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Late-time tails of a Yang–Mills field on Minkowski and Schwarzschild backgrounds

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Abstract

We study the late-time behaviour of spherically symmetric solutions of the Yang–Mills equations on Minkowski and Schwarzschild backgrounds. Using nonlinear perturbation theory we show in both cases that solutions having smooth compactly supported initial data possess tails which decay as t^{-4} at timelike infinity. Moreover, for small initial data on Minkowski background we derive the third-order formula for the amplitude of the tail and numerically confirm its accuracy.

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1. Introduction

In a classical paper [1] Eardley and Moncrief proved that solutions of the Yang–Mills equations on the 3 + 1 Minkowski spacetime starting from smooth initial data remain smooth for all future times. A different proof allowing for initial data with only finite energy was given later by Klainerman and Machedon [2]. Once global regularity was established, the problem of asymptotic behaviour of solutions for $t \to \infty$ was studied by many authors [3–7] who obtained various decay estimates using different techniques and assumptions about initial data. In this paper we are concerned with the simplest possible situation, namely spherically symmetric initial data with compact support. In this case it follows from the conformal method of Christodoulou that the Yang–Mills curvature decays as t^{-4} at timelike infinity [4]. The purpose of this paper is threefold. First, we rederive Christodoulou's result using the nonlinear perturbation theory. The advantage of our approach lies in its wide applicability; in contrast to the conformal method which is very powerful (in the sense of giving sharp decay rates) only for conformally invariant equations.

Second, we go beyond qualitative decay estimates and give the third-order formula for the amplitude of solution which provides a precise *quantitative* information about the tail. We wish to point out that although our results depend crucially on spherical symmetry, the assumption of compact support for initial data is made for simplicity and can be relaxed by imposing a suitable fall-off condition at spatial infinity (which can be implemented via appropriately weighted norms). However, some kind of localization condition is necessary in order to avoid a situation where the tail in time is induced entirely by the tail of initial data at spatial infinity (due, for instance, to nonzero charge).

Third, we argue that the same tail is present in the scattering of spherically symmetric Yang–Mills fields of the Schwarzschild black hole. In this case, the global existence of solutions follows from the work of Chruściel and Shatah [8] who generalized the proof of Eardley and Moncrief to arbitrary globally hyperbolic Lorentzian 4-manifolds. The late-time tail of the Yang–Mills field on the Schwarzschild background was studied in [9], however the fall-off t^{-5} derived there on the basis of the linear perturbation analysis is not correct. As we shall see, the error in [9] is due to the fact that the late-time tail is *not* governed by the linearized evolution. At first sight this might seem odd but upon reflection it is easy to understand. A rough intuitive explanation is that the tail is a far-field effect, hence the flat space tail t^{-4} is expected to persist in any asymptotically flat spacetime as long as the backscattering on the curvature does not produce a more slowly decaying tail. A similar example of the failure of linear perturbation analysis was recently observed in the scattering of skyrmions [10].

2. Minkowski background

We consider the Yang–Mills theory with the gauge group $SU(2)$ and assume the spherically symmetric ansatz for the connection [11]

$$A = w\tau_1\,\mathrm{d}\theta + (\cot\theta\,\tau_3 + w\tau_2)\sin\theta\,\mathrm{d}\phi, \tag{1}$$

where $w = w(t, r)$ and τ_i $(i = 1, 2, 3)$ are the usual generators of $su(2)$. The Yang–Mills equations $d * F = 0$, where $F = dA + A \wedge A$ is the Yang–Mills curvature, reduce then to the semilinear radial wave equation

$$\ddot{w} - w'' - \frac{1}{r^2} w(1 - w^2) = 0, \tag{2}$$

where primes and dots denote derivatives with respect to r and t, respectively. For our purposes, it is convenient to define the function $f(t, r) = (w(t, r) - 1)/r$ and rewrite equation (2) in the following form:

$$\mathcal{L}f := \ddot{f} - f'' - \frac{2}{r}f' + \frac{2}{r^2}f = -f^3 - \frac{3}{r}f^2. \tag{3}$$

Note that $\mathcal{L}$ is the radial wave operator for the $l = 1$ spherical harmonic.

