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Integrability and transcendentality

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Abstract. We derive the two-loop Bethe ansatz for the $\mathfrak{sl}(2)$ twist operator sector of $\mathcal{N} = 4$ gauge theory directly from the field theory. We then analyse a recently proposed perturbative asymptotic all-loop Bethe ansatz in the limit of large spacetime spin at large but finite twist, and find a novel *all-loop* scaling function. This function obeys the Kotikov–Lipatov transcendentality principle and does not depend on the twist. Under the assumption that one may extrapolate back to leading twist, our result yields an all-loop prediction for the large spin anomalous dimensions of twist 2 operators. The latter also appears as an undetermined function in a recent conjecture of Bern, Dixon and Smirnov for the all-loop structure of the maximally helicity violating n-point gluon amplitudes of $\mathcal{N} = 4$ gauge theory. This potentially establishes a direct link between the worldsheet and the spacetime S matrix approach. A further assumption for the validity of our prediction is that perturbative BMN (Berenstein–Maldacena–Nastase) scaling does not break down at four-loop level or beyond. We also discuss *how* the result gets modified *if* BMN scaling does break down. Finally, we show that our result qualitatively agrees at strong coupling with a prediction of string theory.

Keywords: integrable quantum field theory, integrable spin chains (vertex models), quantum integrability (Bethe ansatz)

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1. Introduction and main results

There is mounting evidence that planar $\mathcal{N}=4$ gauge theory might be '*exactly solvable*'. For example, it was recently proposed that higher loop maximally helicity violating (MHV) n-point gluon amplitudes should be iteratively expressible through the (regulated) one-loop amplitudes [1, 2]. On the basis of sophisticated two-loop [3, 1] and three-loop [2] computations, a conjecture for the *all-loop* MHV n-point gluon amplitudes $\mathcal{M}_n$ in $4-2\,\epsilon$ dimensions was formulated in [2]:

$$\mathcal{M}_n = \exp\left[\sum_{\ell=1}^{\infty} a_\epsilon^\ell \left(f^{(\ell)}(\epsilon)\, M_n^{(1)}(\ell\,\epsilon) + C^{(\ell)} + E_n^{(\ell)}(\epsilon)\right)\right]. \tag{1}$$

Here $E_n^{(\ell)}(\epsilon)$ vanishes as $\epsilon \to 0$, $C^{(\ell)}$ are finite constants, and $M_n^{(1)}(\ell\,\epsilon)$ is the $(\ell\,\epsilon)$-regulated

one-loop n-point amplitude. At $\epsilon = 0$ we have $\lim_{\epsilon\to 0}\, a_\epsilon = g^2$ where g^2 is defined as

$$g^2 = \frac{g^2_{\rm YM}\, N}{8\,\pi^2} = \frac{\lambda}{8\,\pi^2}, \tag{2}$$

and λ is the 't Hooft coupling. Finally, the $f^{(\ell)}(\epsilon)$ are generated in the $\epsilon \to 0$ limit by the function

$$f(g) = 4\sum_{\ell=1}^{\infty} g^{2\,\ell}\, f^{(\ell)}(0). \tag{3}$$

This function is in fact related to the large spin anomalous dimension of so-called leading twist operators in the gauge theory [4]. Alternative names are 'soft' anomalous dimension, and 'cusp' anomalous dimension. The simplest representatives of $\mathcal{N} = 4$ twist operators are found in the $\mathfrak{sl}(2)$ sector:

$$\mathrm{Tr}(\mathcal{D}^s \mathcal{Z}^L) + \cdots. \tag{4}$$

Here $\mathcal{D}$ is a light-cone covariant derivative, $\mathcal{Z}$ is one of the three complex scalars of the $\mathcal{N} = 4$ model, s measures the spacetime spin, and the twist L is to equal to, in this sector, one of the $\mathfrak{so}(6)$ R-charges. The leading twist is $L = 2$. The dots in (4) indicate that the true quantum operators are complicated mixtures of states, where the s covariant derivatives may act in all possible ways on the L fields $\mathcal{Z}$. Mixing with multi-trace operators is suppressed in the planar theory. The function $f(g)$ is obtained by considering the large spin $s \to \infty$ limit of the anomalous scaling dimension of the quantum operators (4), which is expected to scale logarithmically as

$$\Delta = s + f(g)\log(s) + \mathrm{O}(s^0). \tag{5}$$

We will call $f(g)$ of (3), (5) the *scaling function*. Note that the scaling structure in (5) is a highly non-trivial structural property of the exact finite s expression for $\Delta = \Delta(s, g)$. Individual Feynman diagrams contributing at intermediate stages of the perturbative calculation of Δ certainly contain higher ($k > 1$) powers $\log^k(s)$. The one-loop $\mathcal{O}(g^2)$ contribution to Δ was first computed in [5, 6] for all s, and indeed behaves as in (5). The $\mathcal{O}(g^4)$ two-loop answer was found in [7, 8], and the $\mathcal{O}(g^6)$ three-loop one, inspired by a fully fledged computation in QCD [9], in [10]. Again, the result indeed scales as in (5), and the state of the art up to now has been [10]

$$f(g) = 4\,g^2 - \tfrac{2}{3}\,\pi^2\, g^4 + \tfrac{11}{45}\,\pi^4\, g^6 + \cdots. \tag{6}$$

Fascinatingly, this agrees via (3) with the three-loop $n = 4$ calculation of (1) by Bern *et al* [2].

There is also mounting evidence that planar $\mathcal{N} = 4$ theory might be *'exactly integrable'*. This means that the *spectral problem*, i.e. the spectrum of all possible scaling dimensions $\{\Delta\}$ of the $\mathcal{N} = 4$ gauge theory, is encoded in a *Bethe ansatz*. This was established for the complete set of possible operators of $\mathcal{N} = 4$ theory at one-loop level [11, 12]. It was then conjectured, on the basis of two- and three-loop computations in the model's $\mathfrak{su}(2)$ sector, that integrability extends to all orders in perturbation theory, and, one hopes, to the non-perturbative level [13]. This was subsequently backed up in various studies [14]–[19], and culminated in a proposal for the *asymptotic all-loop* Bethe equations of the theory [20], further supported by [21]. Here 'asymptotic' means that

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the ansatz of [20] is only expected to correctly yield the anomalous dimension up to, roughly, $\mathcal{O}(g^{2L-2})$, where L is the number of constituent fields (not counting covariant derivatives) of the quantum operators considered[1]. Very recently a proposal was made for circumventing the asymptoticity restriction for the $\mathfrak{su}(2)$ sector by relating the dilatation operator, whose eigenvalues are the dimensions Δ, to a Hubbard Hamiltonian [22]. It would be exciting to also 'Hubbardize' the $\mathfrak{sl}(2)$ sector of twist operators.

Let us stress that 'solvability' and 'integrability' are distinct concepts. The latter is a rather precise but narrow concept referring to the spectral problem of the gauge theory. It means that the spectrum is described by one-dimensional factorized scattering of a set of appropriate elementary excitations. Equivalently, it means that there is a Bethe ansatz. Exactly solving the Bethe ansatz, in a given situation, is rarely easy, and actually generically impossible. Integrability allows one to prove that there will never be a 'plug-in' formula for the spectrum of $\mathcal{N} = 4$ gauge theory!

On the other hand, 'solvability' is a significantly less precise concept. Nevertheless, there is much evidence that in $\mathcal{N} = 4$ gauge theory many quantities beyond the scaling dimensions allow for a precise mathematical description. We have begun our discussion with the conjecture (1), which clearly contains more than just spectral information. Another example is provided by coordinate space correlation functions of more than two local composite operators. Certain intriguing iterative structures were e.g. noticed in four-point functions some time ago [23]–[25]. At the time of writing, the precise relation between the observed solvable structures and the integrable structures is somewhat reminiscent of the well-known paradox of the chicken and the egg.

Recall that a three-loop $\mathfrak{sl}(2)$ Bethe ansatz for gauge theory was conjectured in [18] by taking inspiration from the integrable structures appearing in string theory [26], and indeed reproduced (6). We will further back up the conjecture of [18] by calculating in section 2 the two-loop S matrix of the $\mathfrak{sl}(2)$ sector directly from the field theory. An alternative two-loop derivation, using algebraic methods, was recently presented by Zwiebel [19] (actually, for the bigger sector $\mathfrak{su}(1,1|2)$).

It is amusing to note that (6) is thus reproduced by three completely independent procedures [10, 18, 2], none of them completely rigorous. However, the various approaches, including their assumptions, seem to be completely independent. So (6) is very likely to be correct!

In this paper we will apply the asymptotic all-loop Bethe ansatz of [20] in order to compute all further perturbative corrections to the expression (6). Strictly speaking, the asymptotic ansatz does not apply to leading twist $L = 2$; see above. We will however argue that the scaling function (5) is universal in that it describes the behaviour of the lowest state of any $\mathfrak{sl}(2)$ operator as long as $L \ll s$. For a very recent discussion of this point, at the one-loop level, see [27]. Our argument for the validity of our scaling function, as regards the leading twist operators, is therefore based on *two* assumptions: (1) that it is indeed correct to pick L sufficiently large to stay in the 'asymptotic' regime of the Bethe ansatz, while keeping $L \ll s$, and (2) that the Bethe ansatz of [20] indeed describes the gauge theory, for sufficiently 'long' operators, at four-loop level and beyond. The first assumption is very likely to be true, while the validity of the second is, at the time of

[1] In the case of twist $L = 2$ operators the asymptotic ansatz actually works to $\mathcal{O}(g^6)$ instead of the naively expected $\mathcal{O}(g^2)$ because of superconformal invariance [18, 20].

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writing, much less clear. However, our computation might actually help to decide this issue; see below.

As a highly non-trivial check of our procedure, we will prove that the anomalous dimension Δ is indeed of the expected scaling form (5) to all orders in perturbation theory. We will find the scaling function to be given by the integral representation

$$f(g) = 4\,g^2 - 16\,g^4 \int_0^\infty \mathrm{d}t\,\hat{\sigma}(t)\,\frac{J_1(\sqrt{2}\,g\,t)}{\sqrt{2}\,g\,t}, \tag{7}$$

where the fluctuation density $\hat{\sigma}(t)$ is determined by the solution of the integral equation

$$\hat{\sigma}(t) = \frac{t}{\mathrm{e}^t - 1}\left[\frac{J_1(\sqrt{2}\,g\,t)}{\sqrt{2}\,g\,t} - 2\,g^2 \int_0^\infty \mathrm{d}t'\,\hat{K}(\sqrt{2}\,g\,t, \sqrt{2}\,g\,t')\,\hat{\sigma}(t')\right], \tag{8}$$

with the non-singular kernel

$$\hat{K}(t,t') = \frac{J_1(t)\,J_0(t') - J_0(t)\,J_1(t')}{t - t'}. \tag{9}$$

The functions $J_0(t)$, $J_1(t)$ in the above equations are standard Bessel functions.

We have been unable to find an explicit solution of the integral equation. It would be quite interesting if this could be achieved. It is however straightforward to obtain the weak coupling expansion of the fluctuation density $\hat{\sigma}(t)$ by iterating (8). Using (7) we then obtain the perturbative solution of the scaling function $f(g)$. To e.g. four-loop order one has

$$\begin{aligned} f(g) = {} & 4\,g^2 - 4\,\zeta(2)\,g^4 + (4\,\zeta(2)^2 + 12\,\zeta(4))\,g^6 \\ & - (4\,\zeta(2)^3 + 24\,\zeta(2)\zeta(4) - 4\,\zeta(3)^2 + 50\,\zeta(6))\,g^8 + \cdots. \end{aligned} \tag{10}$$

Using the fact that ζ functions of even argument may be expressed as products of rational numbers and powers of π, this may be simplified to

$$f(g) = 4\,g^2 - \tfrac{2}{3}\,\pi^2\,g^4 + \tfrac{11}{45}\,\pi^4\,g^6 - \left(\tfrac{73}{630}\,\pi^6 - 4\,\zeta(3)^2\right)\,g^8 + \cdots. \tag{11}$$

As a further non-trivial check of our procedure, and thus the validity of (7), we shall find that $f(g)$ obeys the Kotikov–Lipatov principle of maximal transcendentality [7], which was actually used in [10] in order to extract, even at finite spin s, the $\mathcal{N} = 4$ dimensions from the QCD calculation of [9]. When applied to the large s limit, the principle holds that the sum over all the arguments of the products of zeta functions appearing as additive terms at a given loop order ℓ always adds up to $2\,\ell - 2$; see (10), and (81) below.

It would be exciting if the four-loop prediction (11) could be tested by a field theoretic computation, perhaps by way of extending the results of [2] to higher order. Incidentally, even if field theory fails to reproduce (11), we will gain crucial knowledge on the integrable structure of the gauge theory; see section 3.4. The reason is that we are able to predict *how* transcendentality will break down *if* it breaks down. The e.g. four-loop term in the scaling function $f(g)$ in (11) would then get modified to

$$-\left(\tfrac{73}{630}\,\pi^6 - 4\,\zeta(3)^2 + 8\,\beta\,\zeta(3)\right)\,g^8, \tag{12}$$

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where β is an *a priori* unknown number[2]. Furthermore this number would then show up in the four-loop anomalous dimensions of *all* operators of the $\mathcal{N}=4$ theory; see section 3.4. In particular, it would manifest itself in the four-loop dimensions of operators with a large R-charge J, and would in fact induce a perturbative *breakdown* of BMN scaling [28]. The argument may also be turned around: proving that (11) holds as stated would establish that $\beta = 0$, and would therefore be indirect proof that BMN scaling holds up to the four-loop level.

Finally, there is a prediction from string theory [29, 30], assuming the AdS/CFT correspondence, for the *strong coupling* $g \to \infty$ behaviour of the scaling function $f(g)$:

$$f_{\text{string}}(g) = 2\sqrt{2}\,g - \frac{3}{\pi}\log(2) + \mathcal{O}\left(\frac{1}{g}\right), \tag{13}$$

where $2\sqrt{2}\,g = \sqrt{\lambda}/\pi$; cf (2). The leading $\mathcal{O}(g) = \mathcal{O}(\sqrt{\lambda})$ piece is obtained from a classical string spinning with a large angular momentum s on the AdS space [29], while the $\mathcal{O}(g^0) = \mathcal{O}(\lambda^0)$ term is the first quantum correction obtained in [30].

On the other hand, performing the strong coupling limit for our scaling function as defined from the integral equation (8) with (7), (9) (see section 3.5) we do vindicate the $\mathrm{O}(g)$ asymptotics predicted from string theory: the leading contribution to σ is of order $1/g^2$ and eliminates the first term on the rhs of (7). However, our analysis is currently not precise enough to decide whether or not the subleading $\mathrm{O}(g)$ term matches (13). We hope to present a more complete solution of the strong coupling problem in future work.

2. The factorized two-loop $\mathfrak{sl}(2)$ S matrix

In this preliminary chapter we will recall the Bethe ansatz for the $\mathfrak{sl}(2)$ sector of $\mathcal{N}=4$ *twist operators* (4) at one-loop level [12] and beyond [18, 20]. We will then derive it at two-loop level by Feynman diagram computations, successfully checking part of the conjecture of [18]. A complimentary two-loop approach was recently accomplished by Zwiebel [19], who worked out the full (even non-planar) dilatation operator of the bigger $\mathfrak{su}(1,1|2)$ sector by algebraic means, and also demonstrated the emergence of the two-loop two-body S matrix of [18].

The Bethe ansatz is obtained through the diagonalization of an integrable spin chain, whose Hamiltonian is equivalent to the dilatation operator. For a general introduction to this technology see [31]. The states of the spin chain are represented by removing the trace from the gauge theory states. With $s_1 + s_2 + \cdots + s_{L-1} + s_L = s$, one has

$$\not{\mathrm{Tr}}\left((\mathcal{D}^{s_1}\mathcal{Z})(\mathcal{D}^{s_2}\mathcal{Z})\ldots(\mathcal{D}^{s_{L-1}}\mathcal{Z})(\mathcal{D}^{s_L}\mathcal{Z})\right) \longrightarrow |s_1, s_2, \ldots, s_{L-1}, s_L\rangle, \tag{14}$$

corresponding to a chain of length L. The s_i are the spins of the chain, and can, if s is sufficiently large, take on any value due to the non-compact character of the $\mathfrak{sl}(2)$ sector. The Hamiltonian of the chain acts on this state space. Anomalous dimensions Δ are then related to the energies $E(g)$ (i.e. the eigenvalues of the Hamiltonian) through

$$\Delta = L + s + g^2\,E(g). \tag{15}$$

[2] The alert reader will notice that transcendentality could still be preserved if β turned out to be a rational number times $\zeta(3)$. The important point is that our calculation leads to a *detection mechanism* for BMN scaling violation. If a future field theory calculation finds $\beta \neq 0$, BMN scaling breaks down.

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Recall the one-loop Bethe ansatz for $\mathfrak{sl}(2)$, corresponding to a $\mathrm{XXX}_{-\frac{1}{2}}$ nearest neighbour spin chain where the subscript indicates a non-compact spin $-\frac{1}{2}$ representation of $\mathfrak{sl}(2)$:

$$\left(\frac{u_k+(\mathrm{i}/2)}{u_k-(\mathrm{i}/2)}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{s} \frac{u_k-u_j-\mathrm{i}}{u_k-u_j+\mathrm{i}}, \qquad k=1,\ldots,s. \tag{16}$$

The cyclicity constraint and the one-loop energy $E_0 := E(0)$ are

$$\prod_{k=1}^{s} \frac{u_k+(\mathrm{i}/2)}{u_k-(\mathrm{i}/2)} = 1 \qquad \text{and} \qquad E_0 = \sum_{k=1}^{s} \frac{1}{u_k^2+(1/4)}. \tag{17}$$

For a pedagogical derivation of these expressions from the Hamiltonian, using the coordinate space Bethe ansatz, see [18]. A rigorous proof, for any representation of $\mathfrak{sl}(2)$, may be found in [33].

The conjectured asymptotic all-loop Bethe ansatz for $\mathfrak{sl}(2)$ [20] is then obtained by 'deforming' the spectral parameter u, where the deformation parameter is the Yang–Mills coupling constant g:

$$u \pm \frac{\mathrm{i}}{2} = x^{\pm} + \frac{g^2}{2x^{\pm}}. \tag{18}$$

It reads

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{s} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \frac{1-g^2/2x_k^+x_j^-}{1-g^2/2x_k^-x_j^+}, \qquad k=1,\ldots,s, \tag{19}$$

with the new cyclicity constraint and the asymptotic all-loop energy $E(g)$ being given by

$$\prod_{k=1}^{s} \frac{x_k^+}{x_k^-} = 1 \qquad \text{and} \qquad E(g) = \sum_{k=1}^{s} \left(\frac{\mathrm{i}}{x_k^+} - \frac{\mathrm{i}}{x_k^-}\right). \tag{20}$$

It generalizes a three-loop Bethe ansatz first proposed in [18]. Very recently, much additional support of the ansatz was obtained in [21]. It should be noted, however, that we still cannot currently prove that the ansatz (19), (20) really diagonalizes the *gauge theory* at four-loop level and beyond. The reason is that we do not know how to fix possible 'dressing factors' (see [21] and references therein).

