant in Eq.(2.28) to be one.

Smearing the KK-wave to one non-compact dimension, the above supergravity solution is recovered from the σ-model Eq.(2.14) by putting to zero all fields parametrising the positive roots in the Borel representative Eq.(2.10) except the level 0 field ${h_3}^{11}(x^1)$, taking 3 as the time coordinate. To see this, it is convenient to rewrite Eq.(2.10) by disentangling the Cartan generators and the level zero positive step operators in two separate exponentials. One writes

$$\mathcal{V}(x^1) = \exp\left[\sum_{a=2}^{11} {h_a}^a(x^1){K^a}_a\right] \exp\left[{A_3}^{(11)}(x^1){K^3}_{11}\right]\,. \quad (2.29)$$

The expression of $A_3^{(11)}$ in terms of the vielbein defined by Eq.(2.13) is given in Appendix B by Eq.(B.6), namely

$$A_3^{(11)} = -{e_3}^{11}(e^{-1})_{11}^{\ 11}\,. \quad (2.30)$$

The solution is [22], taking into account the embedding relation Eq.(2.17),

$$\begin{aligned}
\text{KKW} \quad : \quad {h_a}^a &= \left\{0\middle|0, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, \frac{-1}{2}\right\} \ln H\,, \\
{h_a}^a {K^a}_a &= \frac{1}{2} \ln H \cdot ({K^3}_3 - {K^{11}}_{11})\,, \\
A_3^{(11)} &= \frac{1}{H} - 1\,,
\end{aligned} \quad (2.31)$$

which, using Eqs.(2.13) and (2.30), is equivalent to the KK-wave solution Eqs.(2.27) and (2.28) of general relativity.

Consider now the KK6-monopole solution. In 11 dimensions, it has 7 longitudinal dimensions (see footnote 5). Taking 11 as the Taub-NUT direction and 4 as the timelike direction, the general relativity solution reads

$$ds^2 = H\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] - (dx^4)^2 + (dx^5)^2 + \cdots + (dx^{10})^2 + H^{-1}\left[dx^{11} - \sum_{i=1}^{3} A_i^{(11)} dx^i\right]^2 \tag{2.32}$$

and

$$F_{ij}^{(11)} \equiv \partial_i A_j^{(11)} - \partial_j A_i^{(11)} = -\varepsilon_{ijk}\, \partial_k H\,, \tag{2.33}$$

where $H(x^1, x^2, x^3)$ is the harmonic function. It can be smeared to 2 spatial dimensions by taking the index j in Eq.(2.33) to label a compact dimension, say 3,

$$\partial_i A_3^{(11)} = \varepsilon_{ik}\, \partial_k H \qquad i, k = 1, 2 \tag{2.34}$$

and $H(x^1, x^2)$ is now harmonic in two dimensions. The metric Eq.(2.32) becomes

$$ds^2 = H\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] - (dx^4)^2 + (dx^5)^2 + \cdots + (dx^{10})^2 + H^{-1}\left[dx^{11} - A_3^{(11)} dx^3\right]^2 \tag{2.35}$$

and is a solution of Einstein's equations.

The smearing to one space dimension is more subtle. As for the magnetic 5-brane, it requires a dual formulation which in this case is defined by the duality relation Eq.(2.19). However the reformulation of the supergravity action is now less straightforward. To understand the dual formulation we first show how to use it for the unsmeared KK6-monopole given by Eqs.(2.32) and (2.33). We rewrite the metric by setting ${e_3}^{11}$, hence $A_3^{(11)}$, to zero and substitute for it the field dual to $A_3^{(11)}$, defined with field strength $\widetilde{F}_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7\nu_8\nu_9|\nu_9} = 9\,\partial_{[\nu_1} A_{\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7\nu_8\nu_9]|\nu_9}$, where the dual field strength $\widetilde{F}$ is defined by Eq.(2.19). The dual (diagonal) metric and the dual potential read

$$\begin{aligned} \text{KK6M} \quad : \quad & g_{11} = g_{22} = g_{33} = H\,, \quad -g_{44} = g_{55} \cdots = g_{10\,10} = 1\,, \quad g_{11\,11} = H^{-1} \\ & A_{4\,5\,6\,7\,8\,9\,10\,11|11} = \frac{1}{H}\,. \end{aligned} \tag{2.36}$$

One verifies that the dual description of the KK6-monopole given by Eq.(2.36) can be derived from an effective action in analogy with the action Eq.(2.22) for the M5. Here the dual field plays the role of a matter field. In the gauge considered here one takes as effective action action

$$\mathcal{S}_{KK6}^{(11)} = \frac{1}{16\pi G_{11}} \int d^{11}x \sqrt{-g^{(11)}} \left[R^{(11)} - \frac{1}{2} \widetilde{F}_{i\,4\,5\,6\,7\,8\,9\,10\,11|11} \widetilde{F}^{i\,4\,5\,6\,7\,8\,9\,10\,11|11}\right]\,, \tag{2.37}$$

where i runs over the three non-compact dimensions $1, 2, 3$. In this dual description we may trivially smear the KK monopole to two or to one non-compact space dimensions by letting the index i in Eq.(2.37) run over the remaining non-compact dimensions. In two non-compact space dimensions, one obtains the dual of the description Eqs.(2.34) and (2.35) and in one dimension one gets in this way a definition of the smeared KK6-monopole which inherits its charge and mass from the parent one with 3 non-compact dimensions.

The charge carried by the KK6-monopoles in three or less non-compact dimensions can be obtained in the dual formulation from the equations of motion of the field $A_{\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7\nu_8\nu_9|\nu_9}$. From Eq.(2.37), one gets

$$\sum_{i,j=1}^{2} \partial_i\big(\sqrt{-g}g^{ij}g^{44}g^{55}g^{66}g^{77}g^{88}g^{99}g^{10\,10}(g^{11\,11})^2\partial_j A_{4\,5\,6\,7\,8\,9\,10\,11|11}\big) = 0\,, \tag{2.38}$$

and using Eq.(2.36) one finds

$$\sum_{i=1}^{2} \partial_i\partial_i H = 0\,, \tag{2.39}$$

outside the source singularities of the harmonic function H. If the latter yields in Eq.(2.39) only δ-function singularities $\sum_k q_k\delta(\vec{r}-\vec{r}_k)$ located in non-compact space points $\vec{r}_k$, one may extend Eq.(2.39) to the whole non-compact space. We write

$$\sum_{i=1}^{2} \partial_i\partial_i H \propto \sum_k q_k\delta(\vec{r}-\vec{r}_k)\,, \tag{2.40}$$

where q_k is the charge of the monopole located at $\vec{r}_k$. For instance a single KK6-monopole located at the origin in 2 non-compact space is described by $H = (q/2\pi)\ln r$.

For 3 or 2 non-compact dimensions, writing Eq.(2.33) or Eq.(2.34) as $F^{(11)}_{\mu\nu}$ one recovers from Eq.(2.40) the charge of the monopoles from the conventional surface integral

$$\int F \equiv \int (1/2)F^{(11)}_{\mu\nu}\, dx^\mu \wedge dx^\nu \propto \sum_k q_k\,, \tag{2.41}$$

where the surface integral enclosed the charges q_k. Eq.(2.41) is equivalent to Eq.(2.40). For KK6-monopoles smeared to one non-compact dimension, the surface integral loses its meaning but the direct definition of charge Eq.(2.40) is still valid. In this way the magnetic KK6-monopole in any number of non-compact transverse dimensions is suitably described (as the magnetic 5-brane by Eq.(2.22)) by a dual effective action Eq.(2.37). As expected for a BPS solution, the charge of the KK6-monopole, smeared or not, is equal to its tension evaluated in string theory, as recalled in Section 4 and in Appendix C.1 where the mass of the KK6-monopole is derived from T-duality.

The above KK6-monopole solution smeared over all dimensions but one can again be obtained as a solution of the σ-model by putting in S^{brane}, with time in 4, all non-Cartan fields to zero except the level 3 component $A_{4\,5\,6\,7\,8\,9\,10\,11|11}$[22]. This is the parameter of the Borel generator[7]

$$\bar{R}_3^{[8,1]} \stackrel{def}{=} R^{4\,5\,6\,7\,8\,9\,10\,11|11}\,, \tag{2.42}$$

[7]We use the bar symbol to distinguish within the same irreducible level 3 A_9 representation the generator $\bar{R}_3^{[8,1]}$ corresponding to a real E_{10} root from the generator $R_3^{[8,1]} = [R_1^{[3]}, R_2^{[6]}]$ pertaining to the degenerate null root, see also below in Section 3.2.

where the subscript labels the A_9 level in the decomposition of the adjoint representation of E_{10}. The solution is, taking into account the embedding relation Eq.(2.17)

$$\begin{aligned} \text{KK6M} \quad : \quad & h_a{}^a = \left\{ \frac{-1}{2} \,\Big|\, \frac{-1}{2}, \frac{-1}{2}, 0, 0, 0, 0, 0, 0, 0, \frac{1}{2} \right\} \ln H \,, \\ & h_a{}^a K^a{}_a = \frac{1}{2} \ln H \, \cdot \, (-K^2{}_2 - K^3{}_3 + K^{11}{}_{11}) \,, \\ & A_{4\,5\ldots10\,11|11} = \frac{1}{H} \,, \end{aligned} \tag{2.43}$$

with $H(x^1)$ harmonic. From Eqs.(2.13) one indeed recovers the KK6-monopole solution of the dual action Eq.(2.37) displayed in Eq.(2.36).

2.2.4 The exotic BPS solutions

As a consequence of the non-commutativity of Weyl reflexions with the temporal involution, the E_{10} σ-model S^{brane} living on E_{10}/K_{10}^{-} was expressed in 10 different ways according to the choice of the time coordinate in Eq.(2.14) in the global signature (1,9). These are related through Weyl transformations of E_{10} from roots of the gravity line. Adding the Weyl reflexion $W_{\alpha_{11}}$ one gets in addition equivalent expressions for S^{brane} where the signatures [23] in Eq.(2.14) are globally different. This equivalence realises in the action formalism the general analysis of Weyl transformations by Keurentjes [33, 35]. Starting with the global signature (1,9) in 10 dimensions, or (1,10) in 11 dimensions, one reaches different signatures $(t, s, \pm)$ in 11 dimensions where t is the number of timelike directions, s is the number of spacelike directions and $\pm$ encodes the sign of the kinetic energy term of the level 1 field in the action Eq.(2.16), $+$ being the usual one and $-$ the 'wrong' one. These are [23][8]: $(1, 10, +)$, $(2, 9, -)$, $(5, 6, +)$, $(6, 5, -)$ and $(9, 2, +)$. The signature changes under the Weyl transformations used in the following sections are presented in Appendix A.

The results obtained in the σ-model Eq.(2.14) are in complete agreement with the interpretation of the Weyl reflexion $W_{\alpha_{11}}$ as a double T-duality in the direction 9 and 10 plus exchange of the two directions [37, 26, 38, 13]. Indeed, it has been shown that T-duality involving a timelike direction changes the signature of space-time leading to the exotic phases of M-theory [39, 40]. The signatures found by Weyl reflexions are thus in perfect agreement with the analysis of timelike T-dualities.

The brane scan of the exotic phases has been studied [41, 42]. Their different BPS branes depend on the signature and the sign of the kinetic term. The number of longitudinal timelike directions for a given brane is constrained. As an example if we consider the so-called M* phase characterised by the signature $(2, 9, -)$, the wrong sign of the kinetic energy term implies that the exotic M2 brane must have even number of timelike directions. There are thus two different M2 branes in M* theory denoted $(0, 3)$ and $(2, 1)$ where the first entry is the number of timelike

[8] The analysis of the different possible signatures related by Weyl reflexions has been extended to all $\mathcal{G}^{++}$ in [36, 34].

longitudinal directions and the second one the number of spacelike longitudinal directions. For instance, the metric of a $(2,1)$ exotic M2 brane with timelike directions 10 , 11, and spacelike direction 9 is

$$\begin{aligned} \mathrm{M2}^* \quad : \quad & g_{11} = g_{22} = H^{1/3} \qquad g_{33} = g_{44} = \cdots = g_{88} = H^{1/3}\,, \\ & g_{99} = -g_{10\,10} = -g_{11\,11} = H^{-2/3}, \end{aligned} \tag{2.44}$$

where H is the harmonic function in the transverse non-compact dimensions.

When smeared in all directions but one this metric is also a solution of the σ-model S^{brane} with the correct identification in Eq.(2.14) of time components and sign shifts in kinetic energy terms. More generally, all the exotic branes smeared to one non-compact space dimension are solutions of this σ-model living on the coset E_{10}/K_{10}^- [23].

3 E_9-branes and the infinite U-duality group

In this section we construct an infinite set of BPS solutions of 11D supergravity and of its exotic counterparts depending on two non-compact space variables. They are related by the Weyl group of E_9 to the basic ones reviewed in Section 2 and constitute an infinite multiplet of U-dualities viewed as Weyl transformations.

3.1 The working hypothesis

Our working hypothesis is that the fields describing BPS solutions of 11D supergravity depending on two non-compact space variables (x^1, x^2) are coordinates in the coset E_{10}/K_{10}^-, in the regular embedding $E_{10} \subset E_{11}$. The coset representatives are taken in the Borel gauge, subject of course to the remark in footnote 4.

We first express in this way the basic solutions of Section 2, smeared to two space dimensions.

Consider the M2 and M5 branes. Their Borel representatives are

$$\mathrm{M2} \quad : \quad \mathcal{V}_1 = \exp\left[\frac{1}{2}\ln H\, h_{11}\right] \exp\left[\frac{1}{H}\, R_1^{[3]}\right] \tag{3.1}$$

$$\mathrm{M5} \quad : \quad \mathcal{V}_2 = \exp\left[\frac{1}{2}\ln H\,(-h_{11} - K^2{}_2)\right] \exp\left[\frac{1}{H}\, R_2^{[6]}\right] . \tag{3.2}$$

Here $R_1^{[3]}$ and $R_2^{[6]}$ are defined in Eqs.(2.23) and (2.24), respectively, and h_{11} was defined in Eq.(2.9). The Cartan fields and the potentials $A_{9\,10\,11}(x_1, x_2)$ for the M2 and $A_{3\,4\,5\,6\,7\,8}(x_1, x_2)$ for the M5 are given by Eqs.(2.25), (2.26) with H now a function of the two variables x^1, x^2. Their metric Eqs.(2.20) and (2.21) are encoded in Eq.(2.13) giving the relation of the Cartan fields to the vielbein and in Eq.(2.17) expressing the embedding of E_{10} in E_{11}. The Hodge duality relations

$$\sqrt{|g|}g^{11}g^{99}g^{10\,10}g^{11\,11}\partial_1 A_{9\,10\,11} = \partial_2 A_{3\,4\,5\,6\,7\,8} \tag{3.3}$$
$$\sqrt{|g|}g^{22}g^{99}g^{10\,10}g^{11\,11}\partial_2 A_{9\,10\,11} = -\partial_1 A_{3\,4\,5\,6\,7\,8}\,, \tag{3.4}$$

reads both for M2 and M5, using Eqs.(2.13), (2.25) and (2.26),

$$\partial_1 H = \partial_2 B \tag{3.5}$$
$$\partial_2 H = -\partial_1 B\,, \tag{3.6}$$

where $B = A_{3\,4\,5\,6\,7\,8}$ for the M2 and $B = A_{9\,10\,11}$ for the M5. In this way, due to the particular choice we made for the tensor components defining the branes, the fields $A_{9\,10\,11}$ and $A_{3\,4\,5\,6\,7\,8}$ are interchanged between the M2 and the M5 when their common value switches from $1/H(x^1, x^2)$ to[9] $B(x^1, x^2)$. Note however that for the M2 (M5) the time in $A_{3\,4\,5\,6\,7\,8}$ ($A_{9\,10\,11}$) is still 9 (3). Eqs.(3.5) and (3.6) are the Cauchy-Riemann relations for the analytic function

$$\mathcal{E}_{(1)} = H + iB\,, \tag{3.7}$$

and H and B are thus conjugate harmonic functions. The duality relations Eqs.(3.5) and (3.6) allow for the replacement of the Borel representatives $\mathcal{V}_1$ and $\mathcal{V}_2$ by

$$\text{M2} \;:\; \mathcal{V}_1' = \exp\left[\frac{1}{2}\ln H\, h_{11}\right]\, \exp\left[B\, R_2^{[6]}\right] \tag{3.8}$$
$$\text{M5} \;:\; \mathcal{V}_2' = \exp\left[\frac{1}{2}\ln H\,(-h_{11} - K^2{}_2)\right]\, \exp\left[B\, R_1^{[3]}\right]\,. \tag{3.9}$$

Note that the representative of the M5 in Eq.(3.9) is, as the representative of the M2 in Eq.(3.1), expressed in terms of the supergravity metric and 3-form potential in two non-compact dimensions.

Consider now the purely gravitational BPS solutions. According to Eqs.(2.31) and (2.43) the Borel representatives are

$$\text{KKW} : \mathcal{V}_0 = \exp\left[\frac{1}{2}\ln H\,(K^3{}_3 - K^{11}{}_{11})\right]\, \exp\left[(H^{-1} - 1)\, K^3{}_{11}\right] \tag{3.10}$$
$$\text{KK6M} : \mathcal{V}_3 = \exp\left[\frac{1}{2}\ln H\,(-K^2{}_2 - K^3{}_3 + K^{11}{}_{11})\right]\, \exp\left[H^{-1}\, R^{4\,5\,6\,7\,8\,9\,10\,11|\,11}\right]\,, \tag{3.11}$$

with $H = H(x^1, x^2)$. Using the duality relations Eq.(2.19) between the [8,1]-form and $A_3^{(11)}$, on may express the Borel representative of the KK6-monopole, as the representative of the KK-wave Eq.(3.10), in two space dimensions in terms of the 11-dimensional metric.

Transforming by E_9 Weyl transformations the Borel representatives of the basic solutions given here, we shall obtain for all E_9 generators associated to its real positive roots, representatives expressed in terms of the harmonic functions $H = H(x^1, x^2)$. These will be transformed

[9] We chose zero for the integration constants of the dual fields $A_{9\,10\,11}$ and $A_{3\,4\,5\,6\,7\,8}$ (cf footnote 6).

through dualities and compensations to Borel representatives expressed in terms of new level 0 fields and 3-form potentials $A_{9\,10\,11}$. This potential and the metric encoded in the level 0 fields through the embedding relation Eq.(2.17) and Eq.(2.13) will be shown to solve the equations of motions of 11D-supergravity. In this way we shall find an infinite set of E_9 BPS solutions related to the M2, M5, KK-waves and KK6-monopoles by U-duality, viewed as E_9 Weyl transformations.

3.2 The M2 - M5 system

3.2.1 The group-theoretical setting

Generalizing the previous notation for generators to all levels by using subscripts denoting the A_9 level, we write $R_1^{[3]} \equiv R^{9\,10\,11}$, $R_{-1}^{[3]} \equiv R_{9\,10\,11}$ and $R_2^{[6]} \equiv R^{3\,4\,5\,6\,7\,8}$, $R_{-2}^{[6]} \equiv R_{3\,4\,5\,6\,7\,8}$. One has

$$\begin{aligned} &[R_1^{[3]}, R_{-1}^{[3]}] = h_{11} \qquad , \qquad [h_{11}, R_1^{[3]}] = 2R_1^{[3]} \\ &[R_2^{[6]}, R_{-2}^{[6]}] = -h_{11} - K^2{}_2 \qquad , \qquad [h_{11}, R_2^{[6]}] = -2R_2^{[6]} \\ &[R_1^{[3]}, R_{-2}^{[6]}] = 0\,. \end{aligned} \tag{3.12}$$

These commutation relations form a Chevalley presentation of a group with Cartan matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}, \tag{3.13}$$

and one verifies that the Serre relations are satisfied. This group is the affine A_1^+ and hence isomorphic to the standard Geroch group which is also the affine extension of $SL(2)$. The central charge is k and derivation d, whose eigenvalues define the affine level, are here given by the embedding of the A_1^+ in E_{10} as

$$k = -K^2{}_2 \quad , \quad d = -\frac{1}{3}K^2{}_2 + \frac{2}{9}(K^4{}_4 + \cdots + K^9{}_9) - \frac{1}{9}(K^3{}_3 + K^{10}{}_{10} + K^{11}{}_{11})\,. \tag{3.14}$$

The level counting operator d is not fixed uniquely by the present embedding.

The multicommutators satisfying the Serre relations form three towers. The positive generators, normalized to one, are

$$R_{1+3n}^{[3]} = 2^{-n} \left[R_1^{[3]}\left[R_1^{[3]}\left[R_2^{[6]}\left[R_1^{[3]}\ldots\left[R_2^{[6]}\left[R_1^{[3]}, R_2^{[6]}\right]\right]\ldots\right]\right.\right.\right. \qquad n \geq 0 \tag{3.15}$$

$$R_{3n}^{[8,1]} = 2^{-(n-1/2)} \left[R_1^{[3]}\left[R_2^{[6]}\left[R_1^{[3]}\ldots\left[R_2^{[6]}\left[R_1^{[3]}, R_2^{[6]}\right]\right]\ldots\right]\right.\right. \qquad n > 0 \tag{3.16}$$

$$R_{-1+3n}^{[6]} = 2^{-(n-1)} \left[R_2^{[6]}\left[R_1^{[3]}\ldots\left[R_2^{[6]}\left[R_1^{[3]}, R_2^{[6]}\right]\right]\ldots\right]\right. \qquad n > 0\,, \tag{3.17}$$

where the affine level n is equal to the number of $R^{[6]}_2$ in the tower[10]. The $R^{[8,1]}_{3n}$ tower correspond to the null roots $n\,\delta$ where

$$\delta = \alpha_3 + 2\alpha_4 + 3\alpha_5 + 4\alpha_6 + 5\alpha_7 + 6\alpha_8 + 4\alpha_9 + 2\alpha_{10} + 3\alpha_{11}\,, \tag{3.18}$$

which has the properties

$$\langle \delta, \delta\rangle = 0 \qquad \langle \delta, \alpha_i\rangle = 0 \quad i = 3, \ldots, 11\,. \tag{3.19}$$

In particular $R^{[8,1]}_3$ is a linear combination $R^{3\,4\,5\,6\,7\,8\,[9\,10,11]}$ of level 3 tensors with all indices distinct. Its height is 30 and thus exceeds the 'classical' limit 29 of [14].

Substituting $h_{11} = -h'_{11} - K^2{}_2$ in Eq.(3.12) one obtains a presentation with $R^{[3]}_1$ and $R^{[6]}_2$ interchanged. While the A_1^+ group in the presentation Eq.(3.12) appears associated with the M2 brane, one could associate the alternate presentation with the M5 brane. The two presentations differ by shifts in the affine level but not by the A_9 level. To avoid complicated notations we keep for the complete M2-M5 system the description given by Eq.(3.12) which is labelled explicitly in terms of the A_9 level. The generators of the A_1^+ group pertaining to the real roots of the M2-M5 system appear in Fig.2a and in Fig.2b.

All the real roots of E_9 can be reached by E_9 Weyl transformations acting on (say) α_{11} defining the generator $R^{[3]}_1$. We shall find convenient for our construction of the infinite set of the E_9 BPS-branes to generate all the real roots from two different real roots, namely α_{11} and $-\alpha_{11}+\delta$ characterising respectively the generators $R^{[3]}_1$ and $R^{[6]}_2$.[11] In this section we obtain the generators of the $R^{[3]}_{1+3n}$ and $R^{[6]}_{-1+3n}$ towers Eqs.(3.15) and (3.17) and their negative counterparts by the Weyl reflexions $W_{\alpha_{11}}$ and $W_{-\alpha_{11}+\delta}$ acting in alternating sequences, starting from their action on the generators $R^{[3]}_1$ and $R^{[6]}_2$. This is depicted in Fig.2a. The Weyl reflexions $W_{\alpha_{11}}$ and $W_{-\alpha_{11}+\delta}$ generate the Weyl group of the affine subgroup A_1^+ of E_9 depicted in Fig.6.[12] Its formal structure is discussed in Appendix G.

For the simply laced algebra considered here, with real roots normed to square length 2, the Weyl reflexion $W_\alpha(\beta)$ in the plane perpendicular to the real root α acting on the arbitrary weight β is given by

$$W_\alpha(\beta) = \beta - \langle\beta, \alpha\rangle\,\alpha \tag{3.20}$$

Consider the Weyl reflexion $W_{-\alpha_{11}+\delta}$ acting on $\alpha_{11} + n\,\delta$, which defines $R^{[3]}_{1+3n}$ for $n \geq 0$ and $R^{[6]}_{1+3n}$ for $n < 0$. One gets

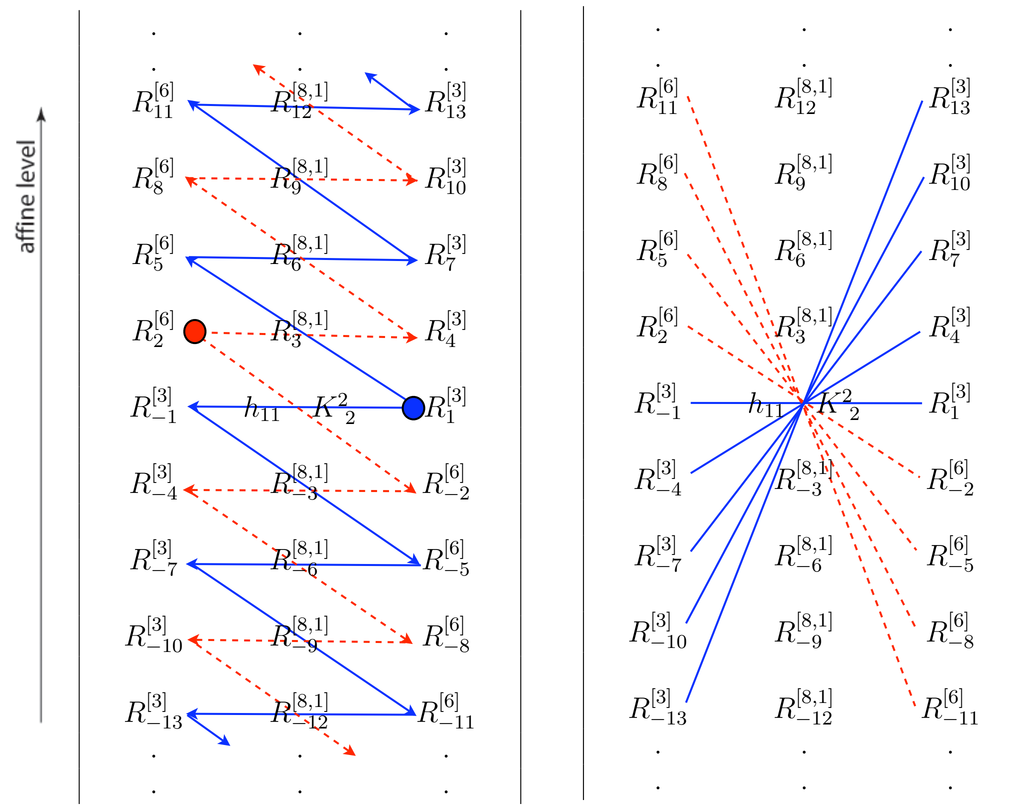
$$W_{-\alpha_{11}+\delta}(\alpha_{11} + n\,\delta) = \alpha_{11} + n\,\delta - \langle\alpha_{11} + n\,\delta, -\alpha_{11} + \delta\rangle(-\alpha_{11} + \delta) = -\alpha_{11} + (n + 2)\,\delta\,, \tag{3.21}$$

where we have used Eq.(3.19). The real root $-\alpha_{11} + (n + 2)\,\delta$ defines the generator $R^{[6]}_{-1+3(n+2)}$ for $n \geq 0$ and $R^{[3]}_{-1+3(n+2)}$ for $n < 0$. The Weyl reflexion has induced an A_9 level increase of four

[10]Shifts in the affine level by one unit corresponds to shifts in A_9 levels by three units, see also [43]. In what follows, when the term level is left unspecified, we always mean the A_9 level.