We consider the late-time evolution of solutions of equation (3) for smooth compactly supported initial data

$$f(0, r) = \varepsilon\alpha(r), \qquad \dot{f}(0, r) = \varepsilon\beta(r). \tag{4}$$

The prefactor ε is added for convenience and to emphasize that our initial data are assumed to be small. Regularity at the origin is ensured by the boundary condition $f(t, r) \sim b(t)r$ for $r \to 0$. As follows from [3] such solutions decay to zero on any compact region of space as $t \to \infty$. To determine the asymptotic behaviour of solutions we define the perturbative expansion

$$f = \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \cdots, \tag{5}$$

where εf_1 satisfies initial data (4) and all f_n with $n > 1$ have zero initial data. Substituting expansion (5) into equation (3) up to the third order we get

$$\mathcal{L} f_1 = 0, \tag{6}$$

$$\mathcal{L} f_2 = -\frac{3}{r} f_1^2, \tag{7}$$

$$\mathcal{L} f_3 = -f_1^3 - \frac{6}{r} f_1 f_2. \tag{8}$$

We solve these equations recursively. The first-order solution is given by the general regular solution of the free radial wave equation for the $l = 1$ spherical harmonic

$$f_1(t, r) = \frac{a'(t-r) + a'(t+r)}{r} + \frac{a(t-r) - a(t+r)}{r^2}, \tag{9}$$

where the function $a(\xi)$ is determined by the initial data

$$a(\xi) = -\frac{1}{2}\xi \int_\xi^\infty \alpha(s)\,\mathrm{d}s + \frac{1}{4}\int_\xi^\infty (s^2 - \xi^2)\beta(s)\,\mathrm{d}s. \tag{10}$$

For compactly supported initial data the function $a(\xi)$ has compact support as well (note that the functions $\alpha(s)$ and $\beta(s)$ in (10) are odd extensions of initial data to the whole line), hence f_1 has no tail in agreement with Huygens' principle.

To solve equations for the higher order perturbations we use the retarded Green's function of the operator $\mathcal{L}$

$$G(t-t', r, r') = [|r - r'| \leqslant t - t' \leqslant r + r'] \frac{r^2 + r'^2 - (t-t')^2}{4r^2}. \tag{11}$$

It follows from (11) that the solution of the inhomogeneous equation $\mathcal{L} f = N(t, r)$ with zero initial data has the form (using null coordinates $u = t' - r'$, $v = t' + r'$) [3]

$$f(t, r) = \frac{1}{8r^2}\int_{|t-r|}^{t+r} \mathrm{d}v \int_{-v}^{t-r} K(t, r; u, v) N(u, v)\,\mathrm{d}u, \tag{12}$$

where the kernel $K(t, r; u, v) = (v-t)(t-u) + r^2$. In the second order, i.e. for equation (7), representation (12) yields

$$f_2(t, r) = -\frac{3}{4r^2}\int_{|t-r|}^{t+r} \mathrm{d}v \int_{-v}^{t-r} K(t, r; u, v) \frac{f_1^2(u, v)}{v-u}\,\mathrm{d}u. \tag{13}$$

Somewhat surprisingly, Huygens' property is preserved in the second order. To see this, let us assume that $a(\xi) = 0$ for $|\xi| \geqslant R$. Then, for $t > r + R$, we may change the order of integration in (13) and rewrite it as (see figure 1)

$$f_2(t, r) = -\frac{3}{r^2}\int_{-R}^{R} \mathrm{d}u \int_{t-r}^{t+r} \frac{(v-t)(t-u) + r^2}{(v-u)^3}\left(a'(u) + \frac{2a(u)}{v-u}\right)^2 \mathrm{d}v. \tag{14}$$

Performing the inner integral we get

$$f_2(t, r) = 8r\int_{-R}^{R} \frac{a(u)}{(t-u)^2 - r^2}\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{a(u)}{(t-u)^2 - r^2}\right)\mathrm{d}u, \tag{15}$$

which after integration gives zero. Thus, $f_2(t, r)$ vanishes identically for $t > r + R$ and consequently there is no tail up to the second order.