Directly proving the higher loop ansatz from the gauge field theory is hard. For all loops it will certainly require ideas that go far beyond 'summing up Feynman diagrams'. To illustrate the complexity we will nevertheless derive the Bethe ansatz at two-loop level by traditional methods. Actually, we will be able to find the S matrix, and we will succeed in checking two-loop factorization of the three-body problem. This is, according to [32], a strong test for integrability. Completing the proof would require us to demonstrate the factorization of the s-body problem for arbitrary s, which we have not attempted to do.

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2.1. One-loop Bethe ansatz and three-body factorization

The $\mathfrak{sl}(2)$ sector contains composite operators built from only one complex scalar field Z of the $\mathcal{N}=4$ SYM set of fields and the Yang–Mills covariant derivative D_μ. The operators are taken to carry symmetric traceless irreps of the Lorentz group. We may project all indices onto the complex direction $z=(x_1+\mathrm{i}x_2)/\sqrt{2}$, which guarantees symmetrization while the trace terms automatically vanish.

Single-trace operators of this type have a natural description as spin chains: each field Z is interpreted as an empty site which may be occupied by any number of derivatives D_z. The spin chain Hamiltonian

$$\mathcal{H}^{(0)}=\sum_{i=1}^{L}\mathcal{H}_i^{(0)} \tag{21}$$

involves a nearest neighbour interaction $\mathcal{H}_i^{(0)}$ that cyclically acts on all sites of the chain of length L. Alternatively, one may consider the asymptotic case, i.e. an open chain of infinite length. The Hamiltonian can transfer derivatives and it is conveniently expressed using matrices containing amplitudes for such processes.

The one-loop Hamiltonian was worked out in [34]: let us denote the number of derivatives on two adjacent sites as $\{s_1,s_2\}$. Then

$$\begin{aligned}&\mathcal{H}_i^{(0)}(\{s_1,s_2\}\to\{s_1,s_2\})=h(s_1)+h(s_2),\\&\mathcal{H}_i^{(0)}(\{s_1,s_2\}\to\{s_1-d,s_2+d\})=-\frac{1}{|d|}\end{aligned} \tag{22}$$

where $h(k)$ is the kth harmonic number. The matrix elements refer to a basis in which $\{s_1,s_2\}$ is divided by $s_1!s_2!$ in order to account for the indistinguishability of the derivatives at each site.

The *Bethe ansatz* rests on the observation that the derivatives D_z behave like particles (or 'magnons') whose motion is governed by a discrete Schrödinger equation. Let us assign a position x_i and a momentum p_i to each magnon. One constructs a wavefunction for each magnon number s:

$$\begin{aligned}&s=1:\sum_{x_1}\Psi^{(0)}(x_1)\,|x_1\rangle,\\&s=2:\sum_{x_1\le x_2}\Psi^{(0)}(x_1,x_2)\,|x_1,x_2\rangle,\\&s=3:\sum_{x_1\le x_2\le x_3}\Psi^{(0)}(x_1,x_2,x_3)\,|x_1,x_2,x_3\rangle\ \ldots.\end{aligned} \tag{23}$$

Here $|x_1\rangle$ denotes a state with a magnon at position x_1 etc. The Hamiltonian reshuffles these 'kets' as it can shift magnons. On the other hand, the kets form a complete set of states whose mutual independence one may use to transform the Schrödinger equation

$$\mathcal{H}^{(0)}\sum_{x_1\le x_2\le\cdots}\Psi^{(0)}(x_1,x_2,\ldots)\,|x_1,x_2,\ldots\rangle=E^{(0)}\sum_{x_1\le x_2\le\cdots}\Psi^{(0)}(x_1,x_2,\ldots)\,|x_1,x_2,\ldots\rangle \tag{24}$$

into a difference equation on the wavefunction $\Psi^{(0)}$. We find e.g. for only one magnon

$$2\,\Psi^{(0)}(x_1)-\Psi^{(0)}(x_1-1)-\Psi^{(0)}(x_1+1)=E^{(0)}\,\Psi^{(0)}(x_1). \tag{25}$$

This can be solved by Fourier transformation:

$$\Psi^{(0)}(x_1) = \mathrm{e}^{\mathrm{i}p_1 x_1}, \qquad E^{(0)} = 4\sin^2\left(\frac{p_1}{2}\right). \tag{26}$$

The one-magnon problem thus defines the dispersion law, i.e. the dependence $E(p)$ of the energy on the momentum of the particle. It is an essential assumption of the Bethe ansatz that the dispersion law for several magnons is simply a sum over the contributions of the individual pseudo-particles given by (26).

For two magnons the arguments of the wavefunction $\Psi^{(0)}(x_1, x_2)$ should obey $x_1 \leq x_2$ in order to avoid overcounting. Since the one-loop Hamiltonian is a two-site interaction, the plane wave solution remains valid when the separation of the magnons is greater than or equal to two. The corresponding difference equation looks in fact like two copies of (25):

$$\begin{aligned} &2\,\Psi^{(0)}(x_1, x_2) - \Psi^{(0)}(x_1 - 1, x_2) - \Psi^{(0)}(x_1 + 1, x_2) \\ &\qquad + 2\,\Psi^{(0)}(x_1, x_2) - \Psi^{(0)}(x_1, x_2 - 1) - \Psi^{(0)}(x_1, x_2 + 1) \\ &\qquad = g^2 M \left(E^{(0)}(p_1) + E^{(0)}(p_2)\right) \Psi^{(0)}(x_1, x_2). \end{aligned} \tag{27}$$

It is a special feature of the Hamiltonian (22) that this equation remains valid when $x_1 = x_2 - 1$. However, we do find a new equation when $x_1 = x_2$ [18]:

$$\begin{aligned} &\tfrac{3}{2}\,\Psi^{(0)}(x_1, x_2) - \Psi^{(0)}(x_1 - 1, x_2) - \tfrac{1}{2}\Psi^{(0)}(x_1 - 1, x_2 - 1) \\ &\qquad + \tfrac{3}{2}\,\Psi^{(0)}(x_1, x_2) - \Psi^{(0)}(x_1, x_2 + 1) - \tfrac{1}{2}\,\Psi^{(0)}(x_1 + 1, x_2 + 1) \\ &\qquad = g^2 M \left(E^{(0)}(p_1) + E^{(0)}(p_2)\right) \Psi^{(0)}(x_1, x_2). \end{aligned} \tag{28}$$

A simple plane wave does not obey this equation, but we can solve this using an ansatz of the form

$$\Psi^{(0)}(x_1, x_2) = \mathrm{e}^{\mathrm{i}p_1 x_1 + \mathrm{i}p_2 x_2} + S^{(0)}(p_2, p_1)\mathrm{e}^{\mathrm{i}p_2 x_1 + \mathrm{i}p_1 x_2}. \tag{29}$$

The physical intuition behind the last formula is that the particles may scatter by exchanging their momenta; the second plane wave is related to this, whereby the factor $S^{(0)}$ is called the *scattering matrix*. It can be determined from (28):

$$S^{(0)}(p_2, p_1) = -\frac{\mathrm{e}^{\mathrm{i}p_1 + \mathrm{i}p_2} - 2\mathrm{e}^{\mathrm{i}p_1} + 1}{\mathrm{e}^{\mathrm{i}p_1 + \mathrm{i}p_2} - 2\mathrm{e}^{\mathrm{i}p_2} + 1}. \tag{30}$$

Note that the two plane waves in (29) (with straight and flipped momenta, respectively) are independent as functions. Equation (28) therefore yields two conditions, although they are equivalent in this case.

For three magnons one writes an ansatz involving a wavefunction $\Psi^{(0)}(x_1, x_2, x_3)$ subject to $x_1 \leq x_2 \leq x_3$ and proceeds to set up difference equations. As before, the magnons do not feel each other when $x_1 + 1 < x_2 < x_3 - 1$. One might expect special behaviour when $x_1 = x_2 - 1$ or $x_3 = x_2 + 1$, but due to the structure of the Hamiltonian this actually does not yield any new conditions. Thus it remains to investigate the cases

$$\text{(i) } x_1 = x_2 < x_3, \qquad \text{(ii) } x_1 < x_2 = x_3, \qquad \text{(iii) } x_1 = x_2 = x_3. \tag{31}$$

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We write an ansatz which straightforwardly generalizes the two-magnon formula (29):

$$\begin{aligned}\Psi^{(0)}(x_1,x_2,x_3) = {} & \mathrm{e}^{\mathrm{i}p_1x_1+\mathrm{i}p_2x_2+\mathrm{i}p_3x_3} + S^{(0)}_{132}\mathrm{e}^{\mathrm{i}p_1x_1+\mathrm{i}p_3x_2+\mathrm{i}p_2x_3} \\ & + S^{(0)}_{213}\mathrm{e}^{\mathrm{i}p_2x_1+\mathrm{i}p_1x_2+\mathrm{i}p_3x_3} + S^{(0)}_{231}\mathrm{e}^{\mathrm{i}p_2x_1+\mathrm{i}p_3x_2+\mathrm{i}p_1x_3} \\ & + S^{(0)}_{312}\mathrm{e}^{\mathrm{i}p_3x_1+\mathrm{i}p_1x_2+\mathrm{i}p_2x_3} + S^{(0)}_{321}\mathrm{e}^{\mathrm{i}p_3x_1+\mathrm{i}p_2x_2+\mathrm{i}p_1x_3}. \end{aligned} \tag{32}$$

If $x_1 = x_2 < x_3$, the difference equation can be separated into three independent pieces according to which momentum multiplies x_3 in the exponentials. The case $x_3 = x_2 > x_1$ obviously allows for a similar distinction w.r.t. x_1. Five of the resulting six equations are independent, so that one may solve

$$\begin{aligned} S^{(0)}_{132} &= S^{(0)}(p_3,p_2), \\ S^{(0)}_{213} &= S^{(0)}(p_2,p_1), \\ S^{(0)}_{231} &= S^{(0)}(p_2,p_1)\,S^{(0)}(p_3,p_1), \\ S^{(0)}_{312} &= S^{(0)}(p_3,p_1)\,S^{(0)}(p_3,p_2), \\ S^{(0)}_{321} &= S^{(0)}(p_2,p_1)\,S^{(0)}(p_3,p_1)\,S^{(0)}(p_3,p_2). \end{aligned} \tag{33}$$

This solution persists when all three magnons coincide, which is again a non-trivial consequence of the structure of the Hamiltonian. We see that the scattering remains non-diffractive, i.e. the momenta are unaltered while they may be exchanged between the magnons. What is more, the three-particle S matrices factor into two-particle processes.

2.2. Bethe ansatz and three-body factorization at two-loop level

The original Bethe ansatz described in the last section may be generalized to higher orders in perturbation theory [18]. To this end one writes a perturbation expansion of all relevant quantities, namely the Hamiltonian, the ingoing wave and the S matrix. The central topic of this section is deriving the two-loop correction to the S matrix in the $\mathfrak{sl}(2)$ sector directly from the $\mathcal{N} = 4$ field theory, and checking three-body factorization to two-loop level.

In the appendices A and B we derive the two-loop Hamiltonian for one, two, and three magnons from a graph calculation using $\mathcal{N} = 2$ superfields [35] and the SSDR scheme (supersymmetric dimensional reduction) [36]. The supergraph formalism is preferable because it minimizes the number of Feynman integrals; for the present purpose the $\mathcal{N} = 2$ formulation is superior to $\mathcal{N} = 1$ supergraphs. We end up with a manageable calculation involving about twenty graphs. SSDR is the best suited regulator since it allows one to treat superfields in a version of dimensional regularization[3].

We attack the problem of calculating quantities with open indices by tensor decomposition and employ the QCD package *Mincer* [37] to evaluate the resulting scalar integrals. The package uses $4 - 2\epsilon$-dimensional vectors so that we explicitly have to symmetrize and take out trace terms. This makes the computer algebra very awkward so that we have limited the scope of the present work to low magnon numbers. The method was detailed in [38] by one of the authors. We will draw heavily upon this reference in the

[3] At the current loop order the scheme cannot lead to ambiguities arising from the ϵ tensor.

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appendices[4]. Appendix A reviews the renormalization of two-loop two-point functions in dimensional regularization. In appendix B we introduce operators $\tilde{D}_1, \tilde{D}_2$ which generate the singular parts of the one- and two-loop two-point functions and we show that the second anomalous dimensions are matrix elements of the combination $\tilde{D}_2 - 1/2\,\tilde{D}_1^2$, thus reproducing the two-loop effective vertex given in [13], where the renormalization of the dilatation operator in dimensional regularization was first discussed. Finally, the $\tilde{D}_i$ are constructed from the supergraphs and the two-loop Hamiltonian is worked out for one, two, and three magnons.

In this section we display the Hamiltonian as it arises from $\tilde{D}_2 - 1/2\,\tilde{D}_1^2$ alone. One can introduce into it a number of gauge parameters which do not appear in the difference equations defining the wavefunction and the S matrix. This freedom is (more than) sufficient for making the Hamiltonian Hermitian and for making the sum of all elements in each row or column disappear, as was the case for the one-loop dilatation operator. In appendix B we also give another set of transfer rules which includes the contribution of a term $-1/4\,[\mathcal{H}^{(0)}_{1\,i}, \mathcal{H}^{(0)}_{0\,i}]$, which is needed when the dilatation operator is required to reproduce the $\mathrm{O}(g^2)$ remixing of the sl(2) sector operators. This term cannot be made Hermitian by the aforementioned gauge transformations and thus from the point of view of the Bethe ansatz it is perhaps best omitted. It is interesting to note, however, that the commutator term does not change the S matrix, while it seems to make redundant any wavefunction renormalization in the Bethe picture.

The disconnected pieces of the two-loop combinatorics do not influence the Hamiltonian. The connected two-loop graphs can stretch over three adjacent sites. The basis elements below denote the number of covariant derivatives at these three sites; we have explicitly indicated a factor $1/(s_1!\,s_2!\,s_3!)$ with which they were rescaled.

Spin 1. Basis: $\{\{1,0,0\},\{0,1,0\},\{0,0,1\}\}$:

$$\mathcal{H}^{(2)}_i(1) = \begin{pmatrix} -\frac{3}{4} & 1 & -\frac{1}{2} \\ 1 & -\frac{3}{2} & 1 \\ -\frac{1}{2} & 1 & -\frac{3}{4} \end{pmatrix}.$$

Spin 2. Basis: $\{\frac{1}{2}\{2,0,0\},\{1,1,0\},\{1,0,1\},\frac{1}{2}\{0,2,0\},\{0,1,1\},\frac{1}{2}\{0,0,2\}\}$:

$$\mathcal{H}^{(2)}_i(2) = \begin{pmatrix} -\frac{15}{32} & \frac{19}{16} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{16} \\ \frac{13}{16} & -\frac{5}{2} & 1 & \frac{23}{16} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & 1 & -\frac{3}{2} & 0 & 1 & -\frac{1}{2} \\ \frac{1}{2} & \frac{17}{16} & 0 & -\frac{63}{16} & \frac{17}{16} & \frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} & 1 & \frac{23}{16} & -\frac{5}{2} & \frac{13}{16} \\ -\frac{1}{16} & -\frac{1}{4} & -\frac{1}{2} & \frac{1}{2} & \frac{19}{16} & -\frac{15}{32} \end{pmatrix}.$$

[4] The calculation of the two-loop anomalous dimension of the twist 2 operators [8] uses similar techniques; it is more fully automated but renounces on the use of supergraphs.

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Spin 3. Basis: $\{\frac{1}{6}\{3,0,0\},\frac{1}{2}\{2,1,0\},\frac{1}{2}\{2,0,1\},\frac{1}{2}\{1,2,0\},\{1,1,1\},\frac{1}{2}\{1,0,2\},\frac{1}{6}\{0,3,0\},\frac{1}{2}\{0,2,1\},\frac{1}{2}\{0,1,2\},\frac{1}{6}\{0,0,3\}\}$:

$$\mathcal{H}_i^{(2)}(3) = \begin{pmatrix} \frac{85}{288} & \frac{115}{144} & -\frac{1}{2} & \frac{43}{72} & -\frac{1}{4} & -\frac{1}{16} & \frac{71}{216} & -\frac{1}{6} & -\frac{1}{24} & -\frac{1}{54} \\ \frac{1}{48} & -\frac{209}{96} & 1 & \frac{29}{24} & -\frac{1}{4} & -\frac{1}{4} & \frac{19}{24} & -\frac{1}{6} & -\frac{7}{48} & -\frac{1}{24} \\ -\frac{1}{2} & 1 & -\frac{39}{32} & 0 & \frac{19}{16} & -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{16} \\ \frac{3}{8} & \frac{29}{24} & 0 & -\frac{247}{48} & \frac{17}{16} & \frac{1}{2} & \frac{109}{48} & -\frac{1}{12} & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{13}{16} & \frac{23}{16} & -\frac{7}{2} & \frac{13}{16} & 0 & \frac{23}{16} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{16} & -\frac{1}{4} & -\frac{1}{2} & \frac{1}{2} & \frac{19}{16} & -\frac{39}{32} & 0 & 0 & 1 & -\frac{1}{2} \\ \frac{71}{216} & \frac{41}{72} & 0 & \frac{215}{144} & 0 & 0 & -\frac{971}{144} & \frac{215}{144} & \frac{41}{72} & \frac{71}{216} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{1}{2} & -\frac{1}{12} & \frac{17}{16} & 0 & \frac{109}{48} & -\frac{247}{48} & \frac{29}{24} & \frac{3}{8} \\ -\frac{1}{24} & -\frac{7}{48} & -\frac{1}{4} & -\frac{1}{6} & -\frac{1}{4} & 1 & \frac{19}{24} & \frac{29}{24} & -\frac{209}{96} & \frac{1}{48} \\ -\frac{1}{54} & -\frac{1}{24} & -\frac{1}{16} & -\frac{1}{6} & -\frac{1}{4} & -\frac{1}{2} & \frac{71}{216} & \frac{43}{72} & \frac{115}{144} & \frac{85}{288} \end{pmatrix}.$$

Let us now focus on the Bethe ansatz. The spin chain Hamiltonian up to two-loop level is

$$\mathcal{H} = \sum_i \mathcal{H}_i^{(0)} + g^2\,\mathcal{H}_i^{(2)} + \cdots, \qquad g^2 = \frac{g_{YM}^2 N}{8\pi^2} \tag{34}$$

and it has energy eigenvalues $E = E^{(0)} + g^2 E^{(2)} + \cdots$. The wavefunctions of the form

$$\Psi(x_1, x_2, \ldots) = \Psi^{(0)}(x_1, x_2, \ldots) + g^2\,\Psi^{(2)}(x_1, x_2, \ldots) + \cdots \tag{35}$$

are contracted on the kets $|x_1, x_2, \ldots\rangle$ defined in section 2.1.

