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The Four-Loop Dressing Phase of $\mathcal{N} = 4$ SYM

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Abstract

We compute the dilatation generator in the $\mathfrak{su}(2)$ sector of planar $\mathcal{N} = 4$ super Yang-Mills theory at four-loops. We use the known world-sheet scattering matrix to constrain the structure of the generator. The remaining few coefficients can be computed directly from Feynman diagrams. This allows us to confirm previous conjectures for the leading contribution to the dressing phase which is proportional to $\zeta(3)$.

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1 Introduction and Overview

The means available for analyzing the AdS/CFT correspondence improved dramatically with the discovery of perturbative integrability of the gauge theory dilatation operator [1–3] and that of classical integrability of the world sheet sigma model [4]. Furthermore, there are arguments [5] on the string theory side of the correspondence that a infinite family of BRST invariant, non-local currents exists at all orders in the inverse 't Hooft coupling expansion suggesting that integrability persists in the quantum theory. In the absence of a definitive and constructive proof of all-order integrability, one may nonetheless assume it and study its consequences.

The fundamental quantity in an integrable (discrete or continuous) theory defined on infinitely extended space-like slices is the scattering matrix of excitations. The S-matrix is constrained by the symmetries of the theory; integrability further requires that no particle production occurs in the scattering process and that the $n \to n$ scattering process is realized by repeated $2 \to 2$ scattering events. A necessary requirement is that the two-particle S-matrix obeys the Yang-Baxter equation.

For the AdS/CFT correspondence the relevant two-particle scattering matrix was introduced in [6]; it turns out that the global symmetries – a centrally extended form of $\mathfrak{psu}(2|2)^2$ determine it up to an overall phase [7]. The Yang-Baxter equation holds automatically in this case. Although initially the S-matrix was determined in the gauge theory framework it was later shown that the tensor structure agreed with perturbative calculations in the gauge-fixed world-sheet theory [8] and that it is consistent with the Zamolodchikov-Faddeev algebra for the string sigma-model [9].

In relativistic quantum field theories the analogous "dressing factor" is determined by crossing symmetry, information on the spectrum of bound states and perhaps perturbative calculations. For the AdS/CFT correspondence both the world sheet and gauge theory integrable systems do not exhibit Lorentz invariance. While on the gauge theory side there is little reason to require an analog of crossing symmetry, on the string theory side two-dimensional Lorentz invariance is only spontaneously broken. As such, one may expect that some form of crossing symmetry survives this breaking.

A crossing-like equation was constructed in [10] and shown in [11] to hold for the known leading [12] and next-to-leading terms [13]. An all-orders solution at a strong coupling expansion was proposed in [14].

An unfortunate feature of this solution is that it is an asymptotic series and thus, without additional information, cannot be directly used to define the dressing phase everywhere in the coupling constant space. In [15] an analytic continuation scheme was described which allowed a guess for the weak-coupling expansion of the dressing phase whose contribution to anomalous dimensions starts at four-loop order where it predicts a transcendental contribution proportional to $\zeta(3)$. This prediction remarkably agrees with the direct calculation of the four-loop cusp anomalous dimension [16, 17]. Subsequently the expansions at weak and strong coupling were shown to be fully consistent [18] and an integral expression for the phase at finite coupling was proposed in [19]

In fact the above agreement is slightly surprising: The four-gluon scattering amplitude of [16] is related to the infinite-spin limit of twist-two anomalous dimensions. Conversely, the analysis of [15] strictly applies to local operators of twist three or higher. Due to the asymptotic nature of the higher-loop Bethe equations the twist-two anomalous dimension can only be predicted reliably up to three loops, see [20] for further recent developments. The agreement thus implies that the cusp anomalous dimension is universal for operators of all twists. In other words, the limiting procedure described in [21, 15] does not suffer from potential order-of-limits ambiguities.

As remarkable as it is, this agreement also presents a puzzle: The universality of the dressing phase implies that all anomalous dimensions of $\mathcal{N} = 4$ SYM have, at four-loop order, a transcendental contribution proportional to $\zeta(3)$. While this is not at all surprising for non-compact subsectors of $\mathcal{N} = 4$ SYM in the large spin limit, it does seem surprising for finite spins and for compact sectors. Indeed, in the infinite spin limit the RG flow mixes an infinite number of operators allowing transcendental numbers to appear even if they are absent at the level of the anomalous dimension matrix. In the latter cases however, the RG flow mixes only finitely many operators and thus precludes the appearance of transcendental numbers. Consequently, for the conjectured dressing phase to be correct, $\zeta(3)$ must appear at the level of the anomalous dimension matrix elements.

Loop integrals may be interpreted – in a first quantized language – as a sum over infinitely many intermediate states producing an analogy with the large-spin $\mathfrak{sl}(2)$ sector operator mixing. From this standpoint, one is entitled to expect the appearance of transcendental numbers at some sufficiently high loop order in any sector. One of the building blocks of the calculation of the renormalization factors of scalar composite operators is the one-loop scalar bubble diagram. It turns out that, in dimensional regularization, its ϵ-expansion contains $\zeta(3)$ at $\mathcal{O}(\epsilon^2)$; consequently, if this bubble is part of a larger diagram and the other momentum integrals yield a third-order pole in the ϵ expansion, $\zeta(3)$ may appear in the residue of a first order pole and thus may contribute to some entry of the anomalous dimension matrix. Counting the required number of inverse powers of the dimensional regulator we immediately reach the conclusion that this mechanism may function first at the four-loop order.

In this paper we shall compute the four-loop dilatation operator in the $\mathfrak{su}(2)$ sector and show that the expectations outlined above are indeed realized. We shall begin in §2 with a review of the constraints imposed by $\mathfrak{su}(2)$ symmetry and Feynman diagrammatics. The unknown coefficients are parametrized in terms of the first nontrivial coefficient of the dressing phase. However, unlike earlier discussions [22, 23] we shall not assume that this operator is part of an integrable Hamiltonian. Instead, we shall determine in §3 the unknown coefficients – and in particular the coefficient related to the dressing phase – by a direct calculation. The calculation is dramatically simplified by the observation that the unknown coefficients may be associated to so-called maximal interactions (i.e. interactions that reshuffle the spins in a maximal way). §4 contains our conclusions. Some technical details as well as some momentum integrals useful for going beyond four loops are included in the appendices.