In the third order, i.e. for equation (8), representation (12) gives $f_3 = f_3^{(1)} + f_3^{(2)}$, where

$$f_3^{(1)}(t, r) = -\frac{1}{8r^2}\int_{|t-r|}^{t+r} \mathrm{d}v \int_{-v}^{t-r} K(t, r; u, v) f_1^3(u, v)\,\mathrm{d}u, \tag{16}$$

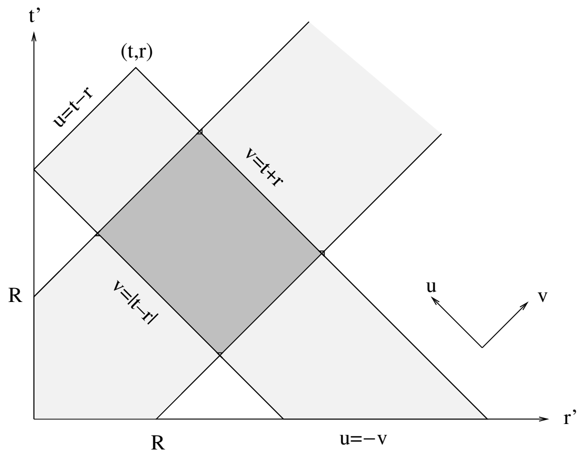

Figure 1. An illustration of the situation in equations (14), (16) and (17). The observation point is located at (t, r), where $t > r + R$. The integration range is given by the intersection of the two shaded regions which depict the domain of dependence of the observation point and the support of the solution $f_1(t', r')$.

$$f_3^{(2)}(t,r) = -\frac{3}{2r^2}\int_{|t-r|}^{t+r} \mathrm{d}v \int_{-v}^{t-r} K(t,r;u,v)\frac{f_1(u,v)f_2(u,v)}{v-u}\,\mathrm{d}u. \tag{17}$$

To calculate $f_3^{(1)}(t,r)$ for $t > r + R$ as above, we change the order of integration and perform the integral over v with the result (using the abbreviation $z = (t-u)^2 - r^2$):

$$f_3^{(1)}(t,r) = 4r\int_{-R}^{R}\left(\frac{a(u)a'(u)^2}{z^2} + \frac{4(t-u)a'(u)a^2(u)}{z^3} + \frac{4\big((t-u)^2 + \frac{1}{5}r^2\big)a^3(u)}{z^4}\right)\mathrm{d}u, \tag{18}$$

which has the following asymptotic behaviour near timelike infinity ($r = \text{const}$ and $t \to \infty$):

$$f_3^{(1)}(t,r) \sim c_1 r t^{-4}, \qquad c_1 = 4\int_{-\infty}^{+\infty} a(u)a'(u)^2\,\mathrm{d}u. \tag{19}$$

In the formula above we replaced R by ∞ in the limits of integration to emphasize that the result holds not only for strictly compactly supported initial data but also for initial data which fall off sufficiently fast at spatial infinity.