For one magnon we may scale away $\Psi^{(2)}$. The Schrödinger equation

$$\sum_i \mathcal{H}_i^{(2)}(1) \sum_{x_1} \Psi^{(0)}(x_1)\,|x_1\rangle = E^{(2)} \sum_{x_1} \Psi^{(0)}(x_1)\,|x_1\rangle \tag{36}$$

leads to the difference condition

$$\begin{aligned} -\tfrac{1}{2}\,\Psi^{(0)}(x_1-2) + 2\,\Psi^{(0)}(x_1-1) - 3\,\Psi^{(0)}(x_1) + 2\,\Psi^{(0)}(x_1+1) - \tfrac{1}{2}\,\Psi^{(0)}(x_1-2) \\ = E^{(2)}\,\Psi^{(0)}(x_1) \end{aligned} \tag{37}$$

which can again be solved by Fourier transformation:

$$\Psi^{(0)}(x_1) = \mathrm{e}^{\mathrm{i}p_1 x_1}, \qquad E^{(2)} = -8\sin^4\left(\frac{p_1}{2}\right). \tag{38}$$

Hence the solution of the two-loop one-magnon problem yields the correction to the one-loop dispersion law (26) for the magnon energy $E(p) = E^{(0)}(p) + g^2\,E^{(2)}(p) + \mathcal{O}(g^4)$. It is identical to that of the $\mathfrak{su}(2)$ [15] and $\mathfrak{su}(1|1)$ [18] sectors, and consistent with the proposed all-loop dispersion law of [16]:

$$E(p) = \frac{1}{g^2}\left(\sqrt{1 + 8\,g^2\sin^2\left(\frac{p}{2}\right)} - 1\right).$$

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The lowest order of the two-magnon problem was discussed in the last section. The two-loop part of the Schrödinger equation reads

$$\begin{aligned}\sum_i \mathcal{H}_i^{(0)} \sum_{x_1\le x_2} \Psi^{(2)}(x_1,x_2)\,|x_1,x_2\rangle + \sum_i \mathcal{H}_i^{(2)} \sum_{x_1\le x_2} \Psi^{(0)}(x_1,x_2)\,|x_1,x_2\rangle \\ = (E^{(0)}(p_1)+E^{(0)}(p_2)) \sum_{x_1\le x_2} \Psi^{(2)}(x_1,x_2)\,|x_1,x_2\rangle \\ + (E^{(2)}(p_1)+E^{(2)}(p_2)) \sum_{x_1\le x_2} \Psi^{(0)}(x_1,x_2)\,|x_1,x_2\rangle. \end{aligned} \tag{39}$$

The resulting difference equations are perhaps not particularly illuminating. We will rather comment on how to solve the system: the interaction length of the two-loop Hamiltonian $\mathcal{H}_i^{(2)}$ is 3. The two magnons must therefore behave as free particles when $x_1 < x_2 - 2$. Thanks to the special form of $\mathcal{H}_i^{(2)}$ the same difference equation still holds when $x_1 = x_2 - 2$. The cases of interest are thus

$$\text{(i) } x_1 = x_2 - 1, \qquad \text{(ii) } x_1 = x_2, \tag{40}$$

which both lead to new equations. In order to satisfy both conditions we must allow for a correction not only to the S matrix but also to the ingoing wavefunction. Let

$$\psi(p_1,p_2) = (1 + g^2\,\delta_{x_1,x_2}\,f(p_1,p_2))\mathrm{e}^{\mathrm{i}p_1x_1+\mathrm{i}p_2x_2}. \tag{41}$$

The wavefunction renormalization ('fudge factor') is local. We write the ansatz

$$\Psi(x_1,x_2) = \psi(p_1,p_2) + S(p_2,p_1)\,\psi(p_2,p_1), \qquad S = S^{(0)} + g^2\,S^{(2)} \tag{42}$$

whose expansion in the coupling constant defines $\Psi^{(0)}, \Psi^{(2)}$.

Case (i) gives a condition relating $f(p_1,p_2)$ to $f(p_2,p_1)$. Substituting this into (ii) makes the fudge factors disappear from the equation so that we can solve for the S matrix:

$$S^{(2)}(p_2,p_1) = -\frac{8\mathrm{i}\,\sin(\frac{p_1}{2})\,\sin(\frac{p_1-p_2}{2})\,\sin(\frac{p_2}{2})\,\left(\sin^2(\frac{p_1}{2})+\sin^2(\frac{p_2}{2})\right)}{(\sin(\frac{p_1-p_2}{2})+2\mathrm{i}\,\sin(\frac{p_1}{2})\,\sin(\frac{p_2}{2}))^2}. \tag{43}$$

This result nicely confirms the conjecture for the two-loop S matrix of the $\mathfrak{sl}(2)$ sector in [18].

The wavefunction renormalization f is not fully determined. It is tempting to assume it to be symmetric under the exchange of p_1, p_2 since the magnons are indistinguishable. In this case we find

$$f(p_1,p_2) = \sin^2\left(\frac{p_1}{2}\right) + \sin^2\left(\frac{p_2}{2}\right) - \frac{1}{2}\,\sin^2\left(\frac{p_1+p_2}{2}\right). \tag{44}$$

The alternative choice for the two-loop Hamiltonian from appendix B yields $f(p_1,p_2) = 0$ if f is symmetric.

The discussion of the two-loop three-magnon scattering combines elements of the one-loop three-magnon case with the two-magnon situation described in the last paragraph. We write for the ingoing wave

$$\begin{aligned}\psi(p_1,p_2,p_3) = (1 + g^2 M\,(\,\delta_{x_1,x_2}\,l(p_1,p_2,p_3) + \delta_{x_2,x_3}\,r(p_1,p_2,p_3) \\ + \delta_{x_1,x_2}\,\delta_{x_2,x_3}\,u(p_1,p_2,p_3)))\mathrm{e}^{\mathrm{i}p_1x_1+\mathrm{i}p_2x_2+\mathrm{i}p_3x_3}\end{aligned} \tag{45}$$

and make the ansatz

$$\Psi(x_1,x_2,x_3) = \psi(p_1,p_2,\,p_3) + S_{132}\,\psi(p_1,p_3,p_2) + S_{213}\,\psi(p_2,p_1,p_3) + S_{231}\,\psi(p_2,p_3,p_1) + S_{312}\,\psi(p_3,p_1,p_2) + S_{321}\,\psi(p_3,p_2,p_1), \tag{46}$$
$$S_{ijk} = S^{(0)}_{ijk} + g^2\,S^{(2)}_{ijk}.$$

As might be expected by now, the free situation must arise when $x_1+2<x_2<x_3-2$, but in fact nothing changes when $x_1+2=x_2$ or $x_2=x_3-2$. We thus have to discuss the cases

$$\begin{aligned}
&\text{(i)} \quad x_1+1=x_2<x_3-1,\\
&\text{(ii)} \quad x_1=x_2<x_3-1,\\
&\text{(iii)} \quad x_1+1<x_2=x_3-1,\\
&\text{(iv)} \quad x_1+1<x_2=x_3,\\
&\text{(v)} \quad x_1+1=x_2=x_3-1,\\
&\text{(vi)} \quad x_1=x_2=x_3-1,\\
&\text{(vii)} \quad x_1+1=x_2=x_3,\\
&\text{(viii)} \quad x_1=x_2=x_3.
\end{aligned} \tag{47}$$

In the first four cases only one x has disappeared, whereby one may use the functional independence of the various exponential factors to organize each difference equation into three separate constraints. Cases (i) and (ii) are equivalent to a two-magnon problem with positions x_1 and x_2: one may solve (i) for three conditions relating $l(p_1,p_2,p_3)$ to $l(p_2,p_1,p_3)$ etc and then substitute the three equations into (ii). This eliminates the left fudge factor l from the equations. Likewise, we can use (iii) to eliminate the right fudge factor r from (iv). We are left with six equations on the five $S^{(2)}_{ijk}$ matrices. A unique solution exists:

$$\begin{aligned}
S^{(2)}_{132} &= S^{(2)}(p_3,p_2),\\
S^{(2)}_{213} &= S^{(2)}(p_2,p_1),\\
S^{(2)}_{231} &= S^{(0)}(p_2,p_1)\,S^{(2)}(p_3,p_1) + S^{(2)}(p_2,p_1)\,S^{(0)}(p_3,p_1),\\
S^{(2)}_{312} &= S^{(0)}(p_3,p_1)\,S^{(2)}(p_3,p_2) + S^{(2)}(p_3,p_1)\,S^{(0)}(p_3,p_2),\\
S^{(2)}_{321} &= S^{(0)}(p_2,p_1)\,S^{(0)}(p_3,p_1)\,S^{(2)}(p_3,p_2) + S^{(0)}(p_2,p_1)\,S^{(2)}(p_3,p_1)\,S^{(0)}(p_3,p_2)\\
&\quad + S^{(2)}(p_2,p_1)\,S^{(0)}(p_3,p_1)\,S^{(0)}(p_3,p_2).
\end{aligned} \tag{48}$$

In other words, the complete S matrix $S=S^{(0)}+g^2\,S^{(2)}$ factors into two-particle processes also at two-loop level.

Once knowing that $\mathcal{H}^{(2)}_i(3)$ reproduces a two-magnon problem when only two arguments coincide, it is natural to put $l(p_1,p_2,p_3) = f(p_1,p_2) = r(p_3,p_1,p_2)$ and so on. With these identifications the cases (i) and (iii) reduce to the condition on the two-magnon fudge factor f found earlier. Of our remaining cases, (v) is empty while the last three all lead to one and the same condition on the ultralocal fudge factor u. There is not enough information at this loop order to solve for u—again, one may speculate that it should be chosen so as to make the ingoing wave symmetric when all three positions

coincide. The solution is then similar to (44) if the two-loop Hamiltonian is as defined in this section, or it vanishes for the alternative choice of $\mathcal{H}_i^{(2)}$ from appendix B.

In conclusion, our analysis confirms the possibility of extending the $\mathfrak{sl}(2)$ sector Bethe ansatz to the two-loop level. It proves the functional form of the two-loop S matrix conjectured in [18], and it shows that the three-magnon S matrix factors into two-particle blocks.

3. The asymptotic all-loop large spin limit

3.1. One-loop large spin limit

Consider the one-loop Bethe equations (16), (17) in the large spin limit $s \to \infty$. This problem was solved in great detail in the context of Reggeized gluon scattering for the very similar case of a non-compact $\mathfrak{sl}(2)$ spin $= 0$ representation, i.e. for an XXX_0 Heisenberg magnet, in [39]. The changes required to treat our present case of non-compact $\mathfrak{sl}(2)$ spin $= -\frac{1}{2}$ are minor. Here we will proceed in a slightly different fashion as compared to [39], where methods involving the Baxter Q function are employed. The reason is that the higher loop generalization of the Baxter function is not yet known. We will therefore directly work with the one-loop Bethe equations (16), (17), which nicely turn into a (singular) integral equation in the large spin limit. Our method will then be extended to the asymptotic all-loop equations (19), (20) in the next section. Interestingly, the effective higher loop integral equation will turn out to be non-singular.

Much intuition may be gained from the fact that the twist $L = 2$ case is, at one-loop level, explicitly solvable for arbitrary spin s; cf appendix C. Studying this solution one finds that the Bethe roots are all real[5] and symmetrically distributed around zero. The root distribution density has a peak at the origin (in particular, there is no gap around zero) and the outermost roots grow linearly with the spin as $\max\{|u_k|\} \to s/2$. We therefore introduce rescaled variables $\bar{u}$, and a density $\bar{\rho}_0(\bar{u})$ normalized to 1:

$$\frac{u_k}{s} \to \bar{u} \qquad \text{with } \bar{\rho}(\bar{u}) = \frac{1}{s}\sum_{k=1}^{s} \delta_0\left(\bar{u} - \frac{u_k}{s}\right) \quad \text{and thus} \quad \int_{-\bar{b}}^{\bar{b}} \bar{\rho}_0(\bar{u}) = 1. \tag{49}$$

We now take the usual logarithm of the Bethe equations (16) and multiply either side by $-\mathrm{i}$:

$$-\mathrm{i}\, L\, \log\left(\frac{u_k + (\mathrm{i}/2)}{u_k - (\mathrm{i}/2)}\right) = 2\,\pi\, n_k - \mathrm{i}\sum_{\substack{j=1\\ j\neq k}}^{s} \log\frac{u_k - u_j - \mathrm{i}}{u_k - u_j + \mathrm{i}}. \tag{50}$$

The integers n_k reflect the ambiguity in the branch of the logarithm, and may be interpreted as (bosonic) quantum mode numbers. In the case of twist $L = 2$ there is only one state. Its root distribution is real and symmetric under $u \leftrightarrow -u$. All positive (negative) roots have mode number $n = 1$ ($n = -1$). In the case of higher twist $L > 2$

[5] It may be shown that, in contrast to the case for the $\mathfrak{su}(2)$ spin $= \frac{1}{2}$ Heisenberg magnet, the roots of the $\mathfrak{sl}(2)$ spin $= -\frac{1}{2}$ Bethe equations are, for all L and s, always real. We thank Kazakov and Zarembo for a discussion of this point.

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there is more than one state[6]. However, for the lowest state the root distribution is again real symmetric with $n = \mathrm{sgn}(u)$. Since s is assumed large, and $u_k = \mathcal{O}(s)$ for nearly all roots, we furthermore expand (50) in $1/u$:

$$\frac{L}{u_k} = 2\,\pi\, n_k - 2 \sum_{\substack{j=1 \\ j\neq k}}^{s} \frac{1}{u_k - u_j}. \tag{51}$$

In this large s limit the rescaled Bethe roots condense onto a smooth cut on the interval $[-\bar{b}, \bar{b}]$ on the real $\bar{u}$-axis. We may therefore take a continuum limit of (51) which yields, using (49),

$$0 = 2\,\pi\,\epsilon(\bar{u}) - 2 \int_{-\bar{b}}^{\bar{b}} \mathrm{d}\bar{u}'\, \frac{\bar{\rho}_0(\bar{u}')}{\bar{u} - \bar{u}'}, \tag{52}$$

where $\epsilon(\bar{u}) = \mathrm{sgn}(\bar{u})$. In particular, the dependence on L in (51) drops out: the lowest state leads to the same large s root distribution, and therefore energy, for *arbitrary* finite twist L.

The singular integral equation (52) is easily solved by inverting the finite Hilbert transform with standard methods. The solution for the rescaled one-loop root density is then found to be

$$\bar{\rho}_0(\bar{u}) = \frac{1}{\pi}\,\log\frac{1+\sqrt{1-4\,\bar{u}^2}}{1-\sqrt{1-4\,\bar{u}^2}} = \frac{2}{\pi}\,\mathrm{arctanh}\left(\sqrt{1-4\,\bar{u}^2}\right), \tag{53}$$

where we have set the interval boundary to $\bar{b} = \frac{1}{2}$, as obtained from the density normalization condition. The result (53) of our procedure agrees with the Baxter Q approach of [39].

Our derivation is closely modelled after the discussion of [40]; in particular, we refer the reader to appendix C of that article. There the 'spinning strings' solutions of (16), (17), where *both* s *and* L are large and of the same order of magnitude, $\mathcal{O}(L) = \mathcal{O}(s)$, were studied. The difference is that in this case the lhs of (51) is not negligible. The ensuing potential L/u on the lhs of (51) opens up a gap $[-\bar{a}, \bar{a}]$ of the root distribution in the vicinity of $\bar{u} = 0$. The resulting density for the lowest state therefore has compact support on two cuts $[-\bar{b}, -\bar{a}]$ and $[\bar{a}, \bar{b}]$ and is expressible through an elliptic integral of the third kind (see equation (C.8) in [40]). One easily checks that when $L \to 0$ the gap disappears, i.e. $\bar{a} \to 0$, and the elliptic density, after rescaling the roots in [40] by $\bar{u} \to (s/L)\bar{u}$ in order to adapt conventions, simplifies to the expression (53), with $\bar{b} \to \frac{1}{2}$.

However, the one-loop anomalous dimension as obtained in [40] does *not* reproduce the expected logarithmic scaling of (5) upon taking the limit $s/L \to \infty$. Instead, it behaves like $\sim \log^2(s)$; cf (E.1) of [40]. This is a classic order-of-limits problem. Assuming s, L large with s/L finite, and subsequently taking $s/L \to \infty$ does not yield the same result as taking s large while keeping L either finite or, at least, $L \ll s$. For a very recent, quite extensive discussion of this fact see [27]. For a recent study of some of the fine structure of the spinning strings limit see [41].

[6] The reader might find it instructive to consult table 2 of [18], where a complete list of the three-loop spectra of the first few states of the $\mathfrak{sl}(2)$ sector may be found.

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The correct result is obtained by a careful derivation of the expression for the energy in the continuum limit $s \to \infty$. From the right equation in (17) we find, using (49),

$$E_0 = \frac{1}{s} \int_{-1/2}^{1/2} d\bar{u}\, \frac{\bar{\rho}_0(\bar{u})}{\bar{u}^2 + (1/4\, s^2)}. \tag{54}$$

Therefore, as opposed to the case for the limit of [40] (see the expression in (C.4)) it is nonsensical to use the unregulated expectation value $\int \mathrm{d}\bar{u}\, \bar{\rho}_0(\bar{u})/\bar{u}^2$ for the energy. The correct expression (54) is actually related to the resolvent $G(\bar{u})$, which is defined for arbitrary complex values of $\bar{u}$ barring the interval $[-1/2, 1/2]$ (this integral is *not* of principal part type) as

$$G(\bar{u}) = \int_{-1/2}^{1/2} \mathrm{d}\bar{u}'\, \frac{\bar{\rho}_0(\bar{u}')}{\bar{u}' - \bar{u}}, \tag{55}$$

through

$$E_0 = \frac{2}{\mathrm{i}}\, G\left(\frac{\mathrm{i}}{2s}\right). \tag{56}$$

Note that this further distinguishes the large spin limit from the 'spinning strings' limit, where the resolvent generates the full set of commuting charges [42]. One then finds from (53) that

$$G(\bar{u}) = \mathrm{i} \log \frac{\sqrt{1 - 4\,\bar{u}^2} + 1}{\sqrt{1 - 4\,\bar{u}^2} - 1}. \tag{57}$$

Using now (56) and taking $s \to \infty$ we find

$$E_0 = 4\, \log(s) + \mathcal{O}(s^0), \tag{58}$$

which is the well-known correct result, as may also be checked directly from the exact finite s result $E = 4\, h(s)$, see (C.4).