[11]We note that in A_1^+ not all real roots are Weyl equivalent but there are two distinct orbits as we will see in more detail below.

[12]This is a Coxeter group whose presentation is $\langle\, W_{\alpha_{11}}, W_{-\alpha_{11}+\delta} \,|\, (W_{\alpha_{11}} W_{-\alpha_{11}+\delta})^\infty = \mathrm{id}\,\rangle$.



a b

Figure 2: A_1^+ *group for the M2-M5 system. (a) The Weyl group: the M2 sequence is depicted by solid lines and the M5 sequence by dashed lines. Horizontal lines represent Weyl reflexions* $W_{\alpha_{11}}$ *and diagonal ones* $W_{-\alpha_{11}+\delta}$. *(b) The* $SL(2)$ *subgroups: the solid lines label* $SL(2)$ *subgroups from the 3-tower Eq.(3.15) and the dashed ones the* $SL(2)$ *subgroups from the 6-tower Eq.(3.17).*

'units' since δ from (3.18) has three units of A_9 level and α_{11} has one. Similarly, acting on the root $-\alpha_{11} + n\,\delta$, which defines the generator $R^{[6]}_{-1+3n}$ for $n > 0$ and $R^{[3]}_{-1+3n}$ for $n \leq 0$, with the Weyl reflexion $W_{\alpha_{11}}$ one gets

$$W_{\alpha_{11}}(-\alpha_{11} + n\,\delta) = -\alpha_{11} + n\,\delta - \langle -\alpha_{11} + n\,\delta, \alpha_{11}\rangle \alpha_{11} = \alpha_{11} + n\,\delta\,. \tag{3.22}$$

This Weyl reflexion induces an A_9 level increase of two units. Thus acting successively with the Weyl reflexions Eqs.(3.21) and (3.22) on any root $\alpha_{11} + n\,\delta$ or in the reverse order on the root $-\alpha_{11} + n\,\delta$ one induces a level increase of six units, or equivalently of two affine levels. Of course interchanging the initial and final roots and the order of the two Weyl reflexions, one decreases the affine level by two units. Thus starting from the roots α_{11} and $-\alpha_{11} + \delta$, we obtain the real roots defining all the generators of the $R^{[3]}_{1+3n}$ and $R^{[6]}_{-1+3n}$ towers Eqs.(3.15) and (3.17) and their negative counterparts. These form two sequences depicted in Fig.2a. The 'M2 sequence' originates from the α_{11} root (and thus from the generator $R_1^{[3]}$) and reads

$$\cdots \overset{-\alpha_{11}+\delta}{\longleftarrow} R_{-7}^{[3]} \overset{\alpha_{11}}{\longleftarrow} R_{-5}^{[6]} \overset{-\alpha_{11}+\delta}{\longleftarrow} R_{-1}^{[3]} \overset{\alpha_{11}}{\longleftarrow} \boxed{R_1^{[3]}} \overset{-\alpha_{11}+\delta}{\longrightarrow} R_5^{[6]} \overset{\alpha_{11}}{\longrightarrow} R_7^{[3]} \overset{-\alpha_{11}+\delta}{\longrightarrow} R_{11}^{[6]} \overset{\alpha_{11}}{\longrightarrow} \cdots \quad (3.23)$$

The 'M5 sequence' originates from $-\alpha_{11} + \delta$ (and thus from the generator $R_2^{[6]}$) and reads

$$\cdots \overset{\alpha_{11}}{\longleftarrow} R_{-8}^{[6]} \overset{-\alpha_{11}+\delta}{\longleftarrow} R_{-4}^{[3]} \overset{\alpha_{11}}{\longleftarrow} R_{-2}^{[6]} \overset{-\alpha_{11}+\delta}{\longleftarrow} \boxed{R_2^{[6]}} \overset{\alpha_{11}}{\longrightarrow} R_4^{[3]} \overset{-\alpha_{11}+\delta}{\longrightarrow} R_8^{[6]} \overset{\alpha_{11}}{\longrightarrow} R_{10}^{[3]} \overset{-\alpha_{11}+\delta}{\longrightarrow} \cdots \quad (3.24)$$

Both sequences are represented in Fig.2a.

The Hodge duality relations Eqs.(3.3) and (3.4) will play an essential role in the determinations of the E_9 BPS-branes. The Hodge dual generators $R_1^{[3]}$ and $R_2^{[6]}$ have commutation relations

$$\left[R_1^{[3]}, R_2^{[6]}\right] = R_3^{[8,1]} \,. \quad (3.25)$$

The roots of the E_9 subalgebra do not contain α_2 when expressed in terms of simple roots. Hence from Eq.(3.19) any Weyl transformation from a E_9 real root leaves invariant (possibly up to a sign) the right hand side of Eq.(3.25). Therefore, the image of the basic pair $R_1^{[3]}, R_2^{[6]}$ by any such E_9 Weyl transformation are pairs whose A_9 level sum is equal to three and we have:

Theorem 1 *The set of Weyl transformations in E_9 mapping the A_1^+ group Eq.(3.12) into itself either transforms the pair $(R_1^{[3]}, R_2^{[6]})$ into itself or into one of the pairs $(R_{1+3p}^{[3]}, R_{-1-3(p-1)}^{[3]})$, $(R_{-1+3(p+1)}^{[6]}, R_{1-3p}^{[6]})$ where p is a positive integer.*

This theorem applies to the above Weyl transformations and is easily checked from Fig.2a.

The A_1^+ group Eq.(3.12) admits two infinite sets of $SL(2)$ subgroups

$$\left[R_{1+3p}^{[3]}, R_{-1-3p}^{[3]}\right] = h_{11} - pK^2{}_2 \quad , \quad \left[h_{11} - pK^2{}_2 \,, R_{\pm(1+3p)}^{[3]}\right] = \pm 2R_{\pm(1+3p)}^{[3]} \quad (p \geq 0)\,, \quad (3.26)$$

$$\left[R_{-1+3p}^{[6]}, R_{1-3p}^{[6]}\right] = -h_{11} - pK^2{}_2 \quad , \quad \left[-h_{11} - pK^2{}_2 \,, R_{\pm(-1+3p)}^{[6]}\right] = \pm 2R_{\pm(-1+3p)}^{[6]} \, (p > 0)\,. \, (3.27)$$

As all Weyl reflexions send opposite roots to opposite transforms, one has

Theorem 2 *The set of Weyl transformations in E_9 mapping the A_1^+ group Eq.(3.12) into itself exchanges the $SL(2)$ subgroups between themselves.*

The subgroups Eq.(3.26) and (3.27) are depicted in Fig.2b.

3.2.2 The M2 sequence

We take as representatives of the M2 sequence all the Weyl transforms of the M2 representative Eq.(3.1). The time coordinate is 9. Following in Fig.2a the solid line towards positive step generators, we encounter Weyl transforms of the $SL(2)$ subgroup generated by $(h_{11}\,, R_1^{[3]}\,, R_{-1}^{[3]})$

represented by a solid line in Fig.2b. Theorem 2 determines from Eqs.(3.26) and (3.27) the Weyl transform of the Cartan generators of Eq.(3.1) and we write

$$\mathcal{V}_{1+6n} = \exp\left[\frac{1}{2}\ln H\,(h_{11} - 2nK^2{}_2)\right]\exp\left[\frac{1}{H}\,R^{[3]}_{1+6n}\right] \qquad n \geq 0 \tag{3.28}$$

$$\mathcal{V}_{-1+6n} = \exp\left[\frac{1}{2}\ln H\,(-h_{11} - 2nK^2{}_2)\right]\exp\left[\frac{1}{H}\,R^{[6]}_{-1+6n}\right] \qquad n > 0\,. \tag{3.29}$$

We shall trade the tower fields $A^{[3]}_{1+6n}, A^{[6]}_{-1+6n}$ parametrising the generators in Eqs.(3.28) and (3.29) in favour of the supergravity potential $A_{9\,10\,11}$ and construct from them BPS solutions of 11D supergravity. We have not indicated sign shifts induced by the Weyl transformations in the tower fields from the sign of the lowest level $n = 0$ field which is taken to be $(+1/H)$. This is here the only relevant sign, as discussed below.

Let us consider explicitly the first two steps. These will introduce the two essential features of our construction: compensation and signature changes.

• From level 1 to level 5: compensation

Following in Fig.2a the solid line towards positive step generators, we first encounter the Weyl reflexion $W_{-\alpha_{11}+\delta}$ sending the level 1 generator $R^{[3]}_1$ to the level 5 generator $R^{[6]}_5$. Eq.(3.29) for $n = 1$ reads

$$\mathcal{V}_5 = \exp\left[\frac{1}{2}\ln H\,(-h_{11} - 2K^2{}_2)\right]\exp\left[\frac{1}{H}\,R^{[6]}_5\right]\,. \tag{3.30}$$

As can be seen in Fig.2a, the Weyl reflexion $W_{-\alpha_{11}+\delta}$ sending $R^{[3]}_1$ to $R^{[6]}_5$ sends its dual $R^{[6]}_2$ to $R^{[6]}_{-2}$, in accordance with Theorem 1. We call $R^{[6]}_{-2}$ the dual generator of $R^{[6]}_5$ and we get by acting with $W_{-\alpha_{11}+\delta}$ on the dual representative for the M2, Eq.(3.8), the 'dual' representative of Eq.(3.30),

$$\mathcal{V}'_5 = \exp\left[\frac{1}{2}\ln H\,(-h_{11} - 2K^2{}_2)\right]\exp\left[-B\,R^{[6]}_{-2}\right]\,. \tag{3.31}$$

The $-$ sign in front of B arises as follows. The generators $-h_{11} - K^2{}_2\,,\ R^{[6]}_2\,,\ R^{[6]}_{-2}$ form an $SL(2)$ group, depicted by a dashed line in Fig.2b. We use the representation[13]

$$h_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad K^2{}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{3.32}$$

with $-h_{11} - K^2{}_2 = h_1, R^{[6]}_2 = e_1\,, R^{[6]}_{-2} = f_1$, where we have also included a representation for the central element $K^2{}_2$. The Weyl reflexion $W_{-\alpha_{11}+\delta}$ is generated by the group conjugation matrix U_5 of $SL(2)$ [31]

$$U_5 = \exp R^{[6]}_{-2}\,\exp(-R^{[6]}_2)\,\exp R^{[6]}_{-2}\,, \tag{3.33}$$

which can be represented by

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{3.34}$$

[13] h_1, e_1 and f_1 are the Chevalley generators of an $SL(2) \subset E_9$.

and thus

$$U_5 R_2^{[6]} U_5^{-1} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = -R_{-2}^{[6]} \,. \tag{3.35}$$

One may verify that the conjugation matrix U_5 acting on the Cartan generator of the representative of the M2 in dual form Eq.(3.8) yields the same Cartan generator in the dual representative $\mathcal{V}_5'$ in Eq.(3.31) as in the direct form $\mathcal{V}_5$, which was obtained from the level 1 representative of the M2 Eq.(3.1).

We now write Eq.(3.31) as an $SL(2)$ matrix times a factor coming from the $K^2{}_2$ contribution. One has

$$\mathcal{V}_5' = H^{-1/2} \begin{bmatrix} H^{1/2} & 0 \\ 0 & H^{-1/2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -B & 1 \end{bmatrix} , \tag{3.36}$$

The negative root in Eq.(3.31) can be transferred to the original Borel gauge by a compensating element of K_{10}^-. To this effect we multiply on the left the matrix Eq.(3.36) by a suitable element of the group $SO(2) = SL(2) \cap K_{10}^-$. For a well chosen θ we get

$$\begin{aligned} \overline{\mathcal{V}}_5' &= H^{-1/2} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} H^{1/2} & 0 \\ 0 & H^{-1/2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -B & 1 \end{bmatrix} \\ &= H^{-1/2} \begin{bmatrix} \left(\dfrac{H}{H^2+B^2}\right)^{-1/2} & 0 \\ 0 & \left(\dfrac{H}{H^2+B^2}\right)^{1/2} \end{bmatrix} \begin{bmatrix} 1 & \dfrac{-B}{H^2+B^2} \\ 0 & 1 \end{bmatrix} \\ &= \exp\left[-\frac{1}{2}\ln(H^2+B^2)\, K^2{}_2\right] \exp\left[\frac{1}{2}\ln\frac{H}{H^2+B^2}\, h_{11}\right] \exp\left[\frac{-B}{H^2+B^2}\, R_2^{[6]}\right] . \end{aligned} \tag{3.37}$$

Using the embedding relation Eq.(2.17) we get from Eq.(2.13) the metric[14] encoded in the representative Eq.(3.37)

$$\begin{aligned} \text{Level } 5 \quad : \quad & g_{11} = g_{22} = (H^2+B^2)\widetilde{H}^{1/3} \qquad g_{33} = g_{44} = \cdots = g_{88} = \widetilde{H}^{1/3} \\ & -g_{99} = g_{10\,10} = g_{11\,11} = \widetilde{H}^{-2/3} , \end{aligned} \tag{3.38}$$

where

$$\widetilde{H} = \frac{H}{H^2+B^2} \,. \tag{3.39}$$

Using this metric and the duality equations Eq.(3.3), (3.4), one obtains the supergravity 3-form potential $A_{9\,10\,11}$ dual to $A_{3\,4\,5\,6\,7\,8} = -B/(H^2+B^2)$

$$A_{9\,10\,11} = \frac{1}{\widetilde{H}} , \tag{3.40}$$

and the dual representative of $\overline{\mathcal{V}}_5'$, expressed in terms of the potential Eq.(3.40) is

$$\overline{\mathcal{V}}_5 = \exp\left[-\frac{1}{2}\ln(H^2+B^2)\, K^2{}_2\right] \exp\left[\frac{1}{2}\ln\widetilde{H}\, h_{11}\right] \exp\left[\frac{1}{\widetilde{H}}\, R_1^{[3]}\right] . \tag{3.41}$$

[14]This solution of 11D supergravity has been derived previously in a different context [44].

The dual pair $\widetilde{H}$ and $\widetilde{B} \equiv -B/(H^2+B^2)$ are the conjugate harmonic functions defined by the analytic function $\mathcal{E}_2$ given by

$$\mathcal{E}_2 = \frac{1}{\mathcal{E}_1} = \widetilde{H} + i\widetilde{B}\,. \tag{3.42}$$

We shall verify later that the metric Eq.(3.38) and the potential Eq.(3.40) solve the equations of motion of 11D supergravity.

• **From level 5 to level 7: signature change**

Pursuing further in Fig.2a the solid line towards positive step generators, we encounter the Weyl reflexion $W_{\alpha_{11}}$ leading from level 5 to the level 7 root $R_7^{[3]}$. Eq.(3.28) reads

$$\mathcal{V}_7 = \exp\left[\frac{1}{2}\ln H\,(h_{11} - 2K^2{}_2)\right]\,\exp\left[\frac{1}{H}\,R_7^{[3]}\right]\,. \tag{3.43}$$

In the computation of the level 5 solution we have followed the sequence of duality transformations and the compensation depicted on the second horizontal line in Fig.3. The sequence of operations required to transform $\mathcal{V}_7$ to a representative expressed in terms of the supergravity 3-form potential parametrizing $R_1^{[3]}$ is depicted in the third line of Fig.3. As discussed below in more details in the analysis of the full M2 sequence, all steps appearing in the figure on the same column at levels 5 and 7 are related by the same Weyl transformation $W_{\alpha_{11}}$. Hence one may short-circuit the first two dualities and the first compensation and evaluate directly the Borel representative pertaining to the last column of the level 5 line in Fig.3. This amounts to take the Weyl transform by $W_{\alpha_{11}}$ of $\overline{\mathcal{V}}_5$ given in Eq.(3.41). The generators $h_{11}\,, R_{-1}^{[3]}\,, R_1^{[3]}$ generate as above an $SL(2)$ group represented here by a solid line in Fig.2b. The Weyl conjugation matrix U_7 sending $R_1^{[3]}$ to $R_{-1}^{[3]}$ is

$$U_7 = \exp R_{-1}^{[3]}\,\exp\left(-R_1^{[3]}\right)\,\exp R_{-1}^{[3]}\,, \tag{3.44}$$

which yields the result corresponding to Eq.(3.35), namely

$$U_7\,R_1^{[3]}\,U_7^{-1} = -R_{-1}^{[3]}\,. \tag{3.45}$$

One gets

$$\overline{\mathcal{V}}_7 = \exp\left[-\frac{1}{2}\ln(H^2+B^2)\,K^2{}_2\right]\,\exp\left[-\frac{1}{2}\ln\widetilde{H}\,h_{11}\right]\,\exp\left[-\frac{1}{\widetilde{H}}\,R_{-1}^{[3]}\right]\,. \tag{3.46}$$

To convert the negative root into a positive one we have to perform a second compensation. Here a new phenomenon appears: the space-time signature changes because the temporal involution Ω does not commute with the above Weyl transformation. As explained in Appendix A.2.1, the signature $(1,10,+)$ becomes $(2,9,-)$ with time coordinates 10 and 11 and negative kinetic energy for the field strength. The signature change affects the compensation matrix. The intersection of the $SL(2)$ group generated by $h_{11}\,, R_{-1}^{[3]}\,, R_1^{[3]}$ with K_{10}^- is not $SO(2)$ but $SO(1,1)$. Indeed the transformed involution Ω' resulting from the action of the Weyl reflexion α_{11} yields $\Omega' R_1^{[3]} = +R_{-1}^{[3]}$, and the combination of step operators invariant under Ω' is the hermitian, hence *non-compact generator* $R_1^{[3]} + R_{-1}^{[3]}$, implying $SL(2)\cap K_{10}^- = SO(1,1)$.

Recall that the field $1/\widetilde{H}$ in Eq.(3.46) is inherited from Eq.(3.41) which was obtained from Eq.(3.37) using Hodge duality. The latter is a differential equation and we have hitherto chosen for simplicity the integration constant to be zero. This choice would lead at level 7 to a singular compensating matrix (see footnote 4) and we therefore will use instead the field $1/\widetilde{H}-1$ (we could keep an arbitrary constant $\gamma \neq 0$ instead of -1 but that would unnecessarily complicate notations). Using the matrices Eq.(3.32) with $h_{11} = h_1\,, R_1^{[3]} = e_1\,, R_{-1}^{[3]} = f_1$, we get for a suitable choice of η the compensated representative

$$\begin{aligned} \overline{\overline{\mathcal{V}}}_7 &= (H^2+B^2)^{-1/2} \begin{bmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{bmatrix} \begin{bmatrix} \widetilde{H}^{-1/2} & 0 \\ 0 & \widetilde{H}^{1/2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\widetilde{H}^{-1}+1 & 1 \end{bmatrix} \\ &= (H^2+B^2)^{-1/2} \begin{bmatrix} (2-\widetilde{H})^{1/2} & 0 \\ 0 & (2-\widetilde{H})^{-1/2} \end{bmatrix} \begin{bmatrix} 1 & 1-(2-\widetilde{H})^{-1} \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

or

$$\overline{\overline{\mathcal{V}}}_7 = \exp\left[-\frac{1}{2}\ln(H^2+B^2)\,{K^2}_2\right]\,\exp\left[\frac{1}{2}\ln(2-\widetilde{H})\,h_{11}\right]\,\exp\left[(1-\frac{1}{2-\widetilde{H}})\,R_1^{[3]}\right]. \tag{3.47}$$

As shown below, this yields a solution of the supersymmetric exotic partner of 11D supergravity with times in 10 and 11 and a negative kinetic energy term. One has

$$A_{9\,10\,11} = -\frac{1}{2-\widetilde{H}}\,, \tag{3.48}$$

where we dropped in the 3-form potential the irrelevant constant (-1). The metric encoded in the representative Eq.(3.47) is, from Eqs.(2.17) and (2.13),

$$\begin{aligned} \text{Level 7}\quad : \quad & g_{11} = g_{22} = (H^2+B^2)\widetilde{\widetilde{H}}^{1/3} \qquad g_{33} = g_{44} = \cdots = g_{88} = \widetilde{\widetilde{H}}^{1/3} \\ & g_{99} = -g_{10\,10} = -g_{11\,11} = \widetilde{\widetilde{H}}^{-2/3}\,, \end{aligned} \tag{3.49}$$

where

$$\widetilde{\widetilde{H}} = 2-\widetilde{H}\,. \tag{3.50}$$

From the duality relations Eqs.(3.3) and (3.4), we see that the field $\widetilde{\widetilde{B}}$ dual to $\widetilde{\widetilde{H}}$ is equal to $-\widetilde{B}$. The dual pair $\widetilde{\widetilde{H}}$ and $\widetilde{\widetilde{B}}$ are conjugate harmonic functions associated to the analytic function

$$\mathcal{E}_3 = 2-\mathcal{E}_2 = \widetilde{\widetilde{H}} + i\widetilde{\widetilde{B}}\,. \tag{3.51}$$

- **The complete M2 sequence**

The M2 sequence is characterized by the generators $R_{-1+6n}^{[6]}, n > 0$ and $R_{1+6n}^{[3]}, n \geq 0$. These are reached by following in Fig.2a the solid line starting from $R_1^{[3]}$ towards the positive roots. The representatives are given in Eqs.(3.28) and Eq.(3.29).

Time	Level	
$(9+)$	1	$\boxed{R_1^{[3]}} \overset{d}{\leftrightarrow} R_2^{[6]}$
		$\downarrow -\alpha_{11}+\delta$
$(9+)$	5	$R_5^{[6]} \overset{d}{\leftrightarrow} R_{-2}^{[6]} \overset{c}{\rightarrow} R_2^{[6]} \overset{d}{\rightarrow} \boxed{R_1^{[3]}}$
		$\downarrow \alpha_{11}$
$(10\,11\,-)$	7	$R_7^{[3]} \overset{d}{\leftrightarrow} R_{-4}^{[3]} \overset{c}{\rightarrow} R_4^{[3]} \overset{d}{\rightarrow} R_{-1}^{[3]} \overset{c}{\rightarrow} \boxed{R_1^{[3]}} \overset{d}{\leftrightarrow} R_2^{[6]}$
		$\downarrow -\alpha_{11}+\delta$
$(10\,11\,-)$	11	$R_{11}^{[6]} \overset{d}{\leftrightarrow} R_{-8}^{[6]} \overset{c}{\rightarrow} R_8^{[6]} \overset{d}{\rightarrow} R_{-5}^{[6]} \overset{c}{\rightarrow} R_5^{[6]} \overset{d}{\leftrightarrow} R_{-2}^{[6]} \overset{c}{\rightarrow} R_2^{[6]} \overset{d}{\rightarrow} \boxed{R_1^{[3]}}$
⋮	⋮	

Figure 3: *The construction of BPS states for the M2 sequence. The superscript d labels a duality and the superscript c labels a compensation. The horizontal rows are connected by Weyl reflexions as indicated.*

A glance on Fig.3 shows that at any level of this sequence one may trade this representative either by performing the Weyl transformation $W_{\alpha_{11}}$ on the preceding level representative expressed in terms of the generator $R_1^{[3]}$ (as was done to reach the level 7 from the level 5 representative), or by the Weyl transformation $W_{-\alpha_{11}+\delta}$ on its dual representative in terms of $R_2^{[6]}$ (as was done to reach the level 5 from the level 1 representative). In this way, one bypasses all the steps depicted on horizontal lines in Fig.3 which involve complicated duality relations and compensations to solely perform a single compensation by a $SO(2)$ or $SO(1,1)$ matrix and a known Hodge duality defined by Eqs.(3.3) and (3.4), as exemplified by the detailed analysis of the two first levels of the sequence. The nature of the compensation needed in the construction of the M2 sequence alternates at each step on a horizontal line of Fig.3 between $SO(2)$ and $SO(1,1)$ as shown in Appendix A.2.1. The representatives of the M2 sequence in terms of the supergravity fields are easily written in terms of complex potentials $\mathcal{E}_n$. These are obtained by operating on the analytic function $\mathcal{E}(z)_1 = H(z,\bar{z}) + iB(z,\bar{z})$, $z = x^1 + ix^2, \bar{z} = x^1 - ix^2$, successively by inversions and translations according to

$$\mathcal{E}(z)_{2n} = \frac{1}{\mathcal{E}(z)_{2n-1}}\,, \qquad \mathcal{E}(z)_{2n+1} = 2 - \mathcal{E}(z)_{2n} \qquad n > 0\,. \tag{3.52}$$

These formulæ summarise the action of the E_9 Weyl group on BPS states which are largely characterised by the harmonic functions $\mathcal{E}$. From Eq.(3.52) one sees that the action consists of inversion and shift in a way very similar to the modular group $SL(2,\mathbb{Z})$. In order to see that there is more than just the action of an $SL(2,\mathbb{Z})$ one must consider the action of the transformations on the full metric, including in particular the conformal factor. For the full solution one gets, defining $\mathcal{F}_{2n-1} = \mathcal{E}_1\mathcal{E}_3\ldots\mathcal{E}_{2n-1}$ for $n>0$ (and $\mathcal{F}_{-1}\equiv 1$), the straightforward generalisation of Eqs. (3.1), (3.41) and (3.47),

$$\mathcal{V}_{1+6n} = \exp\left[-\frac{1}{2}\ln(\mathcal{F}_{2n-1}\bar{\mathcal{F}}_{2n-1})\,K^2{}_2\right]\exp\left[\frac{1}{2}\ln\mathcal{R}e\,\mathcal{E}_{2n+1}\,h_{11}\right]\exp\left[\frac{(-1)^n}{\mathcal{R}e\,\mathcal{E}_{2n+1}}\,R_1^{[3]}\right]$$
$$n \geq 0 \qquad (3.53)$$

$$\mathcal{V}_{-1+6n} = \exp\left[-\frac{1}{2}\ln(\mathcal{F}_{2n-1}\bar{\mathcal{F}}_{2n-1})\,K^2{}_2\right]\exp\left[\frac{1}{2}\ln\mathcal{R}e\,\mathcal{E}_{2n}\,h_{11}\right]\exp\left[\frac{(-1)^{n+1}}{\mathcal{R}e\,\mathcal{E}_{2n}}\,R_1^{[3]}\right]$$
$$n > 0 \qquad (3.54)$$

where $\bar{\mathcal{F}}$ denotes the complex conjugate of $\mathcal{F}$ and the signatures in Eq.(3.53) are $(1, 10, +)$ with time in 9 for n even and $(2, 9, -)$ with times in 10,11 for n odd. In Eq.(3.54) the signatures are $(2, 9, -)$ with times in 10,11 for n even and $(1, 10, +)$ with time in 9 for n odd. The detailed analysis of the signatures for the M2 sequence is done in Appendix A.2.1 and the final results are summarised in Table 3. To interchange the role of even and odd n in the above signatures, one simply builds another M2 sequence starting from the exotic M2 of the theory $(2, 9, -)$ whose metric is given in Eq.(2.44). It has two longitudinal times in 10 and 11 and one longitudinal spacelike direction 9.