2 Long-Range Heisenberg Hamiltonian

A full-fledged field theory calculation of the complete four-loop planar dilatation generator in $\mathcal{N}=4$ SYM is a difficult task whose completion clearly requires new, deep insight in higher-loop technology. The main complications are the extensive combinatorics and the intricate algebra of loop momenta inherent to gauge theories at higher perturbative orders. However, our primary goal is to compute the relevant coefficient for the dressing at this order. The dressing factor can be observed in all closed sectors of the model and we can conveniently restrict to the simplest one, the $\mathfrak{su}(2)$ subsector, cf. [24]. It consists of local operators which are made from just two complex scalars, let us denote them by $\mathcal{Z}$ and ϕ, or, equivalently, *spin up* and *spin down*. Here the planar dilatation operator turns into the Heisenberg $\mathrm{XXX}_{1/2}$ Hamiltonian [1] with perturbative long-ranged deformations [3]

$$\mathcal{H}=\sum_{\ell=0}^{\infty}\left(\frac{\lambda}{16\pi^2}\right)^{\ell}\mathcal{H}_{\ell}. \tag{2.1}$$

Determining this Hamiltonian at the fourth perturbative order would provide us with the leading piece of the dressing phase.

The first few perturbative deformations of the Hamiltonian were obtained in [3]: This construction made use of the fact that the Hamiltonian is some linear combination of all interactions compatible with $\mathfrak{su}(2)$ symmetry which can originate from Feynman diagrams. The coefficients of the interactions could in principle be computed from perturbative field theory. However, such an elaborate calculation was avoided by matching the coefficients to make the spectrum of the Hamiltonian agree with some available data. Together with the further assumption of integrability, a proposal for the Hamiltonian at the third perturbative order could be made. The conjecture has since passed various tests [25, 21] which prove that it is correct.

Here we shall repeat the above procedure to constrain the fourth-order Hamiltonian as much as possible without making any unproven assumptions. The crucial new input that allows us to go to higher orders is the picture of asymptotic excitation states [6] and its scattering matrix [7]. In this picture, spin chain states are replaced by excitations above a ferromagnetic vacuum, the magnons. The ferromagnetic vacuum consists of a long chain of aligned spins, say $\mathcal{Z}$

$$|0\rangle = |\ldots \mathcal{Z}\mathcal{Z}\mathcal{Z} \ldots\rangle. \tag{2.2}$$

This state is protected by a half-BPS condition from receiving quantum corrections to its energy; the complete cancellation of corrections to two-point functions in field theory at two loops is demonstrated explicitly in [26]. A single-magnon state has one of these spins flipped to ϕ, say at position k

$$|k\rangle = |\ldots \mathcal{Z}\overset{\overset{k}{\downarrow}}{\phi}\mathcal{Z} \ldots\rangle. \tag{2.3}$$

Similarly, one can construct states with two or more magnons

$$|k, \ell, \ldots\rangle = |\ldots \mathcal{Z}\overset{\overset{k}{\downarrow}}{\phi}\mathcal{Z} \ldots \mathcal{Z}\overset{\overset{\ell}{\downarrow}}{\phi}\mathcal{Z} \ldots \mathcal{Z}\overset{\overset{\ldots}{\downarrow}}{\phi}\mathcal{Z} \ldots\ldots\ldots\rangle. \tag{2.4}$$

In the asymptotic coordinate space Bethe ansatz [6] the magnons are arranged into momentum eigenstates with an additional phase shift when two magnons move past each other.

The excitation picture is highly constrained by its residual symmetry. It was shown in [7] that the form of the one and two-magnon states is almost completely determined. The only degrees of freedom are a finite redefinition of the coupling constant and the dressing phase. The possibility to redefine coupling constants by a finite amount is inherent to field theories. We can make a suitable choice and all other choices can be recovered from it by substitution. A general analysis [23] shows that the dressing phase starts to contribute at four loops with a single undetermined coefficient $\beta_{2,3}$.

Note that these results are actually not based on the (unproven) assumption of higher-loop integrability: integrability or factorized scattering constrains the scattering of three or more particles. It also implies a constraint on the two-particle scattering matrix which, however, in this case is automatically satisfied [7].

We can now match the coefficients of the Hamiltonian to the zero-, one- and two-particle states. The analysis proceeds along the lines of [23] and the most general result is shown in Tab. 1. The interaction symbols $\{a, b, c, \ldots\}$ represent a sequence of nearest-neighbor interactions $\mathcal{P}_p$ of spins at sites p and $p+1$ summed homogeneously over the spin chain of length L

$$\{a, b, c, \ldots\} = \sum_{p=1}^{L} \mathcal{P}_{p+a}\mathcal{P}_{p+b}\mathcal{P}_{p+c} \cdots . \tag{2.5}$$

As undetermined parameters it contains the coefficient $\beta_{2,3}$ for the dressing phase as well as several irrelevant parameters ϵ. The latter correspond to similarity transformations of the Hamiltonian which do not affect its spectrum. One may change their values by applying the similarity transformation $\mathcal{H} \mapsto \exp(-i\mathcal{X})\mathcal{H}\exp(+i\mathcal{X})$ with the second and

$$
\begin{aligned}
\mathcal{H}_0 &= +\{\} \\
\mathcal{H}_1 &= +2\{\} - 2\{1\} \\
\mathcal{H}_2 &= -8\{\} + 12\{1\} - 2\big(\{1,2\} + \{2,1\}\big) \\
\mathcal{H}_3 &= +60\{\} - 104\{1\} + 4\{1,3\} + 24\big(\{1,2\} + \{2,1\}\big) \\
&\quad - 4i\epsilon_2\{1,3,2\} + 4i\epsilon_2\{2,1,3\} - 4\big(\{1,2,3\} + \{3,2,1\}\big) \\
\mathcal{H}_4 &= +\big(-560 - 4\beta_{2,3}\big)\{\} \\
&\quad + \big(+1072 + 12\beta_{2,3} + 8\epsilon_{3\mathrm{a}}\big)\{1\} \\
&\quad + \big(-84 - 6\beta_{2,3} - 4\epsilon_{3\mathrm{a}}\big)\{1,3\} \\
&\quad - 4\{1,4\} \\
&\quad + \big(-302 - 4\beta_{2,3} - 8\epsilon_{3\mathrm{a}}\big)\big(\{1,2\} + \{2,1\}\big) \\
&\quad + \big(+4\beta_{2,3} + 4\epsilon_{3\mathrm{a}} + 2i\epsilon_{3\mathrm{c}} - 4i\epsilon_{3\mathrm{d}}\big)\{1,3,2\} \\
&\quad + \big(+4\beta_{2,3} + 4\epsilon_{3\mathrm{a}} - 2i\epsilon_{3\mathrm{c}} + 4i\epsilon_{3\mathrm{d}}\big)\{2,1,3\} \\
&\quad + \big(4 - 2i\epsilon_{3\mathrm{c}}\big)\big(\{1,2,4\} + \{1,4,3\}\big) \\
&\quad + \big(4 + 2i\epsilon_{3\mathrm{c}}\big)\big(\{1,3,4\} + \{2,1,4\}\big) \\
&\quad + \big(+96 + 4\epsilon_{3\mathrm{a}}\big)\big(\{1,2,3\} + \{3,2,1\}\big) \\
&\quad + \big(-12 - 2\beta_{2,3} - 4\epsilon_{3\mathrm{a}}\big)\{2,1,3,2\} \\
&\quad + \big(+18 + 4\epsilon_{3\mathrm{a}}\big)\big(\{1,3,2,4\} + \{2,1,4,3\}\big) \\
&\quad + \big(-8 - 2\epsilon_{3\mathrm{a}} - 2i\epsilon_{3\mathrm{b}}\big)\big(\{1,2,4,3\} + \{1,4,3,2\}\big) \\
&\quad + \big(-8 - 2\epsilon_{3\mathrm{a}} + 2i\epsilon_{3\mathrm{b}}\big)\big(\{2,1,3,4\} + \{3,2,1,4\}\big) \\
&\quad - 10\big(\{1,2,3,4\} + \{4,3,2,1\}\big)
\end{aligned}
$$