3.2. Asymptotic all-loop large spin limit

Let us now generalize the analysis of the previous section to the higher loop case. We would therefore like to compute the corrections to the one-loop density (53) and energy (58) as generated by the deformed Bethe equations (19), (20). Compelling arguments for its validity to three-loop level were presented in [18] (in particular the equations reproduce the conjecture of [10] based on the QCD calculation [9], and they agree with [2]). Their all-loop form was conjectured in [20]. See also [21].

We begin by rewriting the asymptotic all-loop Bethe equations (19) with the help of (18) in the following fashion:

$$\left(\frac{u_k + (\mathrm{i}/2)}{u_k - (\mathrm{i}/2)}\right)^L \left(\frac{1 + g^2/2(x_k^-)^2}{1 + g^2/2(x_k^+)^2}\right)^L = \prod_{\substack{j=1 \\ j \neq k}}^{s} \frac{u_k - u_j - \mathrm{i}}{u_k - u_j + \mathrm{i}} \left(\frac{1 - g^2/2x_k^+ x_j^-}{1 - g^2/2x_k^- x_j^+}\right)^2,$$
$$k = 1, \ldots, s. \tag{59}$$

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Let us again take a logarithm on both sides of the equations, and multiply by i:

$$2L\,\arctan(2\,u_k)+\mathrm{i}\,L\,\log\left(\frac{1+g^2/2(x_k^-)^2}{1+g^2/2(x_k^+)^2}\right)=2\,\pi\,\tilde{n}_k-2\sum_{\substack{j=-s/2\\ j\neq 0}}^{s/2}\arctan\,(u_k-u_j)$$
$$+\,2\mathrm{i}\sum_{\substack{j=-s/2\\ j\neq 0}}^{s/2}\log\left(\frac{1-g^2/2x_k^+x_j^-}{1-g^2/2x_k^-x_j^+}\right). \tag{60}$$

Here we have also relabelled the s roots u_k such that the index k runs over the set $k=\pm1,\pm2,\ldots,\pm s/2$. We have furthermore chosen, for convenience, to employ a different choice for the branches of the logarithms as compared to (50). Whereas in (50) the branch cuts run through $u_k=0$ and $u_k=u_j$, in our alternative choice in (60) the arctan functions are analytic at $u_k=0$ and $u_k=u_j$. This replaces the 'bosonic' mode numbers n_k of (50) by 'fermionic' mode numbers $\tilde{n}_k$. For the lowest state (the only one for $L=2$) we have, for even s,

$$\tilde{n}_k=k+\frac{L-3}{2}\,\epsilon(k)\qquad\text{for } k=\pm1,\pm2,\ldots,\pm\frac{s}{2}. \tag{61}$$

To avoid confusion: we are still focusing on the same states, and just chose to change the description.

Let us now proceed in close similarity to the computation of the thermodynamic antiferromagnetic ground state of the Heisenberg magnet (see e.g. [33]). In order to have a uniform spacing between the indices of all roots it is convenient to define $k'=k-\epsilon(k)/2$ such that

$$\tilde{n}_k=k'+\frac{L-2}{2}\,\epsilon(k)\qquad\text{for } k'=\pm\frac{1}{2},\pm\frac{3}{2},\ldots,\pm\frac{s-1}{2}. \tag{62}$$

As $s\to\infty$ we introduce a smooth continuum variable $x=k'/s$. The excitation density may now be defined as $\rho(u)=\mathrm{d}x/\mathrm{d}u$. We divide (60) by s, use (62), replace the sums by integrals, and, finally, take a derivative w.r.t. u. Note that we do *not* rescale u by $1/s$. Then (60) becomes

$$\frac{L}{s}\,\frac{1}{u^2+\frac{1}{4}}+\frac{\mathrm{i}\,L}{s}\,\frac{\mathrm{d}}{\mathrm{d}u}\log\left(\frac{1+g^2/2(x^-(u))^2}{1+g^2/2(x^+(u))^2}\right)$$
$$=2\,\pi\,\rho(u)+\frac{2\,\pi}{s}\,(L-2)\,\delta(u)-2\int_{-b}^{b}\mathrm{d}u'\,\frac{\rho(u')}{(u-u')^2+1}$$
$$+\,2\mathrm{i}\int_{-b}^{b}\mathrm{d}u'\,\rho(u')\,\frac{\mathrm{d}}{\mathrm{d}u}\,\log\left(\frac{1-g^2/2x^+(u)x^-(u')}{1-g^2/2x^-(u)x^+(u')}\right). \tag{63}$$

It is convenient to split the density $\rho(u)$ into a one-loop piece $\rho_0(u)$ and a higher loop piece $\tilde{\sigma}(u)$: $\rho(u)=\rho_0(u)+g^2\,\tilde{\sigma}(u)$. Let us, accordingly, also split off from (63) the one-loop contribution

$$\frac{L}{s}\,\frac{1}{u^2+(1/4)}=2\,\pi\,\rho_0(u)+\frac{2\,\pi}{s}\,(L-2)\,\delta(u)-2\int_{-s/2}^{s/2}\mathrm{d}u'\,\frac{\rho_0(u')}{(u-u')^2+1}, \tag{64}$$

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while the higher loop (two-loop level and beyond) part of (63) becomes

$$0 = 2\pi\,\tilde\sigma(u) - 2\int_{-\infty}^{\infty} \mathrm{d}u'\,\frac{\tilde\sigma(u')}{(u-u')^2+1} + \frac{2\mathrm{i}}{g^2}\int_{-s/2}^{s/2}\mathrm{d}u'\,\rho_0(u')\,\frac{\mathrm{d}}{\mathrm{d}u}\,\log\left(\frac{1-g^2/2x^+(u)x^-(u')}{1-g^2/2x^-(u)x^+(u')}\right) + 2\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}u'\,\tilde\sigma(u')\,\frac{\mathrm{d}}{\mathrm{d}u}\,\log\left(\frac{1-g^2/2x^+(u)x^-(u')}{1-g^2/2x^-(u)x^+(u')}\right). \tag{65}$$

We have dropped the second term on the lhs of (63), as it is easily seen to be suppressed to leading order in the large s limit. This reflects the independence of the large s scaling behaviour of the lowest state on the twist L even beyond the one-loop approximation, as long as $L \ll s$. We have also extended the range of integration of the second and fourth integrals in (65) from $\pm s/2$ to $\pm\infty$, to be justified below.

As a consistency check of our procedure let us rederive the one-loop solution of the previous section from (64). There we used rescaled variables $\bar u = u/s$, and a rescaled density $\bar\rho_0(\bar u) = s\,\rho_0(u)$ such that $\mathrm{d}\bar u\,\rho_0(\bar u) = \mathrm{d}u\,\rho_0(u)$. Using the large s expansions

$$\frac{1}{2\,s}\,\frac{1}{\bar u^2+1/4s^2} = \pi\,\delta(\bar u) + \mathcal{O}\left(\frac{1}{s}\right), \tag{66}$$

$$\frac{1}{s}\,\frac{1}{(\bar u-\bar u')^2+1/s^2} = \pi\,\delta(\bar u-\bar u') + \frac{1}{s}\,\frac{\mathcal{P}}{(\bar u-\bar u')^2} + \mathcal{O}\left(\frac{1}{s^2}\right), \tag{67}$$

where $\mathcal{P}$ indicates a principal part, we find from (64)

$$0 = 4\,\pi\,\delta(\bar u) + 2\,\fint_{-\bar b}^{\bar b}\mathrm{d}\bar u'\,\frac{\bar\rho_0(\bar u')}{(\bar u-\bar u')^2}, \tag{68}$$

which is, since $\epsilon'(\bar u) = 2\,\delta(\bar u)$, precisely the derivative of the one-loop singular integral equation (52). Note that the L dependence has indeed again dropped out. We therefore find the same one-loop result as in the previous section. It should be stressed that, even though the kernel in (64) is of difference form, and the interval boundary values tend to $\pm\infty$, it is *incorrect* to solve this equation by naive Fourier techniques.

Luckily, however, applying a Fourier transform leads to progress with the *higher loop* equation (65). The reason is that the higher loop density fluctuations $\tilde\sigma(u)$ are concentrated in the vicinity of $u = 0$, i.e. $\tilde\sigma(u) \neq 0$ iff $|u| \ll s/2$. This may be verified for twist $L = 2$ operators by using the exact one-loop solution of appendix C, and numerically solving the linear problem of computing the higher loop corrections to the roots of the Hahn polynomials from the Bethe equations (19). We were thus indeed entitled to replace the integral boundaries $\pm s/2$ by $\pm\infty$ in the second and fourth terms on the rhs of (65). The 'scale' of the fluctuations $\tilde\sigma(u)$ is set by the third term on the rhs of (65). Let us calculate it, using $\rho_0(u) = \bar\rho_0(\bar u)/s$, with $\bar\rho_0(\bar u)$ given by (53):

$$\begin{aligned}\frac{2\mathrm{i}}{g^2}\int_{-s/2}^{s/2}\mathrm{d}u'\,\rho_0(u')\,\frac{\partial}{\partial u}\,\log\left(\frac{1-g^2/2x^+(u)x^-(u')}{1-g^2/2x^-(u)x^+(u')}\right)&\\ = -\frac{2\mathrm{i}}{g^2}\sum_{r=1}^{\infty}\frac{1}{r}\left(\frac{g^2}{2}\right)^r\int_{-s/2}^{s/2}\mathrm{d}u'\,\rho_0(u')\,\frac{\partial}{\partial u}\left[\frac{1}{x^+(u)^r}\,\frac{1}{x^-(u')^r} - \frac{1}{x^-(u)^r}\,\frac{1}{x^+(u')^r}\right]&\\ = \frac{E_0}{s}\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}u}\right)\left[\frac{1}{x^+(u)} + \frac{1}{x^-(u)}\right] + \cdots,&\end{aligned} \tag{69}$$

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where we have only kept the leading contribution. Note that only the first, $r = 1$ term in the expansion of the logarithm contributes to this result, and we have used, cf. (55), (56), the relation

$$\int_{-s/2}^{s/2} \mathrm{d}u'\, \rho_0(u')\, \frac{1}{x^{\pm}(u')} = \int_{-s/2}^{s/2} \mathrm{d}u'\, \rho_0(u')\, \frac{1}{u' \pm (\mathrm{i}/2)} + \cdots$$
$$= \frac{1}{s}\, G\left(\frac{\mp \mathrm{i}}{2s}\right) + \cdots = \frac{\mp \mathrm{i}}{2s}\, E_0 + \cdots, \tag{70}$$

which is valid to leading order at large s. It is now clear from (58) that $E_0/s \simeq 4\log(s)/s$ sets the scale of the density fluctuation $\tilde{\sigma}(u)$ in (65). We therefore define $\tilde{\sigma}(u) = -(E_0/s)\,\sigma(u)$, i.e.

$$\rho(u) = \rho_0(u) - g^2\, \frac{E_0}{s}\, \sigma(u). \tag{71}$$

To this leading order, the density fluctuation does not change the density normalization $\int_{-b}^{b} \mathrm{d}u\, \rho(u) = 1$, i.e. $\int_{-b}^{b} \mathrm{d}u\, \rho_0(u) = 1 + \cdots$ since $\lim_{s\to\infty} E_0/s = 0$; see (58). Then (65) becomes

$$\begin{aligned} 0 = {} & 2\pi\, \sigma(u) \\ & - 2 \int_{-\infty}^{\infty} \mathrm{d}u'\, \frac{\sigma(u')}{(u-u')^2 + 1} \\ & - \left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}u}\right) \left[\frac{1}{x^+(u)} + \frac{1}{x^-(u)}\right] \\ & + 2\mathrm{i} \int_{-\infty}^{\infty} \mathrm{d}u'\, \sigma(u')\, \frac{\partial}{\partial u}\, \log\left(\frac{1 - g^2/2x^+(u)x^-(u')}{1 - g^2/2x^-(u)x^+(u')}\right). \end{aligned} \tag{72}$$

We now introduce the Fourier transform $\hat{\sigma}(t)$ of the fluctuation density $\sigma(u)$:

$$\hat{\sigma}(t) = \mathrm{e}^{-t/2} \int_{-\infty}^{\infty} \mathrm{d}u\, \mathrm{e}^{-\mathrm{i}tu}\, \sigma(u), \tag{73}$$

where we have also included a factor $\mathrm{e}^{-t/2}$ for notational convenience. Fourier transforming $\mathrm{e}^{-t/2} \int_{-\infty}^{\infty} \mathrm{d}u\, \mathrm{e}^{\mathrm{i}tu} \times$ equation (72) we find, after some calculation (see appendix D),

$$0 = 2\pi\, \hat{\sigma}(t) - 2\pi \mathrm{e}^{-t}\, \hat{\sigma}(t) - 2\pi \mathrm{e}^{-t}\, \frac{J_1(\sqrt{2}\, g\, t)}{\sqrt{2}\, g} + 4\pi\, g^2\, t \mathrm{e}^{-t} \int_0^{\infty} \mathrm{d}t'\, \hat{K}(\sqrt{2}\, g\, t, \sqrt{2}\, g\, t')\, \hat{\sigma}(t'), \tag{74}$$

where the four terms in (74) correspond, respectively, to the four terms in (72), and the kernel $\hat{K}$ is given in terms of Bessel functions by

$$\hat{K}(\sqrt{2}\, g\, t, \sqrt{2}\, g\, t') = \frac{1}{\sqrt{2}\, g}\, \frac{J_1(\sqrt{2}\, g\, t)\, J_0(\sqrt{2}\, g\, t') - J_0(\sqrt{2}\, g\, t)\, J_1(\sqrt{2}\, g\, t')}{t - t'}. \tag{75}$$

Note that the Fourier transform only diagonalizes the 'main' scattering term in (72), i.e. the kernel $1/((u-u')^2 + 1)$. So we are still left with an integral equation. However, the higher loop equation (74) is, in view of (75), and in contradistinction to the one-loop

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equation (52), *non-singular*. It may be rewritten in the form (8) stated in the introduction. Finally, the all-loop energy is found from (20) to be

$$\begin{aligned} E(g) &= s \int_{-s/2}^{s/2} \mathrm{d}u\, \rho(u) \left(\frac{\mathrm{i}}{x^+(u)} - \frac{\mathrm{i}}{x^-(u)} \right) + \cdots \\ &= E_0 - g^2\, E_0 \int_{-\infty}^{\infty} \mathrm{d}u\, \sigma(u) \left(\frac{\mathrm{i}}{x^+(u)} - \frac{\mathrm{i}}{x^-(u)} \right) + \cdots \end{aligned} \tag{76}$$

to leading order in s. In terms of the Fourier transformed density $\hat{\sigma}(t)$, cf (73), this becomes (see again appendix D)

$$E(g) = E_0 \left(1 - 4\, g^2 \int_0^{\infty} \mathrm{d}t\, \hat{\sigma}(t)\, \frac{J_1(\sqrt{2}\, g\, t)}{\sqrt{2}\, g\, t} \right) + \cdots \tag{77}$$

with $E_0 = 4\log(s) + \cdots$. Notice that, in line with general expectations, we have just shown that the Bethe ansatz of [20] indeed leads to the logarithmic scaling behaviour (5) *to all orders in perturbation theory*, in agreement with general expectations; see e.g. the discussions in [10, 39, 27]. In view of (5), (15), (58) this indeed yields our proposed conjecture for the all-loop scaling function $f(g)$ announced in (7). The proposed scaling function as found from the Bethe ansatz possesses further remarkable properties, to which we will now turn our attention.

3.3. Weak coupling expansion and transcendentality

The Fredholm form of the higher loop integral equation (8) or (74) is ideally suited for the explicit perturbative expansion of the scaling function $f(g)$ of (5) to high orders. Both the inhomogeneous first term and the kernel of (8) have a regular expansion in even powers of g around $g = 0$. We may therefore also expand the transformed density $\hat{\sigma}(t)$ in even powers of g and solve (8) iteratively:

$$\hat{\sigma}(t) = \frac{1}{2}\, \frac{t}{\mathrm{e}^t - 1} - g^2 \left(\frac{1}{8}\, \frac{t^3}{\mathrm{e}^t - 1} + \frac{1}{2}\, \zeta(2)\, \frac{t}{\mathrm{e}^t - 1} \right) + \cdots, \tag{78}$$

where we have used the following representation of the Riemann zeta function:

$$\zeta(n+1) = \frac{1}{n!} \int_0^{\infty} \frac{\mathrm{d}t\, t^n}{\mathrm{e}^t - 1}. \tag{79}$$

Furthermore, the expression for the scaling function (7) may also be expanded in a Taylor series in g^2:

$$f(g) = 4\, g^2 - 4\, g^4 \int_0^{\infty} \frac{\mathrm{d}t\, t}{\mathrm{e}^t - 1} + g^6 \left(2 \int_0^{\infty} \frac{\mathrm{d}t\, t^3}{\mathrm{e}^t - 1} + 4\, \zeta(2) \int_0^{\infty} \frac{\mathrm{d}t\, t}{\mathrm{e}^t - 1} \right) + \cdots. \tag{80}$$

We again use (79) and we find to e.g. six-loop order

$$\begin{aligned} f(g) = {} & 4\, g^2 - 4\, \zeta(2)\, g^4 + (4\, \zeta(2)^2 + 12\, \zeta(4))\, g^6 \\ & - (4\, \zeta(2)^3 + 24\, \zeta(2)\zeta(4) - 4\, \zeta(3)^2 + 50\, \zeta(6))\, g^8 \\ & + (4\, \zeta(2)^4 + 36\, \zeta(2)^2\zeta(4) - 8\, \zeta(2)\zeta(3)^2 + 100\, \zeta(2)\zeta(6) \\ & \quad - 40\, \zeta(3)\zeta(5) + 39\, \zeta(4)^2 + 245\, \zeta(8))\, g^{10} \end{aligned}$$

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$$
\begin{aligned}
&- (4\,\zeta(2)^5 + 48\,\zeta(2)^3\zeta(4) - 12\,\zeta(2)^2\zeta(3)^2 + 150\,\zeta(2)^2\zeta(6) \\
&\quad - 80\,\zeta(2)\zeta(3)\zeta(5) + 114\,\zeta(2)\zeta(4)^2 + 490\,\zeta(2)\zeta(8) - 18\,\zeta(3)^2\zeta(4) \\
&\quad - 210\,\zeta(3)\zeta(7) + 345\,\zeta(4)\zeta(6) - 102\,\zeta(5)^2 + 1323\,\zeta(10))\,g^{12} + \cdots . \qquad (81)
\end{aligned}
$$

It is easy to go to much higher orders if desired (we have expanded to 20-loop order g^{40}). It is seen that the ℓ-loop $\mathcal{O}(g^{2\ell})$ contribution to the anomalous dimension is a sum of products of zeta functions. What is more, the arguments of the zeta functions of each product always add up to the number $2\,\ell - 2$. This is a test of the *'transcendentality principle'* of Kotikov, Lipatov, Onishchenko and Velizhanin as spelled out in [7, 8, 10], and we see that our Bethe ansatz is *consistent* with this principle[7]. Finally it is also seen that the numerical coefficients in front of each zeta function product are integers[8].