Eqs.(3.53) and (3.54) yield from Eqs.(2.13) and (2.17) the metric and three-form potential

$$\begin{aligned} ds^2 &= \mathcal{F}_{2n-1}\bar{\mathcal{F}}_{2n-1}H_{2n+1}^{1/3}[(dx^1)^2 + (dx^1)^2] + H_{2n+1}^{1/3}[(dx^3)^2 + \cdots + (dx^8)^2] + \\ &+ H_{2n+1}^{-2/3}[(-1)^{(n+1)}(dx^9)^2 + (-1)^n(dx^{10})^2 + (-1)^n(dx^{11})^2] \qquad (n \geq 0) \\ A_{9\,10\,11} &= \frac{(-1)^n}{H_{2n+1}} \end{aligned} \qquad (3.55)$$

$$\begin{aligned} ds^2 &= \mathcal{F}_{2n-1}\bar{\mathcal{F}}_{2n-1}H_{2n}^{1/3}[(dx^1)^2 + (dx^1)^2] + H_{2n}^{1/3}[(dx^3)^2 + \cdots + (dx^8)^2] + \\ &+ H_{2n}^{-2/3}[(-1)^n(dx^9)^2 + (-1)^{n+1}(dx^{10})^2 + (-1)^{(n+1)}(dx^{11})^2] \qquad n > 0 \\ A_{9\,10\,11} &= \frac{(-1)^{n+1}}{H_{2n}} \end{aligned} \qquad (3.56)$$

with $H_p = \mathcal{R}e\,\mathcal{E}_p$. We stress that an important effect of the action of the affine Weyl group on the BPS solutions is the change in the conformal factor which is expressed through $\mathcal{F}_{2n-1}\bar{\mathcal{F}}_{2n-1}$.

For each level on the M2-sequence, these equations satisfy the equations of motion of 11D supergravity or of its exotic counterpart outside the singularities of the functions H_p. There, the factor $\mathcal{F}_{2n-1}\bar{\mathcal{F}}_{2n-1}$ can indeed be eliminated by a change of coordinates and the functions H_p are still harmonic functions in the new coordinates. Eqs.(3.55) and (3.56) have then the same dependence on H_p as the M2 metric and 3-form have on $H_1 \equiv H$ and differ thus from the M2 solution only through the choice of the harmonic function. They therefore solve the Einstein equations. This is also discussed more abstractly in section 3.4.

To obtain these results, we have chosen a particular path to reach from level 1 the end of any horizontal line in Fig.3. Along this path, all signs of the fields in the representatives were fixed by the choice $+1/H$ at level 1 and by the Hodge duality relations Eqs.(3.3) and (3.4).

Thus, we do not have to explicitly take into account signs which might affect higher level tower fields in Eqs.(3.28) and (3.29).

The consistency of the procedure used in this Section to obtain solutions related by U-dualities viewed as Weyl transformations rests however on the arbitrariness of the path chosen to reach from level 1 the end of any horizontal line in Fig.3. Dualities for levels $l > 2$ are in principle defined by the Weyl transformations. Consistency is thus equivalent to commuting Weyl transformations with compensations. Compensations and Weyl transformations do indeed commute, as proven in Appendix E.

3.2.3 The M5 sequence

We follow the same procedure as for the M2 sequence. We take as representatives of the M5 sequence all the Weyl transforms of the M5 representative Eq.(3.2) with fields Eq.(2.26) and time coordinate 3. Following in Fig.2a the dashed line towards positive step generators, we encounter Weyl transforms of the $SL(2)$ subgroup generated by $(-h_{11} - K^2{}_2\,, R_2^{[6]}\,, R_{-2}^{[6]})$ represented by a dashed line in Fig.2b. Theorem 2 determines from Eqs.(3.26) and (3.27) the Weyl transform of the Cartan generators of Eq.(3.2) and we write

$$\mathcal{V}_{2+6n} = \exp\left[\frac{1}{2}\ln H\,(-h_{11} - (2n+1)K^2{}_2)\right]\exp\left[\frac{1}{H}\,R^{[6]}_{2+6n}\right] \qquad n \geq 0 \qquad (3.57)$$

$$\mathcal{V}_{-2+6n} = \exp\left[\frac{1}{2}\ln H\,(h_{11} - (2n-1)K^2{}_2)\right]\exp\left[\frac{1}{H}\,R^{[3]}_{-2+6n}\right] \qquad n > 0\,. \qquad (3.58)$$

We shall trade the tower fields $A^{[6]}_{2+6n} = A^{[3]}_{-2+6n}$ in favour of the supergravity potential $A_{9\,10\,11}$ and construct from them BPS solutions of 11D supergravity.

For the M5 itself, we take the dual representative $\mathcal{V}_2'$ expressed in terms of the 3-form potential, which is given in Eq.(3.9). As previously the first two steps, levels 4 and 8, contain the essential ingredients of the whole sequence.

Following in Fig.2a the dashed line towards positive step generators, we first encounter the Weyl reflexion α_{11} sending the level 2 generator $R_2^{[6]}$ to the level 4 generator $R_4^{[3]}$, or equivalently, as exhibited in Fig.4, the dual generator $R_1^{[3]}$ to the generator $R_{-1}^{[3]}$. Applying this Weyl reflexion to Eq.(3.9) and performing an $SO(2)$ compensation we get the representative

$$\overline{\mathcal{V}}_4 = \exp\left[-\frac{1}{2}\ln(H^2+B^2)\,K^2{}_2\right]\exp\left[-\frac{1}{2}\ln\widetilde{H}\,(h_{11} + K^2{}_2)\right]\exp\left[\widetilde{B}\,R_1^{[3]}\right]\,, \qquad (3.59)$$

which yields $A_{9\,10\,11} = \widetilde{B}$ and the metric[15]

$$\begin{aligned}\text{Level 4}\ :\ & g_{11} = g_{22} = (H^2+B^2)\widetilde{H}^{2/3} \qquad -g_{33} = g_{44}\cdots = g_{88} = \widetilde{H}^{-1/3}\\ & g_{99} = g_{10\,10} = g_{11\,11} = \widetilde{H}^{2/3}\,,\end{aligned} \qquad (3.60)$$

[15]This solution of 11 D supergravity has already been derived in a different context [44].

Time	Level	
(3+)	2	$R_2^{[6]} \overset{d}{\leftrightarrow} \boxed{R_1^{[3]}}$
		$\downarrow \alpha_{11}$
(3+)	4	$R_4^{[3]} \overset{d}{\leftrightarrow} R_{-1}^{[3]} \overset{c}{\rightarrow} \boxed{R_1^{[3]}} \overset{d}{\rightarrow} R_2^{[6]}$
		$\downarrow -\alpha_{11}+\delta$
(4 5 6 7 8 +)	8	$R_8^{[6]} \overset{d}{\leftrightarrow} R_{-5}^{[6]} \overset{c}{\rightarrow} R_5^{[6]} \overset{d}{\rightarrow} R_{-2}^{[6]} \overset{c}{\rightarrow} R_2^{[6]} \overset{d}{\leftrightarrow} \boxed{R_1^{[3]}}$
		$\downarrow \alpha_{11}$
(4 5 6 7 8 +)	10	$R_{10}^{[3]} \overset{d}{\leftrightarrow} R_{-7}^{[3]} \overset{c}{\rightarrow} R_7^{[3]} \overset{d}{\rightarrow} R_{-4}^{[3]} \overset{c}{\rightarrow} R_4^{[3]} \overset{d}{\leftrightarrow} R_{-1}^{[3]} \overset{c}{\rightarrow} \boxed{R_1^{[3]}} \overset{d}{\rightarrow} R_2^{[6]}$
.	.	
.	.	

Figure 4: *The construction of BPS states for the M5 sequence. The superscript d labels a duality and the superscript c labels a compensation.*

with $\tilde{H}$ and $\tilde{B}$ as before. The level 4 results are in agreement with the interpretation of the Weyl reflexion $W_{\alpha_{11}}$ as a double T-duality in the directions 9 and 10 plus interchange of the two directions [37, 26, 38, 13]. We recover indeed the level 4 metric and 3-form by applying Buscher's duality rules to the M5 smeared in the directions 9, 10 and 11. This is shown in Appendix D. The next step leads to level 8. As for the computation of the level 7 representative in the M2 sequence, we may skip the two first dualities and the first compensation indicated in the third line of Fig.4. It suffices to perform the Weyl reflexion $W_{-\alpha_{11}+\delta}$ on the dual representative of Eq.(3.59) followed by a $SO(1,1)$ compensation and a Hodge duality. One gets

$$\overline{\overline{\mathcal{V}}}_8 = \exp\left[-\frac{1}{2}\ln(H^2+B^2)\,{K^2}_2\right]\exp\left[-\frac{1}{2}\ln\tilde{\tilde{H}}\,(h_{11}+{K^2}_2)\right]\exp\left[-\tilde{\tilde{B}}\,R_1^{[3]}\right], \tag{3.61}$$

which yields $A_{9\,10\,11} = -\tilde{\tilde{B}}$ and the metric

$$\begin{aligned} \text{Level 8} \quad : \quad & g_{11} = g_{22} = (H^2+B^2)\tilde{\tilde{H}}^{2/3} \qquad g_{33} = -g_{44}\cdots = -g_{88} = \tilde{\tilde{H}}^{-1/3} \\ & g_{99} = g_{10\,10} = g_{11\,11} = \tilde{\tilde{H}} H^{2/3}\,. \end{aligned} \tag{3.62}$$

As shown in Appendix A.2.2 the signature is now $(5,6,+)$ with times in 4,5,6,7,8.

The full M5 sequence is characterized by the roots $R^{[3]}_{-2+6n}, n>0$ and $R^{[6]}_{2+6n}, n\geq 0$. These are reached by following in Fig.2a the dashed line starting at $R_2^{[6]}$ towards the positive roots. The representative is defined by the Cartan generator given in Eq.(3.26) or Eq.(3.27) and by the field $1/H$ multiplying the positive root. As for the M2 sequence, the generalisation to all levels to the lowest ones Eqs.(3.9), (3.59) and (3.61) is straightforward. As indicated in Fig.4, one obtains iteratively the representatives in terms of the supergravity 3-form by solely performing a single compensation by a $SO(2)$ or $SO(1,1)$ matrix and a known Hodge duality defined by Eqs.(3.3) and (3.4). One alternates after two steps between representatives with a single time in 3 and exotic ones with times in 4,5,6,7,8. The nature of the compensation changes at each step. One has

$$\begin{aligned}
\mathcal{V}_{2+6n} &= \exp\left[-\frac{1}{2}\ln(\mathcal{F}_{2n-1}\bar{\mathcal{F}}_{2n-1})\,K^2{}_2\right]\,\exp\left[-\frac{1}{2}\ln\mathcal{R}e\,\mathcal{E}_{2n+1}\,(h_{11}+K^2{}_2)\right] \\
&\cdot\ \exp\left[(-1)^n\mathcal{I}m\,\mathcal{E}_{2n+1}\,R_1^{[3]}\right] \qquad n\geq 0
\end{aligned} \tag{3.63}$$

$$\begin{aligned}
\mathcal{V}_{-2+6n} &= \exp\left[-\frac{1}{2}\ln(\mathcal{F}_{2n-1}\bar{\mathcal{F}}_{2n-1})\,K^2{}_2\right]\,\exp\left[-\frac{1}{2}\ln\mathcal{R}e\,\mathcal{E}_{2n}\,(h_{11}+K^2{}_2)\right] \\
&\cdot\ \exp\left[(-1)^{n+1}\mathcal{I}m\,\mathcal{E}_{2n}\,R_1^{[3]}\right] \qquad n> 0
\end{aligned} \tag{3.64}$$

where in Eq.(3.63) one has the signatures $(1,10,+)$ with time in 3 for n even and $(5,6,+)$ with times in 4,5,6,7,8 for n odd, and in Eq.(3.64) the signatures are $(1,10,+)$ with time in 3 for n odd and $(5,6,+)$ with times in 4,5,6,7,8 for n even. As previously it is always possible to interchange at each pair of levels the two signatures by choosing an exotic M5 to initiate the sequence. The detailed analysis of the signatures for the M5 sequence and of the compensations required is done in Appendix A.2.2 and summarised in Table 5.

These representative yield the metric and 3-form potential for all states on the M5 sequence. We get from Eqs.(3.63) and (3.64)

$$\begin{aligned}
ds^2 &= \mathcal{F}_{2n-1}\bar{\mathcal{F}}_{2n-1}H_{2n+1}^{2/3}[(dx^1)^2+(dx^1)^2]+H_{2n+1}^{2/3}[(dx^9)^2+(dx^{10})^2+(dx^{11})^2] \quad (3.65)\\
&+\ H_{2n+1}^{-1/3}[(-1)^{n+1}(dx^3)^2+(-1)^n(dx^4)^2\cdots+(-1)^n(dx^8)^2] \qquad (n\geq 0)\\
A_{9\,10\,11} &= (-1)^n B_{2n+1}
\end{aligned}$$

$$\begin{aligned}
ds^2 &= \mathcal{F}_{2n-1}\bar{\mathcal{F}}_{2n-1}H_{2n}^{2/3}[(dx^1)^2+(dx^1)^2]+H_{2n}^{2/3}[(dx^9)^2+(dx^{10})^2+(dx^{11})^2] \quad (3.66)\\
&+\ H_{2n}^{-1/3}[(-1)^n(dx^3)^2+(-1)^{n+1}(dx^4)^2+\cdots+(-1)^{n+1}(dx^8)^2] \qquad (n>0)\\
A_{9\,10\,11} &= (-1)^{n+1} B_{2n}
\end{aligned}$$

with $B_p=\mathcal{I}m\,\mathcal{E}_p$.

For each level on the M5-sequence, these equations satisfy the equations of motion of 11D supergravity or of its exotic counterpart outside the singularities of the harmonic functions H_p and B_p. There, the factor $\mathcal{F}_{2n-1}\bar{\mathcal{F}}_{2n-1}$ can indeed be eliminated by a change of coordinates and the functions H_p and B_p are still conjugate harmonic functions of the new coordinates. Eqs.(3.65) and (3.66) have then the same dependence on H_p and B_p as the M5 metric and 3-form have on $H_1\equiv H$ and $B_1\equiv B$ and differ thus from the M5 solution only through the choice of the harmonic functions. They therefore solve the Einstein equations.

3.3 The gravity tower

The affine A_1^+ group generated by $R_1^{[3]}\equiv R^{9\,10\,11}$ and $R_2^{[6]}\equiv R^{3\,4\,5\,6\,7\,8}$ spans three towers of generators. We found BPS solutions for each positive generator of the 3-tower Eq.(3.15) and of the 6-tower Eq.(3.17). All these generators correspond to real roots while those in the third tower Eq.(3.16) generators correspond to null roots of square length zero. Each generator of the third tower at level $3(n+1) \quad n \geq 0$ belongs to an irreducible representation of $A_8 \subset E_9$ whose lowest weight is the real root $\alpha_4 + 2\alpha_5 + 3\alpha_6 + 4\alpha_7 + 5\alpha_8 + 3\alpha_9 + \alpha_{10} + 3\alpha_{11} + n\delta$. We now show that the lowest weight generators belong to a A_1^+ subgroup of E_9 generated by $R^{4\,5\ldots 10\,11|11}$ which sits at level 3 and by $K^3{}_{11}$, which is defined by the level 0 real root $\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10}$.

These two generators are related as follows

$$K^3{}_{11} \quad \leftrightarrow \quad \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10} \equiv \lambda\,, \tag{3.67}$$
$$R^{4\,5\ldots 10\,11|11} \quad \leftrightarrow \quad \alpha_4 + 2\alpha_5 + 3\alpha_6 + 4\alpha_7 + 5\alpha_8 + 3\alpha_9 + \alpha_{10} + 3\alpha_{11} = -\lambda + \delta\,, \tag{3.68}$$

where the last equality in Eq.(3.68) is easily checked using Eq.(3.18). They are Weyl transforms of the generators $R^{9\,10\,11}$ and $R^{3\,4\,5\,6\,7\,8}$ of the the A_1^+ group defined in Eq.(3.12). To see this, first perform the Weyl transformation interchanging 9 and 3. The A_1^+ generators Eq.(3.12) are transformed to (all Weyl transforms of step generators are written up to a sign)

$$R^{9\,10\,11} \quad \rightarrow \quad R^{3\,10\,11} \tag{3.69}$$
$$R^{3\,4\,5\,6\,7\,8} \quad \rightarrow \quad R^{4\,5\,6\,7\,8\,9}\,, \tag{3.70}$$

defined by the roots $\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_{11}$ and $\alpha_4 + 2\alpha_5 + 3\alpha_6 + 4\alpha_7 + 5\alpha_8 + 4\alpha_9 + 2\alpha_{10} + 2\alpha_{11}$. Then perform the Weyl reflexion $W_{\alpha_{11}}$ to get the generators

$$R^{3\,10\,11} \quad \rightarrow \quad K^3{}_9 \tag{3.71}$$
$$R^{4\,5\,6\,7\,8\,9} \quad \rightarrow \quad R^{4\,5\,6\,7\,8\,9\,10\,11|9}\,, \tag{3.72}$$

defined by the roots $\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8$ and $\alpha_4+2\alpha_5+3\alpha_6+4\alpha_7+5\alpha_8+4\alpha_9+2\alpha_{10}+3\alpha_{11}$. Finally perform the Weyl transformation exchanging 9 and 11 to get

$$K^3{}_9 \quad \rightarrow \quad K^3{}_{11} \tag{3.73}$$
$$R^{4\,5\,6\,7\,8\,9\,10\,11|\,9} \quad \rightarrow \quad R^{4\,5\,6\,7\,8\,9\,10\,11|\,11}\,, \tag{3.74}$$

whose defining roots are λ and $-\lambda + \delta$. The transformed Cartan generators are $K^3{}_3 - K^{11}{}_{11}$ and $-K^2{}_2 + K^{11}{}_{11} - K^3{}_3$. Under these transformations, the M2-brane generator is mapped onto the Kaluza-Klein wave generator in the direction 11. The M5-brane generator is mapped to the dual Kaluza-Klein monopole generator $R^{4\,5\,6\,7\,8\,9\,10\,11|\,11}$. These generate the 'gravity A_1^+ group' conjugate in E_9 to the 'brane A_1^+ group' Eq.(3.12).

We now find the BPS solutions of 11D pure gravity (which are of course solution of 11D supergravity) associated to each positive real root of the gravity A_1^+ group. One could redo the analysis of the M2-M5 system starting from the representatives of the KK-wave and KK6-monopole given in Eqs.(3.10) and (3.11) and the duality relations Eq.(2.19). It is however simpler to take advantage of the Weyl mapping of the two A_1^+ subgroups of E_9

$$R^{9\,10\,11} \quad \leftrightarrow \quad K^3{}_{11} \tag{3.75}$$
$$R^{4\,5\,6\,7\,8\,9} \quad \leftrightarrow \quad R^{4\,5\,6\,7\,8\,9\,10\,11|11} \tag{3.76}$$
$$\alpha_{11} \leftrightarrow \lambda \qquad , \qquad \delta \leftrightarrow \delta\,. \tag{3.77}$$

The generators $R^{[3]}_{1+3n}$ of the 3-tower Eq.(3.15) are mapped to generators of level $3n$. We label these generators $R^{[0]}_{3n}$ ($R^{[0]}_0 \equiv K^3{}_{11}$). The generators $R^{[6]}_{-1+3n}$ of the 6-tower Eq.(3.17) are also mapped to generators of level $3n\,(n > 0)$. We label these generators $\bar{R}^{[8,1]}_{3n}$ ($\bar{R}^{[8,1]}_3 \equiv R^{4\,5\,6\,7\,8\,9\,10\,11|11}$). In the mapping the signature changes as shown in Appendix A.3. In particular, the KK-wave $R^{[0]}_0$ yields a single time in 3 and the KK6-monopole $\bar{R}^{[8,1]}_3$ becomes exotic with two times 9 and 10. This mapping of the M2-M5 sequences of Fig.2a to the gravity sequences is illustrated in Fig.5. To the M2 sequence corresponds a wave sequence starting with the KK-wave and to the M5 sequence a monopole sequence staring with the (exotic) KK6-monopole. Note that there is a duplication in each sequence of states with the same level $3n$ for $n > 0$. We shall show that this duplication is spurious in the sense that the two states are related by a switch of coordinates.

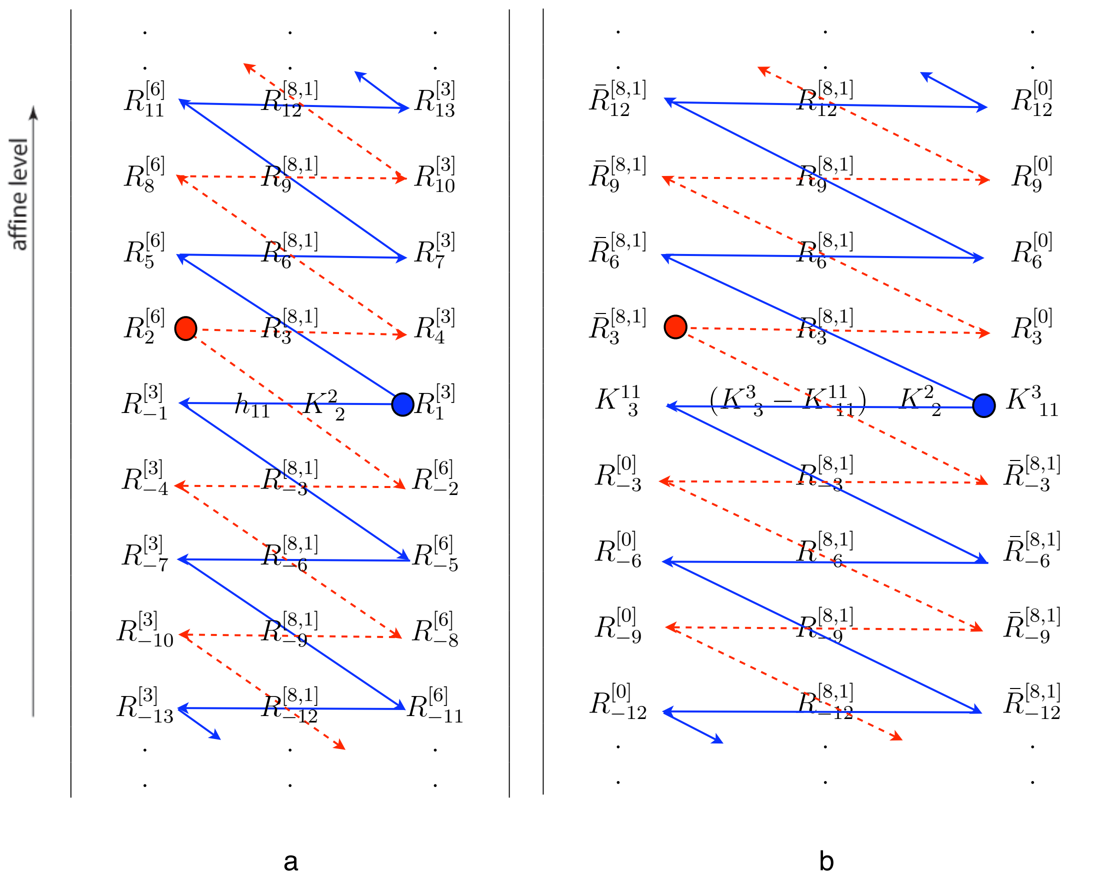


Figure 5: *Mapping of the brane A_1^+ group (Fig.5a) to the gravity A_1^+ group (Fig.5b). M2 and wave sequences are depicted by solid lines, M5 and monopole sequences by dashed lines. In Fig.5a (Fig.5b) horizontal lines represent Weyl reflexions by $W_{\alpha_{11}}$ (W_λ), diagonal lines by $W_{-\alpha_{11}+\delta}$ ($W_{-\lambda+\delta}$).*

From the correspondence we immediately get from the representatives of the M2 sequence (3.28), (3.29), and of the M5 sequences (3.57) and (3.58), the representatives of the KK-wave sequence Eqs.(3.78), (3.79) and of the KK-monopole sequence Eqs.(3.80), (3.81) in terms of the

$R^{[0]}_{3n}$ and $\bar{R}^{[8,1]}_{3n'}$ generators

$$\mathcal{V}_{6n} = \exp\left[\frac{1}{2}\ln H\,(K^3{}_3 - K^{11}{}_{11} - 2nK^2{}_2)\right]\exp\left[\frac{1}{H}\,R^{[0]}_{6n}\right] \qquad n \geq 0 \tag{3.78}$$

$$\mathcal{V}_{6n'} = \exp\left[\frac{1}{2}\ln H\,(-K^3{}_3 + K^{11}{}_{11} - 2n'K^2{}_2)\right]\exp\left[\frac{1}{H}\,\bar{R}^{[8,1]}_{6n'}\right] \qquad n' > 0 \tag{3.79}$$

$$\mathcal{V}_{3+6n'} = \exp\left[\frac{1}{2}\ln H\,(-K^3{}_3 + K^{11}{}_{11} - (2n'+1)K^2{}_2)\right]\exp\left[\frac{1}{H}\,\bar{R}^{[8,1]}_{6n'+3}\right] \quad n' \geq 0 \tag{3.80}$$

$$\mathcal{V}_{-3+6n} = \exp\left[\frac{1}{2}\ln H\,(K^3{}_3 - K^{11}{}_{11} - (2n-1)K^2{}_2)\right]\exp\left[\frac{1}{H}\,R^{[0]}_{6n-3}\right] \quad n > 0\ . \tag{3.81}$$

In these equations we distinguish the representatives of the [0]-tower depicted in the right column of Fig.5b from those of the [8,1]-tower depicted in the left column by labelling the former by n and the latter by n'.