Table 1: The four-loop Hamiltonian. The coefficient $\beta_{2,3}$ is the leading coefficient of the dressing phase at weak coupling. We confirm the prediction $\beta_{2,3} = 4\zeta(3)$ [15] as the principal result of this paper. The coefficients ϵ correspond to similarity transformations and do not influence the spectrum.

third-order contributions to $\mathcal{X}$ given by

$$\begin{aligned}
\mathcal{X}_2 &= \delta\epsilon_2\big(\{1,2\}+\{2,1\}\big),\\
\mathcal{X}_3 &= i\delta\epsilon_{3\mathrm{a}}\big(\{2,1,3\}-\{1,3,2\}\big)+\delta\epsilon_{3\mathrm{b}}\big(\{1,2,3\}+\{3,2,1\}\big)\\
&\quad+\delta\epsilon_{3\mathrm{c}}\{1,3\}+\delta\epsilon_{3\mathrm{d}}\big(\{1,2\}+\{2,1\}\big).
\end{aligned} \tag{2.6}$$

It is worth pointing out that the Hamiltonian at fourth order can be fixed uniquely up to irrelevant terms. In other words, the scattering of three or more magnons is fixed by the scattering of two magnons. This feature is related to the $\mathfrak{su}(2)$ symmetry of the interactions: Interactions at four loops act on at most five adjacent spins. Any elementary interaction among three or more magnons (and therefore at most two vacuum spins) is related to an interaction among at most two magnons (and at least three vacuum spins) by flipping all five interacting spins. Starting at five loops this picture breaks down because interactions of six spins allow for elementary interactions of three magnons which leave no trace on the sector with two or fewer magnons. It turns out that our four-loop Hamiltonian in Tab. 1 is integrable, i.e. it is of the form determined (but not displayed explicitly) in [23]. We have therefore proved four-loop integrability in the $\mathfrak{su}(2)$ sector.

The four-loop Hamiltonian in Tab. 1 is fixed to a large extent. To determine the dressing phase coefficient $\beta_{2,3}$ it suffices to compute only a small number of its coefficients. We see that $\beta_{2,3}$ couples, among others, to the very first and fifth but last interaction structure in Tab. 1. The first structure does not redistribute the spins along the spin chain. There are exceedingly many planar Feynman diagrams which do not change flavor, for example those containing only interactions of gluons and scalars. Therefore, a direct computation of this coefficient seems particularly difficult. In contrast, the coefficients of the five last interactions can be computed relatively easily. They form a class of interactions which reshuffle the spins in a maximal way. At ℓ loops, they contain ℓ permutations of nearest neighbors, see Fig. 1 for a graphical representation of their induced permutations. This is the maximum reshuffling allowed by planar Feynman diagrams [3] and it will turn out to be *generated by the quartic interactions of the scalars only*. In other words, the relevant Feynman diagrams will be those of a ϕ^4-theory.

Moreover, the individual maximal interactions are identifiable by acting on special states: Assume that the Hamiltonian density maps a state

$$\mathcal{H}_4|\ldots\phi\phi\mathcal{Z}\mathcal{Z}\ldots\rangle = c|\ldots\mathcal{Z}\mathcal{Z}\phi\phi\ldots\rangle+\ldots. \tag{2.7}$$

There is a single interaction which achieves this particular reshuffling of spins: $\{2,1,3,2\}$, cf. Fig. 1. Therefore we could infer that c equals the coefficient of this interaction, $c=-12-2\beta_{2,3}-4\epsilon_{3\mathrm{a}}$. The same is true for the other maximal interactions: If all lines going right are associated with ϕ and the others with $\mathcal{Z}$ then a ϕ will move past a $\mathcal{Z}$ towards the right at each elementary crossing. The effect will thus uniquely identify the corresponding interaction.

Being the representation of the dilatation generator on the gauge invariant operators in the $\mathfrak{su}(2)$ sector, the spin chain Hamiltonian in Tab. 1 is also the anomalous dimension matrix of operators in this sector. In any conformal field theory the eigenvalues of the anomalous dimension matrix are independent of the renormalization scheme. Its matrix

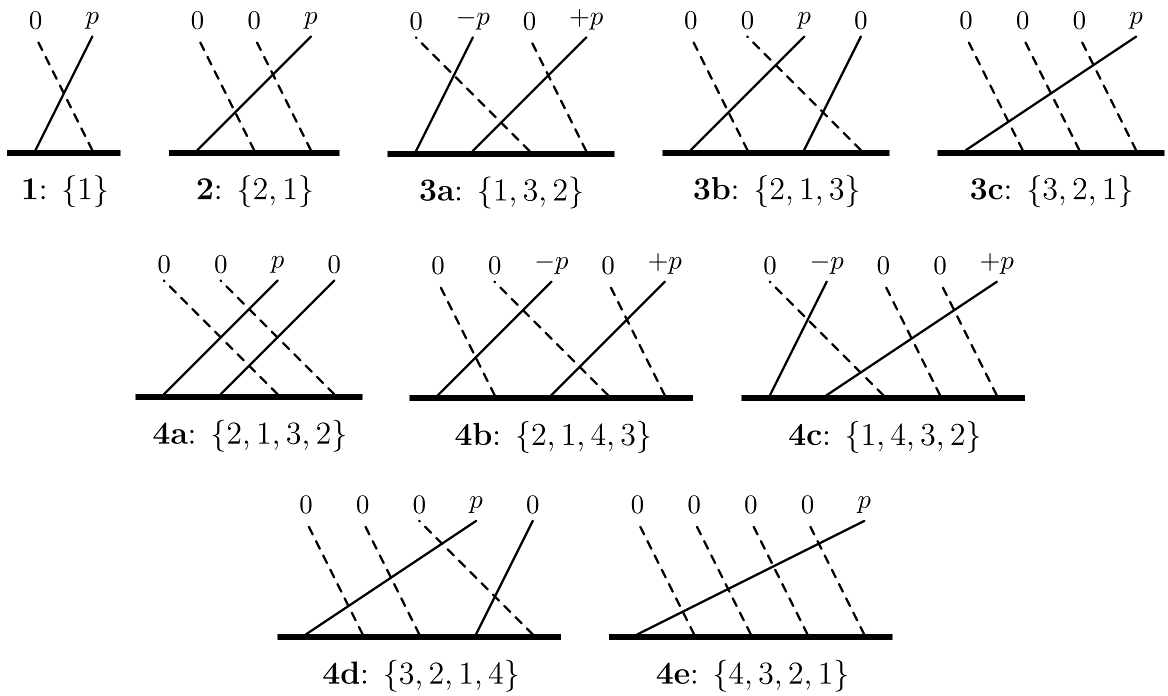


Figure 1: Maximal planar interactions up to four loops. Below the diagrams the permutation symbols are indicated. Solid and dashed lines correspond to two complex scalars in $\mathcal{N} = 4$ SYM. Above the diagrams suitable momenta to remove IR singularities are indicated.