Note that the expansion (81) may be written more compactly when expressing the zeta functions of even arguments through powers of π times rational numbers:

$$
\begin{aligned}
f(g) = 4\,g^2 &\\
&- \tfrac{2}{3}\,\pi^2\,g^4 \\
&+ \tfrac{11}{45}\,\pi^4\,g^6 \\
&- \left(\tfrac{73}{630}\,\pi^6 - 4\,\zeta(3)^2\right)\,g^8 \\
&+ \left(\tfrac{887}{141\,75}\,\pi^8 - \tfrac{4}{3}\,\pi^2\,\zeta(3)^2 - 40\,\zeta(3)\zeta(5)\right)\,g^{10} \\
&- \left(\tfrac{136\,883}{3742\,200}\,\pi^{10} - \tfrac{8}{15}\,\pi^4\,\zeta(3)^2 - \tfrac{40}{3}\,\pi^2\,\zeta(3)\zeta(5)\right. \\
&\qquad \left. - 210\,\zeta(3)\zeta(7) - 102\,\zeta(5)^2\right)\,g^{12} \\
&+ \cdots . \qquad (82)
\end{aligned}
$$

Clearly each π contributes one 'unit' of transcendentality. This however obscures the integer nature of the numerical coefficients (cf footnote 8).

It is instructive to investigate whether the (BMN scaling-preserving) 'AFS' dressing factor [43, 18, 20] for the (approximate; see [44]–[46]) *string*[9] Bethe ansatz (19), (20) is compatible with the transcendentality principle. Possible (BMN scaling violating) gauge dressing factors are briefly treated in section 3.4.

The AFS ansatz leads to a modification, at three-loop level and beyond, of the integral equation (8)

$$
\hat{\sigma}(t) = \frac{t}{\mathrm{e}^t - 1}\left[K'(\sqrt{2}g\,t, 0) - 2\,g^2\int_0^\infty \mathrm{d}t'\,K'(\sqrt{2}g\,t, \sqrt{2}g\,t'), \tilde{\sigma}(t')\right], \qquad (83)
$$

[7] To be more precise, here we have tested a weaker form of the transcendentality principle of [10]. The stronger form applies to the finite s case, and states that the indices of certain harmonic sums add up to $2\ell - 1$. We suspect that our all-loop Bethe ansatz is also consistent with the stronger version; see also [18]. Our finding certainly supports this, as the weaker principle is a consequence of the stronger one. It would be exciting to fully prove the latter from the $L = 2$ finite s Bethe equations (19), (20).

[8] Actually, with our convention (2), higher terms beyond the order we have printed in (81) develop powers of 2 in the denominator. We however checked up to order g^{40} that our scheme does indeed yield integer numbers in front of the zeta functions if g is rescaled as $g \to \sqrt{2}\,g$, which is the Lipatov *et al* convention.

[9] Our motivation here is not so much string theory as such (in particular we investigate the dressing factor at weak coupling, while its original design demands strong coupling), but rather the fact that such dressing factors are known to naturally appear in certain variant, asymptotically integrable spin chains [47]. While these studies were done for compact magnets, it is likely that they may be generalized to the non-compact case of interest in this paper. The variant models tend to violate the Feynman rules of the gauge field theory, which is our main motivation for investigating whether they preserve the transcendentality principle.

where the modified kernel K', see appendix D, reads

$$K'(\sqrt{2}g\,t,\sqrt{2}g\,t') = \hat{K}(\sqrt{2}g\,t,\sqrt{2}g\,t') + \sqrt{2}g\,\tilde{K}(\sqrt{2}g\,t,\sqrt{2}g\,t'), \tag{84}$$

with

$$\tilde{K}(t,t') = \frac{t(J_2(t)\,J_0(t') - J_0(t)\,J_2(t'))}{t^2 - t'^2}, \qquad \tilde{K}(t,0) = \frac{J_2(t)}{t}. \tag{85}$$

The dressing factor then modifies the scaling function $f(g) \to f(g) + \delta f(g)$ in the following fashion:

$$\begin{aligned}\delta f(g) = {} & 0 \times g^2 \\ & 0 \times g^4 \\ & - \; 4\,\zeta(3)\,g^6 \\ & + \; (\tfrac{4}{3}\,\pi^2\,\zeta(3) + 20\,\zeta(5))\,g^8 \\ & - \; (\tfrac{23}{45}\,\pi^4\,\zeta(3) + \tfrac{20}{3}\,\pi^2\,\zeta(5) + 105\,\zeta(7) - 4\,\zeta(3)^2)\,g^{10} \\ & + \; (\tfrac{71}{315}\,\pi^6\,\zeta(3) + \tfrac{79}{30}\,\pi^4\,\zeta(5) + 35\pi^2\,\zeta(7) - 8\,\zeta(3)^3 + 588\,\zeta(9) \\ & - \; 2\,\pi^2\,\zeta(3)^2 - 36\,\zeta(3)\zeta(5))\,g^{12} \\ & + \; \cdots \end{aligned} \tag{86}$$

We see that the integrable modification of the long-range Bethe ansatz of [20] by the 'stringy' AFS [43] dressing factor *violates* the transcendentality principle[10], as now the arguments of the Riemann zeta functions no longer add up to $2\,\ell - 2$.

3.4. Breakdown of BMN scaling and the scaling function

Here we will demonstrate interesting connections between *BMN scaling* [28] on the one hand and our Bethe ansatz method for the scaling function on the other. It is by now rather firmly established that BMN scaling in perturbative gauge theory can only break down, at four-loop level or beyond, through a dressing factor of the general type just discussed; see in particular [47, 21]. This happens in e.g. the plane wave matrix model; see [48].

Let us sketch the quantitative derivation of this effect, restricting ourselves for simplicity to four-loop level, where its detection might still be within reasonable reach of sophisticated field theory methods, perhaps along the lines of [2].

The first modification of the asymptotic Bethe equations of [20] which is still consistent with current knowledge on the integrable structure of $\mathcal{N} = 4$ gauge theory would lead to the following correction of the higher loop Bethe equations (19):

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{s} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-}\,\frac{1 - g^2/2x_k^+x_j^-}{1 - g^2/2x_k^-x_j^+}\,\sigma^2(u_k,u_j) \tag{87}$$

[10] For the gauge theory ansatz the transcendentality principle is a consequence of scaling: the arguments of the potential and kernel in (74) are $\sqrt{2}g\,t, \sqrt{2}g\,t'$ so that the order in g is linked to the total power of t and t' which defines the level of transcendentality. The string theory ansatz (83) breaks the pattern only because of the presence of the extra $\sqrt{2}g$ in front of K' in equation (84). Initially this introduces a mismatch by one unit; by iteration the effect fans out higher up in the perturbative expansion.

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with

$$\sigma^2(u_k, u_j) = \exp(\mathrm{i}\,\beta\, g^6\,(q_2(u_k)\,q_3(u_j) - q_3(u_k)\,q_2(u_j)) + \cdots); \tag{88}$$

see [43, 20, 47, 45, 21] for details[11], and the definition of the charges $q_r(u)$. The dots indicate further terms which might affect the five-loop level and higher levels.

A non-zero value for β leads to 'soft breaking' of BMN scaling: the two-excitation problem can be solved exactly [11], because the momentum constraint implies $u_2 = -u_1$. For spin chain length $L = J$, where J is the BMN R-charge, the different states are distinguished by the Bethe roots $u_{1,n} = \frac{1}{2}\cot(\pi n/(J+1))$. The higher order corrections similarly come out in terms of trigonometric functions. The string spectrum $\Delta - J$ is reproduced by Taylor expanding in $1/J$ when n is small:

$$\begin{aligned}\Delta - J = 2 + 8\,g^2\left(\frac{n\pi}{J}\right)^2 - 16\,g^4\left(\frac{n\pi}{J}\right)^4 + 64\,g^6\left(\frac{n\pi}{J}\right)^6 \\ - 320\,g^8\left(\frac{n\pi}{J}\right)^8 - 512\,g^8\frac{\beta}{J}\left(\frac{n\pi}{J}\right)^6 + \cdots.\end{aligned} \tag{89}$$

The dots stand for higher orders in g^2 and, at any given order, terms subleading in $1/J$. We see the emergence of the effective coupling constant g^2/J^2 in the first four terms of the last formula, while the last term has g^8/J^7, so that it diverges in the BMN limit $g, J \to \infty$ with g/J fixed.

The modified Bethe ansatz (87) requires replacing the kernel in (75) by

$$\hat{K}(\sqrt{2}g\,|t|, \sqrt{2}g\,|t'|) \to \hat{K}(\sqrt{2}g\,|t|, \sqrt{2}g\,|t'|) + 2\,\beta\,(\sqrt{2}g)\,\frac{J_2(\sqrt{2}\,g\,|t|)\,J_1(\sqrt{2}\,g\,|t'|)}{|t\,t'|} + \cdots. \tag{90}$$

The third term on the rhs of (72) becomes

$$-\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}u}\right)\left[\frac{1}{x^+(u)} + \frac{1}{x^-(u)}\right] - \beta\,g^4\,\frac{\mathrm{d}}{\mathrm{d}u}\,q_3(u) + \cdots, \tag{91}$$

or, after Fourier transforming (cf third term in (74)),

$$-2\pi\mathrm{e}^{-t}\left(\frac{J_1(\sqrt{2}\,g\,t)}{\sqrt{2}\,g} + 2\,\beta\,g^2\,J_2(\sqrt{2}\,g\,t)\right) + \cdots. \tag{92}$$

This modifies the four-loop $\mathcal{O}(g^8)$ term of the scaling function (82) to

$$-(\tfrac{73}{630}\,\pi^6 - 4\,\zeta(3)^2 + 8\,\beta\,\zeta(3)). \tag{93}$$

Note that transcendentality is violated unless β is a rational number times $\zeta(3)$ (or π^3). A particularly curious case would be $\beta = \frac{1}{2}\zeta(3)$, which would lead to the much simpler four-loop answer $-\frac{73}{630}\,\pi^6$.

Note that such a modified Bethe ansatz would also change the anomalous dimensions of all operators in other sectors. For example, in the $\mathfrak{su}(2)$ sector we would find for the

[11] The detailed argumentation which allows us to draw this conclusion is actually rather subtle and requires putting together various results. The main steps are: (1) the three-loop Bethe ansatz is solidly known; (2) the structure of the four-loop Bethe ansatz is also known, up to the term involving β in (87), in the $\mathfrak{su}(2)$ sector [47]; (3) the multiplicative modification affecting the $\mathfrak{su}(2)$ sector as in (87) must also multiplicatively affect in the same fashion the $\mathfrak{sl}(2)$ sector, as first conjectured in [18] and later proved in [21].

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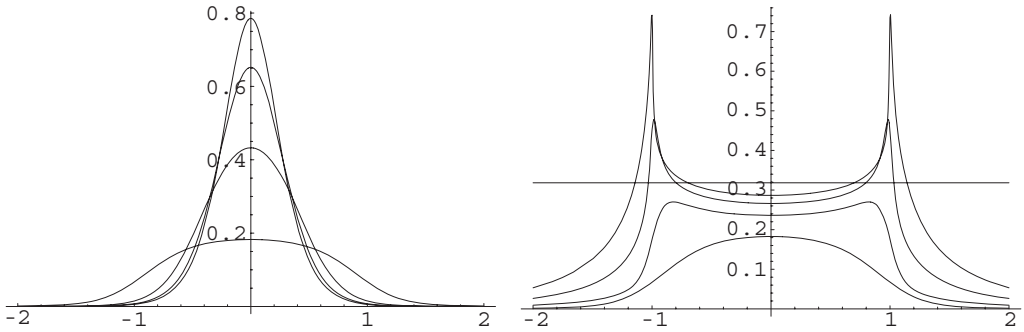

Figure 1. The density of Bethe roots at weak (left) and strong coupling (right).

length $L = 5$ operator $\mathrm{Tr}\, X^2\, Z^3 + \cdots$ (this case is actually equivalent to the $\mathfrak{sl}(2)$ twist 3 operator $\mathrm{Tr}\, D^2\, Z^3 + \cdots$) to four-loop level

$$E(g) = 4 - 6\, g^2 + 17\, g^4 - (\tfrac{115}{2} + 8\, \beta)\, g^6 + \cdots . \tag{94}$$

It would be very interesting if the modification were non-rational[12]. Incidentally, we see that $\beta \neq 0$ would also rule out the Hubbard Hamiltonian as a candidate for the $\mathfrak{su}(2)$ dilatation operator beyond three-loop order; cf equation (68) in [22].

3.5. Strong coupling expansion and string theory

The Fourier transformed integral equation (74) does not lend itself to strong coupling analysis due to the oscillatory nature of the kernel (75). We rather return to the configuration space integral equation (72).

The two diagrams in figure 1 give a series of plots of the root density for progressively higher values of the coupling constant. The left picture shows the weak coupling regime; the graphs depict the root density at $\sqrt{2}g = 0, 1/4, 1/2, 1$, respectively. The $\sqrt{2}g = 0$ distribution is the tallest peak. It is given by the Fourier back-transform of the first term in (78):

$$\sigma_0(u) = \frac{\pi}{4}\, \frac{1}{\cosh^2(\pi\, u)}. \tag{95}$$

All other curves are numerical solutions of (72). Augmenting the coupling constant makes the peak around $u = 0$ become wider and flatter.

In the second diagram we plotted $2g^2\, \sigma(u/(\sqrt{2}g))$ for $\sqrt{2}g = 1, 4, 16, 64$. With increasing coupling the graphs rise; they develop peaks at ± 1 while the middle parts tend to $1/\pi$. On undoing the scaling we would nevertheless recover the tendency seen at weak coupling, i.e. the support of the root density roughly stretches to the interval

[12] Clearly the Bethe ansätze [20] proposed so far also lead to a transcendentality principle at weak coupling: if we assign, in accordance with the meaning of the word, transcendentality degree 0 to rational or algebraic numbers, then weak coupling dimensions of operators carrying finite charges (i.e. without taking limits of large R-charges or large spin quantum numbers) are always of zero degree in the currently proposed ansätze. On the other hand, zeta functions do appear naturally in individual higher loop Feynman diagrams, and, from this point of view, might well appear in high order contributions to anomalous dimensions.

$[-\sqrt{2}g, \sqrt{2}g]$ within which the density tends to

$$\sigma_\infty(u) = \frac{1}{2\pi g^2}. \tag{96}$$

Note that the constant function $\sigma(u) = 1/(2\pi g^2)$ is an exact solution of (72) if the support is extended to the entire real axis (likewise $\sigma(t) = \delta(t)/g^2$ is a solution of (74)). Furthermore, $\sigma_\infty(u)$ would exactly cancel the leading $\mathrm{O}(g^2)$ contribution to the scaling function (77), thus yielding the $\mathrm{O}(g)$ asymptotics expected from string theory.

Numerically, we could confirm the cancellation of the $\mathrm{O}(g^2)$ part of the scaling function up to an error of a few per cent, but reliable predictions for subleading terms remained out of reach. It is indispensable to understand the strong coupling regime by analytic means. We hope to clarify the issue in future work.

Acknowledgments

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Appendix A. Two-point functions in the $\mathfrak{sl}(2)$ sector

A.1. Perturbative CFT in the dimensional reduction scheme

We shall restrict our attention to leading N (planar) two-point functions of single-trace operators in the $\mathfrak{sl}(2)$ sector. For any given spin chain with length = twist L there are many distinct operators differing in the total number of derivatives and their positioning on the sites of the chain.

Renormalization must be done in such a way as to unmix these states and to make their correlators finite. The theory is then seen to be conformally invariant; for example the two-point function of a renormalized primary operator of spin s has the form [49]

$$\langle P^s(1)\bar{P}^s(2)\rangle = \frac{c(g^2)J_{\mu_1\nu_1}(x_{12})\dots J_{\mu_s\nu_s}(x_{12})}{(x_{12}^2)^{\Delta(g^2)}}, \tag{A.1}$$

where

$$J_{\mu\nu}(x) = \eta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2} \tag{A.2}$$

is the inversion tensor, and the μ and ν indices are separately made traceless and symmetric. Knowledge of any one term in the product of inversion tensors is sufficient for reconstructing the full correlator. In [38] we considered the term with no η symbol, because we were interested in a minimal set of graphs (trace terms are potentially more divergent and there are also a few Feynman diagrams which always carry at least one power of η on dimensional grounds). In the present work we wish to construct the asymptotic two-loop

dilatation operator in the $\mathfrak{sl}(2)$ sector. The task is greatly simplified by focusing on the pure trace terms, because these obviously cannot exist between operators of different spin. This property becomes important when subtracting out disconnected parts.

As before, we use $\mathcal{N} = 2$ superfields and regularize by SSDR (supersymmetric dimensional reduction) [36] in x-space. This amounts to doing the superalgebra as in four dimensions, while the underlying scalar propagator is modified as in standard dimensional regularization:

$$\langle Z(1)\bar{Z}(2)\rangle = \frac{c_0}{x_{12}^2}(\mu x_{12}^2)^\epsilon, \qquad c_0 = -\frac{1}{4\pi^2}, \qquad \Box_1\langle Z(1)\bar{Z}(2)\rangle = \delta(x_{12}), \tag{A.3}$$

although we suppress the mass scale μ throughout the paper[13]. The tree level correlators of operators of length L and spin s thus contain the x-space structure

$$X(L,s) = \frac{N^L \eta_{z\bar{z}}^s}{(-4\pi^2)^L (x_{12}^2)^{L(1-\epsilon)+s}} \tag{A.4}$$

and a whole series of terms with x_{12} with open indices, which may be recovered by appealing to conformal invariance.

In order to extract the one- and two-loop anomalous dimensions we must keep track of the leading and subleading order in the ϵ expansion of the bare correlators:

$$\begin{aligned}\langle\, \mathcal{O}_i\, \bar{\mathcal{O}}_j\,\rangle = X(L,s)\Bigg[&(T_{0\,ij} + \epsilon\, T_{1\,ij}) + g^2\left(A_{11\,ij}\,\frac{1}{\epsilon} + A_{10\,ij}\right)(x_{12}^2)^\epsilon \\ &+ g^4\left(A_{22\,ij}\,\frac{1}{\epsilon^2} + A_{21\,ij}\,\frac{1}{\epsilon} + A_{20\,ij}\right)(x_{12}^2)^{2\epsilon} + \cdots\Bigg]\end{aligned} \tag{A.5}$$

where the Yang–Mills coupling constant is dressed by[14]

$$g^2 = \frac{g_{YM}^2 N}{8\pi^2} \tag{A.6}$$

and the fractional powers of x_{12}^2 arise from the integration measure in the Feynman graphs defining the one- and two-loop contributions.