To get the representatives for the wave sequence in terms of the gravitational potential $A^{(11)}_3$ given by Eq.(2.30), we apply the mapping Eqs.(3.75), (3.76) and (3.77) to the representative of the M2 sequence in terms of the supergravity 3-form potential[16], Eqs.(3.53) and (3.54)

$$\begin{aligned}\mathcal{V}_{6n} &= \exp\left[-\frac{1}{2}\ln(\mathcal{F}_{2n-1}\bar{\mathcal{F}}_{2n-1})\,K^2{}_2\right]\exp\left[\frac{1}{2}\ln\mathcal{R}e\,\mathcal{E}_{2n+1}\,(K^3{}_3 - K^{11}{}_{11})\right]\\ &\quad\exp\left[(-1)^n\left(\frac{1}{\mathcal{R}e\,\mathcal{E}_{2n+1}} - 1\right)K^3{}_{11}\right] \qquad n \geq 0\end{aligned} \tag{3.82}$$

$$\begin{aligned}\mathcal{V}_{6n'} &= \exp\left[-\frac{1}{2}\ln(\mathcal{F}_{2n'-1}\bar{\mathcal{F}}_{2n'-1})\,K^2{}_2\right]\exp[\frac{1}{2}\ln\mathcal{R}e\,\mathcal{E}_{2n'}\,(K^3{}_3 - K^{11}{}_{11})]\\ &\quad\exp\left[(-1)^{n'+1}\left(\frac{1}{\mathcal{R}e\,\mathcal{E}_{2n'}} - 1\right)K^3{}_{11}\right] \qquad n' > 0\end{aligned} \tag{3.83}$$

where in Eq.(3.82) one has the signatures $(1,10,+)$ with time in 3 for n even and in 11 for n odd, and in Eq.(3.83) the signatures are $(1,10,+)$ with time in 3 for n' odd and in 11 for n' even (see Appendix A.3).

It is proven in Appendix F that the KK-wave sequence contains a redundancy of the solutions for $n > 0$, namely Eq.(3.82) and Eq.(3.83) lead to identical metric up to interchange of the time coordinates 3 and 11. The full wave sequence for $n > 0$ has metric:

$$\begin{aligned}ds^2_{6n'} &= \mathcal{F}_{2n'-1}\bar{\mathcal{F}}_{2n'-1}\Big[(dx^1)^2 + (dx^2)^2\Big] + (-1)^{n'}H^{-1}_{2n'}(dx^3)^2 + \Big[(dx^4)^2\cdots + (dx^{10})^2\Big]\\ &+ (-1)^{n'+1}H_{2n'}\Big[dx^{11} - \Big((-1)^{n'+1}H^{-1}_{2n'} + (-1)^{n'}\Big)dx^3\Big]^2,\end{aligned} \tag{3.84}$$

where $H_p = \mathcal{R}e\,\mathcal{E}_p$. For $n = 0$ it is given by Eq.(2.27). All metrics in the KK-wave are solutions of 11D supergravity. The factor $\mathcal{F}_{2n'-1}\bar{\mathcal{F}}_{2n'-1}$ can again be eliminated by a (singular) coordinate change, preserving the harmonic character of H_p.

[16] We have added an integration constant -1 to the field $A^{(11)}_3$ as in the discussion below Eq.(2.28).

From the representatives of the M5 sequence in terms of the supergravity 3-form potential, Eqs.(3.63) and (3.64), we get the representatives for the monopole sequence in terms of the gravitational potential ${A_3}^{(11)}$

$$\begin{aligned}\mathcal{V}_{3+6n'} &= \exp\left[-\frac{1}{2}\ln(\mathcal{F}_{2n'-1}\bar{\mathcal{F}}_{2n'-1})\,{K^2}_2\right]\,\exp\left[-\frac{1}{2}\ln\mathcal{R}e\,\mathcal{E}_{2n'+1}\,({K^3}_3-{K^{11}}_{11}+K^2_2)\right]\\ &\quad\exp\left[(-1)^{n'}\mathcal{I}m\,\mathcal{E}_{2n'+1}\,{K^3}_{11}\right]\qquad n'\geq 0\end{aligned}\tag{3.85}$$

$$\begin{aligned}\mathcal{V}_{-3+6n} &= \exp\left[-\frac{1}{2}\ln(\mathcal{F}_{2n-1}\bar{\mathcal{F}}_{2n-1})\,{K^2}_2\right]\,\exp\left[-\frac{1}{2}\ln\mathcal{R}e\,\mathcal{E}_{2n}\,({K^3}_3-{K^{11}}_{11}+K^2_2)\right]\\ &\quad\exp\left[(-1)^{n+1}\mathcal{I}m\,\mathcal{E}_{2n}\,{K^3}_{11}\right]\qquad n>0\end{aligned}\tag{3.86}$$

where in Eq.(3.85) one has the signatures $(2,9,-)$ with time in 9,10 for n' even and $(5,6,+)$ with time in 4,5,6,7,8 for n' odd, and in Eq.(3.86) the signatures are $(2,9,-)$ with time in 9,10 for n odd and $(5,6,+)$ with time in 4,5,6,7,8 for n even (see Appendix A.3).

In analogy with the KK-wave sequence, the metric in Eqs.(3.85) and (3.86) are equivalent up to a redefinition of the time coordinates (see Appendix F). There is thus only one gravity tower, the left and the right tower of Fig.5b are equivalent, each of them contains the full wave and monopole sequences.

The full monopole sequence has the metric:

$$\begin{aligned}ds^2_{3+6n'} &= \mathcal{F}_{2n'-1}\bar{\mathcal{F}}_{2n'-1}H_{2n'+1}\Big[(dx^1)^2+(dx^2)^2\Big]+H_{2n'+1}(dx^3)^2+(-1)^{n'}\Big[(dx^4)^2\cdots+(dx^8)^2\Big]\\ &+(-1)^{n'+1}\Big[(dx^9)^2+(dx^{10})^2\Big]+H^{-1}_{2n'+1}\Big[dx^{11}-\Big((-1)^{n'}B_{2n'+1}\Big)dx^3\Big]^2,\end{aligned}\tag{3.87}$$

where $B_p=\mathcal{I}m\,\mathcal{E}_p$.
Again the metric of the monopole sequence solve the Einstein equations.

The generators $R^{[3]}_{1+3p}$, $R^{[6]}_{2+3p}$, $\bar{R}^{[8,1]}_{3+3p}$, $p\geq 0$, and ${K^3}_{11}$ span the M2, M5 and gravity towers for positive real roots and define distinct BPS solutions. All positive real roots of E_9 can be reached from these by permuting coordinate indices in A_8 or equivalently by performing Weyl transformations W_{α_i} from the gravity line depicted in Fig.1 with nodes 1 and 2 deleted. In this way we reach all E_9 positive real roots and the related BPS solutions. In what follows we shall keep the above notation for all towers of positive real roots differing by A_8 indices, and specify the coordinates when needed.

3.4 Analytic structure of BPS solutions and the Ernst potential

We have obtained an infinite U-duality multiplet of E_9 BPS solutions of 11D supergravity depending on two non-compact space variables. This was achieved by analysing various $A_1^+\equiv A_1^{(1)}$ subalgebras of E_9, which allow us to reach all positive roots within such a subalgebra from sequences of Weyl reflexions starting from basic BPS solutions reviewed in Section 2. A striking feature of the method is that each solution is determined by a pair of conjugate harmonic functions H_p and B_p which can be combined into an analytic function $\mathcal{E}_p = H_p + iB_p$, where p characterises the level of the solution. This feature emerges from the action of the affine A_1^+ subgroup on the representatives and is clearly not restricted to supergravity. In this section, we establish the link with another A_1^+ subgroup of E_9, namely the Geroch group of general relativity. As is well known [27, 28, 29], the latter acts on stationary axisymmetric (or colliding plane wave) solutions in four space-time dimensions (which can be embedded consistently into 11D supergravity) via 'non-closing dualities' generating infinite towers of higher order dual potentials. Here we explain the action of the Geroch group on BPS solutions, for which the so-called *Ernst potential* (see (3.95) below) is an analytic function, and hence is entirely analogous to the function $\mathcal{E}$ encountered above. As we will see this action (so far not exhibited in the literature to the best of our knowledge) 'interpolates' between free field dualities and the full non-linear action of the Geroch group on non-analytic Ernst potentials — exactly as for the M2-M5 sequence discussed in section 3.2. To keep the discussion simple we will restrict attention to four-dimensional Einstein gravity with two commuting Killing vectors, that is, depending only on two (spacelike) coordinates.

Before we specialise to the case of BPS solutions we present the more general formalism. The general line element in this case is of the form

$$ds^2 = H^{-1}e^{2\sigma}(dx^2 + dy^2) + (-\rho^2 H^{-1} + H\widehat{B}^2)dt^2 + 2H\widehat{B}dt\,dz + Hdz^2. \tag{3.88}$$

Here, ∂_t and ∂_z are Killing vectors, hence the metric coefficients depend only on the space coordinates $(x, y) \equiv (x^1, x^2)$. Furthermore, we have adopted a conformal frame for the (x, y) components of the metric, with conformal factor $e^{2\sigma}$. $\widehat{B}$ is the called the Matzner–Misner potential and related to the Ehlers potential B through the duality relation

$$\epsilon_{ij}\partial_j B = \rho^{-1}H^2\partial_i\widehat{B}, \tag{3.89}$$

where $i, j = 1, 2$. There is no need to raise or lower indices, as the metric in (x, y) space is the flat Euclidean metric, with $\epsilon_{12} = \epsilon^{12} = 1$. Therefore the inverse duality relation is $\epsilon_{ij}\partial_j\widehat{B} = -\rho H^{-2}\partial_i B$.

The vacuum Einstein equations for the line element Eq.(3.88) in terms of the Matzner–Misner potential $\widehat{B}$ read

$$\begin{aligned} H\partial_i(\rho\partial_i H) &= \rho\left(\partial_i H\partial_i H - \rho^{-2}H^4\partial_i\widehat{B}\partial_i\widehat{B}\right) \\ \rho H^{-1}\partial_i(\rho\partial_i\widehat{B}) &= 2\rho\partial_i\left(\frac{\rho}{H}\right)\partial_i\widehat{B}\,, \end{aligned} \tag{3.90}$$

Rewritten in terms of the Ehlers potential B these give, using Eq.(3.89),

$$\begin{aligned} H\partial_i(\rho\partial_i H) &= \rho\left(\partial_i H\partial_i H - \partial_i B\partial_i B\right) \\ H\partial_i(\rho\partial_i B) &= 2\rho\partial_i H\partial_i B\,, \end{aligned} \tag{3.91}$$

where the two sets of equations Eqs.(3.90) and (3.91) are related by the so-called Kramer–Neugebauer transformation $B \leftrightarrow \widehat{B}, H \leftrightarrow \rho/H$. In addition, there are equations for ρ and the conformal factor σ. These are two (compatible) first order equations for the conformal factor

$$\begin{aligned} \rho^{-1}\partial_{(x}\rho\partial_{y)}\sigma &= \frac{1}{4}(H^{-1}\partial_x H)(H^{-1}\partial_y H) + \frac{1}{4}(H^{-1}\partial_x B)(H^{-1}\partial_y B), \\ \rho^{-1}\partial_x\rho\partial_x\sigma - \rho^{-1}\partial_y\rho\partial_y\sigma &= +\frac{1}{4}(H^{-1}\partial_x H)^2 + \frac{1}{4}(H^{-1}\partial_x B)^2 \\ &\quad -\frac{1}{4}(H^{-1}\partial_y H)^2 - \frac{1}{4}(H^{-1}\partial_y B)^2 \,, \end{aligned} \tag{3.92}$$

while ρ satisfies the two-dimensional Laplace equation without source

$$\partial_i\partial_i\rho = 0\,. \tag{3.93}$$

A second order equation for σ can be deduced by varying ρ, or alternatively from the constraints and the dynamical equations for the metric Eq.(3.90) [or Eq.(3.91)]; it reads

$$\partial_i\partial_i\sigma = -\frac{1}{4}\rho H^{-2}\big(\partial_i H\partial_i H + \partial_i B\partial_i B\big), \tag{3.94}$$

If ρ is different from a constant (as is the case generally with axisymmetric stationary or colliding plane wave solutions), we can integrate the first order Eqs.(3.92), which determine the conformal factor up to one integration constant; the second order equation Eq.(3.94) is then automatically satisfied as a consequence of the other equations of motion. On the other hand, as we will see below, the BPS solutions are characterized by $\rho =$ constant, for which the l.h.s. of Eq.(3.92) vanishes identically (whence the r.h.s. must also vanish identically). In this case, we are left with the second order Eq.(3.94), and the conformal factor is only determined modulo a harmonic function in (x, y).

The equations of motion Eq.(3.91) can be rewritten conveniently in terms of the *complex Ernst potential* [cf .Eq.(3.7)]

$$\mathcal{E} = H + iB, \tag{3.95}$$

satisfying the Ernst equation

$$H\partial_i(\rho\partial_i\mathcal{E}) = \rho\partial_i\mathcal{E}\partial_i\mathcal{E}. \tag{3.96}$$

As we will see below this equation is trivially satisfied for BPS solutions in the sense that both sides vanish identically.

3.4.1 BPS solutions

In our analysis of the 11-dimensional gravity tower, the lowest level BPS solution is the KK-wave Eq.(2.27). It stems from the generator ${K^3}_{11}$ with time in 3. In 4D gravity with x^1, x^2 as non compact space variables, the corresponding wave solution is associated to the Chevalley generator ${K^3}_4$ depicted in Fig.6 by the node 3. Taking the timelike direction to be 3, we get $[(x, y) \equiv (x^1, x^2),\, (t, z) \equiv (x^3, x^4)]$

$$ds^2 = dx^2 + dy^2 + (H-2)dt^2 - 2(1-H)dt\,dz + Hdz^2. \tag{3.97}$$

Here, $H = H(x, y)$ is a *harmonic function* in x, y, which, for the brane with a source at $x = y = 0$ we choose to be $H = \frac{1}{2}\ln(x^2 + y^2) = \ln|\zeta|$ in terms of the complex coordinate

$$\zeta = x + iy. \tag{3.98}$$

Comparing Eq.(3.97) with Eq.(3.88), we see that for this BPS solution the general fields $\widehat{B}, \sigma$ and ρ are expressed in terms of H as[17]

$$e^{2\sigma} = H, \qquad \widehat{B} = b - H^{-1}, \qquad \rho = 1. \tag{3.99}$$

Using the duality relation Eq.(3.89) one obtains the Ehlers potential B up to an integration constant. Indeed, as already mentioned above, with Eq.(3.99), the duality relations Eq.(3.89) just become the Cauchy–Riemann equations for the Ernst potential (3.95), to wit

$$\partial_x B = -\partial_y H \quad , \quad \partial_y B = \partial_x H\,, \tag{3.100}$$

or, in short notation, $\epsilon_{ij}\partial_j H = -\partial_i B$. Conversely, for Eq.(3.89) to reduce to the Cauchy–Riemann relations, we must have $\widehat{B} = 1/H +$ constant and ρ constant. *Therefore, the Cauchy–Riemann equations for the Ernst potential are equivalent to the BPS ('no force') condition and may thus be taken as the defining equations for BPS solutions.* In a supersymmetric context these (first order) equations would be equivalent to the Killing spinor conditions defining the BPS solution.

For $H = \frac{1}{2}\ln(x^2 + y^2)$, we immediately obtain

$$B = \arctan\left(\frac{y}{x}\right) + \text{constant} = \arg(\zeta) + \text{constant}\,, \tag{3.101}$$

whence the Ernst potential is simply

$$\mathcal{E}(\zeta) = \ln|\zeta| + i\arg(\zeta) + \text{constant} = \ln\zeta + \text{constant}\,, \tag{3.102}$$

and so is an *analytic* function of ζ. It is then easy to see that the equations of motion and the constraint equations are satisfied for *any* analytic Ernst potential $\mathcal{E}$ if ρ is constant (and in particular, with $\rho = 1$). Namely both the Ernst equation Eq.(3.96) as well as Eq.(3.92) reduce to the identity $0 = 0$ for all such solutions. Because Eq.(3.92) is void, the conformal factor must then be determined from the second order equation Eq.(3.94). For holomorphic $\mathcal{E}$, Eq.(3.94) can be rewritten as

$$\partial_\zeta\partial_{\bar\zeta}\sigma = -\frac{1}{2}\rho\,\frac{\partial_\zeta\mathcal{E}\partial_{\bar\zeta}\bar{\mathcal{E}}}{(\mathcal{E} + \bar{\mathcal{E}})^2} \tag{3.103}$$

and only in this case the solution to this equation can be given in closed form. It reads

$$\sigma(\zeta, \bar\zeta) = \frac{1}{2}\ln\left(\mathcal{E} + \bar{\mathcal{E}}\right) \tag{3.104}$$

[17]The constant b for $\widehat{B}$ should be chosen as 1 in order to obtain an asymptotically flat solution in more than four space-time dimensions. From the point of view of the two-dimensional reduction, however, it does not matter and can be chosen arbitrarily. Note also, that constant shifts of $\widehat{B}$ are part of the Matzner-Misner $SL(2)$, see below.

The ambiguity involving harmonic functions left by Eq.(3.94) is related to the covariance of the equations of motion under *conformal analytic coordinate transformations* of the complex coordinate $\zeta = x + iy$, which leave the 2-metric in diagonal form, viz.

$$\zeta \longrightarrow \zeta' = f(\zeta) \ . \tag{3.105}$$

As is well known, the conformal factor transforms as

$$\sigma(\zeta,\bar{\zeta}) \longrightarrow \sigma(\zeta,\bar{\zeta}) + \frac{1}{2}\ln\left|f(\zeta)\right|^2 , \tag{3.106}$$

under such transformations, where the second term on the r.h.s. is indeed harmonic. We already used this fact when we removed the conformal factors $\mathcal{F}_{2n-1}\bar{\mathcal{F}}_{2n-1}$ to prove that the metric in Eqs.(3.55), (3.56), (3.65), (3.66), (3.84) and (3.87) solve the Einstein equations outside the singularities of the harmonic functions.

3.4.2 Action of Geroch group

The Geroch group for (3+1)-dimensional gravity in the stationary axi-symmetric case discussed here is affine $\widehat{SL(2)}$ with central extension ($\equiv A_1^+$); this is the same structure we encountered for the M2 and M5 towers. It is depicted in Fig.6 by the Dynkin diagram formed by the nodes 3 and 4. Extending the diagram with node 2 to the overextended A_1^{++}, we may as previously identify the central charge with a Cartan generator of A_1^{++} ($-K^2{}_2$ in $E_{10} \equiv E_8^{++}$). Adding the node 1 leads to the very extended A_1^{+++} which is the pure gravity counterpart (in $D = 4$) of E_{11}.

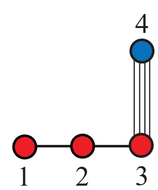


Figure 6: *Dynkin diagram of A_1^+ and its 'horizontal' extensions A_1^{++} and A_1^{+++}. The Matzer–Misner $SL(2)$ is represented by the node 3 and the Ehlers $SL(2)$ by the node 4.*

The two distinguished $SL(2)$ subgroups of A_1^+ corresponding to the Matzner–Misner and the Ehlers cosets appear in different real forms. The timelike Killing vector ∂_t turns the Matzner–Misner coset into $SL(2)/SO(1,1)$ (with non-compact denominator group), whereas the Ehlers coset is $SL(2)/SO(2)$ (with compact denominator group). With 3 or 4 as time direction, the temporal involution Eq.(2.8) leaves indeed invariant the Lorentz generator $\Omega\,(K^3{}_4 + K^4{}_3) = K^3{}_4 + K^4{}_3$ in the Matzer–Misner $SL(2)$, but preserves the rotation generator $\Omega\,(R^{4|4} - R_{4|4}) = R^{4|4} - R_{4|4}$ in the Ehlers $SL(2)$. Here $R^{4|4}$ is the simple positive step operator corresponding in four dimensions to the 11-dimensional generator $R^{4\,5\,6\,7\,8\,9\,10\,11|\,11}$ of Section 3.3.

For the underlying split real algebra A_1^+ we use a simple Chevalley–Serre basis consisting of e_i, f_i, h_i ($i = 3, 4$) and derivation d (see also Appendix G). The central element is $c = h_3 + h_4$. The index '4' refers to the Ehlers $SL(2)$ and the index '3' refers to the Matzner–Misner $SL(2)$, as depicted in Fig.6.

- **Ehlers group**

The Ehlers $SL(2)$ acts by Möbius transformations on the (analytic) Ernst potential. Using the standard notation for Möbius transformation generators[18]

$$L_{-1} = -\frac{\partial}{\partial \mathcal{E}}, \qquad L_0 = -\mathcal{E}\frac{\partial}{\partial \mathcal{E}}, \qquad L_1 = -\mathcal{E}^2\frac{\partial}{\partial \mathcal{E}}, \tag{3.107}$$

the Ehlers generators are[19]

$$e_4 = iL_{-1}, \qquad h_4 = 2L_0, \qquad f_4 = iL_1, \tag{3.108}$$

with the resulting transformation of the real and imaginary components of $\mathcal{E}$

$$\begin{aligned} f_4 H &= 2HB, & f_4 B &= B^2 - H^2, \\ h_4 H &= -2H, & h_4 B &= -2B, \\ e_4 H &= 0, & e_4 B &= -1. \end{aligned} \tag{3.109}$$

- **Matzner–Misner group**

The infinitesimal action of the Matzner–Misner group on general $SL(2)/SO(1,1)$ coset fields H and $\widehat{B}$ is (for $\rho = 1$)

$$\begin{aligned} f_3 H &= -2H\widehat{B}, & f_3 \widehat{B} &= \widehat{B}^2 + H^{-2}, \\ h_3 H &= 2H, & h_3 \widehat{B} &= -2\widehat{B}, \\ e_3 H &= 0, & e_3 \widehat{B} &= -1. \end{aligned} \tag{3.110}$$

In order to compute its action on the Ernst potential $\mathcal{E}$, we have to exploit the duality relation Eq.(3.89) between the potentials $\widehat{B}$ and B. It is straightforward to work out the action of e_2 and h_2, with the result

$$h_3 B = 2B, \qquad e_3 B = 0, \tag{3.111}$$

where some constants have been fixed from the commutation relations. Finally, the action of f_2 on B follows from

$$\epsilon_{ij}\partial^j(f_3 B) = f_3(H^2\partial_i\widehat{B}) = -2H^2\widehat{B}\partial_i\widehat{B} + H^2\partial_i(H^{-2}). \tag{3.112}$$

[18]Note that these are Möbius transformations on $\mathcal{E}$ and *not* on the complex coordinate ζ. The system admits an additional *conformal symmetry* acting on the complex coordinate ζ [45], which shows again that *any* analytic Ernst potential solves the equations of motion.

[19]The factors i are understood from studying the invariant bilinear form for (L_{-1}, L_0, L_1) which is non-standard from the Kac–Moody point of view.

Using $\widehat{B} = b - H^{-1}$ and the Cauchy–Riemann equation, this yields

$$\epsilon_{ij}\partial^j(f_3 B) = -2b\partial_i H = -2b\epsilon_{ij}\partial^j B. \tag{3.113}$$

Therefore we find

$$f_3 B = -2bB, \tag{3.114}$$

setting an integration constant equal to zero. We note that in order to satisfy $[e_3, f_3] = h_3$ on B we need to have a non-trivial action of e_3 on the integration constant b, namely $e_3 b = -1$ which is consistent with the general shift property of the Matzner–Misner group Eq.(3.110) and $\widehat{B} = b - H^{-1}$ for this solution. In a sense, one can view the Matzner–Misner group as acting via Möbius transformations on the variable b. However, closing this action with the Ehlers Möbius transformations on $\mathcal{E}$ leads one to introduce new constants (notably in $f_4\widehat{B}$) which transform non-trivially under the remaining generators. For completeness we note the transformation rules

$$e_4\widehat{B} = 0, \qquad h_4\widehat{B} = 2\widehat{B}, \tag{3.115}$$

which are true generally and

$$f_4\widehat{B} = 2BH^{-1} + \lambda\,, \tag{3.116}$$

where λ is an example of a new constant. This last relation is true *on the solution* $\widehat{B} = b - H^{-1}$; generally the result would be some non-local expression.

Combining Eq.(3.110) and Eq.(3.114), the action of f_2 on the full Ernst potential is

$$f_3\mathcal{E} = -2b\mathcal{E} - 2\,. \tag{3.117}$$

This shows that the action of f_3 on $\mathcal{E}$ does not yield a new transformation, but simply a linear combination of previous ones (to wit, e_4 and h_4). Hence, under the action of the Matzner–Misner $SL(2)$, a BPS solution will remain a BPS solution.[20] The formula Eq.(3.117) agrees with our findings in Eq.(3.52).

This almost 'trivial' action of the Geroch group on the BPS solutions – which essentially acts only via Möbius transformations on the Ernst potential $\mathcal{E}$ – confirms our previous finding for M2 and M5 branes (3.52), but is in marked contrast to its action on non-BPS stationary axisymmetric solutions [29]. There $\mathcal{E}(x, y)$ is *not* analytic, and $\rho(x, y)$ is a non-constant function, often identified with a radial coordinate (so-called Weyl canonical coordinates). When starting from the vacuum solution to obtain say, the Schwarzschild or Kerr solution, the (x, y) dependence of the Ernst potential is precisely the one induced by the (x, y) dependence of the spectral parameter whose coordinate dependence, in turn, hinges on the coordinate dependence of ρ. Since we have $\rho = 1$ for BPS solutions, this mechanism does not work, confirming our

[20] The constant parameters $b, \lambda, \ldots$ do not influence the analyticity of the solution, although they are essential for the action of the Geroch group.

conclusion that the action of the Geroch group cannot turn an analytic Ernst potential into a non-analytic one, hence leaves the class of BPS solutions stable.

The results of this section can be summarised by saying that the Weyl group of A_1^+ acts via shifts and inversions on the complex Ernst potential and at the same time transforms the conformal factor but leaves invariant the set of analytic Ernst potentials.

4 Dual formulation of the E_9 multiplet

4.1 Effective actions

We showed in Section 2.2 that the basic magnetic BPS solutions of 11D supergravity (M5 and KK6-monopole) smeared in all directions but one are expressible in terms of the dual potentials $A_{3\,4\,5\,6\,7\,8}$ and $A_{4\,5\,6\,7\,8\,9\,10\,11|11}$ parametrising the Borel generators $R_2^{[6]}$ and $\bar{R}_3^{[8,1]}$. In higher non-compact transverse space dimensions these potentials are related by Hodge duality to the supergravity fields $A_{9\,10\,11}$ and ${A_i}^{(11)}$. The dual potentials take on the solutions, up to an integration constant, the same value $1/H$ as do the fields $A_{9\,10\,11}$ and ${A_3}^{(11)}$ for the basic electric BPS solutions, namely the M2-branes at level 1 and the KK-waves at level 0. H is, in any number of non-compact transverse spacelike directions, a harmonic function with δ-function singularities at the location of the sources.