To calculate the contribution to the tail coming from $f_3^{(2)}(t,r)$ we need to know both the leading and the subleading terms in the asymptotic expansion of $f_2(u,v)$ near null infinity ($u = \text{const}$ and $v \to \infty$). This calculation is deferred to the appendix where we show that near null infinity

$$f_2(u,v) = \frac{h'(u)}{v-u} + \frac{2h(u)}{(v-u)^2} + \frac{2g(u)}{(v-u)^2} + \mathcal{O}(v^{-3}), \tag{20}$$

where $h(u)$ and $g(u)$ are defined by (A.5) and (A.6), respectively. Note that the first two terms in (20) represent the 'free' part of the iterate $f_2(t,r)$; as we shall see in a moment this part does *not* affect the behaviour of $f_3^{(2)}(t,r)$ at timelike infinity. Substituting (20) into (17) and proceeding along the same lines as in the derivation of expression (19), we obtain the following asymptotic behaviour near timelike infinity:

$$\begin{aligned} &f_3^{(2)}(t,r) \sim c_2 r t^{-4}, \\ &c_2 = 4\int_{-\infty}^{+\infty}\left[\frac{\mathrm{d}}{\mathrm{d}u}(h(u)a(u)) + g(u)a'(u)\right]\mathrm{d}u = -12\int_{-\infty}^{+\infty} a(u)a'(u)^2\,\mathrm{d}u, \end{aligned} \tag{21}$$

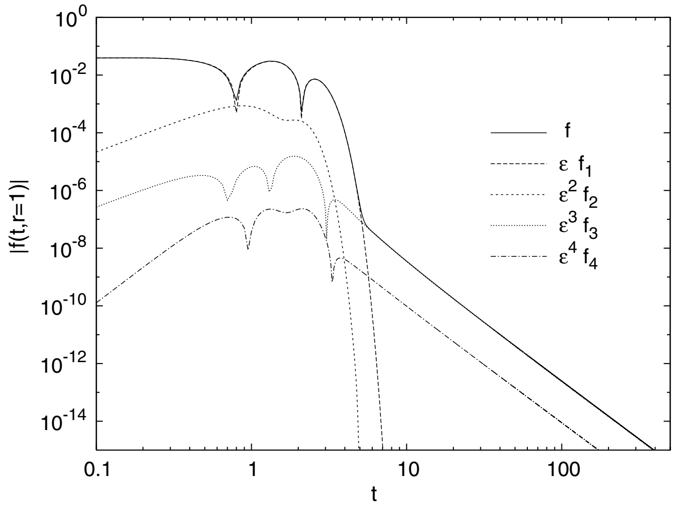

Figure 2. We plot (on log–log scale) the numerical solution $f(t, r = 1)$ of the initial value problem (3)–(4) for $\alpha(r) = r\exp(-r^2)$, $\beta(r) = r(2 - r^2)\exp(-r^2)$ and $\varepsilon = 0.1$, and compare it with the third-order approximation $\varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3$ (produced by solving numerically the perturbation equations (6)–(8)). These two functions are indistinguishable at the scale of the figure so both are depicted by the single solid line. The contributions of the individual iterates are superimposed to demonstrate that the tail comes from f_3. The fourth-order iterate f_4 serves as the estimation of the error. Note that f_4 has the same late-time slope (decay rate) as f_3, in agreement with remark 2. The fit of the function $Ct^{-\gamma}\exp(A/t)$ to the full solution $f(t, 1)$ for late times gives $\gamma \simeq 4.001$ and $C \simeq 2.391 \times 10^{-5}$ which is around 4% off the third-order prediction $\varepsilon^3 c = \varepsilon^3 155\sqrt{3\pi}/20736 \approx 2.295 \times 10^{-5}$ obtained by evaluating the integral in (22).

where the last expression follows from (A.6) and integration by parts. Putting equations (19) and (21) together we finally get the leading asymptotic behaviour near timelike infinity

$$f_3(t, r) \sim crt^{-4}, \qquad c = -8\int_{-\infty}^{+\infty} a(u)a'(u)^2\,\mathrm{d}u. \tag{22}$$

This is our main result. We claim that expression (22) provides a very good approximation of the tail for solutions having sufficiently small initial data. More precisely, we conjecture that for any given smooth compactly supported functions $\alpha(r)$ and $\beta(r)$, one can choose ε such that for each fixed $r > 0$ and $t \to \infty$ the remainder $|t^4 f(t, r) - \varepsilon^3 cr|$ is as small as one pleases. The numerical evidence supporting this conjecture is shown in figure 2. The obvious issue remains as to whether the perturbation expansion corresponding to given initial data is convergent for sufficiently small values of ε. Without a proof of convergence, our analysis is not mathematically rigorous, nevertheless we believe that it gives a rather convincing and, most importantly, *quantitative* description of the late-time tail.