Consistency of $\mathcal{N} = 4$ as a conformal field theory grants that T_0, A_{11}, A_{22} are simultaneously diagonalizable. In a diagonal basis $\{O_i\}$ they obey

$$A_{11} = \Gamma_1\, T_0, \qquad A_{22} = \tfrac{1}{2}\,\Gamma_1^2\, T_0. \tag{A.7}$$

Here Γ_1 is also diagonal and contains the one-loop anomalous dimensions $\gamma_{1\,i}$.

The divergences are removed by introducing Z matrices of the form

$$Z = R + g^2\, B + g^4\left(C_1\,\frac{1}{\epsilon} + C_0\right) + \cdots \tag{A.8}$$

where R is diagonal and has as its entries the Z factors for the individual operators

$$\mathcal{Z}_i = 1 + g^2\,\frac{z_{11\,i}}{2\epsilon} + g^4\left(\frac{z_{22\,i}}{4\epsilon^2} + \frac{z_{21\,i}}{4\epsilon}\right) + \cdots \tag{A.9}$$

[13] Our discussion of the renormalization of conformal correlators in x-space using the SSDR scheme is built upon the works [17, 50, 38].

[14] We deviate from the convention in [38] by a coupling constant rescaling so as to be more in line with the literature.

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while B, C have zero on the diagonal. The Z factors and the anomalous dimensions are determined from the bare two-point functions by imposing

$$F = Z \, \langle \mathcal{O} \, \bar{\mathcal{O}} \rangle \, Z^\dagger \tag{A.10}$$

where F is again diagonal and is defined by the renormalized two-point functions

$$f_i = X(L,s)|_{\epsilon=0} \, (a_{0\,i} + g^2 \, a_{1\,i} + g^4 \, a_{2\,i}) \, (x_{12}^2)^{-g^2 \, \gamma_{1\,i} - g^4 \, \gamma_{2\,i}} + \cdots . \tag{A.11}$$

To be more precise, we demand that both sides be equal at each order in g^2 up to positive powers of ϵ. The resulting system of equations does not completely fix C_1, C_0, so that we limit our scope to the determination of $R, B, a_{0i}, a_{1i}, \gamma_{1i}, \gamma_{2i}$. We may thus drop the constant part A_{20} of the g^4 two-point functions from our analysis.

A.2. Graphs

We exploit the $\mathcal{N} = 2$ superfield formalism in order to minimize the number of Feynman diagrams. For a quick review of the essentials of the formalism and expressions for the graphs we would like to refer the reader to [38], where two-loop two-point functions of operators of length 3 are discussed. Our notation and conventions are in fact borrowed from that work; in particular, the article contains a list of graphs upon which we draw here. However, in [38] the $(x_z x_{\bar{z}})^s$ term of the two-point functions was used, so that some graphs could be omitted because they always come with $\eta_{z\bar{z}}$.

At order g^2, we additionally have to take into account a graph F (see figure A.2) in which a free vector line goes from the connection in D_z on the left end of the two-point function to that in $D_{\bar{z}}$ on the right (the Feynman gauge vector propagator is proportional to η). Correspondingly, there is an $\mathrm{O}(g^4)$ graph consisting of the same free line paired with the divergent one-loop graph G_0. On the other hand, we do not need to consider the combination of the free vector line with the 'BPS-like' $\mathrm{O}(g^2)$ integral B_0 since this configuration stays finite. Next, in [38] we were able to drop the product $G_0 * B_0$ as G_0 only has a simple pole (in x-space) while the part of B_0 without η is a contact term also when there are partial derivatives on the outer legs, i.e. it is always $\mathrm{O}(\epsilon)$. Terms in B_0 which involve $\eta_{z\bar{z}}$ are finite, i.e. $\mathrm{O}(1)$, so that in the present context the product $G_0 * B_0$ becomes relevant.

With respect to the genuine two-loop integrals there are not many changes: the finiteness of some terms which we dropped from graph G_3 remains guaranteed and hence we may take over the simplified sum $G_3 + G_4$ given in formula (61) in [38]. The 'BPS-like' graphs behave in the same manner as B_0: the part without η is a contact term and the other parts are finite. They can still safely be omitted.

A first difference is that graphs G_{10} and G_{11} start to contribute: before, the poles from these graphs cancelled in the sum over all diagrams within each class; this is not the case in the new situation. We point out an error in formula (55) in the original version of [38]: two parts of the integral were added with a wrong relative sign. The cancellation of the associated poles in the calculation of [38] can be verified for both 'halves' on their own so that the mistake did not show. The correct expression for G_{10} is

$$G_{10} = (12)[\partial_{\nu 14}\partial_{\mu 23} + \eta_{\mu\nu}\Box_{34}/4 - (1 - (12)(1^-2^-))\partial_{\mu 13}\partial_{\nu 24}]. \tag{A.12}$$

But there are also three genuinely new graphs (see figure A.1).

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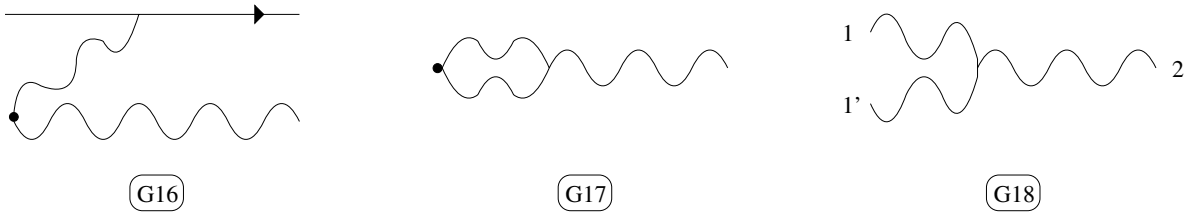

Figure A.1. Additional graphs at order g^4.

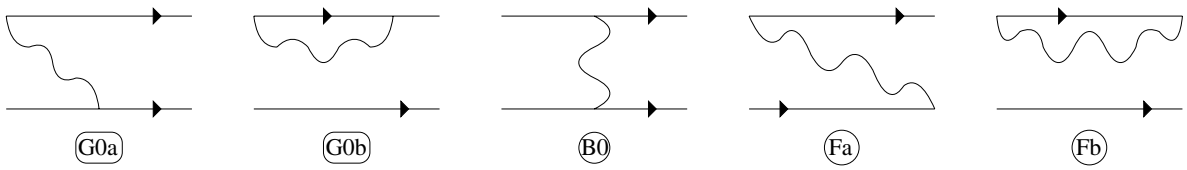

Figure A.2. Graphs defining the one-loop dilatation operator.

Like in the pictures in [38] we have omitted free matter propagators. Point 1 is on the left and point 2 on the right of the graphs. The connection carries the indices μ and ν there, respectively, while the connection at $1'$ has index ρ. The lines are split only for convenience of drawing—the notation $1'$ in G_{18} does not refer to a new point. It was introduced in order to distinguish the two vector propagators joining the cubic vertex from the left.

After the evaluation of Grassmann and $SU(2)$ integrations we find

$$G_{16} = (12)\,[-\eta_{\mu\nu}/4], \tag{A.13}$$

$$G_{17} = \eta_{\mu\nu}/2, \tag{A.14}$$

$$G_{18} = \mathrm{i}[(\partial_1 - \partial_{1'})_\nu\,\eta_{\mu\rho}/4 - (\partial_1 - \partial_2)_\rho\,\eta_{\mu\nu}/4 + (\partial_{1'} - \partial_2)_\mu\,\eta_{\nu\rho}/4\,]. \tag{A.15}$$

In the same way as graphs G_3, G_4 in [38] occur together, we may add G_{16} and its mirror image $\tilde{G}_{16}$ into G_{11} because their combinatorics are equal:

$$\begin{aligned} G_{11} + G_{16} + \tilde{G}_{16} = (12)\,[&-\partial_{\nu 13}\partial_{\mu 23} + \eta_{\mu\nu}(\partial_{13} - \partial_{23})\cdot(\partial_{14} - \partial_{24})/4 \\ &+ \eta_{\mu\nu}(\Box_{14} + \Box_{24})/4 + \cdots]. \end{aligned} \tag{A.16}$$

(The dots indicate omitted finite terms.)

The rest of the calculation proceeds along the same lines as before (appropriately adapted to the new tensor component), i.e. the reconstruction of the Fourier transform of integrals with open indices from projections with the total momentum q and the η symbol, which are built into *Mathematica* and evaluated by the *Mincer* package [37].

A.3. The length 3, spin 3 mixing problem

As an illustration of what has been said before we re-examine the mixing of the length 3 operators:

$$\{s_1, s_2, s_3\} = \mathrm{Tr}((D_z^{s_1} Z)(D_z^{s_2} Z)(D_z^{s_3} Z)) \tag{A.17}$$

at leading order in N. In particular, the spin 3 mixing problem involves the operators

$$\mathcal{B} = \{\{3,0,0\},\{2,1,0\},\{1,2,0\},\{1,1,1\}\}. \tag{A.18}$$

The one-loop logarithms and the constant order T_0 of the tree level correlators are diagonalized by choosing the directions

$$\begin{aligned}
\mathcal{O} &= \{1,3,3,2\},\\
\mathcal{K} &= \{1,-1,-1,-2\},\\
\mathcal{V}_1 &= \{2,-9,-9,24\},\\
\mathcal{V}_2 &= \{0,1,-1,0\}
\end{aligned} \tag{A.19}$$

relative to the basis $\mathcal{B}$. Note that $\mathcal{V}_1, \mathcal{V}_2$ have identical first anomalous dimension and therefore the eigenspace may be spanned by any two independent directions. We have made a splitting into an even and an odd part under reversal of the trace; as a consequence $\mathcal{V}_2$ decouples from the other operators. Renormalization in the $\overline{MS}$ scheme outlined above yields the anomalous dimensions

$$\begin{aligned}
\gamma_{\mathcal{O}} &= 0,\\
\gamma_{\mathcal{K}} &= g^2 4 - g^4 6,\\
\gamma_{\mathcal{V}_1} &= g^2\,\tfrac{15}{2} - g^4\,\tfrac{225}{16},\\
\gamma_{\mathcal{V}_2} &= g^2\,\tfrac{15}{2} - g^4\,\tfrac{225}{16},
\end{aligned} \tag{A.20}$$

up to terms of $\mathrm{O}(g^6)$. The individual $\mathcal{Z}_i$ are given by the anomalous dimensions in the standard way. As explained above, the system does not entirely determine the C matrices, while we can fix B:

$$B = \begin{pmatrix} 0 & 0 & 0 & 0\\ -\frac{1}{2} & 0 & 0 & 0\\ \frac{3}{4} & -\frac{165}{28} & 0 & \alpha\\ 0 & 0 & -315\,\alpha & 0 \end{pmatrix}. \tag{A.21}$$

The parameter α is not calculable from our system of equations because the anomalous dimensions of $\mathcal{V}_1, \mathcal{V}_2$ are degenerate. We may put it to zero, bearing in mind that an arbitrary remixing of the two operators is possible.

The anomalous dimensions and the entries of B are independent of whether we calculate the $(\eta_{z\bar{z}})^3$ terms as outlined in this paper or the $(x_z\, x_{\bar{z}})^3$ part of the correlators as in [38], although now in $\overline{MS}$.[15]

The operators $\mathcal{V}_1$ and $\mathcal{V}_2$ are conformal primaries of spin 3. The numerator of their renormalized two-point functions should contain three powers of the inversion tensor $J_{z\bar{z}} = \eta_{z\bar{z}} - 2x_z x_{\bar{z}}/x^2$, and correspondingly we find that the normalization of the $(x_z\, x_{\bar{z}})^3$ terms differs by -8 from that of the $(\eta_{z\bar{z}})^3$ part. The operator $\mathcal{K}$ is a first derivative of the primary $\mathcal{K}_6$ [38]. The normalizations of the two terms in $\langle \mathcal{K}\,\bar{\mathcal{K}}\rangle$ are indeed consistent with being derivatives of a common spin 2 two-point function; similarly for the protected operator $\mathcal{O} = 1/3\, D_z^3\,\{0,0,0\}$. In conclusion, in this example renormalization works in the same way for these two components of the tensor structure. Conformal invariance is manifest.

[15] In [38] we deviated from the strict $\overline{MS}$ prescription by choosing the basis $\mathcal{B}$ in an ϵ dependent way, so that T_1 became diagonal as well. This had the advantage of decoupling the protected operator $\mathcal{O}$.

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Appendix B. The dilatation operator and renormalization

B.1. Matrix elements of the dilatation operator in dimensional regularization

Suppose there are linear operators $\tilde{D}_1, \tilde{D}_2$

$$\tilde{D}_1\,\mathcal{O}_i = \left(\frac{1}{\epsilon}\,D_{11\,ij} + D_{10\,ij}\right)\mathcal{O}_j, \tag{B.1}$$

$$\tilde{D}_2\,\mathcal{O}_j = \left(\frac{1}{\epsilon^2}\,D_{22\,ij} + \frac{1}{\epsilon}\,D_{21\,ij}\right)\mathcal{O}_j, \tag{B.2}$$

such that

$$\langle \mathcal{O}_i\,\bar{\mathcal{O}}_j\rangle_{g^2} = \langle(\tilde{D}_1\mathcal{O}_i)\,\bar{\mathcal{O}}_j\rangle_{g^0} = \left(\frac{1}{\epsilon}\,D_{11}\,T_0 + (D_{10}\,T_0 + D_{11}\,T_1)\right)_{ij}, \tag{B.3}$$

$$\langle \mathcal{O}_i\,\bar{\mathcal{O}}_j\rangle_{g^4} = \langle(\tilde{D}_2\mathcal{O}_i)\,\bar{\mathcal{O}}_j\rangle_{g^0} = \left(\frac{1}{\epsilon^2}\,D_{22}\,T_0 + \frac{1}{\epsilon}\,(D_{21}\,T_0 + D_{22}\,T_1)\right)_{ij}. \tag{B.4}$$

The eigenvectors of D_{11} constitute the aforementioned diagonal basis $\mathcal{O}_i$. In this frame $D_{11} = -\Gamma_1$, by which token the pole part of $\tilde{D}_1$ is the negative of the *one-loop dilatation operator*.

We will now consider the epsilon expansion of equation (A.10) order by order in g^2 up to $\mathrm{O}(\epsilon)$. For the rest of this section we assume the operators to be eigenvectors of D_{11}.

We may take $X(L,s)$ out of our system of equations: any set of renormalization factors that renders finite the bare correlators without the $X(L,s)$ factor remains a solution on multiplication by $X(L,s)$ because the latter is not singular in ϵ.

From the constant part at g^0 we immediately identify $a_{0\,i} = t_{0\,ii}$. At $\mathrm{O}(g^2)$ the epsilon expansion yields simple logarithms, simple poles and a constant part. From the first two sets of terms and the diagonal of the third we learn that

$$\gamma_{1\,i} = z_{11\,i} = -D_{11\,ii}, \qquad a_{1\,i} = D_{10\,ii}\,t_{0\,ii}, \tag{B.5}$$

while the off-diagonal part of the constant term constrains B but is not sufficient to fix it completely; hermiticity of the two-point function on the lhs of (B.3) halves the number of independent equations. (This places constraints on D_{10}. Similarly D_{21} is constrained by the hermiticity of the lhs of (B.4).)

At $\mathrm{O}(g^4)$ there are a number of conditions to solve: the double pole and the double logarithm in the epsilon expansion of (A.10) yield two equations implying that

$$z_{22\,i} = D_{22\,ii} = \tfrac{1}{2}\gamma_{1\,i}^2 \tag{B.6}$$

while the $\log(x_{12}^2)/\epsilon$ terms give nothing new. The diagonals of the simple logarithm and simple pole parts lead to

$$\gamma_{2\,i} = z_{21\,i} = -2\,(D_{21\,ii} - D_{10\,ii}\,D_{11\,ii}). \tag{B.7}$$

The rhs of the last equation is actually the action of a combination of $\tilde{D}_1, \tilde{D}_2$:

$$\frac{1}{\epsilon}\,(D_{21\,ii} - D_{10\,ii}\,D_{11\,ii}) = \left(\left(\tilde{D}_2 - \frac{1}{2}\,\tilde{D}_1^2\right)\mathcal{O}_i\right)_i. \tag{B.8}$$

The off-diagonal entries of the simple logarithm part depend on B and those of the simple pole part on B and C_1. The matrix C_1 cannot yet be fixed uniquely, but we now have

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enough equations for computing B. The resulting matrix equation is the off-diagonal part of

$$B\,D_{11} - D_{11}\,B = -2\,(D_{21} - \tfrac{1}{2}(D_{11}\,D_{10} + D_{10}\,D_{11}) - \tfrac{1}{4}(D_{11}\,D_{10} - D_{10}\,D_{11})). \tag{B.9}$$

Remarkably, the last term in this expression does not contribute on the diagonal, because D_{11} is diagonal. Hence the matrix

$$D_2 = -2\,(D_{21} - \tfrac{1}{2}\{D_{11}, D_{10}\} - \tfrac{1}{4}[D_{11}, D_{10}]) \tag{B.10}$$

has $\gamma_{2\,i}$ on its diagonal and it determines B through (B.9).

It was shown in [51] that the *two-loop dilatation generator* acts in precisely this way: suppose that the dilatation operator has an expansion

$$\Delta = 1 + g^2\,\Delta_1 + g^4\,\Delta_2 + \cdots. \tag{B.11}$$

We want to solve the eigenvalue problem

$$\Delta\,(\mathcal{O} + g^2\,B\,\mathcal{O} + \cdots) = (1 + g^2\,\Gamma_1 + g^4\,\Gamma_2 + \cdots)(\mathcal{O} + g^2\,B\,\mathcal{O} + \cdots). \tag{B.12}$$

Here Γ_1, Γ_2 are diagonal matrices containing the anomalous dimensions of the individual operators, and the lowest order remixing of the operators is named B. The dilatation operator acts on the vector of operators $\mathcal{O}$ as a linear map

$$\Delta_1\,\mathcal{O} = D_1\,\mathcal{O}, \qquad \Delta_2\,\mathcal{O} = D_2\,\mathcal{O}. \tag{B.13}$$

Once again, we choose the basis for the operators to be the set of eigenvectors of D_1, so that $\Delta_1\,\mathcal{O} = D_1\,\mathcal{O} = \Gamma_1\,\mathcal{O}$. The eigenvalue problem at order g^4 yields

$$D_2 = \Gamma_2 + (\Gamma_1\,B - B\,\Gamma_1) \tag{B.14}$$

exactly like D_2 from (B.10). Note that the diagonal of B remains undetermined—it corresponds to trivial operator rescalings and may be put to zero.

We have thus identified the matrix elements of the two-loop dilatation operator from the renormalization procedure in dimensional regularization. The next section addresses the construction of the dilatation operator itself.