In Section 3 we constructed BPS solutions of 11D supergravity in two transverse spacelike directions for all E_9 positive real roots. We shall label such description of the BPS states in terms of the supergravity metric and 3-form the 'direct' description. Each solution was obtained by relating through dualities and compensations the 'generalised dual potential' $1/H$ parametrising an $E_9 \subset E_{10}$ positive root in the Borel representative of E_{10} to the supergravity metric and 3-form.

In this Section, we will present a space-time description of the BPS states directly in terms of the generalised dual potentials. We label it the 'dual' description. We shall show that the dual description of the BPS solutions can be derived from gauge fixed effective actions

$$\mathcal{S}^{(11)}_{\{q\}} = \frac{1}{16\pi G_{11}} \int d^{11}x \sqrt{|g|} \left(R^{(11)} - \epsilon \frac{1}{2} F_{i\,\{q\}} F^{i\,\{q\}} \right) , \tag{4.1}$$

where i runs over the two non-compact dimensions $1, 2$ and ϵ is $+1$ if the action involves a single time coordinate (or an odd number of time coordinates) and -1 if the number of time coordinates is even. $F_{i\,\{q\}} = \partial_i A_{\{q\}}$ where $\{q\}$ stands for the tensor indices of the $A_p^{[N]}$ potential multiplying $R_p^{[N]}$ in the Borel representative. Here p is the level and $[N]$ labels a tower [3], [6], [0] or $[8, 1]$ for any set of $A_8 \subset E_9$ tensor indices. The set of indices is fixed by the A_1^+ group selected by a Weyl transformed in A_8 of the A_1^+ subgroup of E_{10} chosen in Eq.(3.12).

We first consider the M2-M5 system of Section 3.2. Explicitly $A^{[3]}_{1+3n} = A_{9\,10\,11\,,\,[3\,4\,\ldots\,10\,11].n}$ for the 3-tower depicted in the right column of Fig.2a and $A^{[6]}_{2+3n} = A_{3\,4\,5\,6\,7\,8\,,\,[3\,4\,\ldots\,10\,11].n}$ for the 6-tower depicted in the left column. Here the symbol n is the number of times the antisymmetric set of indices [3 4 … 10 11] must be taken. That this is the correct tensor structure follows from the structure of the 'gradient representations' in [14, 16].

For all BPS states in the M2-M5 system, we take

$$A_p^{[N]} = 1/H \,. \tag{4.2}$$

The metric associated to $A_p^{[N]}$ is encoded in the Borel representatives of the M2 sequence Eqs.(3.28), (3.29) and the M5 sequence (3.57), (3.58). We combine Eqs.(3.28) and (3.58) to form the 3-tower Eq.(3.15) and Eqs.(3.29) and (3.57) to form the 6-tower Eq.(3.17) . We have

$$\mathcal{V}_{1+3n} = \exp\left[\frac{1}{2}\ln H\,(h_{11} - nK^2{}_2)\right]\exp\left[\frac{1}{H}\,R^{[3]}_{1+3n}\right] \qquad n \geq 0 \tag{4.3}$$

$$\mathcal{V}_{2+3n} = \exp\left[\frac{1}{2}\ln H\,(-h_{11} - (n+1)K^2{}_2)\right]\exp\left[\frac{1}{H}\,R^{[6]}_{2+3n}\right] \qquad n \geq 0\,, \tag{4.4}$$

which yield for the 3-tower Eq.(4.3) the metric Eq.(4.5) and for the 6-tower Eq.(4.4) the metric Eq.(4.6)

$$\begin{aligned} &|g_{11}| = |g_{22}| = H^{1/3+n} \\ &|g_{33}| = |g_{44}| = \cdots = |g_{88}| = H^{1/3} \qquad |g_{99}| = |g_{10\,10}| = |g_{11\,11}| = H^{-2/3}\,, \end{aligned} \tag{4.5}$$

$$\begin{aligned} &|g_{11}| = |g_{22}| = H^{2/3+n} \\ &|g_{33}| = |g_{44}| = \cdots = |g_{88}| = H^{-1/3} \qquad |g_{99}| = |g_{10\,10}| = |g_{11\,11}| = H^{2/3}\,. \end{aligned} \tag{4.6}$$

For the time components of the metric one multiplies the absolute values of the metric components by a minus sign. The time components for the 3- and 6-towers are specified in Section 3.2 for the Weyl orbits initiated by the M2 with time in 9 and by the M5 with time in 3. Note that we could as well take the Weyl orbit initiated by the M2 with times in 9, 10 and by the M5 with times in 4, 5, 6, 7, 8. Alternatively we could mix the two orbits to avoid for instance at all level exotic solutions and have always time in 9 for the M2 sequence and in 3 for the M5 sequence depicted in Fig.2a. Note that in all cases, climbing the 3-tower or the 6-tower by steps of one unit of n amounts to alternate between BPS states on the M2 sequence and the M5 sequence. We shall comment on this feature in Section 6.

We now verify that the matter term in Eq.(4.1) solves the Einstein equations with $A_p^{[N]} = 1/H$ and the metric given in Eqs.(4.5) and (4.6). For the 3-tower the matter Lagrangian reads

$$\begin{aligned} \mathcal{L} &= -\epsilon\frac{1}{2}\sqrt{|g|}F_{i\,\{q\}}F^{i\,\{q\}} \\ &= -\epsilon\frac{1}{2}\sum_{i=1}^{2}\sqrt{|g|}g^{ii}g^{99}g^{10\,10}g^{11\,11}[g^{33}g^{44}\ldots g^{11\,11}]^n\left(\partial_i A^{[3]}_{9\,10\,11\,,\,[3\,4\,\ldots\,10\,11].n}\right)^2\,, \end{aligned} \tag{4.7}$$

while for the 6-towers one gets

$$\mathcal{L} = -\epsilon\frac{1}{2}\sum_{i=1}^{2}\sqrt{|g|}g^{ii}g^{33}g^{44}g^{55}g^{66}g^{77}g^{88}[g^{33}g^{44}\ldots g^{11\,11}]^n\left(\partial_i A^{[6]}_{3\,4\,5\,6\,7\,8\,,\,[3\,4\,\ldots\,10\,11].n}\right)^2\,. \tag{4.8}$$

One computes from Eqs.(4.7) and (4.8) the energy-momentum tensors for the 3-tower

$$
\begin{aligned}
&T^1_1 = -T^2_2 = -\frac{1}{4}H^{-7/3-n}\,\left[(\partial_1 H)^2 - (\partial_2 H)^2\right]\,, \qquad T^1_2 = T^2_1 = -\frac{1}{2}H^{-7/3-n}\,[\partial_1 H \partial_2 H]\\
&T^3_3 = T^4_4 = \ldots T^8_8 = -\frac{2n-1}{4}H^{-7/3-n}\,[(\partial_1 H)^2 + (\partial_2 H)^2]\\
&T^9_9 = T^{10}_{10} = T^{11}_{11} = -\frac{2n+1}{4}H^{-7/3-n}\,\left[(\partial_1 H)^2 + (\partial_2 H)^2\right]\,,
\end{aligned}
\tag{4.9}
$$

and for the 6-tower

$$
\begin{aligned}
&T^1_1 = -T^2_2 = -\frac{1}{4}H^{-8/3-n}\,\left[(\partial_1 H)^2 - (\partial_2 H)^2\right]\,, \qquad T^1_2 = T^2_1 = -\frac{1}{2}H^{-8/3-n}\,[\partial_1 H \partial_2 H]\\
&T^3_3 = T^4_4 = \ldots T^8_8 = -\frac{2n+1}{4}H^{-8/3-n}\,\left[(\partial_1 H)^2 + (\partial_2 H)^2\right]\\
&T^9_9 = T^{10}_{10} = T^{11}_{11} = -\frac{2n-1}{4}H^{-8/3-n}\,\left[(\partial_1 H)^2 + (\partial_2 H)^2\right]\,.
\end{aligned}
\tag{4.10}
$$

The fact that T^μ_ν in Eqs.(4.9) and (4.10) does not depend on ϵ results from the cancellation between negative signs arising from the kinetic energy term in the action Eq.(4.1) and from the concomitant even numbers of time metric components. From the metric Eqs.(4.5) and (4.6) one easily verifies that the Einstein equations

$$
R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R = T^\mu_\nu\,,
\tag{4.11}
$$

with T^μ_ν given by Eqs.(4.9) and (4.10), are satisfied. This result holds for $\epsilon = \pm 1$ because the left hand side of Eq.(4.11) turns out to be independent of ϵ.

Using the mapping Eqs.(3.75), (3.76), (3.77) of the brane A_1^+ group Eq.(3.12) to the gravity towers A_1^+ group, one maps the 3-tower to the 0-tower and the 6-tower to the [8,1]-tower depicted respectively on the right and left columns of Fig.5b. We combine Eqs.(3.79) and (3.80) to form the [8,1]-tower. One has

$$
\mathcal{V}_{3+3n} = \exp\left[\frac{1}{2}\ln H\,(-K^3{}_3 + K^{11}{}_{11} - (n+1)K^2{}_2)\right]\,\exp\left[\frac{1}{H}\,\bar{R}^{[8,1]}_{3n+3}\right] \qquad n \geq 0\,.
\tag{4.12}
$$

The corresponding dual metric are

$$
\begin{aligned}
&|g_{11}| = |g_{22}| = H^{n+1}\\
&|g_{33}| = H \qquad |g_{11\,11}| = H^{-1}\\
&|g_{aa}| = 1 \qquad a \neq 1,2,3,11\,.
\end{aligned}
\tag{4.13}
$$

As expected, up to the interchange of the coordinates 3 and 11, the same result holds for the redundant 0-tower (except for the level 0 of E_{10}, which is the KK-wave). The verification of the Einstein equations derived from the actions Eq.(4.1) duplicates that of the M2-M5 system. Note that in this generalized dual formulation, the metric of the gravity tower are all diagonal.

The above results for the 3- and 6-towers and for the gravity tower have been established for a chosen set of tensor indices, namely the set determined by the choice of the A_1^+ subgroup of E_{10} for which $R_1^{[3]}$ is identified with $R^{9\,10\,11}$. The validity of the effective action Eq.(4.1) for all $E_9 \subset E_{10}$ fields associated to its positive real roots follows from permuting the A_8 tensor indices, that is from performing Weyl transformations of the gravity line.

4.2 Charges and masses

The charge content of the E_9 BPS solutions is easier to analyse in the dual description because the dual potential is not mixed with fields arising from the compensations. *Outside the sources*, the equation of motion for the dual field $A_{\{q\}}$ is from Eq.(4.1)

$$\sum_{i=1}^{2} \partial_i \left(\sqrt{|g|} g^{\{q\}} \partial_i A_{\{q\}} \right) = 0 \,, \qquad (4.14)$$

where

$$g^{\{q\}} = g^{ii} g^{99} g^{10\,10} g^{11\,11} [g^{33} g^{44} \dots g^{11\,11}]^n \qquad \text{3-tower : level } (1{+}3n) \qquad (4.15)$$

$$g^{\{q\}} = g^{ii} g^{33} g^{44} g^{55} g^{66} g^{77} g^{88} [g^{33} g^{44} \dots g^{11\,11}]^n \qquad \text{6-tower : level } (2{+}3n) \qquad (4.16)$$

$$g^{\{q\}} = g^{ii} g^{44} g^{55} \dots g^{10\,10} (g^{11\,11})^2 [g^{33} g^{44} \dots g^{11\,11}]^n \qquad \text{[8,1]-tower : level } (3{+}3n) \,, \qquad (4.17)$$

with $n \geq 0$. The appearance of the n-fold blocks of antisymmetric metric factors $g^{33} g^{44} \cdots g^{11\,11}$ is again due to the embedding of E_9 in E_{10}, see [43]. From Eqs.(4.5), (4.6), (4.13), and from the embedding relation in E_{11} Eq.(2.17), we get for all towers and hence, by permutation of tensor indices in A_8, for all E_9 BPS states $\sqrt{|g|} g^{\{q\}} = \pm H^2$. As, up to an integration constant, one has always $A_{\{q\}} = 1/H$, the field equation Eq.(4.14) reduces to

$$\sum_{i=1}^{2} \partial_i \partial_i H = 0 \,. \qquad (4.18)$$

Here, as for the KK-monopole discussed in Section 2.2, Eq.(4.18) is valid outside the sources and the latter are determined by fixing the singularities of the function H. Labelling the positions of the smeared M2 by x_k^1, x_k^2 and their charges by q_k, one takes

$$H(x^1, x^2) = \sum_k \frac{q_k}{2\pi} \ln \sqrt{(x^1 - x_k^1)^2 + (x^2 - x_k^2)^2} \,, \qquad (4.19)$$

and, in analogy with Eq.(2.40), the extension of Eq.(4.18) including the sources reads[21] *for all E_9 BPS solutions*

$$\sum_{i=1}^{2} \partial_i \partial_i H = \sum_k \frac{q_k}{2\pi} \delta(\mathbf{x} - \mathbf{x_k}) \,. \qquad (4.20)$$

[21]We fix the M5 charges by the Weyl transformation relating the M2 to the M5, which for convenience was not explicitly used in our general derivation of the E_9 BPS solutions in Section 3.

Thus we obtain, for all BPS states, the same charge value as for the M2, as expected from U-dualities viewed as E_9 Weyl reflexions. Our identification of q_k with a charge is however not the conventional one as long as our solutions with 2 non-compact space dimensions have not been identified with static solutions in 2+1 space-time dimensions. This raises the question whether we are allowed to decompactify the time. This will be examined in the following Sections 4.2.1 and 4.2.2. We wish to stress that decompactification of time or space dimensions is *not* the same as 'unsmearing'. The latter term refers to the undoing of the smearing process by which the dependence of the harmonic functions characterising our BPS solutions is reduced by one or more variables through compactification. Thus unsmearing implies decompactification of space dimensions but the converse is not necessarily true. When it is true we call the decompactified dimensions 'transverse'. For the basic BPS solution smeared to two space dimensions, unsmearing is of course possible up to the space dimensions of the defining solution given in Section 2.2 (8 for the M2, 5 for the M5, 9 for the KK-wave and 3 for the KK-monopole). *For all higher level BPS states in 2 non-compact space dimensions, unsmearing is impossible.* It is indeed straightforward to show that the Einstein equation Eq.(4.11) is not satisfied if the harmonic function H entering the right hand side of the equation is extended to three dimensions.

One verifies that in the dual formulation all E_9 BPS states can be smeared to one space dimension, the charge being still defined by Eq.(4.20) with i equal to 1. These solutions are also solutions of the σ-model S^{brane} Eq.(2.14).

A criterion for decompactification of longitudinal spacelike directions and of timelike directions will be obtained from the requirement that the tensions should be finite. These quantities will be evaluated in the string context from string dualities and uplifting to eleven dimensions. For each BPS state characterised by a dual potential $A_p^{[N]}$ we define an action $\mathcal{A}$ given in Planck units by the product of all spatial and temporal compactification radii, each of them at a power equal to the number of times the corresponding index occurs in $A_p^{[N]}$. One gets from Eqs.(4.15), (4.16) and (4.17) the action $\mathcal{A}_l$ of the level l solution[22]

$$\mathcal{A}_{1+3n} = \frac{1}{l_p^{9n+3}} R_9 R_{10} R_{11} [R_3 R_4 \dots R_{11}]^n \qquad \text{3-tower : level } (1+3n) \tag{4.21}$$

$$\mathcal{A}_{2+3n} = \frac{1}{l_p^{9n+6}} R_3 R_4 R_5 R_6 R_7 R_8 [R_3 R_4 \dots R_{11}]^n \qquad \text{6-tower : level } (2+3n) \tag{4.22}$$

$$\mathcal{A}_{3+3n} = \frac{1}{l_p^{9(n+1)}} R_4 R_5 \dots R_{10} (R_{11})^2 [R_3 R_4 \dots R_{11}]^n \qquad \text{[8,1]-tower : level } (3+3n)\,, \tag{4.23}$$

where l_p is the 11-dimensional Planck constant ($l_p^9 = 8\pi G_{11}$). We identify for non-exotic states $\mathcal{A}$ to MR_t where M is the mass of the source and R_t the compactification time radius. We derive in Appendix C.5 the actions $\mathcal{A}$, Eqs.(4.21), (4.22) and (4.23), from the interpretation in the context of string theory of the Weyl reflexions used to construct the BPS solutions, both for exotic and non-exotic states. Requiring finiteness of the action density implies that $A_{\{q\}}$ be

[22] Similar actions were considered also in [46].

linear in the radii for those directions, spatial or temporal, which can be decompactified. For non-exotic states this is equivalent to requirement of finite tension.

It is immediately checked that for the basic branes $n = 0$ one obtains the correct mass formula for the M2 (time in 9) Eq.(C.11), the M5 (time in 3) Eq.(C.3) and the KK-monopole (time in 4) Eq.(C.5). The KK-monopole mass is in agreement with the calculation of the ADM mass of the unsmeared KK6-monopole [47]. Our criterion confirms that time and all longitudinal space dimensions can be taken to be non-compact except for the Taub-Nut direction 11 of the KK6-monopole which occurs quadratically in Eq.(4.23) and hence cannot be decompactified. This fact is in agreement with the fact that the KK monopole solution in 3 transverse space dimensions, characterised by an harmonic function $H = 1+q/r$ where $r^2 \equiv (x^1)^2+(x^2)^2+(x^9)^2$, has, in order to avoid a conical singularity, its radius $R_{11} \propto q$ [48, 49, 50] and hence finite.

We now examine further the nature of the BPS solutions for level higher than 3, that is outside the realm of the basic BPS solutions of Section 2.2.

4.2.1 From level 4 to level 6

The actions $\mathcal{A}_l$ defined in Eqs.(4.21), (4.22) and (4.23) are in agreement with the computation of the masses for level 4 Eq.(C.6) with time in 3, level 5 Eq.(C.12) with time in 9, level 6 Eq.(C.13) with time in 3 obtained in Appendix C in the string context. Thus time occur linearly in $\mathcal{A}$ and can be non-compact. Examining the dependence of $\mathcal{A}$ in the spatial radii, we see that the spacelike directions that can be decompactified are $(4, 5, 6, 7, 8)$ for $l = 4$, $(10, 11)$ for $l = 5$ and none for $l = 6$.

U-duality requires that the dimensionally reduced metric in 2+1 dimensions be identical for these solutions and in addition be equivalent to the (2+1)-dimensional metric for the basic BPS solutions of Section 2.2.[23] We now show that this requirement is fulfilled, both in the direct and in the dual formalism.

To perform the dimensional reduction we write in general the 11-dimensional metric in the following form

$$ds^2 = g_{\mu\nu} dx^\mu\, dx^\nu + \sum_r h_{rr}(dx^r)^2\,, \tag{4.24}$$

where $\mu, \nu = 1, 2, a$ and $r \neq 1, 2, a$, labelling a as the time coordinate. To find the canonical Einstein action in $d = 3$ dimensions, the components of the reduced metric have to be Weyl rescaled

$$\widetilde{g}_{\mu\nu} = g_{\mu\nu}\, h^{\frac{1}{d-2}} = g_{\mu\nu}\, h\,, \tag{4.25}$$

where $h = \det\, h_{rs}$.

We first consider the level 4 state. In the direct formulation the level 4 metric is given by Eq.(3.60) and for the dimensional reduction to 3 dimensions we get using Eq.(4.25) with

[23] All BPS solutions for $l \leq 6$ should then form a multiplet of E_8 which is the symmetry of 11D supergravity reduced to (2+1) dimensions.

$h = \widetilde{H}^{1/3}$ and time in 3

$$ds^2_{l=4,3D} = \widetilde{g}_{\mu\nu} dx^\mu\, dx^\nu = H[(dx^1)^2 + (dx^2)^2] - (dx^3)^2\,. \tag{4.26}$$

The same metric is obtained in the reduction of the dual metric Eq.(4.5) with $n = 1$, where now $h = H^{-1/3}$. Similarly the level 5 solution with time in 9, given in the direct formulation by Eq.(3.38) and in the dual one by Eq.(4.6) with $n = 1$ with respectively $h = \widetilde{H}^{2/3}$ and $h = H^{-2/3}$, yields from Eq.(4.25)

$$ds^2_{l=5,3D} = \widetilde{g}_{\mu\nu} dx^\mu\, dx^\nu = H[(dx^1)^2 + (dx^2)^2] - (dx^9)^2\,. \tag{4.27}$$

The level 6 solution with the timelike direction 3 is given in the direct formulation by Eq.(3.84) with $n' = 1$, namely

$$\begin{aligned} ds^2_{l=6} &= (H^2 + B^2)[(dx^1)^2 + (dx^2)^2] - \widetilde{H}^{-1}(dx^3)^2 + [(dx^4)^2 \cdots + (dx^{10})^2] \\ &+ \widetilde{H}[dx^{11} - (\widetilde{H}^{-1} - 1)dx^3]^2\,. \end{aligned} \tag{4.28}$$

and in the dual formulation by Eq.(4.13) with $n = 1$, that is

$$ds^2_{l=6} = H^2[(dx^1)^2 + (dx^2)^2] - H(dx^3)^2 + [(dx^4)^2 \cdots + (dx^{10})^2] + H^{-1}(dx^{11})^2\,, \tag{4.29}$$

Reducing the level 6 metric Eqs.(4.28) and (4.29) to 3 dimensions, we again find

$$ds^2_{l=6,3D} = \widetilde{g}_{\mu\nu} dx^\mu\, dx^\nu = H[(dx^1)^2 + (dx^2)^2] - (dx^3)^2\,. \tag{4.30}$$

One easily checks that the same 3-dimensional metric (with suitable time coordinate) are recovered for all basic BPS solutions recalled in Section 2.

We have thus verified that all the BPS solutions of 11D supergravity, for levels $l \leq 6$ are equivalent in 2+1 dimensions. These solutions constitute an $E_8 \subset E_9$ multiplet of branes (see footnote 23). The E_8 multiplet is the same as the one studied some time ago algebraically as a consequence of M-theory compactified on T^8 for which the masses of the different BPS states of the multiplet has been derived (see [37] and in particular Table 11), and their space-time interpretation was obtained in reference [44]. We recover here these results from the Weyl group of E_9 endowed with the temporal involution, and from the interpretation of these Weyl transformation in the context of string theory. This E_9 containing a timelike direction is the correct setting to describe all the BPS solutions with two unsmeared spacelike directions in a group theoretical language. We summarise below for all the levels $0 < l \leq 6$ its mass content[24] in the form $\mathcal{A}_l = MR_t$, where $\mathcal{A}_l$ is the level l action Eqs.(4.21), (4.22) and (4.23), M the mass and R_t the time radius (which can be taken to ∞), to exhibit their striking relation with the E_9 dual potentials.

[24]For the level 3 KK-monopole potential we have put the time in 4 as in Section 2 instead of 9, 10 in the general metric Eq.(3.87) to avoid here exotic states.

l	1	2	3	4	5	6
time	9	3	4	3	9	3
E_9 field	$A_{9\,10\,11}$	$A_{3\ldots 8}$	$A_{4\ldots 10\,11,11}$	$A_{3\ldots 11,9\,10\,11}$	$A_{3\ldots 11,3\ldots 8}$	$A_{3\ldots 11,4\ldots 11,11}$
$\mathcal{A}_l = MR_t$	$\frac{R_9 R_{10} R_{11}}{l_p^3}$	$\frac{R_3 \ldots R_8}{l_p^6}$	$\frac{R_4 \ldots R_{11}.R_{11}}{l_p^9}$	$\frac{R_3 \ldots R_{11}.R_9 R_{10} R_{11}}{l_p^{12}}$	$\frac{R_3 \ldots R_{11}.R_3 \ldots R_8}{l_p^{15}}$	$\frac{R_3 \ldots R_{11}.R_4 \ldots R_{11}.R_{11}}{l_p^{18}}$

These BPS states already reach levels beyond the classical levels $l \leq 3$ for which a dictionary between E_{10} fields and space-time fields depending on one coordinate exists [14]. In the next section we discuss the solutions for $l > 6$.

4.2.2 The higher level solutions

The action formulae Eqs.(4.21), (4.22) and (4.23) are all in agreement with the evaluation of $\mathcal{A}$ in the string context, as seen from Section C.5. No time or longitudinal space radius occurs linearly in $\mathcal{A}$. Thus time is compact and the only non-compact space radii are the transverse two dimensions.

All these states can only be reached from basic non-exotic solutions through a timelike T-duality.

U-duality does no more imply that the dimensionally reduced metric in (2+1) dimensions should be identical to that of the basic ones and one indeed verifies that they are distinct. However metric and the induced dilaton field are expected to be identical when reduced to 2 dimensions. This is indeed the case as we now show.

Reducing all the solutions down to three dimensions[25] 1, 2 and r, with $r \in 3 \ldots 11$, we get[26]

$$ds^2_{l,3D} = \widetilde{g}_{11}(H, B)[(dx^1)^2 + (dx^2)^2] + (dx^r)^2. \tag{4.31}$$

Because for all the solutions we have unity in front of $(dx^r)^2$, reducing on x^r down to two dimensions and performing a Weyl rescaling all the metric reduce to a flat two-dimensional space with zero dilaton field.

5 Transcending 11D supergravity

We have seen in Section 4 that all E_9 BPS solutions (including the basic ones discussed in Section 2.2) can be smeared to one space dimension in the dual formalism. There is in fact no such description in the direct formalism, except for the M2 and the KK-wave. More generally, the dual formalism in one non-compact space dimension is equivalent to the σ-model Eq.(2.14) restricted to a single $l > 0$ field. In that case indeed the matter term is the same in both

[25] The spacelike or timelike nature of x^r is irrelevant for this argument.
[26] In the dual formalism this formula holds with $\widetilde{g}_{11}$ depending only on H.

formalisms as no covariant derivative arises in the σ-model [14, 23] and the Einstein term [38] coincides with its level zero.

We now consider BPS solutions obtained from positive real roots of E_{10} not present in E_9. These solutions can be obtained by performing Weyl transformations on E_9 BPS solutions smeared to one dimension. They depend on one non-compact space variable and may have no counterpart in 11D supergravity. We illustrate the construction of such solutions by one example.