A few remarks are in order.

Remark 1. It should be clear from the above derivation that the simplicity of the final result (22) is due to some amazing cancellations (notably those occurring in equations (15) and (21)) which in turn are attributed to the particular form of the nonlinearity of the Yang–Mills equations. In this respect, the Yang–Mills equations are exceptional and particularly interesting mathematically; for most other nonlinear perturbations of the wave equation the tail is a second-order phenomenon which is much easier to analyse (e.g., see [10]).

Remark 2. Note that all iterates f_k behave as $\mathcal{O}[(v-u)^{-1}]$ near null infinity and therefore at each order of the perturbation expansion the sources behave as $\mathcal{O}[(v-u)^{-3}]$. For such sources, one might expect by dimensional arguments that there would be a t^{-3} tail. Fortunately, due to the identity

$$\int_{t-r}^{t+r} \frac{(v-t)(t-u)+r^2}{(v-u)^3}\,\mathrm{d}v = 0, \tag{23}$$

all coefficients of hypothetical t^{-3} tails vanish identically and thus all higher order terms in the perturbation expansion decay as t^{-4}. This fact is crucial since otherwise the third-order approximation would break down for late times; for instance a nonzero fourth-order term $\sim\varepsilon^4/t^3$ would make formula (22) useless for times $t \gtrsim 1/\varepsilon$.

Remark 3. Note that equation (3) has the scaling symmetry: if $f(t,r)$ is a solution, so is $f_\lambda(t,r) := \lambda f(\lambda t, \lambda r)$. Under this scaling the energy scales as $E(f_\lambda) = \lambda E(f)$, hence given any finite energy initial datum one can scale it down to an arbitrarily small amplitude and energy. Note, however, that for compactly supported initial data such a rescaling spreads the support by a factor $1/\lambda$ and for this reason it cannot make large data smaller in the sense of our perturbation expansion. This follows immediately from the fact that all iterates $f_k(t,r)$ in (5) scale in the same way. In other words, the rescaling does not change the convergence properties of the perturbation expansion.

3. Schwarzschild background

On the exterior Schwarzschild spacetime

$$\mathrm{d}s^2 = -\left(1-\frac{2m}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \qquad r > 2m, \tag{24}$$

the spherically symmetric Yang–Mills equation corresponding to the ansatz (1) takes the form

$$\left(1-\frac{2m}{r}\right)^{-1}\ddot{w} - \left(\left(1-\frac{2m}{r}\right)w'\right)' - \frac{1}{r^2}w(1-w^2) = 0. \tag{25}$$

When $m = 0$ this equation reduces of course to (2). In terms of the new variables

$$x = r + 2m\ln\left(\frac{r}{2m}-1\right), \qquad h(t,x) = w(t, r(x)) - 1, \tag{26}$$

equation (25) becomes

$$\ddot{h} - \frac{\mathrm{d}^2h}{\mathrm{d}x^2} + \left(1-\frac{2m}{r}\right)\frac{2}{r^2}h = -\left(1-\frac{2m}{r}\right)\frac{1}{r^2}(3h^2+h^3), \tag{27}$$