B.2. The one-loop dilatation operator

In the planar limit the combinatorics for the two-point functions $\langle \mathcal{X}\,\bar{\mathcal{Y}}\rangle$ has the following features:

- At tree level, we find a cyclic sum over, say, site 1 in $\mathcal{X}$ joining site i in $\bar{\mathcal{Y}}$. All other lines are parallel.
- At loop level, the interaction is between adjacent sites. It can occur at any site in each part of the tree level configuration.

The $\mathrm{O}(g^2)$ contribution to the correlator $\langle \mathcal{X}\,\bar{\mathcal{Y}}\rangle$ originates from the $\mathcal{N} = 2$ supergraphs (see figure A.2) where, of course, the underlying Feynman integral is the same in G_{0a}, G_{0b}. It was called G_0 in [38] and is one-loop divergent. The 'BPS-like' graph B_0 is finite. The third structure F simply has a free vector line; it involves no loop integration. The configurations G_{0a}, G_{0b} occur with the gauge line emanating from any of the four end points; likewise, F_a, F_b must be joined by the opposite constellations.

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It is natural to interpret the one-loop interaction as a sum over a two-site 'Hamiltonian' shifting over all sites in $\mathcal{X}$, which is then contracted on $\bar{\mathcal{Y}}$ much as in the tree level correlator. The combinatoric factors for the Feynman graphs can be found by looking at the correlator

$$F_1(s_1, s_2, s_3, s_4) = \langle \mathrm{Tr}(T^a\, D_z^{s_1} Z\, D_z^{s_2} Z)(1)\, \mathrm{Tr}(D_{\bar{z}}^{s_4} \bar{Z}\, D_{\bar{z}}^{s_3} \bar{Z}\, T^b)(2) \rangle \tag{B.15}$$

at leading order in N (i.e. N^2), which is in a manner of speaking the one-loop interaction excised from the full correlator $\langle \mathcal{X}\, \bar{\mathcal{Y}} \rangle$. We find a -2 for the 'disconnected parts' G_{0b}, F_b and a 1 otherwise. The disconnected diagrams can be attributed to the two-site interaction to their left or to their right, so that we scale by $1/2$ in order to avoid overcounting.

If the interaction connects sites $i, i+1$ in $\mathcal{X}$ to $j, j+1$ in $\mathcal{Y}$, then the other fields in the operators are joined by parallel free lines:

$$\Pi(s_1, s_2) = \partial_{z\,1}^{s_1} \partial_{\bar{z}\,2}^{s_2}\, \Pi_{12} = -\delta_{s_1,s_2}\, \frac{\eta_{z\bar{z}}^{s_2}\, 2^{s_2}\, s_2! \sum_{k=1}^{s_2} (k-\epsilon)}{4\pi^2 (x_{12}^2)^{(s2+1-\epsilon)}} + \cdots, \tag{B.16}$$

where the omitted terms contain x_z or $x_{\bar{z}}$. The key observation is that the $X(L, s)$ term in the complete correlator can only exist when all free lines have the same spin at both ends [34]. Coupling between sites with different spins is only possible where the interaction is; since we want no x_{12} with free indices the interaction can at most 'transfer' a derivative from one of the two sites to the other. In particular, it must conserve the total spin.

Let us normalize by the inverse of the tree level. This will simply remove all the free lines and scale down F_1:

$$\mathcal{H}_i^{(0)} = \hat{F}_1(s_i, s_{i+1}, s_j, s_{j+1}) \frac{1}{\Pi(s_j, s_j)\, \Pi(s_{j+1}, s_{j+1})}, \tag{B.17}$$

where $\hat{F}_1$ is F_1 with the overcounting corrected and the group factor $N^2 \delta^{ab}$ stripped off.

Without any derivatives, the graphs G_{0a}, G_{0b}, F_a, F_b are absent while $B_0 = \mathrm{O}(\epsilon)$, whence $\hat{F}_1(0,0,0,0) \to 0$. When the total spin is not zero, $\mathcal{H}_i^{(0)}(s)$ is conveniently given as a matrix:

At *spin 1* we can have

$$\{s_i, s_{i+1}\}, \{s_j, s_{j+1}\} \in \{\{1,0\}, \{0,1\}\} \tag{B.18}$$

and our set of graphs produces

$$\mathcal{H}_i^{(0)}(1) = -\frac{1}{\epsilon} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{B.19}$$

At *spin 2* we have the basis[16]

$$\{s_i, s_{i+1}\}, \{s_j, s_{j+1}\} \in \{\tfrac{1}{2}\{2,0\}, \{1,1\}, \tfrac{1}{2}\{0,2\}\} \tag{B.20}$$

and the rules for transferring derivatives are

$$\mathcal{H}_i^{(0)}(2) = -\frac{1}{\epsilon} \begin{pmatrix} \frac{3}{2} & -1 & -\frac{1}{2} \\ -1 & 2 & -1 \\ -\frac{1}{2} & -1 & \frac{3}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}. \tag{B.21}$$

[16] The normalization of the basis elements reflects the fact that several derivatives at the same site are indistinguishable.

At *spin 3* the basis elements are

$$\{s_i, s_{i+1}\}, \{s_j, s_{j+1}\} \in \{\tfrac{1}{6}\{3,0\}, \tfrac{1}{2}\{2,1\}, \tfrac{1}{2}\{1,2\}, \tfrac{1}{6}\{0,3\}\} \tag{B.22}$$

while the derivatives may be transferred according to

$$\mathcal{H}_i^{(0)}(3) = -\frac{1}{\epsilon}\begin{pmatrix} \frac{11}{6} & -1 & -\frac{1}{2} & -\frac{1}{3} \\ -1 & \frac{5}{2} & -1 & -\frac{1}{2} \\ -\frac{1}{2} & -1 & \frac{5}{2} & -1 \\ -\frac{1}{3} & -\frac{1}{2} & -1 & \frac{11}{6} \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{2} & \frac{1}{2} & -\frac{2}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{B.23}$$

The pole part of these rules accurately reproduces the result of [34]: the diagonal entries are $h(s_i) + h(s_{i+1})$ where $h(n)$ are the harmonic numbers, and the off-diagonal entries are $-1/d$ where d counts the number of transferred derivatives. The finite part could doubtlessly also be fitted: we observed that the contribution from B_0 apparently always equals that of F_a, F_b which is trivial to compute. Graphs G_{0a}, G_{0b} contain only a one-loop integral, so that a result can be obtained in closed form. In contrast, at the two-loop level this is not easy due to the complexity of the integrals. Consequently, we limit the scope of this work to the first few cases obtained by direct calculation.

The one-loop dilatation operator is defined as

$$\tilde{D}_1 = \sum_{i=1}^{l} \mathcal{H}_i, \tag{B.24}$$

i.e. the 'Hamiltonian' runs over all sites in an operator $\mathcal{X}$, mapping it to a sum of terms with a new distribution of the derivatives over the sites in the chain. By construction,

$$\langle (\tilde{D}_1\, \mathcal{X})\, \bar{\mathcal{Y}} \rangle_{g^0} = \langle \mathcal{X}\, \bar{\mathcal{Y}} \rangle_{g^2}. \tag{B.25}$$

We conclude the section with two remarks. First, the definition of $\mathcal{H}_i^{(0)}$ in (B.17) is necessarily asymmetric because we have normalized from the right. Correspondingly, the constant parts of the transfer rules are not symmetric matrices. On the other hand, the pole part is symmetric, because in terms of complete two-point functions the matrices T_0 and Γ_1 must be simultaneously diagonalizable. Second, it should be stressed that the $X(L,s)$ terms are by far better suited to the construction of the interaction Hamiltonian $\mathcal{H}_i$ than for example the terms with no traces considered in [38]: those allow non-vanishing free lines between $D_z^{s_1} Z(1)$ and $D_{\bar{z}}^{s_2} \bar{Z}(2)$ for unequal spins $s_1 \neq s_2$, and the interaction need not conserve the total spin either. While the pole part of the one-loop dilatation operator is correctly obtained in this picture, we found it problematic to consistently subtract out disconnected parts at two-loop level.

B.3. The two-loop dilatation operator

In analogy to (B.15) we try to read off the operator $\tilde{D}_2$ from the $\mathrm{O}(g^4)$ contribution to

$$F_2(s_1, s_2, s_3, s_4, s_5, s_6) = \langle \mathrm{Tr}(T^a\, D_z^{s_1} Z\, D_z^{s_2} Z\, D_z^{s_3} Z)(1)\, \mathrm{Tr}(D_{\bar{z}}^{s_6} \bar{Z}\, D_{\bar{z}}^{s_5} \bar{Z}\, D_{\bar{z}}^{s_4} \bar{Z}\, T^b)(2) \rangle. \tag{B.26}$$

In doing so we should remember that matrix elements of the two-loop dilatation operator were defined by several terms; most prominently γ_2 came about as a matrix element of the combination $\tilde{D}_2 - \tilde{D}_1^2/2$—see equation (B.7) and the comment after it. We fall on

the renormalization scheme of [13]: the two-loop effective vertex has to be corrected by subtracting the square of the one-loop vertex. Explicitly, we take out

$$\begin{aligned}\tfrac{1}{2}\,\tilde{D}_1^2 &= \tfrac{1}{2}\sum_i \mathcal{H}_i^{(0)} \sum_j \mathcal{H}_j^{(0)} \\ &= \sum_{i+1<j} \mathcal{H}_i^{(0)}\,\mathcal{H}_j^{(0)} + \tfrac{1}{2}\sum_i (\tfrac{1}{2}\,(\mathcal{H}_i^{(0)})^2 + \mathcal{H}_i^{(0)}\mathcal{H}_{i+1}^{(0)} + \mathcal{H}_{i+1}^{(0)}\mathcal{H}_i^{(0)} + \tfrac{1}{2}\,(\mathcal{H}_{i+1}^{(0)})^2).\end{aligned} \tag{B.27}$$

(The derivation of the dilatation operator presented here is 'asymptotic' in that it assumes the existence of disconnected pieces.) The first term in the last formula corresponds to the situation where the two one-loop Hamiltonians do not overlap, and thus all terms are disconnected. If both pairs $\{i, i+1\}, \{j, j+1\}$ are outside our 'window' F_2, they will simply cancel disconnected parts that we do not see in the excised part. Likewise, if only one of $\mathcal{H}_i^{(0)}, \mathcal{H}_j^{(0)}$ touches the excised part, we would see an order g^2 contribution, which we need not consider. Thus the cases of interest are (we put the left of F_2 at position i)

$$\text{(i) } \mathcal{H}_i^{(0)}\mathcal{H}_{i+2}^{(0)}, \qquad \text{(ii) } \mathcal{H}_{i-1}^{(0)}\mathcal{H}_{i+1}^{(0)}, \qquad \text{(iii) } \mathcal{H}_{i-1}^{(0)}\mathcal{H}_{i+2}^{(0)}, \tag{B.28}$$

whose relevant g^4 diagrams may be directly subtracted from the set of graphs in F_2. The second term in (B.27) is unfortunately not amenable to this treatment: by way of example we do not have a diagram that identically equals two consecutive contributions of G_{0a}.

Our strategy thus starts by setting up an operator J_i from the g^4 graphs in F_2 with the subtraction of disconnected parts described in the last paragraph, whereas the overlapping part of $(\tilde{D}_1)^2/2$ will be dealt with later on. To avoid overcounting we have to rescale contributions with free lines: in complete analogy to the one-loop case we scale down, by a factor $1/2$, such graphs that connect two matter lines but leave the right or left line free. Note that no rescaling is needed when the free line is the central one; this situation is particular to exactly one position of the Hamiltonian. Configurations with two free lines can be arbitrarily shifted between the three positions within the Hamiltonian because the dilatation operator will involve a sum over positions. In order to compensate for overcounting we choose to scale by $1/4$ if the interaction is concentrated on one of the outer lines, and by $1/2$ if it is on the central line. We define

$$J_i = \hat{F}_2(s_i, s_{i+1}, s_{i+2}, s_j, s_{j+1}, s_{j+2}) \frac{1}{\Pi(s_j, s_j)\,\Pi(s_{j+1}, s_{j+1})\,\Pi(s_{j+2}, s_{j+2})}, \tag{B.29}$$

with $\hat{F}_2$ being F_2 after the appropriate modification of the set of graphs and once again after omission of the group factor $N^3\delta^{ab}$.

The connected part in (B.27) can be derived from the transfer rules for derivatives given in the last section. Recall that according to equation (B.10) the matrix elements of the two-loop dilatation operator also contain the term $-\frac{1}{4}(D_{11}D_{10} - D_{10}D_{11})$, when the dilatation operator is made to reproduce the $\mathrm{O}(g^2)$ remixing B. By splitting the one-loop transfer rules into a pole part $\mathcal{H}_{1\,i}^{(0)}$ and a constant piece $\mathcal{H}_{0\,i}^{(0)}$, we can construct this term as an operator in much the same way as the connected part of $\tilde{D}_1^2$. Note that $-\frac{1}{4}[\mathcal{H}_{1,i}^{(0)}, \mathcal{H}_{0,j}^{(0)}]$ has no disconnected part since $\mathcal{H}_1^{(0)}$ and $\mathcal{H}_0^{(0)}$ commute when they do not overlap.

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Finally, the full two-loop dilatation operator takes the form

$$D_2 = \sum_{i=1}^{l} \mathcal{H}_i^{(2)}|_{\epsilon^{-1}}, \tag{B.30}$$

with the two-loop Hamiltonian

$$\begin{aligned}\mathcal{H}_i^{(2)} = J_i - \tfrac{1}{2}\,(\tfrac{1}{2}\,(\mathcal{H}_i^{(0)})^2 + \mathcal{H}_i^{(0)}\mathcal{H}_{i+1}^{(0)} + \mathcal{H}_{i+1}^{(0)}\mathcal{H}_i^{(0)} + \tfrac{1}{2}\,(\mathcal{H}_{i+1}^{(0)})^2) \\ - \tfrac{1}{4}(\tfrac{1}{2}\,\mathcal{H}_{1\,i}^{(0)}\mathcal{H}_{0\,i}^{(0)} + \mathcal{H}_{1\,i}^{(0)}\mathcal{H}_{0\,i+1}^{(0)} + \mathcal{H}_{1\,i+1}^{(0)}\mathcal{H}_{0\,i}^{(0)} + \tfrac{1}{2}\,\mathcal{H}_{1\,i+1}^{(0)}\mathcal{H}_{0\,i+1}^{(0)} - (\mathcal{H}_1^{(0)} \leftrightarrow \mathcal{H}_0^{(0)})).\end{aligned} \tag{B.31}$$

The Hamiltonian $\mathcal{H}_i^{(2)}$ has in fact a non-vanishing $1/\epsilon^2$ part, but the second-order poles are distributed over the matrices in such a way that they drop out in the sum over all positions. The transfer rules below and in the main text describe the $1/\epsilon$ part.

In this appendix we give the transfer rules corresponding to the full Hamiltonian including the terms in the last line of (B.31). These come from the commutator

$$-1/4\,[\mathcal{H}_{1\,i}^{(0)}, \mathcal{H}_{0\,j}^{(0)}].$$

Note that this term is anti-Hermitian, so that the transfer rules cannot be transformed into symmetric matrices. In the main text we omit the commutator term, since in the context of the Bethe ansatz it is preferable to have a Hermitian Hamiltonian. In any case, the exact resolution of the mixing is not easy to obtain in the Bethe ansatz picture which projects out the descendants. Surprisingly, the formulae below do apply to the length 3, spin 3 mixing problem although they were derived for longer chains for which disconnected pieces have to be subtracted.

The explicit bases and two-loop transfer rules up to spin 3 are:

Spin 1. Basis: $\{\{1,0,0\},\{0,1,0\},\{0,0,1\}\}$:

$$\mathcal{H}_i^{(2)}(1) = \begin{pmatrix} -\frac{3}{4} & 1 & -\frac{1}{2} \\ 1 & -\frac{3}{2} & 1 \\ -\frac{1}{2} & 1 & -\frac{3}{4} \end{pmatrix}.$$

Spin 2. Basis: $\{\frac{1}{2}\{2,0,0\},\{1,1,0\},\{1,0,1\},\frac{1}{2}\{0,2,0\},\{0,1,1\},\frac{1}{2}\{0,0,2\}\}$:

$$\mathcal{H}_i^{(2)}(2) = \begin{pmatrix} -\frac{19}{32} & \frac{17}{16} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{16} \\ \frac{21}{16} & -\frac{9}{4} & 1 & \frac{29}{16} & 0 & -\frac{1}{8} \\ -\frac{3}{4} & 1 & -\frac{3}{2} & -\frac{1}{2} & 1 & -\frac{3}{4} \\ \frac{1}{4} & \frac{11}{16} & 0 & -\frac{67}{16} & \frac{11}{16} & \frac{1}{4} \\ -\frac{1}{8} & 0 & 1 & \frac{29}{16} & -\frac{9}{4} & \frac{21}{16} \\ -\frac{1}{16} & -\frac{1}{4} & -\frac{1}{2} & \frac{1}{2} & \frac{17}{16} & -\frac{19}{32} \end{pmatrix}.$$

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Spin 3. Basis: $\{\frac{1}{6}\{3,0,0\},\frac{1}{2}\{2,1,0\},\frac{1}{2}\{2,0,1\},\frac{1}{2}\{1,2,0\},\{1,1,1\},\frac{1}{2}\{1,0,2\},\frac{1}{6}\{0,3,0\},\frac{1}{2}\{0,2,1\},\frac{1}{2}\{0,1,2\},\frac{1}{6}\{0,0,3\}\}$:

$$\mathcal{H}_i^{(2)}(3) = \begin{pmatrix} \frac{25}{288} & \frac{11}{18} & -\frac{1}{2} & \frac{77}{144} & -\frac{1}{4} & -\frac{1}{16} & \frac{71}{216} & -\frac{1}{6} & -\frac{1}{24} & -\frac{1}{54} \\ \frac{43}{72} & -\frac{63}{32} & 1 & \frac{3}{2} & 0 & -\frac{1}{8} & \frac{149}{144} & 0 & -\frac{1}{16} & \frac{1}{72} \\ -\frac{5}{6} & 1 & -\frac{43}{32} & -\frac{1}{2} & \frac{17}{16} & -\frac{3}{4} & -\frac{1}{3} & \frac{1}{2} & -\frac{3}{8} & -\frac{7}{48} \\ \frac{91}{144} & \frac{5}{4} & 0 & -\frac{81}{16} & \frac{11}{16} & \frac{1}{4} & \frac{191}{72} & 0 & 0 & -\frac{1}{18} \\ -\frac{7}{24} & 0 & \frac{21}{16} & \frac{29}{16} & -3 & \frac{21}{16} & -\frac{1}{6} & \frac{29}{16} & 0 & -\frac{7}{24} \\ -\frac{7}{48} & -\frac{3}{8} & -\frac{3}{4} & \frac{1}{2} & \frac{17}{16} & -\frac{43}{32} & -\frac{1}{3} & -\frac{1}{2} & 1 & -\frac{5}{6} \\ -\frac{1}{216} & \frac{19}{144} & 0 & \frac{19}{18} & 0 & 0 & -\frac{1031}{144} & \frac{19}{18} & \frac{19}{144} & -\frac{1}{216} \\ -\frac{1}{18} & 0 & \frac{1}{4} & 0 & \frac{11}{16} & 0 & \frac{191}{72} & -\frac{81}{16} & \frac{5}{4} & \frac{91}{144} \\ \frac{1}{72} & -\frac{1}{16} & -\frac{1}{8} & 0 & 0 & 1 & \frac{149}{144} & \frac{3}{2} & -\frac{63}{32} & \frac{43}{72} \\ -\frac{1}{54} & -\frac{1}{24} & -\frac{1}{16} & -\frac{1}{6} & -\frac{1}{4} & -\frac{1}{2} & \frac{71}{216} & \frac{77}{144} & \frac{11}{18} & \frac{25}{288} \end{pmatrix}.$$