IIA	M-theory	1	2	3	4	5	6	7	8	9	10	11
D6	KK6		×	×	×	×	×	×	×			×
D8	M9		×	×	×	×	×	×	×	×	×	×

Table 1: ***The D8 brane obtained from the D6 brane by performing the Weyl reflexion $W_{\alpha_{11}}$, i.e. a double T-duality plus exchange of the directions x^9 and x^{10}.***

We shall obtain the M9 (namely the 'uplifting'[27] of the D8 brane of massive Type IIA supergravity)[28] by performing a Weyl transformation on the KK6-monopole smeared in all directions but one described in Section 2.2.3. We start with a D6 along the spatial directions $2, 3, \ldots, 7$ and we choose 8 as time coordinate. The D6 is smeared in the directions 9 and 10 and thus depends only on the non-compact variable x^1. The uplifting to M-theory of the D6 yields a KK6-monopole with Taub-NUT direction 11 (see Table 1). To obtain the D8 and its uplifting M9, we perform the Weyl reflexion $W_{\alpha_{11}}$ which may be viewed as a double T-duality in the directions 9 and 10 plus exchange of the two radii [37, 26, 38, 13].

The KK6-monopole, in the longitudinal directions $2, \ldots, 7$ with timelike direction 8 and Taub-NUT direction 11, smeared in 9 and 10, is described in the σ-model by a level 3 generator[29] $R^{2345678\,11|11}$. The solution is given in Eq.(2.43) up to a permutation of indices. Defining $p^a = -h_a{}^a$ one has

$$\begin{aligned}
&p^1 = p^9 = p^{10} = \frac{1}{2}\ln H(x^1)\\
&p^{11} = -\frac{1}{2}\ln H(x^1)\\
&p^i = 0 \qquad i = 2\ldots 8\\
&A_{2345678\,11|11} = \frac{1}{H(x^1)}.
\end{aligned} \tag{5.1}$$

[27]There is a no-go theorem stating that massive 11-dimensional supergravity does not exist [51]. The concept of uplifting the D8 seems thus puzzling. However a definition of 'massive 11D supergravity' has been proposed [52] for a background with an isometry generated by a spatial Killing vector. In that theory the M9 solution does exist [53]. Existence of the M9 is also suggested by the study of the central charges of the M-theory superalgebra [54, 55].

[28]The metric of the D8 brane has previously been discussed in the context of E_{11} in [24].

[29]This level 3 step operator contains the index 2 and belongs to a E_9 conjugate in E_{10} to the E_9 we used in the previous sections.

The level 3 root $\alpha^{(3)}$ corresponding to $R^{2345678\,11|11}$ is $\alpha^{(3)} = \alpha_2+2\alpha_3+3\alpha_4+4\alpha_5+5\alpha_6+6\alpha_7+7\alpha_8+4\alpha_9+\alpha_{10}+3\alpha_{11}$. Performing the Weyl reflexion $W_{\alpha_{11}}$, we get $W_{\alpha_{11}}(\alpha^{(3)}) \equiv \alpha^{(4)} = \alpha^{(3)}+\alpha_{11}$. The root $\alpha^{(4)}$ is the lowest weight of the A_9 irreducible representation of level 4 [16] whose Dynkin labels are $(2,0,0,0,0,0,0,0,0)$. This A_9 representation in the decomposition of the adjoint representation of E_{10} is not in E_9. The action of $W_{\alpha_{11}}$ on the Cartan fields is [13]

$$p'^a = p^a + \frac{1}{3}(p^9 + p^{10} + p^{11}) \qquad a = 2\ldots 8 \tag{5.2}$$

$$p'^a = p^a - \frac{2}{3}(p^9 + p^{10} + p^{11}) \qquad a = 9, 10, 11. \tag{5.3}$$

Using the embedding relation Eq.(2.17), the Weyl transform of Eqs.(5.1) yields the solution

$$p'^1 = \frac{4}{6}\ln H(x^1) \tag{5.4}$$

$$p'^a = \frac{1}{6}\ln H(x^1) \qquad a = 2\ldots 10 \tag{5.5}$$

$$p'^{11} = -\frac{5}{6}\ln H(x^1). \tag{5.6}$$

$$A_{23456789\,10\,11|11|11} = \frac{1}{H(x^1)}\,. \tag{5.7}$$

The level 4 field Eq.(5.7) contains the antisymmetric set of indices $2, 3\ldots 11$. These are not apparent in its A_9 Dynkin labels given above but are needed in the 11-dimensional metric stemming from the embedding $E_{10} \subset E_{11}$ encoded in Eq.(2.17). The $A_{10} \subset E_{11}$ Dynkin labels of this field in this embedding are indeed $(2,0,0,0,0,0,0,0,0,1)$. From Eq.(2.13) one gets the 11-dimensional metric

$$ds^2_{M9} = H^{4/3}(dx^1)^2 + H^{1/3}[(dx^2)^2 + \cdots - (dx^8)^2 + (dx^9)^2 + (dx^{10})^2] + H^{-5/3}(dx^{11})^2\,. \tag{5.8}$$

One verifies the validity of the dual formalism equation Eq.(4.14) for the field Eq.(5.7), namely

$$\frac{d}{dx^1}\left(\sqrt{|g|}g^{22}g^{33}\ldots g^{10\,10}[g^{11\,11}]^3\frac{d}{dx^1}A_{23456789\,10\,11|11|11}\right) = \frac{d^2}{d(x^1)^2}H(x^1) = 0\,. \tag{5.9}$$

The metric Eq.(5.8) describes the M9 [53], which reduced to 10 dimensions gives the D8 [56] of massive type IIA. The computation of the M9-mass in the string context Eq.(C.1) agrees with our general action formula

$$\mathcal{A}_{M9} = R_8 M_{M9} = \frac{R_2R_3R_4R_5R_6R_7R_8R_9R_{10}R_{11}^3}{l_p^{12}}\,, \tag{5.10}$$

indicating that R_{11} is compact.

6 Summary and comments

We have constructed an infinite E_9 multiplet of BPS solutions of 11D supergravity and of its exotic counterparts depending on two non-compact variables. These solutions are related by U-dualities realised as Weyl transformations of the E_9 subalgebra of E_{11} in the regular embedding $E_9 \subset E_{10} \subset E_{11}$. Each BPS solution stems from an E_9 potential $A_p^{[N]}$ multiplying the generator $R_p^{[N]}$ in the Borel representative of the coset space E_{10}/K_{10}^- where K_{10}^- is invariant under a temporal involution. $A_p^{[N]}$ is related to the supergravity 3-form and metric through dualities and compensations. This E_9 multiplet of states split into three classes according to the level l. For $0 \leq l \leq 3$ we recover the basic BPS solutions, namely, the KK-wave, the M2, the M5 and the KK-monopole. For $4 \leq l \leq 6$, the solutions have 8 longitudinal space dimensions. We argue that for higher levels, all 9 longitudinal directions, including time, are compact. Each BPS solution can be mapped to a solution of a dual effective action of gravity coupled to matter expressed in terms of the E_9 potential $A_p^{[N]}$. In the dual formulation the BPS solutions can be smeared to one non-compact space dimension and coincides then with solutions of the E_{10} σ-model build upon E_{10}/K_{10}^-. The σ-model yields in addition an infinite set of BPS space-time solutions corresponding to all real roots of E_{10} which are not roots of E_9. These appear to transcend 11D supergravity, as exemplified by the lowest level ($l = 4$) solution which is identified to the M9.

The relation between the 11D supergravity 3-form and metric, and the E_9 potentials $A_p^{[N]}$ has significance beyond the realm of BPS solutions. To see this we first recall that the E_9 potentials can be organised in towers defined by decomposing $E_9 \subset E_{10}$ into A_1^+ subgroups with central charge $-K^2{}_2 \in E_{10}$. Each of these A_1^+ subgroup contains two 'brane' towers $[N] = [3], [6]$ or one 'gravity tower' $[N] = [8, 1]$ of real roots (the two gravity towers $[N] = [8, 1]$ and $[0]$ are redundant except for the lowest level representing the KK-wave). We first examine the brane towers.

The recurrences of the 3-tower $A^{[3]}_{1+3n}$ alternate in nature at each step: they switch from states on the M2 sequence to those on the M5 sequence. This feature is illustrated in Fig.2a where the 3-tower recurrences are depicted on the right column. On the other hand, each recurrence of the 3-tower is related by *duality-compensation pairs* to the supergravity 3-form potential which we denote by $(A_1^{[3]})_q$, as seen in the horizontal lines of both Fig.3 and Fig.4. Here we designated by the integer q the number of duality-compensation pairs needed to reach the field $(A_1^{[3]})_q$ from $A^{[3]}_{1+3n}$. In the realm of BPS states studied in this paper each field $A^{[3]}_{1+3n}$ defines a different BPS solution of 11D supergravity defined by $(A_1^{[3]})_q$ and the related metric. Comparing Fig.2a with Fig.3 and 4 we see that q is equal to n. This relation expresses the fact that *the number of steps needed to climb the 3-tower up to the field* $A^{[3]}_{1+3n}$ *is equal to the number of duality-compensation pairs needed to reach from* $A^{[3]}_{1+3n}$ *the 11D supergravity 3-form defining the corresponding BPS solution.* However the relation $q = n$ does not rely on the BPS character of the solution and hence has general significance, which can be pictured as follows.

Were all the compensations matrices put equal to unity no new solutions could be generated by Weyl transformations from any solution of 11D supergravity defined by its 3-form $A_1^{[3]}$ (or from its Hodge dual) and metric. Indeed, because of the Weyl equivalence of all dualities depicted in Fig.3 and Fig.4, one would simply get

$$A^{[3]}_{1+3n} \stackrel{\mathrm{I}}{=} A^{[3]}_{1} \qquad n\,\text{even} \tag{6.1}$$

$$A^{[3]}_{1+3n} \stackrel{\mathrm{I}}{=} A^{[6]}_{2} \qquad n\,\text{odd}\,, \tag{6.2}$$

where $A^{[6]}_{2}$ is taken to be the Hodge dual of $A^{[3]}_{1}$ and the superscript I means that all compensations have been formally equated to unity. A similar analysis of the 6-tower depicted in the left column of Fig.2a would yield

$$A^{[6]}_{2+3n} \stackrel{\mathrm{I}}{=} A^{[6]}_{2} \qquad n\,\text{even} \tag{6.3}$$

$$A^{[6]}_{2+3n} \stackrel{\mathrm{I}}{=} A^{[3]}_{1} \qquad n\,\text{odd}\,. \tag{6.4}$$

The same phenomenon would appear in the gravity tower, where it is somewhat hidden in the redundant 0-tower of Fig.5b.

The non-trivial content of the E_9 tower potentials is entirely due to compensations. These prevent 'duals of duals' to be equivalent to unity and one may view the E_9 towers as defining 'non-closing dualities', familiar from the standard Geroch group. They translate through the compensation process the genuine non-linear structure of gravity.

Acknowledgments

We are greatly indebted to Philippe Spindel for many interesting and fruitful discussions and we are grateful to Marc Henneaux for his support and encouragement. Laurent Houart thanks Riccardo Argurio for many illuminating conversations. François Englert thanks Sasha Belavin for useful comments. François Englert and Laurent Houart are grateful to the Max-Planck-Institut für Gravitationsphysik at Potsdam, and Axel Kleinschmidt and Hermann Nicolai to the Université Libre de Bruxelles, for their warm hospitality.

This work was supported in part by IISN-Belgium (convention 4.4511.06 and convention 4.4505.86) by the European Commission FP6 RTN programme MRTN-CT-2004-005104, and by the Belgian Federal Science Policy Office through the Interuniversity Attraction Pole P VI/11.

A Signature changes and compensations

Expressing a Weyl transformation W as a conjugation by a group element U_W of E_{11} (E_{10}), one defines the involution Ω' operating on the conjugate elements by

$$\Omega'(T') = U_W\,\Omega(\underbrace{U_W^{-1}T'U_W}_{T})\,U_W^{-1}\,, \tag{A.1}$$

where T and T' are any conjugate pair of generators in E_{11} (E_{10}). The subgroup invariant under Ω is conjugate to the subgroup invariant under Ω'. However, Weyl reflexions in general do not commute with the temporal involution [33, 35].

A.1 The gravity line

First consider the gravity line of E_{11} endowed with the temporal involution Eq.(2.8). The Weyl reflexion W_{α_1} in the hyperplane perpendicular to α_1 changes the time index in the A_{10} tensors from 1 to 2. Indeed applying Eq.(A.1) to the Weyl reflexion W_{α_1} generates from $\Omega_1 \equiv \Omega$ a new involution $\Omega_2 \equiv \Omega'$ such that

$$\begin{aligned}
&U_1\,\Omega_1 K^2{}_1\,U_1^{-1} = \rho K^2{}_1 = \rho\Omega_2 K^1{}_2\,,\\
&U_1\,\Omega_1 K^1{}_3\,U_1^{-1} = \sigma K^3{}_2 = \sigma\Omega_2 K^2{}_3\,,\\
&U_1\,\Omega_1 K^i{}_{i+1}\,U_1^{-1} = -\tau K^{i+1}_{\ i} = \tau\Omega_2 K^i{}_{i+1} \quad i > 2\,.
\end{aligned} \tag{A.2}$$

Here ρ, σ, τ are plus or minus signs which may arise as step operators are representations of the Weyl group up to signs. Eq.(A.2) illustrate the general result that such signs always cancel in the determination of Ω' because they are identical in the Weyl transform of corresponding positive and negative roots, as their commutator is in the Cartan subalgebra which forms a true representation of the Weyl group. The content of Eq.(A.2) is represented in Table 2. The signs below the generators of the gravity line indicate the sign in front of the negative step operator obtained by the involution: a minus sign is in agreement with the conventional Chevalley involution and indicates that the indices in $K^m{}_{m+1}$ are both either space or time indices while a plus sign indicates that one index must be time and the other space.

gravity line	$K^1{}_2$	$K^2{}_3$	$K^3{}_4$	$\cdots$	$K^{D-1}{}_D$	time coordinate
Ω_1	$+$	$-$	$-$	$-$	$-$	1
Ω_2	$+$	$+$	$-$	$-$	$-$	2

Table 2: *Involution switches from Ω_1 to Ω_2 in E_{11} due to the Weyl reflexion W_{α_1}*

Table 2 shows that the time coordinates in E_{11} must now be identified either with 2, or with all indices $\neq 2$. We choose the first description, which leaves unaffected coordinates attached to planes invariant under the Weyl transformation. More generally, performing Weyl reflexions from roots of the gravity line, we can identify the time index to any A_{10} tensor index.

The Weyl transformations on the gravity line of E_{11} (or E_{10}) simply changes the time coordinate but do not modify the global signature $(1,10)$ (or $(1,9)$). This need not be the case for Weyl transformations from roots pertaining to higher levels. We shall determine the different signatures for the M2 and M5 sequences. In order to do that, we have to study the effect of the Weyl reflexions $W_{\alpha_{11}}$ and $W_{-\alpha_{11}+\delta}$ on the involutions characterising the M2 and the M5 we started with. We consider separately the two sequences. We will also see that the nature of the compensation transformations ($SO(2)$ or $SO(1,1)$) is determined and follows from this analysis. Finally we shall consider the signatures induced on the gravity tower by the mapping Eqs.(3.75), (3.76) and (3.77).

A.2 The brane towers

A.2.1 Signatures of the M2 sequence

We start with the conventional M2 described by the solution Eq.(3.1) of 11-dimensional supergravity with the signature (1,10, +). Here the first entry denotes the number of timelike directions (in our case the single direction 9), the second denotes the number of spacelike directions and the third gives the sign of the kinetic energy term in the action (+ being the usual one). We want to determine the space-time signature for all the solutions of the M2 sequence. The M2 of level 1 is characterised by the involution $\Omega_{l_1} \equiv \Omega_9$ fixing 9 as a time coordinate.

To determine the signature at level 5, we perform the Weyl transformation $W_{-\alpha_{11}+\delta}$ to find the corresponding new involution Ω_{l_5}. The generators of the gravity line affected by this reflexion are ${K^2}_3$ and ${K^8}_9$. From Eq.(A.1) we have (from now on, we drop irrelevant signs ρ, σ, τ appearing in Eq.(A.2))

$$\begin{aligned}
\Omega_{l_5} {K^2}_3 &= U_{W_{-\alpha_{11}+\delta}}\, \Omega_{l_1} R^{2\,4\,5\,6\,7\,8}\, U^{-1}_{W_{-\alpha_{11}+\delta}} \\
&= -{K^3}_2 \qquad \text{(A.3)}\\
\Omega_{l_5} {K^8}_9 &= U_{W_{-\alpha_{11}+\delta}}\, \Omega_{l_1} R_{3\,4\,5\,6\,7\,9}\, U^{-1}_{W_{-\alpha_{11}+\delta}} \\
&= +{K^9}_8\,. \qquad \text{(A.4)}
\end{aligned}$$

The signature of the level 5 solution is thus unchanged, the only time coordinate is still 9. The action of the involution Ω_{l_5} on $R^{9\,10\,11}$ follows from $W_{-\alpha_{11}+\delta}(\alpha_{11}) = 2\delta - \alpha_{11}$. We get

$$\Omega_{l_5} R^{9\,10\,11} = U_{W_{-\alpha_{11}+\delta}}\, \Omega_{l_1} R^{[6]}_5\, U^{-1}_{W_{-\alpha_{11}+\delta}} = +R_{9\,10\,11}\,. \qquad \text{(A.5)}$$

As 9 is a timelike coordinate, this yields the usual sign for this generator and one sticks to the signature $(1,10,+)$ as it should. We have $\Omega_{l_5}\,(R^{[6]}_2 - R^{[6]}_{-2}) = R^{[6]}_2 - R^{[6]}_{-2}$ and the compensation for the $SL(2)$ of level 5 (see Eqs.(3.36)-(3.37)) lies in its $SO(2)$ subgroup.

We now perform the Weyl reflexion $W_{\alpha_{11}}$ to reach level 7. We have

$$
\begin{aligned}
\Omega_{l_7} K^8{}_9 &= U_{W_{\alpha_{11}}}\, \Omega_{l_5} R^{8\,10\,11}\, U^{-1}_{W_{\alpha_{11}}} \\
&= -K^9{}_8 && \text{(A.6)}\\
\Omega_{l_7} R^{9\,10\,11} &= U_{W_{\alpha_{11}}}\, \Omega_{l_5} R_{9\,10\,11}\, U^{-1}_{W_{\alpha_{11}}} \\
&= +R_{9\,10\,11}\,. && \text{(A.7)}
\end{aligned}
$$

From Eq.(A.6), we deduce immediately that the time coordinates are now 10 and 11 and from Eq.(A.7) we deduce that the sign of the kinetic terms is the 'wrong' one, namely it corresponds to the (2,9,-) theory as it should [39, 40]. We have $\Omega_{l_7}\,(R^{9\,10\,11} + R_{9\,10\,11}) = R^{9\,10\,11} + R_{9\,10\,11}$ and the compensation for the $SL(2)$ of level 7 solution Eq.(3.47) lies in its $SO(1,1)$ subgroup.

levels ($n > 0$)	times	$(t, s, \pm)$	compensation
1	9	$(1, 10, +)$	$--$
5	9	$(1, 10, +)$	$SO(2)$
1+6n , n odd	10,11	$(2, 9, -)$	$SO(1, 1)$
-1+6(n+1), n odd	10,11	$(2, 9, -)$	$SO(2)$
1+6n , n even	9	$(1, 10, +)$	$SO(1, 1)$
-1+6(n+1), n even	9	$(1, 10, +)$	$SO(2)$

Table 3: *Involution switches from Ω_{l_1} to $\Omega_{l_{\pm 1+6n}}$ in the M2 sequence due to the application of the successive Weyl reflexions $W_{-\alpha_{11}+\delta}$ and $W_{\alpha_{11}}$*

We can now repeat the analysis to all levels of the M2 sequence. We use $W_{-\alpha 11+\delta}$ to go from level $1+6n$ to level $-1+6(n+1)$. Replacing in Eqs.(A.3), (A.4) and (A.5) Ω_{l_1} by $\Omega_{l_{1+6n}}$ and Ω_{l_5} by $\Omega_{l_{-1+6(n+1)}}$, we see that the signature of the theory is unchanged. Furthermore, analysing the action of $\Omega_{l_{-1+6(n+1)}}$ on $R_2^{[6]}$, we conclude that the compensation for level $-1+6(n+1)$ is always an $SO(2)$ one. We use $W_{\alpha_{11}}$ to go from level $-1+6(n+1)$ to level $1+6(n+1)$. Replacing in Eqs.(A.6) and (A.7) Ω_{l_5} by $\Omega_{l_{-1+6(n+1)}}$ and Ω_{l_7} by $\Omega_{l_{1+6(n+1)}}$, we see that theories $(1, 10, +)$ and $(2, 9, -)$ are interchanged. The action of $\Omega_{l_{1+6(n+1)}}$ on $R_1^{[3]}$ shows that the compensation at level $1 + 6(n + 1)$ lies always in $SO(1, 1)$. The results are summarised in Table 3.

A.2.2 Signatures of the M5 sequence

We start with the non-exotic M5 described by Eq.(3.2) solution of 11D supergravity with the signature (1,10, +) and time in 3. We want to determine the space-time signature of all the solutions of the M5 sequence depicted in Fig.2. The M5 of level 2 is characterised by the involution $\Omega_{l_2} \equiv \Omega_3$ fixing 3 as a time coordinate.

To determine the signature of level 4, we perform the Weyl reflexion $W_{\alpha_{11}}$ to find the new involution Ω_{l_4}. The generator of the gravity line affected by this reflexion is $K^8{}_9$. From Eq.(A.1) we have

$$\Omega_{l_4} K^8{}_9 = U_{W_{\alpha_{11}}}\, \Omega_{l_2} R^{8\,10\,11}\, U^{-1}_{W_{\alpha_{11}}} = -K^9{}_8 \tag{A.8}$$

$$\Omega_{l_4} R^{9\,10\,11} = U_{W_{\alpha_{11}}}\, \Omega_{l_2} R_{9\,10\,11}\, U^{-1}_{W_{\alpha_{11}}} = -R_{9\,10\,11}\,, \tag{A.9}$$

From Eq.(A.8), we deduce immediately that there is no change of signature and thus 3 remains the only timelike direction. From Eq.(A.9) the sign of the kinetic terms is unchanged and the phase is still (1,10,+) as it should. We have $\Omega_{l_4}(R^{9\,10\,11} - R_{9\,10\,11}) = R^{9\,10\,11} - R_{9\,10\,11}$. Hence the coset characterising the level 4 solution is $SL(2)/SO(2)$ and the compensation at level 4 (see Eq.(3.59)) lies in $SO(2)$.

level	$K^2{}_3$	$K^3{}_4$	$K^4{}_5$	$K^5{}_6$	$K^6{}_7$	$K^7{}_8$	$K^8{}_9$	$K^9{}_{10}$	$K^{10}{}_{11}$	times	$(t,s,\pm)$
4	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	3	$(1,10,+)$
8	$-$	$+$	$-$	$-$	$-$	$-$	$+$	$-$	$-$	4,5,6,7,8	$(5,6,+)$

Table 4: *Involutions at level 4 and 8.*

To determine the signature of level 8, we perform the Weyl reflexion $W_{-\alpha 11+\delta}$ to find the new involution Ω_{l_8}. We have

$$\begin{aligned} \Omega_{l_8} K^2{}_3 &= U_{W_{-\alpha_{11}+\delta}}\, \Omega_{l_4} R^{2\,4\,5\,6\,7\,8}\, U^{-1}_{W_{-\alpha_{11}+\delta}} = -K^3{}_2 \\ \Omega_{l_8} K^8{}_9 &= U_{W_{-\alpha_{11}+\delta}}\, \Omega_{l_4} R_{3\,4\,5\,6\,7\,9}\, U^{-1}_{W_{-\alpha_{11}+\delta}} = +K^9{}_8\,. \end{aligned} \tag{A.10}$$

The flip of sign in $K^2{}_3$ and $K^8{}_9$ illustrated in Table 4 shows that the resulting theory comprises the 5 time coordinates 4,5,6,7,8. The involution Ω_{l_8} acts on $R^{9\,10\,11}$ according to

$$\Omega_{l_8} R^{9\,10\,11} = U_{W_{-\alpha_{11}+\delta}}\, \Omega_{l_4} R^{[6]}_5\, U^{-1}_{W_{-\alpha_{11}+\delta}} = -R_{9\,10\,11}\,. \tag{A.11}$$

The directions 9,10,11 being spacelike, the action of the involution on $R^{9\,10\,11}$ yields the 'right' kinetic energy terms. The new theory is $(5,6,+)$ as it should [39, 40]. We have $\Omega_{l_8}(R^{[6]}_2 + R^{[6]}_{-2}) = R^{[6]}_2 + R^{[6]}_{-2}$. Hence the compensation at level 8 (see Eq.(3.61)) lies in $SO(1,1)$.

levels $(n>0)$	times	$(t,s,\pm)$	compensation
2	3	$(1,10,+)$	$--$
4	3	$(1,10,+)$	$SO(2)$
2+6n , n odd	4,5,6,7,8	$(5,6,+)$	$SO(1,1)$
-2+6(n+1), n odd	4,5,6,7,8	$(5,6,+)$	$SO(2)$
2+6n , n even	3	$(1,10,+)$	$SO(1,1)$
-2+6(n+1), n even	3	$(1,10,+)$	$SO(2)$

Table 5: *Involution switches from Ω_{l_2} to $\Omega_{l_{\pm 2+6n}}$ in the M5 sequence due to the application of the successive Weyl reflexions $W_{\alpha_{11}}$ and $W_{-\alpha_{11}+\delta}$.*

We can repeat the analysis to find the signature of all the levels of the M5 sequence. Again we find only the two signatures found at the lower levels alternating every two steps while, as for the M2 sequence, the nature of the compensation alternates at each step. The results are summarised in Table 5.

A.3 The gravity towers

The gravity towers were obtained by performing Weyl transformations on the brane towers. In section 3.3. we have showed that the M2 sequence is mapped to the wave sequence and the M5 sequence to the monopole sequence. We shall take advantage of this Weyl mapping to find the signatures of the gravity towers.

The brane towers comprise 4 different signatures:

- $(1, 10, +)$ with time in 9 and $(2, 9, -)$ with time in 10 and 11 for the M2 sequence,
- $(1, 10, +)$ with time in 3 and $(5, 6, +)$ with time in $4, 5, 6, 7, 8$ for the M5 sequence.

We perform the Weyl mapping on the 4 signatures in three steps. The first Weyl transformation $W_{(1)}$ interchanges 9 and 3 on the gravity line. The second Weyl transformation is the Weyl reflexion $W_{(2)} \equiv W_{\alpha_{11}}$ and the last Weyl transformation $W_{(3)}$ interchanges 9 and 11 on the gravity line.