where $r = r(x)$. Dropping the nonlinear terms on the right-hand side of (27) one gets the linear (1+1)-dimensional wave equation on the real axis $-\infty < x < \infty$ with the effective potential $V(x) = 2/r^2 - 4m/r^3$. This equation describes the propagation of the dipole $(l = 1)$ electromagnetic perturbation of the Schwarzschild black hole. For *intermediate* times the linearized approximation is very good; this stage of evolution has the form of exponentially damped oscillations dominated by the fundamental (i.e., least damped) quasinormal mode. We recall that quasinormal modes are solutions of the linearized equation satisfying the outgoing wave conditions $h(t,x) \sim \mathrm{e}^{-\mathrm{i}k(t\mp x)}$ for $x \to \pm\infty$. In the case at hand the fundamental quasi-normal mode has the eigenvalue $k = 0.496\,53 - 0.184\,98i$ (in units where $2m = 1$) [12].

The quasi-normal mode decays exponentially, so for late times it becomes negligible and eventually a polynomial tail is uncovered. Since the pioneering work of Price [13] it has been

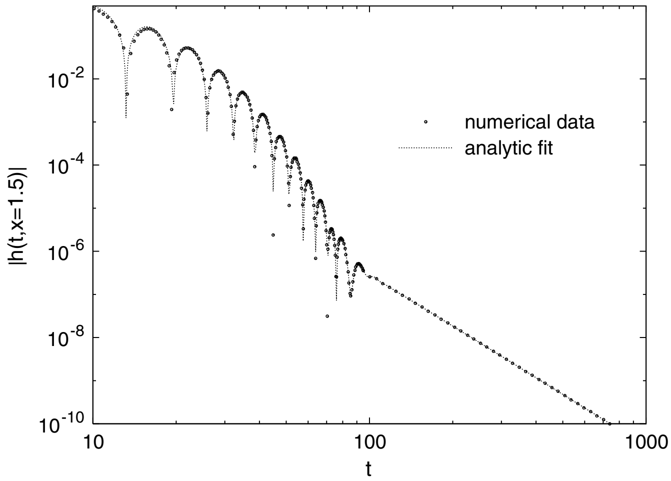

Figure 3. Scattering of the Yang–Mills wave off the Schwarzschild black hole (with $2m = 1$). We plot (on a log–log scale) the numerical solution $h(t, x = 1.5)$ of equation (27) for the initial data of the form of the 'ingoing' Gaussian $h(0, x) = A\exp(-(x - x_0)^2/s^2)$, with $A = 0.85, x_0 = 3, s = 1.5$. Fitting the exponentially damped sinusoid $Q(t) = Be^{-\Gamma t}\sin(\Omega t + \delta)$ on the interval $30 < t < 60$ we get $\Gamma = 0.184$ and $\Omega = 0.495$, in perfect agreement with the known parameters of the fundamental quasinormal mode. The fit of the power law decay $P(t) = Ct^{-\gamma}\exp(D/t + E/t^2)$ for times $t > 300$ gives $\gamma = 3.9997$. The sum $|Q(t) + P(t)|$ (depicted by the dashed line) provides a remarkably good approximation of the full solution for *all* $t \gtrsim 20$. It should be pointed out, however, that our initial data were tuned a bit to maximize the effect of the nonlinearity. If the subdominant t^{-5} tail coming from the potential has a large coefficient, i.e. the tail behaves as $Ct^{-4} + \tilde{C}t^{-5}$ with $C \ll \tilde{C}$, then one has to wait for a long time before the true asymptotic behaviour sets in (which might be misleading without an analytic insight).

known that the tail of the lth multipole decays as t^{-2l-3}, thus for the dipole the linearized theory predicts the tail t^{-5} and this is exactly the result derived in [9]. We claim that this prediction is incorrect and the actual tail behaves in the same manner as in Minkowski spacetime, namely it decays as t^{-4}. Regarding equation (25) as the perturbation (for $r \gg 2m$) of equation (2), one can see from dimensional considerations that the failure of linearization is due to the fact that the linear terms in (25) corresponding to nonzero curvature (proportional to m) are of *shorter range* (using PDE jargon) than the nonlinear terms. Thus, the presence of the black hole should not alter the flat space tail t^{-4}. The numerical substantiation of this handwaving argument is shown in figure 3.