elements however do not generically have this property. At the level of the Hamiltonian in Tab. 1 this is reflected by the fact that the undetermined coefficients ϵ do not affect its eigenvalues [22].

3 Four-Loop Calculation

With these preparations we are now in a position to compute the undetermined coefficients that appear in the spin chain Hamiltonian. Among the various approaches to finding the anomalous dimension matrix we shall consider the renormalization of composite operators. It was successfully used in [27] to determine the two-loop and (under the assumption of proper BMN scaling for one-excitation BMN states) the all-loop dispersion relation.

The renormalization of composite operators and the subtraction of subdivergences proceeds by introducing renormalization factors and counterterm diagrams analogous to the Bogoliubov R-operation. For our purpose this procedure was systematized in [3] where an iterative subtraction scheme was developed that allows the subtraction of entire subdiagrams. This is the scheme we shall use.

We are therefore to compute Feynman diagrams with one vertex being the composite operator of interest and additional vertices dictated by the $\mathcal{N} = 4$ SYM Lagrangian. As described in detail in the previous section, our goal is to find the entries of the four-loop anomalous dimension matrix that reshuffle scalar fields in a maximal way. Besides scalar fields, the internal lines of these diagrams may *a priori* also be fermions and gauge fields.

Two simple observations imply however that the situation is substantially simpler.

By inspection of the diagrams in Fig. 1 it is easy to see that Feynman diagrams containing gauge fields cannot lead to such maximal reshuffling (in the sense described previously) of scalar fields. Indeed, using the fact that the gauge field interactions are flavor-blind, one may see that replacing any of the four-point vertices by scalar-vector interactions leads to diagrams not exhibiting maximal reshuffling.

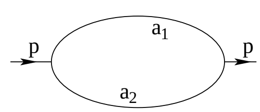
Let us consider next scalar-fermion interactions. R-charge conservation implies that any diagram with external fermion fields and an insertion of an operator in the $\mathfrak{su}(2)$ sector vanishes identically. Diagrams with internal fermion lines have a similar fate. To see this let us note that the Yukawa interactions of the $\mathcal{N}=4$ SYM Lagrangian are proportional to the SO(6) Dirac matrices. Since the fields of the $\mathfrak{su}(2)$ sector are complex, the Dirac matrices appearing in these vertices will also carry complex vector indices. Their algebra, $\{\Gamma^{\bar{a}}, \Gamma^{b}\} = \eta^{\bar{a}b}$, implies that holomorphic matrices square to zero. Therefore the flavor of scalar fields coupling to fermionic loops must alternate. It is then easy to see that for $2n$ flavors at most n pairs can be interchanged. This does not lead to a maximal permutation and we can thus disregard fermion loops.

The conclusion is therefore that the Feynman diagrams contributing to the entries of the four-loop anomalous dimension matrix describing a maximal reshuffling of spins are scalar diagrams in which each vertex contains two types of scalar fields and the interaction interchanges them. These are the diagrams listed in Fig. 1.

To compute the contribution to the anomalous dimension matrix we need to compute the amplitudes in Fig. 1 and isolate the overall ultraviolet divergence by subtracting all their UV subdivergences. While in general it is convenient to use a variant of dimensional regularization which preserves supersymmetry, in the context of our calculation making a definite choice is not an issue since all our diagrams have scalar internal lines. However, since all fields are massless we must be careful to separate the UV divergences from IR divergences. To this end we shall assign off-shell momenta to some of the external fields;[1] they are chosen such that the number of momenta is minimal while all IR divergences are eliminated. It turns out that up to four-loop order it suffices that only two of the external fields carry momentum; an appropriate choice is depicted in Fig. 1.

All momentum integrals may be computed easily by reduction to a small set of master integrals. Common building blocks are bubble diagrams with arbitrary exponents for the two propagators; their expressions are (here and elsewhere the dimensionality of space-time is $d = 4 - 2\epsilon$) [2]

$$L(a_1, a_2) \equiv (p^2)^{a_{12}-d/2} \times$$



$$
\begin{aligned}
&= \int \frac{d^d q}{(2\pi)^d} \, \frac{(p^2)^{a_{12}-d/2}}{(q^2)^{a_1}((q+p)^2)^{a_2}} \\
&= (4\pi)^{-d/2} \, \frac{\Gamma(a_{12}-\frac{d}{2})}{\Gamma(a_1)\Gamma(a_2)} \, \frac{\Gamma(\frac{d}{2}-a_1)\Gamma(\frac{d}{2}-a_2)}{\Gamma(d-a_{12})}
\end{aligned}
\qquad (3.1)
$$

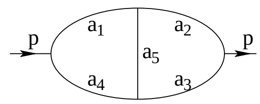
[1] The external fields can either belong to the operator being renormalized or be attached to the vertices of the $\mathcal{N}=4$ SYM Lagrangian.

[2] We extract the overall momentum dependence for the sake of notational convenience.

where $a_{12} = a_1 + a_2$. Indeed, diagrams **1**, **2**, **3b**, **3c**, **4d** and **4e** may be computed exactly by repeated identification of one-loop bubble subdiagrams. Once a bubble subintegral is evaluated, the exponent of the propagator carrying the momentum p flowing through the bubble is shifted by an integer multiple of the dimensional regulator.