The first and last Weyl transformations $W_{(1)}$ and $W_{(3)}$ permute tensor indices and do not alter the global signature. Only $W_{(2)}$ can change the global signature $(t, s, \pm)$. There are 2 simple roots affected by $W_{(2)} \equiv W_{\alpha_{11}}$: α_8 and α_{11} defining respectively the generators ${K^8}_9$ and $R^{9\,10\,11}$. From Eq.(A.1) we get, dropping irrelevant signs,

$$\Omega' \, {K^8}_9 = s_1 \, {K^9}_8 = U_{W_{(2)}} \underbrace{\Omega \, R^{8\,10\,11}}_{s_1 \, R_{8\,10\,11}} U^{-1}_{W_{(2)}} \tag{A.12}$$

$$\Omega' \, R^{9\,10\,11} = s_2 \, R_{9\,10\,11} = U_{W_{(2)}} \underbrace{\Omega \, R_{9\,10\,11}}_{s_2 \, R_{9\,10\,11}} U^{-1}_{W_{(2)}} \, . \tag{A.13}$$

where s_1 and s_2 are signs. The possible change of signatures by the Weyl transformation $W_{(2)}$ will be deduced from the signs s_1 and s_2.

A.3.1 Signatures of the wave sequence

• Mapping of the signature $(1, 10, +)$ with time in 9

The global signature $(1, 10, +)$ is not modified by $W_{(1)}$ but the time coordinate is no longer in 9 but in 3. From Eqs.(A.12) and (A.13), with $s_1 = -1$ and $s_2 = -1$, we deduce that the signature is unchanged by the second Weyl reflexion $W_{(2)}$. The last transformation $W_{(3)}$ does not change the signature either.

The signature $(1, 10, +)$ with time in 9 is thus mapped by the three successive Weyl transformations to the signature $(1, 10, +)$ with time in 3.

• Mapping of the signature $(2, 9, -)$ with time in 10 and 11

The first Weyl transformation $W_{(1)}$ does not modify the signature. We then perform the Weyl transformation $W_{(2)}$. From Eq.(A.12) with $s_1 = +1$ we find that the time coordinate becomes 9 and from Eq.(A.13) with $s_2 = +1$, we deduce the sign of the kinetic term. This sign is the 'usual' one and the signature becomes $(1, 10, +)$ with time coordinate 9. The last Weyl reflexion $W_{(3)}$ does not change the global signature but puts the time coordinate in 11.

The signature $(2, 9, -)$ with times in 10 and 11 is thus mapped by the three successive Weyl transformations to the signature $(1, 10, +)$ with time in 11.

- **Signatures of the wave sequence**

levels $(n, n' > 0)$	times	$(t, s, \pm)$
0	3	$(1, 10, +)$
6n , n odd	11	$(1, 10, +)$
6n, n even	3	$(1, 10, +)$
6n' , n' odd	3	$(1, 10, +)$
6n', n' even	11	$(1, 10, +)$

Table 6: *Signatures of the wave sequence*

The coset representatives of the M2 sequence $\mathcal{V}_{1+6n}$ Eq.(3.53) and $\mathcal{V}_{-1+6n}$ Eq.(3.54) are mapped respectively to the coset representatives of the wave sequence $\mathcal{V}_{6n}$ Eq.(3.82) and $\mathcal{V}_{6n'}$ Eq.(3.83). All the signatures of the wave sequence are summarised in Table 6.

A.3.2 Signatures of the monopole sequence

- **Mapping of the signature** $(1, 10, +)$ **with time in** 3

The global signature $(1, 10, +)$ is not modified by $W_{(1)}$ but the time coordinate is no longer in 3 but in 9. We then perform the Weyl transformation $W_{(2)}$. From Eq.(A.12) with $s_1 = -1$, we find that the time coordinates become 10 and 11 and from Eq.(A.13) with $s_2 = +1$ we deduce the sign of the kinetic term. This sign is the 'wrong' one and the signature becomes $(2, 9, -)$. The last Weyl reflexion $W_{(3)}$ does not change the global signature $(2, 9, -)$ but puts the time coordinates in 9 and 10.

The signature $(1, 10, +)$ with time in 3 is mapped by the three successive Weyl transformations to the signature $(2, 9, -)$ with times in 9 and 10.

- **Mapping of the signature** $(5, 6, +)$ **with times in** $4, 5, 6, 7, 8$

The first Weyl transformation $W_{(1)}$ does not modify the signature. From Eqs.(A.12) and (A.13) with $s_1 = +1$ and $s_2 = -1$ we see that the signature is invariant under the second Weyl reflexion $W_{(2)}$. The last Weyl reflexion $W_{(3)}$ also leaves the signature unchanged.

The signature $(5, 6, +)$ with time in $4, 5, 6, 7, 8$ is left invariant by the Weyl mapping.

- **Signatures of the monopole sequence**

The coset representatives of the M5 sequence $\mathcal{V}_{2+6n}$ Eq.(3.63) and $\mathcal{V}_{-2+6n}$ Eq.(3.64) are

levels $(n, n' > 0)$	times	$(t, s, \pm)$
3	9,10	$(2, 9, -)$
3+ 6n' , n' odd	4,5,6,7,8	$(5, 6, +)$
3+6n', n' even	9,10	$(2, 9, -)$
-3+6n , n odd	9,10	$(2, 9, -)$
-3+6n, n even	4,5,6,7,8	$(5, 6, +)$

Table 7: *Signatures of the monopole sequence*

mapped respectively to the coset representatives of the monopole sequence $\mathcal{V}_{3+6n'}$ Eq.(3.85) and $\mathcal{V}_{-3+6n}$ Eq.(3.86). All the signatures of the monopole sequence are summarised in Table 7.

B Coset representatives of the gravity tower

We want to rewrite

$$\mathcal{V}_1 = \exp[h_3{}^3(K^3{}_3 - K^{11}{}_{11}) + h_3{}^{11}K^3{}_{11}]\,, \tag{B.1}$$

in terms of a product of two exponentials, the first one containing only the Cartan generators, namely we want to determine $A_3{}^{(11)}$ in

$$\mathcal{V}_2 = \exp[h_3{}^3(K^3{}_3 - K^{11}{}_{11})]\exp[A_3{}^{(11)}\,K^3{}_{11}]\,. \tag{B.2}$$

In terms of the $SL(2)$ matrices defined in Eq.(3.32), $K^3{}_3 - K^{11}{}_{11} = h_1$ and $K^3{}_{11} = e_1$. One has

$$\mathcal{V}_1 = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} h_3{}^3 & h_3{}^{11} \\ 0 & -h_3{}^3 \end{bmatrix}^n = \begin{bmatrix} e_{11}{}^{11} & -e_3{}^{11} \\ 0 & e_3{}^3 \end{bmatrix} \tag{B.3}$$

$$e_3{}^3 = e^{-h_3{}^3} \qquad e_{11}{}^{11} = e^{h_3{}^3} \qquad e_3{}^{11} = -\frac{h_3{}^{11}}{2h_3{}^3}\,(e^{h_3{}^3} - e^{-h_3{}^3})\,, \tag{B.4}$$

where the $e_\mu{}^n$ are the vielbein Eq.(2.13) characterising the KK-solution (see also Appendix B of [22]). On the other hand, from Eq.(B.4), one gets

$$\mathcal{V}_2 = \begin{bmatrix} e_{11}{}^{11} & e_{11}{}^{11}\,A_3{}^{(11)} \\ 0 & e_3{}^3 \end{bmatrix}\,. \tag{B.5}$$

Equating $\mathcal{V}_1$ Eq.(B.3) and $\mathcal{V}_2$ Eq.(B.5), we have

$$A_3{}^{(11)} = -e_3{}^{11}(e_{11}{}^{11})^{-1}\,. \tag{B.6}$$

and the metric corresponding to the representative Eq.(B.1) with time in 3 is thus

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^4)^2 + \cdots + (dx^{10})^2 - (e_3{}^3)^2(dx^3)^2 + (e_{11}{}^{11})^2\left[dx^{11} - A_3{}^{(11)}dx^3\right]^2\,. \tag{B.7}$$

C Masses and U-duality

We first review the KK6-monopole mass and then derive all the masses or actions $\mathcal{A}_l$ of the E_9 multiplet in the M-theory context from the Weyl reflexions interpreted as T-dualities. Their spacelike or timelike nature is discussed.

C.1 The level 3 solution

The KK6-monopole mass can be derived from the M5 mass using the relation between 11-dimensional supergravity and type IIA theory and T-duality. We recall that the relations between the 11-dimensional parameters R_{11} and the Planck length l_p and the string parameters g and l_s are (we neglect all numerical factors):

$$l_p = g^{1/3}\, l_s \qquad , \qquad R_{11} = g\, l_s\,. \tag{C.1}$$

On the other hand if one compactifies a direction of type IIA theory on a circle of radius R, the T-duality along this direction acts as:

$$R \to \frac{l_s^2}{R} \qquad , \qquad g \to \frac{g l_s}{R}. \tag{C.2}$$

We start with a M5 along the directions $4 \ldots 8$ and 3 is the longitudinal timelike direction. The mass of this elementary M5 is given by

$$M_{l=2} = \frac{R_4 \ldots R_8}{l_p^6}. \tag{C.3}$$

We smear this M5 along the directions 10 and 11. Reducing along the eleventh direction, using Eq.(C.1), one obtains a NS5 in type IIA then performing a T-duality along the direction 10, using Eq.(C.2) one get the KK5-monopole of type IIA with Taub-NUT direction 10

$$M_{kk5} = \frac{R_4 \ldots R_8 R_{10}^2}{g_s^2\, l_s^8}\,. \tag{C.4}$$

Uplifting back to eleven dimension, using Eq.(C.1), one obtains a KK6 monopole with longitudinal directions $4 \ldots 8, 11$ and with Taub-NUT direction 10 . Using Eqs.(C.1), (C.2), one find the mass of this KK6-monopole[30]

$$M_{l=3} = \frac{R_4 \ldots R_8 R_{11} R_{10}^2}{l_p^9}. \tag{C.5}$$

[30] The KK6-monopole discussed here has timelike direction is 3 and the Taub-NUT direction is 10. The level 3 KK6-monopole of the gravity tower in the mapping from the 6-tower in Sections 3 and 4 have timelike directions 9 and 10 and Taub-NUT direction 11. Note that in Section 2.2.3 the non-exotic KK6-monopole has its single timelike direction in 4 and Taub-NUT direction in 11.

C.2 The level 4 solution

To go to level 4 from the M5 considered in Eq.(C.3) we perform as in Section 3.2.3 the Weyl reflexion $W_{\alpha_{11}}$ interpreted here as a double T-duality plus exchange in the direction 9 and 10. We thus smear the KK5-monopole with mass given in Eq.(C.4) in the direction 9 and perform a further T-duality in this direction. From Eqs.(C.1), (C.2), we find the mass of the level 4 solution

$$M_{l=4} = \frac{R_4 \ldots R_8 R_9^2 R_{10}^2 R_{11}^2}{l_p^{12}}. \tag{C.6}$$

C.3 The level 5 solution

As in Section 3.2.2, to reach the level 5 solution we perform, the Weyl reflexion $W_{-\alpha_{11}+\delta} \equiv W_{-\alpha_{11}+\delta}$ sending the level 1 generator $R_1^{[3]}$ to the level 5 generator $R_5^{[6]}$.

We first decompose the Weyl reflexion $W_{-\alpha_{11}+\delta}$ in terms of simple Weyl reflexions which have an interpretation in terms of permutations of coordinates and double T-duality plus exchange of the directions 9 and 10. This decomposition will permit us to compute the mass of the level 5 solution and also to check that the timelike direction 9 is unaffected by $W_{-\alpha_{11}+\delta}$. We write

$$W_{-\alpha_{11}+\delta} = s_i \, s_j \ldots s_k \,, \tag{C.7}$$

where $s_i \equiv s_{\alpha_i}$ is the simple Weyl reflexion corresponding to the simple root α_i. To perform this decomposition, we can use the following lemma (whose proof is straightforward computing both sides of the equality) :

Lemma *If a real root γ can be written as the sum of two real roots $\gamma = \gamma_1 + \gamma_2$ such that $< \gamma_1, \gamma_2 >= -1$, then the Weyl reflexion s_γ can be decomposed as $s_\gamma = s_{\gamma_1} \, s_{\gamma_2} \, s_{\gamma_1}$.*

One may write the root $-\alpha_{11} + \delta$ as the sum $\gamma_1 + \gamma_2$ where γ_1 is the root associated to R^{345} and γ_2 is the root associated to $R^{6\,7\,8}$. Using the lemma, we get

$$W_{-\alpha_{11}+\delta} = s_{\gamma_1+\gamma_2} = s_{\gamma_1} \, s_{\gamma_2} \, s_{\gamma_1} \,, \tag{C.8}$$

with

$$\begin{aligned} s_{\gamma_1} &= \underbrace{T_{39} \, T_{4\,10} \, T_{5\,11}}_{\mathcal{R}_1} \, s_{\alpha_{11}} T_{5\,11} \, T_{4\,10} \, T_{39} & \text{(C.9)} \\ s_{\gamma_2} &= \underbrace{T_{69} \, T_{7\,10} \, T_{8\,11}}_{\mathcal{R}_2} \, s_{\alpha_{11}} T_{8\,11} \, T_{7\,10} \, T_{69} \,, & \text{(C.10)} \end{aligned}$$

where T_{ij} is the Weyl reflexion of the gravity line permuting the i and j indices namely it permutes the compactified radii $R_i \leftrightarrow R_j$ and $s_{\alpha_{11}}$ is the simple Weyl reflexion with respect to α_{11} interpreted in type IIA as a double T-duality in the directions 9 and 10 followed by an exchange of the two radii. We can directly check that there are no timelike T-dualities when we perform the Weyl reflexion $W_{\alpha_{11}}$. The reason is that the T_{69} and T_{39} replace always the time coordinate 9 by 3 or 6 before any double T-duality.

We can now compute the mass of the level $l = 5$ solutions by applying the Weyl reflexion $W_{-\alpha_{11}+\delta}$ to the expression Eq.(C.11) for the mass of the M2 with longitudinal spacelike directions 10, 11 and smeared in all spacelike directions but two 1 and 2 . The M2 mass is given by

$$M_{M2} = \frac{R_{10}\, R_{11}}{l_p^3} \,. \tag{C.11}$$

To perform the sequence of simple Weyl reflexions in the decomposition of $W_{-\alpha_{11}+\delta}$, we insist on two points :

- the permutations of the radii $R_i \leftrightarrow R_j$ are always performed in 11 dimensions ;
- the T-dualities are performed in 10 dimensions.

Then, when reaching any $s_{\alpha_{11}}$ in a sequences of Weyl reflexions, we reduce the 11-dimensional theory to type IIA to perform the double T-duality plus exchange of radii, and then do an uplifting to 11 dimensions. Performing successive permutations of radii, reduction on type IIA, T-duality, exchange of radii, uplifting to 11 dimensions, and so on, we find the mass of the level 5 solution

$$M_{M2} = \frac{R_{10}\, R_{11}}{l_p^3} \quad \stackrel{W_{-\alpha_{11}+\delta}}{\longrightarrow} \quad M_{l=5} = \frac{(R_3\, R_4 \ldots R_8)^2 R_{10}\, R_{11}}{l_p^{15}} \,. \tag{C.12}$$

C.4 The level 6 solution

To obtain the mass of the level 6 solution we start from the expression for the level 5 mass Eq.(C.12) and we use the brane to gravity tower map Eqs.(3.75)-(3.77). In the string language this amounts to perform 3 steps: first, exchange the radii R_9 and R_3 in 11 dimensions, second, reduce to type IIA and, using Eqs.(C.1)-(C.2), perform a double T-duality in the directions 9 and 10 plus exchange of the radii R_9 and R_{11}, finally uplift back to 11 dimensions and exchange the radii R_9 and R_{11}. The result is[31]

$$M_{l=6} = \frac{(R_4 \ldots R_{10})^2\, R_{11}^3}{l_p^{18}} \,. \tag{C.13}$$

C.5 Beyond level 6

To cope with time-like T-dualities and exotic states involving more than one timelike direction, we first compactify time to a radius R_t for the non-exotic states. We define the action $\mathcal{A} = M R_t$,

[31]Note that as the previous cases no timelike T-duality has been performed.

where M is the mass. Applying the relations Eqs.(C.1) and (C.2) to $\mathcal{A}_l = M_l R_t$ for the U-dualities performed in Sections C.1-C.4 leaves all computations of masses unchanged. As $\mathcal{A}$ treats space and time symmetrically it is natural to assume that U-duality can be extended to timelike T-dualities by applying the relations Eqs.(C.1) and (C.2) directly to $\mathcal{A}$ for both space and time radii.

We can now compute A_l for any level l from the $l \leq 6$ non-exotic solutions: we use the double T-duality plus inversion of radii encoded in the Weyl transformations $W_{\alpha_{11}}$ and $W_{-\alpha_{11}+\delta}$ used in Section 3 to reach *all* E_9 BPS states. It is easy to show by recurrence that one recovers in this way for all levels the results Eqs.(4.21), (4.22) and (4.23).

Indeed, consider first the brane towers depicted in Fig.2a. Perform the Weyl transformation $W_{\alpha_{11}}$ and assume that the formula Eq.(4.22) is true for level $2+3m$. Using Eqs.(C.1) and (C.2) one obtains Eq.(4.21) for $n = m+1$. Perform the Weyl transformation $W_{-\alpha_{11}+\delta}$ and assume that the formula Eq.(4.21) is valid for level $1+3m$. Using the decomposition of $W_{-\alpha_{11}+\delta}$ described in Section C.3 one finds Eq.(4.22) for $n = m+1$. The validity of assumption at the levels $l < 6$ yields Eqs.(4.21) and (4.22).

For the [8,1]-gravity tower, assume that the formula Eq.(4.22) is valid for all level $2+3m$ and use the gravity tower map Eqs.(3.75)-(3.77) translated in string language as in Section C.4. One finds Eq.(4.23) for $n = m$, QED.

D Level 4 by Buscher's duality

We review the Buscher formulation [57, 58] of T-duality in 10-dimensional superstring theories for backgrounds admitting one Killing vector[32] . In Buscher's construction one starts with a manifold $\mathcal{M}$ with metric g_{ij} in the string frame, dilaton background ϕ and NS-NS background potential b_{ij}. If the background is invariant under x^{10} translations, it becomes under T-duality in the direction 10 ($a, b = 1 \dots 9$)

$$\begin{aligned}
&\widetilde{g}_{10\,10} = 1/g_{10\,10}, \qquad \widetilde{g}_{10\,a} = b_{10\,a}/g_{10\,10} \\
&\widetilde{g}_{ab} = g_{ab} - (g_{10\,a}g_{10\,b} - b_{10\,a}b_{10\,b})/g_{10\,10} \\
&\widetilde{b}_{10\,a} = g_{10\,a}/g_{10\,10} \\
&\widetilde{b}_{ab} = b_{ab} - (g_{10\,a}b_{10\,b} - b_{10\,a}g_{10\,b})/g_{10\,10} \\
&\widetilde{\phi} = \phi - \frac{1}{2}\ln g_{10\,10}\,.
\end{aligned} \tag{D.1}$$

We apply these transformations to a M5 brane with longitudinal spacelike directions $4 \dots 8$ and longitudinal timelike direction 3 to generate the level 4 BPS solution. We smear the M5 in the directions 9, 10, 11. We perform a double T-duality in the directions 9 and 10.

[32]Here we are interested in NS-NS backgrounds as it was discussed originally [57, 58], thus the formula apply not only for type II but also to the bosonic string in 26 dimensions. We will not need here the generalisation to R-R backgrounds [59].

Upon dimensional reduction along x^{11} to type IIA, this M5 yields a NS5 brane smeared in the directions 9,10 with non-compact transverse directions are 1 and 2. The smeared NS5 is given in the string frame by

$$
\begin{aligned}
& ds^2_{NS5} = -(dx^3)^2 + (dx^4)^2 + \cdots + (dx^8)^2 + H\left[(dx^1)^2 + (dx^2)^2 + (dx^9)^2 + (dx^{10})^2\right] \\
& H(r) = \ln r, \qquad r^2 \equiv (x^1)^2 + (x^2)^2, \\
& e^{\phi} = H^{1/2} \\
& \widetilde{F}_{r345678} = \partial_r(1/H)\,,
\end{aligned}
\tag{D.2}
$$

where $\widetilde{F}_{r345678}$ is the Hodge dual of the 4-form NS field strength db. We use the dual because the smearing procedure is always performed in the 'electric' description of a brane.

We first perform the T-duality in the direction 10. Performing a T-duality on a direction transverse to a NS5 yields a KK5 monopole with the Taub-NUT direction in this transverse direction. Thus the T-duality on the configuration Eqs.(D.2) generate a KK5 monopole with Taub-NUT direction 10 and smeared along 9. To find from Buscher's rule the transformed configuration, we need the value of the non-zero b field. Using the Hodge duality and Eqs.(D.2), we find that the non-zero component of b is

$$
b_{9\,10} = \mathrm{arctg}(x^2/x^1) \equiv B\,. \tag{D.3}
$$

Using Eqs.(D.1), we find the metric of the smeared KK5 monopole,

$$
ds^2_{skk5} = -(dx^3)^2 + (dx^4)^2 + \cdots + (dx^8)^2 + H\left[(dx^1)^2 + (dx^2)^2 + (dx^9)^2\right] + H^{-1}\left[(dx^{10}) - B(dx^9)\right]^2 \tag{D.4}
$$

and $\phi = 0$, as it should. We then perform the second T-duality in the direction 9, applying Eqs.(D.1) on Eq.(D.4). We get (in the string frame)

$$
\begin{aligned}
& ds^2 = -(dx^3)^2 + (dx^4)^2 + \cdots + (dx^8)^2 + H\left((dx^1)^2 + (dx^2)^2\right) + \widetilde{H}\left((dx^9)^2 + (dx^{10})^2\right) \\
& e^{\phi} = \widetilde{H}^{1/2} \\
& b_{9\,10} = \widetilde{B},
\end{aligned}
\tag{D.5}
$$

where $\widetilde{H} = H/(H^2 + B^2)$ and $\widetilde{B} = -B/(H^2 + B^2)$. Finally uplifting the configuration Eq.(D.5) back to eleven dimension, we find

$$
\begin{aligned}
& ds^2 = H\widetilde{H}^{-1/3}\left((dx^1)^2 + (dx^2)^2\right) + \widetilde{H}^{-1/3}\left(-(dx^3)^2 + (dx^4)^2 + \cdots + (dx^8)^2\right) + \\
& + \widetilde{H}^{2/3}\left((dx^9)^2 + (dx^{10})^2 + (dx^{11})^2\right) \\
& A_{9\,10\,11} = \widetilde{B}\,.
\end{aligned}
\tag{D.6}
$$

The solution Eq.(D.6) of 11D supergravity is exactly the level 4 solution Eq.(3.60) obtained starting with the level 2 solution (describing the double smeared M5) and performing a Weyl reflexion $W_{\alpha_{11}}$ to go to level 4. This result is in agreement with the interpretation of the Weyl reflexion $W_{\alpha_{11}}$ as a double T-duality in the directions 9,10 plus the interchange of the two directions [37, 26, 38, 13].

E Weyl transformations commute with compensations

We will show that the set of dualitites, compensations and Weyl transformations needed to express, in the M2 and M5 sequences, the Borel representative at a given level in terms of the level 1 supergravity field does not depend on the path chosen in Fig.3 and Fig.4. Equivalently we will prove that Weyl transformations and compensations do commute. The same proof can be done for the gravity tower.

We note that the nature of the compensation matrix, i.e. $SO(2)$ or $SO(1,1)$, is unaltered by the Weyl reflexions $W_{\alpha_{11}}$ or $W_{-\alpha_{11}+\delta}$. In other words it is the same along a column in Fig.3 and Fig.4. Indeed Eq.(A.1) shows that the Weyl reflexion mapping the level k generator to the level $k+n$ acts on the involution $\Omega R_k = \epsilon R_{-k}$ where ϵ is a sign, to yield $\Omega' R_{k+n} = U \Omega R_k U^{-1} = \epsilon R_{-k-n}$ with the same sign (irrelevant signs in the Weyl transformed have been dropped). Taking this fact into account, we will analyse simultaneously the $SO(2)$ and $SO(1,1)$ compensations.

We start at a given level from the Borel representative given by Eqs.(3.28) and (3.29) for the M2 sequence and by Eqs.(3.57) and (3.58) for the M5 sequence. After a number of dualities and compensations we reach $\boldsymbol{R_i} \equiv R^{[6]}_{1-3n}$ or $\boldsymbol{R_i} \equiv R^{[3]}_{-1-3n}$ defining a Borel representative $\mathcal{V}_i$. Both cases are shown in Table 8. We will show that $\mathcal{V}_f$ defined by $\boldsymbol{R_f}$ in the Table is independent of the path joining $\boldsymbol{R_i}$ to $\boldsymbol{R_f}$. Hence compensations and Weyl transformations do commute.

levels	compensation					
-1+3p	$R^{[6]}_{-1+3p}$	...	$\boldsymbol{R_i} \equiv R^{[6]}_{1-3n}$	$\rightarrow$	$R^{[6]}_{-1+3n}$	...
$W_{\alpha_{11}}$			$\Downarrow$		$\downarrow$	
1+3p	$R^{[3]}_{1+3p}$	...	$R^{[3]}_{-1-3n}$	$\Rightarrow$	$\boldsymbol{R_f} \equiv R^{[3]}_{1+3n}$	...
.			.		.	
.			.		.	
.			.		.	
1+3r	$R^{[3]}_{1+3r}$	...	$\boldsymbol{R_i} \equiv R^{[3]}_{-1-3n}$	$\rightarrow$	$R^{[3]}_{1+3n}$	...
$W_{-\alpha_{11}+\delta}$			$\Downarrow$		$\downarrow$	
-1+3 (r+2)	$R^{[6]}_{-1+3(r+2)}$	...	$R^{[6]}_{1-3(n+2)}$	$\Rightarrow$	$\boldsymbol{R_f} \equiv R^{[6]}_{-1+3(n+2)}$	...

Table 8: *Commutation of compensations and Weyl transformations: the paths depicted by simple arrows "$\rightarrow$" and by double arrows "$\Rightarrow$" lead to the same result. The table applies to both M2 and M5 sequences described in Fig.3 and Fig.4. The Weyl transformations are* $W_{\alpha_{11}}$ *and* $W_{-\alpha_{11}+\delta}$, *and* $1 < p$, $0 < r$, $n < p, r$.

.

E.1 case 1: $R_i \equiv R^{[6]}_{1-3n}$ and $R_f \equiv R^{[3]}_{1+3n}$

Let $\mathcal{V}_a$ be a coset representative along a path joining $\boldsymbol{R_i}$ to $\boldsymbol{R_f}$. We write the Cartan contribution $\mathcal{V}_a^{(0)}$ to $\mathcal{V}_a$ as

$$\mathcal{V}_a^{(0)} = \exp\left[\big(\frac{1}{2}\ln \mathcal{R}e\,\mathcal{E}_a\big)\;\mathcal{C}_a + \Xi\, K^2{}_2\right] . \tag{E.1}$$

Here we isolated in $\mathcal{V}_a^{(0)}$ a contribution $\Xi\, K^2{}_2$ left invariant along the paths by both compensations and Weyl transformations. The relevant contribution of the Cartan generators is the transformed $[(1/2)\ln \mathcal{R}e\,\mathcal{E}_a\,\mathcal{C}_a]$ of $[(1/2)\ln \mathcal{R}e\,\mathcal{E}_i)\;\mathcal{C}_i]$ at $\boldsymbol{R_i}$ where $\mathcal{C}_i$ is the linear combination of the Cartan generators h_{11} and $K^2{}_2$ pertaining to the $SL(2)$ subgroup containing $R^{[6]}_{1-3n}$. From Eq.(3.27), we have $\mathcal{C}_i = \alpha(-h_{11} - nK^2{}_2)$, and $\alpha = +1$ $[-1]$ if the compensation is in $SO(2)$ $[SO(1,1)]$. The Ernst potential $\mathcal{E}_a$ is invariant along a column of Table 8.