Appendix C. Explicit solution for twist 2

The one-loop Bethe equation (16) may be recast as a second-order difference equation for the Baxter Q function $Q_s(u)$:

$$T_s(u)\,Q_s(u) = \left(u+\frac{\mathrm{i}}{2}\right)^L Q_s(u+\mathrm{i}) + \left(u-\frac{\mathrm{i}}{2}\right)^L Q_s(u-\mathrm{i}), \tag{C.1}$$

where $Q_s(u)$ is a polynomial of degree s in the variable u, whose algebraic roots are the Bethe roots $\{u_k\}$,

$$Q_s(u) = C_s \prod_{k=1}^{s}(u-u_k), \tag{C.2}$$

i.e. the solutions of (16), and C_s is a, for our purposes, irrelevant normalization constant. For twist $L=2$ the excitation number s has to be even, and the Baxter equation (C.1) is exactly solvable in terms of a hypergeometric function:

$$Q_s(u) = {}_3F_2[-s, s+1, \tfrac{1}{2}-\mathrm{i}u; 1, 1; 1] \qquad \text{with } T_s(u) = 2\,u^2 - s^2 - s - \tfrac{1}{2}. \tag{C.3}$$

The hypergeometric series terminates if s is an even natural number, and therefore generates the explicit polynomial solution of the twist 2 Baxter equation[17]. The roots are all real and their distribution is even, i.e. the $Q_s(u)$ in (C.3) are actually polynomials in u^2. Therefore the cyclicity constraint in (17) is automatically satisfied. The energy is found from (17), (C.2) to be

$$E_s = 2\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}u}\left[\log Q_s\left(u+\frac{\mathrm{i}}{2}\right)\right]_{u=0} = 4\,(\psi(s+1)-\psi(1)) = 4\,h(s). \tag{C.4}$$

Here $h(s) = \sum_{j=1}^{s} 1/j$ are the harmonic numbers, which may also be expressed through the logarithmic derivative of the gamma function $\psi(s) = \mathrm{d}/\mathrm{d}s \log\Gamma(s)$. In practice, the roots are easily found with a root finder. For example, with `Mathematica` one may define

[17] The details of the solution (C.3) were worked out by Virginia Dippel (unpublished) by adapting the method of [39] to the present case. The polynomials (C.3) belong to the family of so-called *Hahn polynomials* [39].

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```
Hahn[s_, u_] :=
Expand[HypergeometricPFQ[{-s, s + 1, 1/2 - I u}, {1, 1}, 1]]
```

and generate a table of all Bethe roots up to spin s, with an accuracy of k digits,

```
utable[s_] := Table[Flatten[NSolve[Hahn[2 t, u] == 0, u, k]], {t, 1, s/2}]
```

This is suitable, without further refinements, for finding the Bethe roots up to spin $s \sim 70$ with an accuracy, if desired, of hundreds of digits.

Appendix D. Fourier transforms

D.1. The gauge theory ansatz

In this appendix we find the Fourier transform of the fourth term on the rhs of (72), in which the density $\sigma(u')$ is integrated against the kernel:

$$K(u,u') = \mathrm{i}\,\partial_u \log\left(\frac{1-g^2/2\,x^+(u)\,x^-(u')}{1-g^2/2\,x^-(u)\,x^+(u')}\right)^2. \tag{D.1}$$

The definitions used in the last formula are

$$u = x(u) + \frac{g^2}{2\,x(u)}, \qquad x(u) = \frac{u}{2}\left(1+\sqrt{1-\frac{2g^2}{u^2}}\right), \tag{D.2}$$

$$u^\pm = u \pm \frac{\mathrm{i}}{2}, \qquad x^\pm(u) = x(u^\pm). \tag{D.3}$$

The branch cut of the square root is defined by the principal branch of the logarithm. In the following we parametrize using

$$\tilde{u}^+ = \tfrac{1}{2} - \mathrm{i}\,u = -\mathrm{i}\,u^+, \qquad \tilde{u}^- = \tfrac{1}{2} + \mathrm{i}\,u = \mathrm{i}\,u^-, \tag{D.4}$$

which obey the relation

$$\sqrt{(\tilde{u}^\pm)^2} = \tilde{u}^\pm \tag{D.5}$$

because both $\tilde{u}^+, \tilde{u}^-$ have positive real part. Further, let

$$y(u) = \sqrt{1+\frac{2g^2\lambda^2}{u^2}} \tag{D.6}$$

and

$$K_0^\pm(u) = \frac{1}{\tilde{u}^\pm\, y(\tilde{u}^\pm)}, \qquad K_1^\pm(u) = \frac{1}{\sqrt{2}g\lambda}\left(1-\frac{1}{y(\tilde{u}^\pm)}\right). \tag{D.7}$$

Since we are, in this paper, exclusively interested in *symmetric* densities, we will consider a $u' \leftrightarrow -u'$ symmetrized version of the kernel. Our principal equation is

$$\begin{aligned} -\mathrm{i}\,g^2 \int_0^1 \mathrm{d}\lambda\,\lambda\,[&\partial_u\,(K_0^+(u)-K_0^-(u))\,(K_0^+(u')\,+K_0^-(u')) \\ &+\partial_u\,(K_1^+(u)-K_1^-(u))\,(K_1^+(u')+K_1^-(u'))] \\ &= \frac{\mathrm{i}}{2}\,\partial_u \log\left(\frac{(1-g^2/2\,x^+(u)\,x^-(u'))(1+g^2/2\,x^+(u)\,x^+(u'))}{(1-g^2/2\,x^-(u)\,x^+(u'))(1+g^2/2\,x^-(u)\,x^-(u'))}\right)^2. \end{aligned} \tag{D.8}$$

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To prove this, we first do the parameter integrals on the left-hand side:

$$\begin{aligned}
&\int \mathrm{d}\lambda\, \frac{\lambda}{y(\tilde{u})\, y(\tilde{u}')} = \frac{\tilde{u}\,\tilde{u}'}{2\, g^2} \log(\tilde{u}\, y(\tilde{u}) + \tilde{u}'\, y(\tilde{u}')),\\
&\int \mathrm{d}\lambda\, \frac{1}{\lambda\, y(\tilde{u})\, y(\tilde{u}')} = \log(\lambda) - \log(y(\tilde{u}) + y(\tilde{u}')),\\
&\int \mathrm{d}\lambda\, \frac{1}{\lambda y(\tilde{u})} = \log(\lambda) - \log(1 + y(\tilde{u})).
\end{aligned} \tag{D.9}$$

Here we rely on (D.5) to simplify. Next, we change back to the original variables $u^{\pm}$. We express the roots in terms of $u, x(u)$ using the second relation in (D.2) in the form

$$\sqrt{1 - \frac{2g^2}{u^2}} = \frac{2\, x(u)}{u} - 1 \tag{D.10}$$

and finally eliminate u in favour of $x(u), g^2$ by the first relation in (D.2). In a last step we collect all terms into one logarithm and factor the argument. As long as g is small this will not shift the logarithm by some multiple of π; one may check that the Fourier transform below commutes with the Taylor expansion in g.

Next, we observe

$$K_j^{\pm}(u) = \int_0^{\infty} \mathrm{d}t\, \mathrm{e}^{\pm \mathrm{i}\, u\, t} \mathrm{e}^{-t/2}\, J_j(\sqrt{2} g \lambda\, t), \qquad j = 0, 1 \tag{D.11}$$

and hence

$$\begin{aligned}
&K_j^{+}(u) + K_j^{-}(u) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\, u\, t} \mathrm{e}^{-|t|/2}\, J_j(\sqrt{2} g \lambda\, |t|),\\
&-\mathrm{i}\, \partial_u \left(K_j^{+}(u) - K_j^{-}(u)\right) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\, u\, t}\, |t|\, \mathrm{e}^{-|t|/2}\, J_j(\sqrt{2} g \lambda\, |t|).
\end{aligned} \tag{D.12}$$

Summing up, we have shown that

$$\begin{aligned}
&\frac{\mathrm{i}}{2}\, \partial_u \log \left(\frac{(1 - g^2/2\, x^{+}(u)\, x^{-}(u'))(1 + g^2/2\, x^{+}(u)\, x^{+}(u'))}{(1 - g^2/2\, x^{-}(u)\, x^{+}(u'))(1 + g^2/2\, x^{-}(u)\, x^{-}(u'))} \right)^2\\
&\qquad = g^2 \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\, u\, t} \int_{-\infty}^{\infty} \mathrm{d}t'\, \mathrm{e}^{\mathrm{i}\, u'\, t'}\, |t| \mathrm{e}^{-(|t| + |t'|)/2}\, \hat{K}(\sqrt{2} g\, |t|, \sqrt{2} g\, |t'|),
\end{aligned} \tag{D.13}$$

where

$$\begin{aligned}
\hat{K}(t, t') &= \int_0^1 \mathrm{d}\lambda\, \lambda\, [J_0(\lambda\, t)\, J_0(\lambda\, t') + J_1(\lambda\, t)\, J_1(\lambda\, t')]\\
&= \frac{J_1(t)\, J_0(t') - J_0(t)\, J_1(t')}{t - t'}.
\end{aligned} \tag{D.14}$$

We conclude that for symmetric $\sigma(u')$

$$\begin{aligned}
&\mathrm{e}^{-|t|/2} \int_{-\infty}^{\infty} \mathrm{d}u\, \mathrm{e}^{-\mathrm{i}\, t\, u} \int_{-\infty}^{\infty} \mathrm{d}u'\, K(u, u')\, \sigma(u')\\
&\qquad = 2\, \pi\, g^2\, |t| \mathrm{e}^{-|t|} \int_{-\infty}^{\infty} \mathrm{d}t'\, \hat{K}(\sqrt{2} g\, |t|, \sqrt{2} g\, |t'|) \left[\mathrm{e}^{-|t'|/2} \int_{-\infty}^{\infty} \mathrm{d}u'\, \mathrm{e}^{\mathrm{i}\, u'\, t'}\, \sigma(u') \right]\\
&\qquad = 2\, \pi\, g^2\, |t| \mathrm{e}^{-|t|} \int_{-\infty}^{\infty} \mathrm{d}t'\, \hat{K}(\sqrt{2} g\, |t|, \sqrt{2} g\, |t'|)\, \hat{\sigma}(t'),
\end{aligned} \tag{D.15}$$

where we have used (73). Now, $\hat{\sigma}(t')$ is an even function if $\sigma(u')$ is. We may thus reduce to the positive half-axis, which yields the final form of the last term in (74).

One has, similar to formula (D.11),

$$\int_0^\infty \mathrm{d}t\, \mathrm{e}^{\pm \mathrm{i}\, u\, t}\, \mathrm{e}^{-t/2}\, \frac{J_j(\sqrt{2} g\, t)}{\sqrt{2} g\, t} = \frac{(\sqrt{2}\, g)^{j-1}}{j} \left(\tilde{u}^{\pm} \left(1 + \sqrt{1 + 2\, g^2/(\tilde{u}^{\pm})^2} \right) \right)^{-j}, \qquad j \geq 1. \tag{D.16}$$

From this one can easily derive the following pretty result for the Fourier transforms $\hat{q}_r(t)$ of the eigenvalues $q_r(u)$ of the commuting operators of the integrable magnet. The expression [16]

$$q_r(u) = \frac{1}{r-1} \left(\frac{\mathrm{i}}{x^+(u)^{r-1}} - \frac{\mathrm{i}}{x^-(u)^{r-1}} \right), \tag{D.17}$$

turns into[18]

$$\hat{q}_r(t) = \int_{-\infty}^{\infty} \mathrm{d}u\, \mathrm{e}^{-\mathrm{i}\, t\, u}\, q_r(u) = 4\, \pi \left(\frac{\sqrt{2}}{\mathrm{i}\, g} \right)^{r-2} \mathrm{e}^{-|t|/2}\, \frac{J_{r-1}(\sqrt{2} g\, t)}{\sqrt{2} g\, t}.$$

In particular, using this result for $r = 2$ we obtain the expression (77) for the energy $E(g)$ in Fourier space. As a further corollary we find the Fourier transform of the third term on the rhs of (72), as stated in (74):

$$\mathrm{e}^{-|t|/2} \int_{-\infty}^{\infty} \mathrm{d}u\, \mathrm{e}^{-\mathrm{i}\, t\, u} \frac{1}{2}\, \partial_u \left(\frac{1}{x^+(u)} + \frac{1}{x^-(u)} \right) = 2\, \pi \mathrm{e}^{-|t|}\, \frac{J_1(\sqrt{2} g\, |t|)}{\sqrt{2} g}. \tag{D.18}$$

D.2. The string dressing factor

In order to include the *dressing factor* for the 'string Bethe ansatz' [43, 20], as needed in the discussion at the end of section 3.3, we replace in equation (72) the kernel $K(u, u')$ from the gauge theory Bethe ansatz by

$$K_s(u, u') = -\partial_u\, (u - u')\, \log \left(\frac{(1 - g^2/2\, x^+(u)\, x^-(u'))(1 - g^2/2\, x^-(u)\, x^+(u'))}{(1 - g^2/2\, x^+(u)\, x^+(u'))(1 - g^2/2\, x^-(u)\, x^-(u'))} \right)^2, \tag{D.19}$$

which will again be needed in a $u' \leftrightarrow -u'$ symmetrized form. In view of the analysis in the last section, the question arises of whether this expression can also be written as a one-parameter integral over pairs of the form $K_j^{\pm}(u)\, K_j^{\pm}(u')$. As we shall see shortly, this is indeed the case.

Quite clearly we have to deal with two distinct pieces, namely the part involving $\partial_u\, u$ and that with $\partial_u\, u'$. In the first case, the expression is explicitly symmetrized in u' whereas $\partial_u\, u$ on the whole is also even with respect to the integrals on the half-axis that we may expect to find. We are led to look for combinations involving $K_j^+(u) + K_j^-(u)$ and likewise in u'. We remark that under the Fourier transform, $\partial_u\, u \leftrightarrow -t\, \partial_t$. The differential operator can thus be incorporated at no expense. Surprisingly, the $K_0^{\pm}$ alone

[18] These expressions were first obtained by Didina Serban (2005, unpublished).

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suit our purpose: in the same fashion as before we may demonstrate that

$$-\frac{1}{2}\,\partial_u\, u\,\log\left(\frac{(1-g^2/2\,x^+(u)\,x^-(u'))(1-g^2/2\,x^-(u)\,x^+(u'))}{(1-g^2/2\,x^+(u)\,x^+(u'))(1-g^2/2\,x^-(u)\,x^-(u'))}\right)^2+(u'\leftrightarrow -u')$$
$$=-g^2\int_{-\infty}^{\infty}\mathrm{d}t\,\mathrm{e}^{\mathrm{i}\,u\,t}\int_{-\infty}^{\infty}\mathrm{d}t'\,\mathrm{e}^{\mathrm{i}\,u'\,t'}$$
$$\times\,2\,|t|\,\partial_{|t|}\,\mathrm{e}^{-(|t|+|t'|)/2}\int_0^1\mathrm{d}\lambda\,\lambda\,J_0(\sqrt{2}g\lambda\,|t|)\,J_0(\sqrt{2}g\lambda\,|t'|). \tag{D.20}$$

For the second piece we must try $K_j^+(u)-K_j^-(u)$ and similarly for u', because the simple derivative in u is odd while the extra power of u' forces antisymmetrization on the log factor. In a beautifully symmetric way we can realize the term as a parameter integral this time over antisymmetric combinations of only K_1:

$$\frac{1}{2}\,\partial_u\, u'\,\log\left(\frac{(1-g^2/2\,x^+(u)\,x^-(u'))(1-g^2/2\,x^-(u)\,x^+(u'))}{(1-g^2/2\,x^+(u)\,x^+(u'))(1-g^2/2\,x^-(u)\,x^-(u'))}\right)^2+(u'\leftrightarrow -u')$$
$$=-g^2\int_{-\infty}^{\infty}\mathrm{d}t\,\mathrm{e}^{\mathrm{i}\,u\,t}\int_{-\infty}^{\infty}\mathrm{d}t'\,\mathrm{e}^{\mathrm{i}\,u'\,t'}$$
$$\times\,2\,|t|\,\partial_{|t'|}\,\mathrm{e}^{-(|t|+|t'|)/2}\int_0^1\mathrm{d}\lambda\,\lambda\,J_1(\sqrt{2}g\lambda\,|t|)\,J_1(\sqrt{2}g\lambda\,|t'|). \tag{D.21}$$

In the right-hand sides of the last two formulae the derivatives can either fall upon the exponential or on the Bessel functions. Accordingly, we reproduce the Fourier transformed gauge theory kernel $\hat{K}$ and an additional piece $\sqrt{2}\,g\,\tilde{K}$, defined as

$$\tilde{K}(t,t')=-2\int_0^1\mathrm{d}\lambda\,\lambda\,[\partial_t\,J_0(\lambda\,t)\,J_0(\lambda\,t')+\partial_{t'}\,J_1(\lambda\,t)\,J_1(\lambda\,t')]$$
$$=\frac{t\,[J_2(t)\,J_0(t')-J_0(t)\,J_2(t')]}{(t-t')(t+t')}. \tag{D.22}$$

Here one should first do the parametric integration in both terms separately and then differentiate and simplify.

Equation (D.15) is replaced by

$$\mathrm{e}^{-|t|/2}\int_{-\infty}^{\infty}\mathrm{d}u\,\mathrm{e}^{-\mathrm{i}\,t\,u}\int_{-\infty}^{\infty}\mathrm{d}u'\,K_s(u,u')\,\sigma(u')$$
$$=2\,\pi\,g^2\,|t|\mathrm{e}^{-|t|}\int_{-\infty}^{\infty}\mathrm{d}t'\,[\hat{K}(\sqrt{2}g|t|,\sqrt{2}g|t'|)+\sqrt{2}g\,\tilde{K}(\sqrt{2}g|t|,\sqrt{2}g|t'|)]\,\hat{\sigma}(t'). \tag{D.23}$$

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