A similar iterative identification of bubble subintegrals reduces the diagrams **3a** and **4c** to special cases of the two-loop master bubble integral

$$T(a_1, a_2, a_3, a_4, a_5) \equiv (p^2)^{a_{12345}-d} \times \qquad (3.2)$$



with some arbitrary powers of propagators. For integer exponents $a_{1,\ldots,5}$ such integrals have been computed in the past (e.g. [28]). We are however interested in situations when some of the exponents are not integers (special cases have been previously analyzed in [29]), being dependent on the dimensional regulator. Perhaps the most effective way of computing such integrals is to use the Mellin-Barnes parametrization [28]. The identity

$$\frac{1}{(a+b)^\nu} = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dw\, \frac{\Gamma(-w)\Gamma(w+\nu)}{\Gamma(\nu)}\, \frac{a^w}{b^{\nu+w}} \qquad (3.3)$$

allows a straightforward evaluation of the Feynman parameter integrals and expresses the result of the momentum integral in terms of multiple contour integrals which can be evaluated through the residue theorem. This method has the advantage of producing explicit integral representations for the coefficients of the various powers of the dimensional regulator. The algorithm of [30] for the analytic continuation $\epsilon \to 0$ as well as the numerical evaluation of the resulting coeffcients has been successfully automated [31]. An MB parametrization of $T(a_1, a_2, a_3, a_4, a_5)$ is

$$\begin{aligned} T(a_1, a_2, a_3, a_4, a_5) &= \frac{(4\pi)^{-d}}{\Gamma(a_1)\Gamma(a_4)\Gamma(a_5)\Gamma(d-a_{145})} \qquad (3.4)\\ &\times \int_{-i\infty}^{+i\infty} \frac{dw_1 dw_2}{(2\pi i)^2} \frac{\Gamma(a_{145}-\frac{d}{2}+w_{12})}{\Gamma(a_{1245}-\frac{d}{2}+w_{12})} \Gamma(-w_1)\Gamma(-w_2)\Gamma(a_4+w_{12})\Gamma(d-a_{1245}-w_{12})\\ &\quad \frac{\Gamma(\frac{d}{2}-a_{14}-w_1)\Gamma(a_{12345}-d+w_1)}{\Gamma(\frac{3d}{2}-a_{12345}-w_1)} \frac{\Gamma(\frac{d}{2}-a_{45}-w_2)\Gamma(\frac{d}{2}-a_3+w_2)}{\Gamma(a_3-w_2)} \end{aligned}$$

with the notation $a_{ijk\ldots} = a_i + a_j + a_k + \ldots$ and similarly for $w_{ijk\ldots}$. While this parametrization does not manifestly exhibit the symmetries of the original diagram, they are restored after the remaining integrals are performed. It is possible (though perhaps less efficient in terms of the necessary number of MB parameters) to construct a Mellin-Barnes parametrization manifestly exhibiting the $(\mathbb{Z}_2)^2$ symmetries of (3.2). The ϵ-expansions of the two-loop integrals we shall require read:

$$\begin{aligned} T(1,1,1,1,\epsilon) &= L(1,1)^2 \left(\tfrac{1}{3} + \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon^2 + \left(-\tfrac{7}{3} + \tfrac{14}{3}\zeta(3)\right)\epsilon^3 + \ldots\right),\\ T(1,1,1,\epsilon,1) &= L(1,1)^2 \left(\tfrac{1}{6} + \tfrac{1}{2}\epsilon + \tfrac{13}{6}\epsilon^2 + \left(+\tfrac{55}{6} - \tfrac{23}{3}\zeta(3)\right)\epsilon^3 + \ldots\right),\\ T(1,1,1,1,2\epsilon) &= L(1,1)^2 \left(\tfrac{1}{6} + \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon^2 + \left(-\tfrac{17}{3} + \tfrac{31}{3}\zeta(3)\right)\epsilon^3 + \ldots\right), \end{aligned}$$

$$T(1,1,1,1+\epsilon,\epsilon) = L(1,1)^2\Big(\tfrac{5}{24}+\tfrac{5}{12}\epsilon+\tfrac{25}{24}\epsilon^2+\big(+\tfrac{5}{12}+\tfrac{19}{6}\zeta(3)\big)\epsilon^3+\ldots\Big),$$
$$T(1,1,1,2\epsilon,1) = L(1,1)^2\Big(\tfrac{1}{12}+\tfrac{5}{12}\epsilon+\tfrac{29}{12}\epsilon^2+\big(+\tfrac{161}{12}-\tfrac{71}{6}\zeta(3)\big)\epsilon^3+\ldots\Big). \quad (3.5)$$

Here it was convenient to factor out two powers of the one-loop bubble $L(1,1)$ which has the expansion

$$\begin{aligned} L(1,1) &= (4\pi)^\epsilon \frac{\Gamma(\epsilon)\Gamma^2(1-\epsilon)}{16\pi^2\Gamma(2-2\epsilon)} \\ &= \frac{1}{16\pi^2\epsilon}\left(4\pi e^{-\gamma}\right)^\epsilon\Big(1+2\epsilon+\big(4-\tfrac{1}{12}\pi^2\big)\epsilon^2+\big(8-\tfrac{1}{6}\pi^2-\tfrac{7}{3}\zeta(3)\big)\epsilon^3+\ldots\Big). \end{aligned} \quad (3.6)$$

In both of the last two integrals **4a** and **4b** it is trivial to isolate a factor $L(1,1)$. The remaining three-loop integrals may be computed in several ways. One approach makes use of integration by parts identities, known in this case as the triangle rule, to reduce them to combinations of one- and two-loop bubble integrals with various exponents (see App. §A for details). A second approach directly evaluates the three-loop integrals and in the process tests that the infrared region is non-singular. We list the necessary Mellin-Barnes integrals and the ϵ-expansions of all diagrams in App. §B. Needless to say, the two calculations lead to the same answer.

For a vector of operators $\mathbf{O}$, the relation between the bare and renormalized operators is given by the renormalization factor $\mathbf{Z}$

$$\mathbf{O}^{\mathrm{bare}} = \mathbf{Z}\cdot\mathbf{O}^{\mathrm{ren}}\ . \quad (3.7)$$

The ℓ-loop contribution to $\mathbf{Z}$ is found from the overall divergence of ℓ-loop diagrams with exactly one insertion of a member of the vector $\mathbf{O}$. To isolate the overall divergence it is necessary to include counterterm diagrams which are generated recursively by the lower-loop renormalization factor. These diagrams also eliminate the non-local momentum dependence. The relation between the renormalization factor and the anomalous dimension matrix (a.k.a. dilatation generator or spin chain Hamiltonian) $\mathcal{H}$ is standard:

$$\delta\mathcal{H} = \lim_{\epsilon\to 0}\epsilon\,\mathbf{Z}^{-1}\frac{d}{d\ln g_{\mathrm{YM}}}\mathbf{Z}\ . \quad (3.8)$$

This expression implies an exponential-like structure for the renormalization factor $\mathbf{Z}$;[3] in particular, the derivative of $\mathbf{Z}$ must be left-proportional to $\mathbf{Z}$ and, in order that the $\delta\mathcal{H}$ be well defined in the $\epsilon\to 0$ limit, the factor of proportionality can only have additional simple poles.[4]

A subtraction scheme that enforces these constraints and at each loop order isolates directly the contribution to the anomalous dimension matrix was described in [3] for the use in two-point functions. An adapted version for the use in operator renormalization diagrams is presented in App. §C where the explicit rules are given and then applied to the relevant diagrams.