We now examine the transformations of $\mathcal{V}_a^{(0)}$ along the paths.

- path "$\rightarrow$"

  After compensation, both for the $SO(2)$ and $SO(1,1)$ compensations, one gets[33] $\mathcal{C}_c = -\,\mathcal{C}_i = \alpha(h_{11} + nK^2{}_2)$. Performing the Weyl transformation $W_{\alpha_{11}}$, $K^2{}_2$ is left invariant and the sign of h_{11} changes. The final expression for $\mathcal{C}_a$ is thus $\mathcal{C}_f = \alpha(-h_{11} + nK^2{}_2)$.

- path "$\Rightarrow$"

  Performing the Weyl reflexion $W_{\alpha_{11}}$, we get $\mathcal{C}_{W_{\alpha_{11}}} = \alpha(h_{11} - nK^2{}_2)$. To perform the subsequent compensation, we must identify the Cartan generator of $SL(2)$ subgroup containing $R^{[3]}_{-1-3n}$. From Eq.(3.26) this is precisely $\mathcal{C}_{W_{\alpha_{11}}}$. After compensation we then obtain $\mathcal{C}_f = -\,\mathcal{C}_{W_{\alpha_{11}}} = \alpha(-h_{11} + nK^2{}_2)$.

The two different paths yield the same $\mathcal{C}_f$, *QED*.

The proof of path equivalence for the contribution of the step operators to $\mathcal{V}_a$ is immediate except for possible sign shifts in the Weyl transformations. Taking these into account, one easily verifies that this affects in the same way both columns of Fig.8 connecting $\boldsymbol{R_i}$ to $\boldsymbol{R_f}$, as these columns list generators corresponding to opposite roots (see discussion after Eq.(A.2)). This completes the proof.

E.2 case 2: $R_i \equiv R^{[3]}_{-1-3n}$ and $R_f \equiv R^{[6]}_{-1+(3n+2)}$

Our starting point is Eq.(E.1). In the relevant contribution of the Cartan generators, $\mathcal{C}_i$ is here the linear combination of h_{11} and $K^2{}_2$ pertaining to the $SL(2)$ subgroup containing $R^{[3]}_{-1-3n}$. From Eq.(3.26), we have $\mathcal{C}_i = \alpha(h_{11} - nK^2{}_2)$.

[33]The action of the compensation on $\mathcal{C}_i$ is obtained by straightforward generalisation of the compensations performed in Sections 3.2.2 and 3.2.3.

- path "→"

  After compensation, both for the $SO(2)$ and $SO(1,1)$ compensations, one gets $\mathcal{C}_c = -\mathcal{C}_i = \alpha(-h_{11} + nK^2{}_2)$. Performing the Weyl transformation $W_{-\alpha_{11}+\delta}$, $K^2{}_2$ is left invariant and $h_{11} \to -h_{11} - 2\,K^2{}_2$. The final expression for $\mathcal{C}_a$ is thus $\mathcal{C}_f = \alpha(h_{11} + (n+2)K^2{}_2)$.

- path "⇒"

  Performing the Weyl reflexion $W_{-\alpha_{11}+\delta}$, we get $\mathcal{C}_{W_{-\alpha_{11}+\delta}} = \alpha(-h_{11} - (n+2)K^2{}_2)$. To perform the subsequent compensation, we must identify the Cartan generator of $SL(2)$ subgroup containing $R^{[6]}_{1-3(n+2)}$. From Eq.(3.27) this is precisely $\mathcal{C}_{W_{-\alpha_{11}+\delta}}$. After compensation we then obtain $\mathcal{C}_f = -\mathcal{C}_{W_{-\alpha_{11}+\delta}} = \alpha(h_{11} + (n+2)K^2{}_2)$.

The two different paths yield the same $\mathcal{C}_f$. It can also be checked that not only the actions on the generators commute but that also the corresponding fields are transformed in the same way. *QED.*

F Redundancy of the two gravity towers

We will show that there are redundancies at each level 3n of the wave sequence and of the monopole sequence.

Let us consider first the wave sequence. The metric of the wave sequence solutions follow from the representatives Eqs.(3.82) and (3.83). One has

$$\begin{aligned} ds^2_{6n} &= g_{ab}\,dx^a\,dx^b \\ &= \mathcal{F}_{2n-1}\bar{\mathcal{F}}_{2n-1}\Big[(dx^1)^2 + (dx^2)^2\Big] + (-1)^{n+1}H^{-1}_{2n+1}(dx^3)^2 + \Big[(dx^4)^2 \cdots + (dx^{10})^2\Big] \\ &+ (-1)^n H_{2n+1}\Big[dx^{11} - \Big((-1)^n H^{-1}_{2n+1} + (-1)^{n+1}\Big)dx^3\Big]^2 , \end{aligned} \tag{F.1}$$

$$\begin{aligned} ds^2_{6n'} &= \widetilde{g}_{ab}\,dx^a\,dx^b \\ &= \mathcal{F}_{2n'-1}\bar{\mathcal{F}}_{2n'-1}\Big[(dx^1)^2 + (dx^2)^2\Big] + (-1)^{n'}H^{-1}_{2n'}(dx^3)^2 + \Big[(dx^4)^2 \cdots + (dx^{10})^2\Big] \\ &+ (-1)^{n'+1} H_{2n'}\Big[dx^{11} - \Big((-1)^{n'+1} H^{-1}_{2n'} + (-1)^{n'}\Big)dx^3\Big]^2 , \end{aligned} \tag{F.2}$$

where $H_p = \mathcal{R}e\,\mathcal{E}_p$.

We now show that these metric are identical at each level $6n = 6n'$ up to interchange of the coordinates 3 and 11. Using the relation

$$\mathcal{E}(z)_{2n+1} = 2 - \mathcal{E}(z)_{2n} \qquad \Longrightarrow \qquad H_{2n+1} = 2 - H_{2n}\,, \tag{F.3}$$

in Eq.(F.1) and performing the following coordinate transformation

$$\begin{cases} x'_3 = -x_{11} & , \\ x'_{11} = x_3 & , \\ x'_a = x_a & a \neq 3, 11\,, \end{cases} \tag{F.4}$$

we get the transformed metric

$$
\begin{aligned}
ds'^2_{6n} &= \mathcal{F}_{2n-1}\bar{\mathcal{F}}_{2n-1}\Big[(dx^1)^2+(dx^2)^2\Big]+(-1)^{n+1}(2-H_{2n})^{-1}(dx^{11})^2+\Big[(dx^4)^2\cdots+(dx^{10})^2\Big] \\
&+ (-1)^n(2-H_{2n})\Big[dx^3+\Big((-1)^n(2-H_{2n})^{-1}+(-1)^{n+1}\Big)dx^{11}\Big]^2 . \qquad \text{(F.5)}
\end{aligned}
$$

We thus conclude that $ds'^2_{6n}=ds^2_{6n'}$ for $n=n'$:

$$
\begin{array}{lll}
g'_{11\,11} & =(-1)^{n+1}H_{2n} & =\widetilde{g}_{11\,11}\,, \\
g'_{3\,3} & =(-1)^n(2-H_{2n}) & =\widetilde{g}_{3\,3}\,, \\
g'_{3\,11} & =-1+H_{2n} & =\widetilde{g}_{3\,11}\,, \\
g'_{a\,a} & =\widetilde{g}_{a\,a}\quad a\neq 3,11\,. &
\end{array} \qquad \text{(F.6)}
$$

Let us now consider the monopole sequence. The metric of the monopole sequence solutions follow from the representatives Eqs.(3.85) and (3.86). One has

$$
\begin{aligned}
ds^2_{3+6n'} &= g_{ab}\,dx^a\,dx^b \\
&= \mathcal{F}_{2n'-1}\bar{\mathcal{F}}_{2n'-1}H_{2n'+1}\Big[(dx^1)^2+(dx^2)^2\Big]+H_{2n'+1}(dx^3)^2+(-1)^{n'}\Big[(dx^4)^2\cdots+(dx^8)^2\Big] \\
&+(-1)^{n'+1}\Big[(dx^9)^2+(dx^{10})^2\Big]+H^{-1}_{2n'+1}\Big[dx^{11}-\Big((-1)^{n'}B_{2n'+1}\Big)dx^3\Big]^2 \qquad \text{(F.7)}
\end{aligned}
$$

$$
\begin{aligned}
ds^2_{-3+6n} &= \widetilde{g}_{ab}\,dx^a\,dx^b \\
&= \mathcal{F}_{2n-1}\bar{\mathcal{F}}_{2n-1}H_{2n}\Big[(dx^1)^2+(dx^2)^2\Big]+H_{2n}(dx^3)^2+(-1)^{n+1}\Big[(dx^4)^2\cdots+(dx^8)^2\Big] \\
&+(-1)^{n}\Big[(dx^9)^2+(dx^{10})^2\Big]+H^{-1}_{2n}\Big[dx^{11}-\Big((-1)^{n+1}B_{2n}\Big)dx^3\Big]^2 . \qquad \text{(F.8)}
\end{aligned}
$$

where $B_p=\mathcal{I}m\,\mathcal{E}_p$.

We now show that these metric are identical at each level $3+6n'=-3+6n$, i.e. $n=n'+1$, up to interchange of the coordinates 3 and 11. Replacing in Eq.(F.8) n by $n'+1$, we get

$$
\begin{aligned}
ds^2_{-3+6(n'+1)} &= \mathcal{F}_{2n'-1}\bar{\mathcal{F}}_{2n'-1}\mathcal{E}_{2n'+1}\bar{\mathcal{E}}_{2n'+1}H_{2n'+2}\Big[(dx^1)^2+(dx^2)^2\Big] \\
&+ H_{2n'+2}(dx^3)^2+(-1)^{n'}\Big[(dx^4)^2\cdots+(dx^8)^2\Big] \\
&+ (-1)^{n'+1}\Big[(dx^9)^2+(dx^{10})^2\Big]+H^{-1}_{2n'+2}\Big[dx^{11}-\Big((-1)^{n'}B_{2n'+2}\Big)dx^3\Big]^2 . \qquad \text{(F.9)}
\end{aligned}
$$

Using the relation

$$
\mathcal{E}(z)_{2n'+2}=(\mathcal{E}(z)_{2n'+1})^{-1} \qquad \Longrightarrow \qquad H_{2n'+2}=\frac{H_{2n'+1}}{\mathcal{E}_{2n'+1}\bar{\mathcal{E}}_{2n'+1}}\,, \qquad \text{(F.10)}
$$

$$
\Longrightarrow \qquad B_{2n'+2}=-\frac{B_{2n'+1}}{\mathcal{E}_{2n'+1}\bar{\mathcal{E}}_{2n'+1}}\,, \qquad \text{(F.11)}
$$

in Eq.(F.9) and performing the transformation of coordinates Eq.(F.4), we get the transformed metric

$$
\begin{aligned}
ds^{2\prime}_{-3+6(n'+1)} &= \mathcal{F}_{2n'-1}\bar{\mathcal{F}}_{2n'-1}H_{2n'+1}\Big[(dx^1)^2+(dx^2)^2\Big]+\frac{H_{2n'+1}}{\mathcal{E}_{2n'+1}\bar{\mathcal{E}}_{2n'+1}}(dx^{11})^2\\
&+ (-1)^{n'}\Big[(dx^4)^2\cdots+(dx^8)^2\Big]+(-1)^{n'+1}\Big[(dx^9)^2+(dx^{10})^2\Big]\\
&+ \big(\frac{H_{2n'+1}}{\mathcal{E}_{2n'+1}\bar{\mathcal{E}}_{2n'+1}}\big)^{-1}\Big[dx^3+\Big((-1)^{n'+1}\frac{B_{2n'+1}}{\mathcal{E}_{2n'+1}\bar{\mathcal{E}}_{2n'+1}}\Big)dx^{11}\Big]^2 .
\end{aligned}
\tag{F.12}
$$

We thus conclude that $ds^{2\prime}_{-3+6(n'+1)} = ds^2_{3+6n'}$:

$$
\begin{array}{lll}
g'_{11\,11} & = H^{-1}_{2n'+1} & = g_{11\,11}\,,\\
g'_{3\,3} & = H^{-1}_{2n'+1}(\mathcal{E}_{2n'+1}\bar{\mathcal{E}}_{2n'+1}) & = g_{3\,3}\,,\\
g'_{3\,11} & = (-1)^{n'+1}B_{2n'+1}H^{-1}_{2n'+1} & = g_{3\,11}\,,\\
g'_{a\,a} & = g_{a\,a} \quad a\neq 3,11. &
\end{array}
\tag{F.13}
$$

We see that there is only one gravity tower, the left and the right tower of Fig.5b being equivalent (except for the level 0 KK-wave) as each of them contains the full wave and monopole sequences.

G Structure of the A_1^+ U-duality group

Our U-duality group in two non-compact dimensions is the infinite order Weyl group $\mathcal{W}$ of an affine group. The structure of such Weyl groups is known in terms of translations and the finite Weyl group of the underlying finite group of rank r [31]. The affine Weyl group is the semi-direct product of translations $\mathbb{Z}$ and the finite Weyl group. For the case of affine A_1^+, which features prominently in this paper, the affine Weyl group is simply

$$\mathcal{W} = \mathbb{Z}_2 \ltimes \mathbb{Z}. \tag{G.14}$$

To derive this fact it is useful to denote the two simple roots of A_1^+ by α_1 and α_2. These can be identified to α_{11} and $-\alpha_{11}+\delta$ for the M2 and the M5 sequences, and to λ and $-\lambda+\delta$ for the KK-wave and the KK-monopole sequences. The simple roots α_1 and α_2 are a basis of the root lattice (they span the ladder diagrams of Fig.2 and Fig.5). To describe the fundamental reflexions W_1, W_2 in these two roots, it is sufficient to give their action on a basis:

$$
\begin{aligned}
W_1(\alpha_1) &= -\alpha_1, & W_1(\alpha_2) &= \alpha_2+2\alpha_1,\\
W_2(\alpha_1) &= \alpha_1+2\alpha_2, & W_2(\alpha_2) &= -\alpha_2.
\end{aligned}
\tag{G.15}
$$

We take the $\mathbb{Z}_2$ to be generated by the horizontal Matzner–Misner reflexion W_2. The Coxeter relations for this Weyl group are $(W_1W_2)^\infty = \text{id}$ and $(W_2W_1)^\infty = \text{id}$, in other words there are no mixed relations. Therefore all Weyl group elements are of the form

$$(W_1W_2)^n, \qquad (W_2W_1)^n, \qquad W_1(W_2W_1)^n, \qquad W_2(W_1W_2)^n, \qquad \text{id}\,, \tag{G.16}$$

for some $m, n \geq 0$. Defining $T^n = (W_1 W_2)^n$ for $n \geq 0$ and $T^n = (W_2 W_1)^{(-n)}$ for $n \leq 0$ one deduces for the $\mathbb{Z}_2$ generated by W_2 the structure

$$W_2 T^n = T^{-n} W_2 \,, \tag{G.17}$$

illustrating the semi-directness of the product in this case. The set of all elements of the infinite order Weyl group is thus

$$\mathcal{W} = \{T^n : n \in \mathbb{Z}\} \cup \{W_2 T^n : n \in \mathbb{Z}\} \,, \tag{G.18}$$

with relations

$$\begin{aligned} W_2 W_2 &= 1 \\ W_2 T^n &= T^{-n} W_2 \\ T^n T^m &= T^{n+m}. \end{aligned} \tag{G.19}$$

The translations T act vertically in the diagrams of Fig.2a, Fig.3, Fig. 4, Fig.5a and Fig.5b. They connect the points lying both on the same tower and on the same sequence.

We can act with the Weyl group on any integrable representation $\rho : A_1^+ \to End(V)$, by letting [31]

$$U_i = \exp(\rho(f_i)) \exp(-\rho(e_i)) \exp(\rho(f_i)) \in GL(V). \tag{G.20}$$

Here e_i and f_i are the simple Chevalley generators. It is not hard to see that this definition implies that U_i is actually an element of $SO(V)$ in the sense that $U_i U_i^T = \mathrm{id}_V$ for $i = 1, 2$ where the *Chevalley* transposed element is $U_i^T = \exp(\rho(e_i)) \exp(-\rho(f_i)) \exp(\rho(e_i))$.[34]

From Eq.(G.20) it is straightforward to show that Weyl reflexions are always elements of the compact subgroup of the split real form of the associated group, so in our case this means K_{10}^+.

[34]The definition Eq.(G.20) does not necessarily imply $U_i U_i = \mathrm{id}_V$. In order to arrive at the proper Weyl group one has to factor out the subgroup generated by $U_i U_i$ from the $GL(V)$ subgroup generated by the U_i, see §3.8 in [31].

References

[1] E. Cremmer, B. Julia, and J. Scherk. Supergravity theory in 11 dimensions. *Phys. Lett.*, B76:409–412, 1978.

[2] E. Cremmer and B. Julia. The SO(8) supergravity. *Nucl. Phys.*, B159:141, 1979.

[3] S. Ferrara, J. Scherk, and B. Zumino. Algebraic properties of extended supergravity theories. *Nucl. Phys.*, B121:393, 1977.

[4] E. Cremmer, J. Scherk, and S. Ferrara. SU(4) invariant supergravity theory. *Phys. Lett.*, B74:61, 1978.

[5] B. Julia. Group disintegrations. Lecture in Applied Mathematics, Vol. 21 (1985) AMS-SIAM.

[6] B. Julia. Infinite Lie algebras in physics. Invited talk given at Johns Hopkins Workshop on Current Problems in Particle Theory, Baltimore, Md., May 25-27, 1981.

[7] E. Cremmer and B. Julia. The N=8 supergravity theory. 1. The lagrangian. *Phys. Lett.*, B80:48, 1978.

[8] J. H. Schwarz and P. C. West. Symmetries and transformations of chiral N=2 D = 10 supergravity. *Phys. Lett.*, B126:301, 1983.

[9] E. Cremmer, B. Julia, H. Lu, and C. N. Pope. Higher-dimensional origin of D = 3 coset symmetries. 1999, hep-th/9909099.

[10] T. Damour, S. de Buyl, M. Henneaux, and C. Schomblond. Einstein billiards and overextensions of finite-dimensional simple Lie algebras. *JHEP*, 08:030, 2002, hep-th/0206125.

[11] M. R. Gaberdiel, D. I. Olive, and P. C. West. A class of Lorentzian Kac-Moody algebras. *Nucl. Phys.*, B645:403–437, 2002, hep-th/0205068.

[12] N. D. Lambert and P. C. West. Coset symmetries in dimensionally reduced bosonic string theory. *Nucl. Phys.*, B615:117–132, 2001, hep-th/0107209.

[13] F. Englert, L. Houart, A. Taormina, and P. West. The symmetry of M-theories. *JHEP*, 09:020, 2003, hep-th/0304206.

[14] T. Damour, M. Henneaux, and H. Nicolai. E_{10} and a 'small tension expansion' of M theory. *Phys. Rev. Lett.*, 89:221601, 2002, hep-th/0207267.

[15] P. West. E_{11} and M theory. *Class. Quant. Grav.*, 18:4443–4460, 2001, hep-th/0104081.

[16] H. Nicolai and T. Fischbacher. Low level representations for E_{10} and E_{11}. 2003, hep-th/0301017.

[17] P. West. Very extended E_8 and A_8 at low levels, gravity and supergravity. *Class. Quant. Grav.*, 20:2393–2406, 2003, hep-th/0212291.

[18] A. Kleinschmidt, I. Schnakenburg, and P. West. Very-extended Kac-Moody algebras and their interpretation at low levels. *Class. Quant. Grav.*, 21:2493–2525, 2004, hep-th/0309198.

[19] T. Damour and M. Henneaux. E_{10}, BE_{10} and arithmetical chaos in superstring cosmology. *Phys. Rev. Lett.*, 86:4749–4752, 2001, hep-th/0012172.

[20] T. Damour, M. Henneaux, B. Julia, and H. Nicolai. Hyperbolic Kac-Moody algebras and chaos in Kaluza-Klein models. *Phys. Lett.*, B509:323–330, 2001, hep-th/0103094.

[21] T. Damour, M. Henneaux, and H. Nicolai. Cosmological billiards. *Class. Quant. Grav.*, 20:R145–R200, 2003, hep-th/0212256.

[22] F. Englert and L. Houart. $\mathcal{G}^{+++}$ invariant formulation of gravity and M-theories: Exact BPS solutions. *JHEP*, 01:002, 2004, hep-th/0311255.

[23] F. Englert, M. Henneaux, and L. Houart. From very-extended to overextended gravity and M-theories. *JHEP*, 02:070, 2005, hep-th/0412184.

[24] P. West. The IIA, IIB and eleven dimensional theories and their common E_{11} origin. *Nucl. Phys.*, B693:76–102, 2004, hep-th/0402140.

[25] C. M. Hull and P. K. Townsend. Unity of superstring dualities. *Nucl. Phys.*, B438:109–137, 1995, hep-th/9410167.

[26] N. A. Obers and B. Pioline. U-duality and M-theory, an algebraic approach. 1998, hep-th/9812139.

[27] R. Geroch. A method for generating solutions of Einstein's equations. *J. Math. Phys.*, 12:918–924, 1971.

[28] R. Geroch. A method for generating new solutions of Einstein's equation. 2. *J. Math. Phys.*, 13:394–404, 1972.

[29] P. Breitenlohner and D. Maison. On the Geroch group. *Ann. Poincare*, 46:215, 1987.

[30] H. Nicolai and A. Nagar. Infinite dimensional symmetries in gravity. In: Ciufolini, I. (ed.) et al.: Gravitational waves, IoP Publishing Ltd. 2001.

[31] V. Kac. Infinite dimensional Lie algebras. 1995.

[32] A. Kleinschmidt and H. Nicolai. E_{10} cosmology. *JHEP*, 01:137, 2006, hep-th/0511290.

[33] A. Keurentjes. E_{11}: Sign of the times. *Nucl. Phys.*, B697:302–318, 2004, hep-th/0402090.

[34] A. Keurentjes. Poincare duality and G^{+++} algebras. 2005, hep-th/0510212.

[35] A. Keurentjes. Time-like T-duality algebra. *JHEP*, 11:034, 2004, hep-th/0404174.

[36] S. de Buyl, L. Houart, and N. Tabti. Dualities and signatures of $\mathcal{G}^{++}$ invariant theories. *JHEP*, 06:084, 2005, hep-th/0505199.

[37] S. Elitzur, A. Giveon, D. Kutasov, and E. Rabinovici. Algebraic aspects of matrix theory on T^d. *Nucl. Phys.*, B509:122–144, 1998, hep-th/9707217.

[38] T. Banks, W. Fischler, and L. Motl. Dualities versus singularities. *JHEP*, 01:019, 1999, hep-th/9811194.

[39] C. M. Hull. Timelike T-duality, de Sitter space, large N gauge theories and topological field theory. *JHEP*, 07:021, 1998, hep-th/9806146.

[40] C. M. Hull. Duality and the signature of space-time. *JHEP*, 11:017, 1998, hep-th/9807127.

[41] C. M. Hull and R. R. Khuri. Branes, times and dualities. *Nucl. Phys.*, B536:219–244, 1998, hep-th/9808069.

[42] R. Argurio and L. Houart. On the exotic phases of M-theory. *Phys. Lett.*, B450:109–118, 1999, hep-th/9810180.

[43] A. Kleinschmidt and H. Nicolai. $K(E_9)$ from $K(E_{10})$. hep-th/0611314.

[44] E. Lozano-Tellechea and T. Ortin. 7-branes and higher Kaluza-Klein branes. *Nucl. Phys.*, B607:213–236, 2001, hep-th/0012051.

[45] B. Julia and H. Nicolai. Conformal internal symmetry of 2d sigma-models coupled to gravity and a dilaton. *Nucl. Phys.*, B482:431–465, 1996, hep-th/9608082.

[46] J. Brown, O. J. Ganor, and C. Helfgott. M-theory and E_{10}: Billiards, branes and imaginary roots. *JHEP*, 08:063, 2004, hep-th/0401053.

[47] L. Bombelli, R. K. Koul, G. Kunstatter, J-H. Lee, and R. D. Sorkin. On energy in five-dimensional gravity and the mass of the Kaluza-Klein monopole. *Nucl. Phys.*, B289:735, 1987.

[48] S. W. Hawking. Gravitational instantons. *Phys. Lett.*, A60:81, 1977.

[49] G. W. Gibbons and S. W. Hawking. Classification of gravitational instanton symmetries. *Commun. Math. Phys.*, 66:291–310, 1979.

[50] A. Sen. Dynamics of multiple Kaluza-Klein monopoles in M and string theory. *Adv. Theor. Math. Phys.*, 1:115–126, 1998, hep-th/9707042.

[51] K. Bautier, S. Deser, M. Henneaux, and D. Seminara. No cosmological D = 11 supergravity. *Phys. Lett.*, B406:49–53, 1997, hep-th/9704131.

[52] E. Bergshoeff, Y. Lozano, and T. Ortin. Massive branes. *Nucl. Phys.*, B518:363–423, 1998, hep-th/9712115.

[53] E. Bergshoeff and J.P. van der Schaar. On M-9-branes. *Class. Quant. Grav.*, 16:23–39, 1999, hep-th/9806069.

[54] P. K. Townsend. M-theory from its superalgebra. 1997, hep-th/9712004.

[55] C. M. Hull. Gravitational duality, branes and charges. *Nucl. Phys.*, B509:216–251, 1998, hep-th/9705162.

[56] E. Bergshoeff, M. de Roo, M. B. Green, G. Papadopoulos, and P. K. Townsend. Duality of type II 7-branes and 8-branes. *Nucl. Phys.*, B470:113–135, 1996, hep-th/9601150.

[57] T. H. Buscher. A symmetry of the string background field equations. *Phys. Lett.*, B194:59, 1987.

[58] T. H. Buscher. Path integral derivation of quantum duality in nonlinear σ-models. *Phys. Lett.*, B201:466, 1988.

[59] E. Bergshoeff, C. M. Hull, and T. Ortin. Duality in the type II superstring effective action. *Nucl. Phys.*, B451:547–578, 1995, hep-th/9504081.