Unfortunately, for the Schwarzschild background we were not able to derive a quantitative formula, analogous to (22), relating the amplitude of the tail to initial data. An attempt to repeat the perturbation analysis from section 2 encounters serious difficulties on Schwarzschild background which are caused by the violation of Huygens' principle in 1 + 1 dimensions and the presence of the potential. It would be interesting to pursue this problem further, perhaps borrowing ideas from an approach proposed some time ago by Barack [14]. Although Barack considered only the linear wave equation, we wish to emphasize that there are many similarities between his work and our analysis in section 2.

4. Conclusions

Using third-order nonlinear perturbation theory, we determined the late-time tail of spherically symmetric Yang–Mills equations on Minkowski background. We also gave heuristic arguments that the same tail is present on Schwarzschild background. In both cases we provided numerical evidence supporting our results. We hope that our approach will trigger more rigorous mathematical analyses of these physically important phenomena.

We remark that the ideas presented here can be applied to other nonlinear wave equations. For example, one can show by similar methods that for the semilinear wave equation $g^{\mu\nu}\nabla_\mu\nabla_\nu\phi+|\phi|^p=0$ on Minkowski background the tail decays as t^{1-p} for $p>1+\sqrt{2}$, while on the Schwarzschild background the tail changes its character at $p=4$ from linear (Price's law $\phi\sim t^{-3}$ for $p\geqslant 4$ [15]) to nonlinear ($\phi\sim t^{1-p}$ for $1+\sqrt{2}<p<4$). A systematic analysis of the competition between linear and nonlinear effects in scattering for semilinear wave equations in Minkowski spacetime will be given elsewhere [16].

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Appendix

We derive here equation (20). Our starting point is equation (13) in which we relabel coordinates $(u,v)\to(u',v')$ and use the retarded time $u=t-r$:

$$f_2(u,r)=-\frac{3}{4r}\int_{|u|}^{u+2r}\mathrm{d}v'\int_{-v'}^{u}\frac{u'+v'-2u+(uu'+uv'-u'v'-u^2)/r}{v'-u'}f_1^2(u',v')\,\mathrm{d}u'. \tag{A.1}$$

We let $\epsilon=1/r$ and define the quantity

$$I(u,\epsilon):=-\frac{3}{2}\int_{|u|}^{u+2/\epsilon}\mathrm{d}v'\int_{-v'}^{u}\frac{u'+v'-2u+\epsilon(uu'+uv'-u'v'-u^2)}{v'-u'}f_1^2(u',v')\,\mathrm{d}u'. \tag{A.2}$$

Expanding this in Taylor's series $I(u,\epsilon)=A(u)+B(u)\epsilon+\mathcal{O}(\epsilon^2)$, we obtain

$$A(u):=I(u,0)=-\frac{3}{2}\int_{|u|}^{\infty}\mathrm{d}v'\int_{-v'}^{u}\frac{u'+v'-2u}{v'-u'}f_1^2(u',v')\,\mathrm{d}u', \tag{A.3}$$

$$B(u):=\left.\frac{\partial I(u,\epsilon)}{\partial\epsilon}\right|_{\epsilon=0}:=g(u)+h(u), \tag{A.4}$$

where

$$h(u)=-\frac{3}{2}\int_{|u|}^{\infty}\mathrm{d}v'\int_{-v'}^{u}\frac{uu'+uv'-u'v'-u^2}{v'-u'}f_1^2(u',v')\,\mathrm{d}u' \tag{A.5}$$

$$g(u)=3\int_{-\infty}^{u}a'(u')^2\,\mathrm{d}u'. \tag{A.6}$$

An elementary calculus exercise shows that

$$h'(u)=A(u). \tag{A.7}$$

Putting all the above equations together and noting that $r=(v-u)/2$, we get equation (20).

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