[3]For operators which do not mix under RG flow the relation is $Z = \exp\big[\epsilon^{-1}\int_0^1 dt\, t^{-1}\gamma(tg_{\mathrm{YM}})\big]$.

[4]It is in principle possible that in a different renormalization scheme individual matrix elements could have divergent terms; however these terms should be removable by similarity transformations.

Finally, in this scheme the equation (3.8) reduces to the simple operation of picking the residue of the $1/\epsilon$ pole of the subtracted diagrams

$$\tilde{I} = 2(16\pi^2)^\ell \lim_{\epsilon\to 0} \epsilon\, \bar{I}(\epsilon)\ . \tag{3.9}$$

The factor of $(16\pi^2)^\ell$ corresponds to the normalization of the ℓ-loop Hamiltonian $\mathcal{H}_\ell$ and allows for a direct comparison of the quantity $\tilde{I}$ to the coefficients in (2.1). In our case this leads to

$$\begin{aligned}
\tilde{I}_1 &= -2, & \tilde{I}_{4\mathrm{a}} &= -\ \ 4 + \ \ 4\,\zeta(3),\\
\tilde{I}_2 &= -2, & \tilde{I}_{4\mathrm{b}} &= +10 - 12\,\zeta(3),\\
\tilde{I}_{3\mathrm{a}} &= +4, & \tilde{I}_{4\mathrm{c}} &= +\ \ 2 + \ \ 8\,\zeta(3),\\
\tilde{I}_{3\mathrm{b}} &= -4, & \tilde{I}_{4\mathrm{d}} &= -10 + \ \ 4\,\zeta(3),\\
\tilde{I}_{3\mathrm{c}} &= -4, & \tilde{I}_{4\mathrm{e}} &= -10,
\end{aligned} \tag{3.10}$$

which represent the coefficients relating the structures listed in Fig. 1 and the spin chain Hamiltonian (cf. Tab. 1). Clearly, $\tilde{I}_1$, $\tilde{I}_2$, $\tilde{I}_{3\mathrm{c}}$ and $\tilde{I}_{4\mathrm{e}}$ reproduce the coefficients of $\{1\}$, $\{2,1\}$, $\{3,2,1\}$ and $\{4,3,2,1\}$, respectively. The coefficients undetermined by symmetry considerations are fixed by our calculation to be[5]

$$i\epsilon_2 = -1, \qquad \epsilon_{3a} = -2 - 3\zeta(3), \qquad i\epsilon_{3b} = -3 - \zeta(3) \tag{3.11}$$

and

$$\beta_{2,3} = 4\zeta(3). \tag{3.12}$$

In particular we are able to uniquely fix the leading coefficient $\beta_{2,3}$ for the dressing phase. It is in full agreement with the results of [16, 15, 17].

4 Conclusions and Outlook

In this paper we have computed the four-loop dilatation operator in the $\mathfrak{su}(2)$ sector of $\mathcal{N} = 4$ SYM. The main observation which led to substantial technical simplifications is that the coefficients undetermined by symmetry constraints can be chosen to correspond to "maximal interactions" – i.e. interactions that reshuffle the spins in a maximal way. For appropriately chosen gauge theory operators these interactions are entirely determined by Feynman diagrams with only scalar interactions. We found that, starting at four-loop order, the anomalous dimensions of long operators become transcendental; this may be traced to the dilatation operator acquiring transcendental coefficients. We have extracted the relevant coefficient of the dressing phase and found it identical to the one reproducing the four-loop cusp anomalous dimension computed in [16, 17]. Our result confirms the particular analytic continuation used to guess the dressing phase at weak coupling [15].

[5] The conventional factors of i indicate that the Hamiltonian is not manifestly hermitian. With a proper choice of scalar product, however, it becomes quasi-hermitian as it should.

The main obstacle for computing higher-loop anomalous dimensions in any sector of $\mathcal{N}=4$ SYM and thus directly computing the S-matrix dressing phase is, as in all off-shell calculations, the proliferation of Feynman diagrams. In compact sectors the symmetries of the theory restrict (sometimes substantially) the structure of the anomalous dimension matrix. At any loop order the maximal interactions enjoy the same technical simplifications as the ones employed in the calculations described here; moreover, it is possible that some of the relevant momentum integrals exhibit a recursive structure.[6] It would be interesting to identify and compute them, thus providing a direct evaluation of important parts of the dressing phase.

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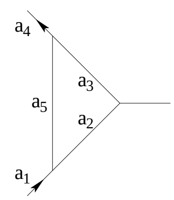
A The Triangle Rule

Consider the Feynman integral

$$F(a_1,a_2,a_3,a_4,a_5) \equiv$$



. (A.1)

This integral may be part of a larger Feynman diagram and the labels $a_{1,\ldots,5}$ represent the exponents of the propagators of the corresponding internal lines. Inserting the operator $l^\mu \partial/\partial l^\mu$ in the integral representing this diagram and equating the results of the action of the derivative on the original integrand and the result of the integration by parts leads to

$$(d-a_2-a_3-2a_5)F = \left(a_2\,\mathbf{2}^+\left(\mathbf{5}^- - \mathbf{1}^-\right) + a_3\,\mathbf{3}^+\left(\mathbf{5}^- - \mathbf{4}^-\right)\right)F \tag{A.2}$$

where, for example, $\mathbf{1}^\pm F(a_1,a_2,a_3,a_4,a_5) = F(a_1\pm 1,a_2,a_3,a_4,a_5)$.

The various terms in such a decomposition may acquire however spurious infrared divergences which are regularized by the dimensional regulator and – provided that the IR of the original integral was properly regularized – cancel when all terms are assembled.

The triangle rule together with the straightforward evaluation of bubble integrals

[6] A candidate for this property is $\{m,\ldots,1,m+1,\ldots,2,\ldots,n+m-1,\ldots,n-1,n+m\ldots n\}$.

leads to the following expressions for the diagrams in Fig. 1:[7]

$$
\begin{aligned}
I_1 &= (p^2)^{-\epsilon}\, L(1,1),\\
I_2 &= (p^2)^{-2\epsilon}\, L(1,1)\, L(1+\epsilon,1),\\
I_{3\mathrm{a}} &= (p^2)^{-3\epsilon}\, L(1,1)\, T(1,1,1,1,\epsilon),\\
I_{3\mathrm{b}} &= (p^2)^{-3\epsilon}\, L(1,1)^2 L(1+\epsilon,1+\epsilon),\\
I_{3\mathrm{c}} &= (p^2)^{-3\epsilon}\, L(1,1)\, L(1+\epsilon,1)\, L(1+2\epsilon,1),\\
I_{4\mathrm{a}} &= (p^2)^{-4\epsilon}\, \frac{L(1,1)}{1-3\epsilon}\Big[L(2,\epsilon)\, T(1,1,1,2\epsilon,1) - L(2,3\epsilon)\, T(1,1,1,\epsilon,1)\Big],\\
I_{4\mathrm{b}} &= (p^2)^{-4\epsilon}\, L(1,1)^2 T(1,1,1,1+\epsilon,\epsilon),\\
I_{4\mathrm{c}} &= (p^2)^{-4\epsilon}\, \frac{L(1,1)}{1-3\epsilon}\Big[L(2,\epsilon)\, T(1,1,1,1,2\epsilon) - L(2,1)\, T(1,1,1,1+\epsilon,\epsilon)\\
&\qquad + \epsilon\, L(1,1+\epsilon)\, T(1,1,1,1,2\epsilon) - \epsilon\, L(1,1+\epsilon)\, T(1,1,1,2\epsilon,1)\Big],\\
I_{4\mathrm{d}} &= (p^2)^{-4\epsilon}\, L(1,1)^2 L(1+\epsilon,1)\, L(1+2\epsilon,1+\epsilon),\\
I_{4\mathrm{e}} &= (p^2)^{-4\epsilon}\, L(1,1)\, L(1+\epsilon,1)\, L(1+2\epsilon,1)\, L(1+3\epsilon,1).
\end{aligned}
\tag{A.3}
$$

Note that the integrals $T(1,1,1,\epsilon,1)$ and $T(1,1,1,2\epsilon,1)$ can be evaluated further using the triangle rule

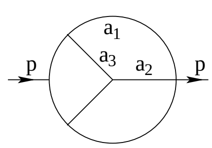
$$
T(1,1,1,\epsilon,1) = \frac{L(1,1)}{1-3\epsilon}\Big(L(2,\epsilon) - L(2,2\epsilon) + \epsilon\, L(1,1+\epsilon) - \epsilon\, L(1+\epsilon,1+\epsilon)\Big), \tag{A.4}
$$
$$
T(1,1,1,2\epsilon,1) = \frac{L(1,1)}{1-4\epsilon}\Big(L(2,2\epsilon) - L(2,3\epsilon) + 2\epsilon\, L(1,1+2\epsilon) - 2\epsilon\, L(1+\epsilon,1+2\epsilon)\Big).
$$

For compactness, we left them unexpanded in (A.3); The integral $I_{4\mathrm{a}}$ can thus be evaluated as an analytic expression in ϵ.

B Three-Loop Integrals

The three-loop integral that remains after one identifies a one-loop bubble subintegral in $I_{4\mathrm{a}}$ may be evaluated directly thus testing the application of the triangle rule and the correct infrared regularization of the contributions to the anomalous dimension matrix. A Mellin-Barnes parametrization of a master integral containing $I_{4\mathrm{a}}$ is

$$
BM(a_1,a_2,a_3) \equiv (p^2)^{a_{123}+4-3d/2} \times
$$



$$
\begin{aligned}
&= (4\pi)^{-3d/2} \int_{-i\infty}^{i\infty} \frac{dw_1 dw_2 dw_3 dw_4}{(2\pi i)^4}\\
&\quad \times \frac{\Gamma(-w_1)\Gamma(-w_2)\Gamma(-w_3)\Gamma(-w_4)\Gamma(a_{123}-\frac{d}{2}+w_{12})}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(d-a_{123})}
\end{aligned}
$$

[7] It is trivial to identify in the expressions of $I_{4\mathrm{a}}$ and $I_{4\mathrm{c}}$ the IR divergent components mentioned above.

$$
\begin{aligned}
&\times \frac{\Gamma(\frac{d}{2}-a_{13}-w_1)\Gamma(\frac{d}{2}-a_{12}-w_2)\Gamma(a_1+w_{12})}{\Gamma(1-w_1)\Gamma(3\frac{d}{2}-3-a_{123}-w_2)\Gamma(1+a_{123}-\frac{d}{2}+w_{12})}\\
&\times \frac{\Gamma(d-2-a_{123}-w_{123})\Gamma(\frac{d}{2}-1+w_3)}{\Gamma(1-w_3)}\\
&\times \frac{\Gamma(d-2-a_{123}-w_{24})\Gamma(\frac{d}{2}+w_{24})}{\Gamma(-w_{24})}\\
&\times \frac{\Gamma(3+a_{123}-d+w_{234})\Gamma(1+a_{123}-\frac{d}{2}+w_{1234})\Gamma(1-\frac{d}{2}-w_{234})}{\Gamma(1-w_3)\Gamma(d-1+w_{234})}\ .
\end{aligned}
\tag{B.1}
$$

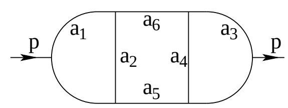
Then,

$$
I_{4\mathrm{a}} = (p^2)^{-4\epsilon}\, L(1,1) BM(1,\epsilon,1) \tag{B.2}
$$

and its evaluation leads to the result listed in equation (B.6).

Similarly, $I_{4\mathrm{c}}$ is a special case of the master integral

$$
BL(a_1,a_2,a_3,a_4,a_5)\ \equiv (p^2)^{a_{12345}+3-3d/2} \times
$$



. (B.3)

A Mellin-Barnes parametrization is

$$
\begin{aligned}
&BL(a_1,a_2,a_3,a_4,a_5) = (4\pi)^{-3d/2} \int_{-i\infty}^{i\infty} \frac{dw_1 dw_2 dw_3 dw_4}{(2\pi i)^4}\\
\times\ & \frac{\Gamma(-w_1)\Gamma(-w_2)\Gamma(1-\frac{d}{2}+a_{12}+w_{12})\Gamma(1+w_{12})}{\Gamma(a_1)\Gamma(a_2)\Gamma(d-1-a_{12})}\\
\times\ & \frac{\Gamma(-w_3)\Gamma(-w_4)\Gamma(1-\frac{d}{2}+a_{34}+w_{34})\Gamma(1+w_{34})}{\Gamma(a_3)\Gamma(a_4)\Gamma(d-1-a_{34})}\\
\times\ & \Gamma(\tfrac{d}{2}-1-a_1-w_1)\Gamma(\tfrac{d}{2}-1-a_2-w_2)\Gamma(\tfrac{d}{2}-1-a_3-w_3)\Gamma(\tfrac{d}{2}-1-a_4-w_4)\\
\times\ & \frac{\Gamma(2+a_{123456}-\frac{3d}{2}+w_{13})}{\Gamma(2d-2-a_{123456}-w_{13})}\frac{\Gamma(\frac{d}{2}-a_5+w_{24})}{\Gamma(a_5-w_{24})}\frac{\Gamma(\frac{3d}{2}-2-a_{12346}-w_{1234})}{\Gamma(2+a_{12346}-d+w_{1234})}\ .
\end{aligned}
\tag{B.4}
$$

Then

$$
I_{4\mathrm{c}} = (p^2)^{-4\epsilon}\, L(1,1) \lim_{\nu\to 0} BL(1,\epsilon,1,1,\nu,1) \tag{B.5}
$$

whose evaluation leads to the result listed in equation (B.6).

The integrals listed here are useful for the calculation of the coefficients of the higher-loop Hamiltonian in the $\mathfrak{su}(2)$ sector. The resulting ϵ-expansions of the integrals in Fig. 1 read

$$
\begin{aligned}
I_1 &= \frac{1}{16\pi^2\epsilon}\left(\frac{4\pi e^{-\gamma}}{p^2}\right)^{\epsilon}\Big(1+2\epsilon+\big(4-\tfrac{1}{12}\pi^2\big)\epsilon^2+\big(8-\tfrac{1}{6}\pi^2-\tfrac{7}{3}\zeta(3)\big)\epsilon^3+\ldots\Big),\\
I_2 &= (I_1)^2\Big(\tfrac{1}{2}+\tfrac{1}{2}\epsilon+\tfrac{3}{2}\epsilon^2+\big(\tfrac{9}{2}-3\zeta(3)\big)\epsilon^3+\ldots\Big),\\
I_{3\mathrm{a}} &= (I_1)^3\Big(\tfrac{1}{3}+\tfrac{1}{3}\epsilon+\tfrac{1}{3}\epsilon^2+\big(-\tfrac{7}{3}+\tfrac{14}{3}\zeta(3)\big)\epsilon^3+\ldots\Big),\\
I_{3\mathrm{b}} &= (I_1)^3\Big(\tfrac{1}{3}+\tfrac{2}{3}\epsilon+\tfrac{8}{3}\epsilon^2+\big(\tfrac{32}{3}-\tfrac{22}{3}\zeta(3)\big)\epsilon^3+\ldots\Big),
\end{aligned}
$$

$$
\begin{aligned}
I_{3\mathrm{c}} &= (I_1)^3\Big(\tfrac{1}{6}+\tfrac{1}{2}\epsilon+\tfrac{13}{6}\epsilon^2+\big(\tfrac{55}{6}-\tfrac{11}{3}\zeta(3)\big)\epsilon^3+\dots\Big),\\
I_{4\mathrm{a}} &= (I_1)^4\Big(\tfrac{1}{12}+\tfrac{1}{3}\epsilon+\tfrac{19}{12}\epsilon^2+\big(\tfrac{43}{6}-\tfrac{10}{3}\zeta(3)\big)\epsilon^3+\dots\Big),\\
I_{4\mathrm{b}} &= (I_1)^4\Big(\tfrac{5}{24}+\tfrac{5}{12}\epsilon+\tfrac{25}{24}\epsilon^2+\big(\tfrac{5}{12}+\tfrac{19}{6}\zeta(3)\big)\epsilon^3+\dots\Big),\\
I_{4\mathrm{c}} &= (I_1)^4\Big(\tfrac{1}{8}+\tfrac{1}{3}\epsilon+\tfrac{9}{8}\epsilon^2+\big(\tfrac{10}{3}-\tfrac{3}{2}\zeta(3)\big)\epsilon^3+\dots\Big),\\
I_{4\mathrm{d}} &= (I_1)^4\Big(\tfrac{1}{8}+\tfrac{1}{2}\epsilon+\tfrac{21}{8}\epsilon^2+\big(\tfrac{27}{2}-\tfrac{13}{2}\zeta(3)\big)\epsilon^3+\dots\Big),\\
I_{4\mathrm{e}} &= (I_1)^4\Big(\tfrac{1}{24}+\tfrac{1}{4}\epsilon+\tfrac{37}{24}\epsilon^2+\big(\tfrac{107}{12}-\tfrac{13}{6}\zeta(3)\big)\epsilon^3+\dots\Big).
\end{aligned}
\tag{B.6}
$$

C Subtraction Scheme

Here we describe the subtraction scheme used to extract the contributions to the anomalous dimensions without having to insert couterterms at each stage of the calculation. For each connected diagram drawn with the composite operator as the lower-most vertex, one

- partitions it in all possible connected subdiagrams (including the trivial partition into a single subdiagram) and interprets those diagrams as contributing to the renormalization of a composite operator,
- discards all partitions which are interconnected horizontally (all partial diagrams must be "dropped" onto the composite operator from above in a well-defined sequence, in similarity to a famous arcade game),
- discards all partitions for which there are two or more top-most diagrams,
- evaluates the momentum integrals for the of the remaining partitions,
- sums the products of the momentum integrals for each partition weighted by $\ell(-1)^n$ (n is the number of partial diagrams and ℓ is the loop number of the top-most diagram in the partition).[8]

Applying this scheme to the diagrams in Fig. 1 we find the following subtracted integrals $\tilde{I}$

$$
\begin{aligned}
\bar{I}_1 &= -I_1,\\
\bar{I}_2 &= -2I_2+I_1^2,\\
\bar{I}_{3\mathrm{a}} &= -3I_{3\mathrm{a}}+2I_2I_1,\\
\bar{I}_{3\mathrm{b}} &= -3I_{3\mathrm{b}}+4I_2I_1-I_1^3,\\
\bar{I}_{3\mathrm{c}} &= -3I_{3\mathrm{c}}+3I_2I_1-I_1^3,\\
\bar{I}_{4\mathrm{a}} &= -4I_{4\mathrm{a}}+I_{3\mathrm{a}}I_1+3I_{3\mathrm{b}}I_1+4I_2^2-6I_2I_1^2+I_1^4,\\
\bar{I}_{4\mathrm{b}} &= -4I_{4\mathrm{b}}+3I_{3\mathrm{a}}I_1+I_{3\mathrm{b}}I_1+2I_2^2-2I_2I_1^2,
\end{aligned}
$$

[8] It is easy to see that this weight is related to a derivative with respect to the loop-counting parameter, as in equation (3.8).

$$
\begin{aligned}
\bar{I}_{4c} &= -4I_{4c} + I_{3a}I_1 + I_{3c}I_1 + 2I_2^2 - I_2I_1^2, \\
\bar{I}_{4d} &= -4I_{4d} + 3I_{3b}I_1 + 3I_{3c}I_1 + 2I_2^2 - 5I_2I_1^2 + I_1^4, \\
\bar{I}_{4e} &= -4I_{4e} + 4I_{3c}I_1 + 2I_2^2 - 4I_2I_1^2 + I_1^4.
\end{aligned} \tag{C.1}
$$

It is not hard to find that the the quantities $\tilde{I}$ exhibit only a simple pole in the ϵ-expansion. A strong crosscheck of the correctness of the subtraction is the cancellation of non-local and divergent momentum-dependence that arises in (B.6).

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