Calc. Var. (2006) 25(4): 535–540
DOI 10.1007/s00526-005-0371-4 **Calculus of Variations**

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Generalized Korn's inequality and conformal Killing vectors

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Abstract Korn's inequality plays an important role in linear elasticity theory. This inequality bounds the norm of the derivatives of the displacement vector by the norm of the linearized strain tensor. The kernel of the linearized strain tensor are the infinitesimal rigid-body translations and rotations (Killing vectors). We generalize this inequality by replacing the linearized strain tensor by its trace free part. That is, we obtain a stronger inequality in which the kernel of the relevant operator are the conformal Killing vectors. The new inequality has applications in General Relativity.

1 Introduction

Let Ω be a domain in $\mathbb{R}^n$, with $n \geq 2$ and let u^i be a vector field in Ω, with $i = 1, \ldots, n$. We denote the Euclidean inner product by $u_i u^i$, where the summation convention with respect to repeated indices is used and the indices are moved with the Kronecker delta δ_{ij} (i.e; $u_j = \delta_{ij} u^i$). Let $H^1(\Omega)$ be the standard Sobolev space of vectors fields with norm

$$||u||_{H^1(\Omega)} = \left(\int_\Omega u_i u^i \, d\mu \right)^{1/2} + \left(\int_\Omega \partial_i u_j \partial^i u^j \, d\mu \right)^{1/2}, \tag{1}$$

where ∂_i denotes partial derivative with respect to the coordinate x_i and $d\mu$ is the Euclidean volume element.

For all functions $u \in H^1(\Omega)$, there exists a constant C, independent on u, such that the following inequality holds

$$||u||^2_{H^1(\Omega)} \leq C \int_\Omega (u^i u_i + e_{ij}(u) e^{ij}(u)) \, d\mu, \tag{2}$$

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where

$$e_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i). \tag{3}$$

Inequality (2) is known as Korn's inequality. This inequality has a long history. It plays a central role in elasticity, see the review [8] and also the recent articles [3, 14] and [4].

In linear elasticity u^i is the displacement vector and the tensor (3) is known as the linearized strain tensor. The solutions u^i of $e_{ij}(u) = 0$ are the infinitesimal generator of rigid-body rotations and translations (Killing vectors of the flat metric), which are precisely the only (infinitesimal) displacements which do not change the shape of the body. The dimension of the kernel is $n(n+1)/2$.

The energy of the elastic body is given by (see, for example, [12])

$$E(u) = \int_\Omega (c^{ijkl} e_{ij}(u) e_{kl}(u) - F^i u_i)\, d\mu, \tag{4}$$

where F^i is the external force and c_{ijkl} are certain bounded functions which depend on the particular material. They satisfy $c_{ijkl} = c_{klij}$, $c_{ijkl} = c_{jikl}$, $c_{ijlk} = c_{ijkl}$ and the positivity condition

$$c^{ijkl} e_{ij}(u) e_{kl}(u) \geq C_0 e_{ij}(u) e^{ij}(u), \tag{5}$$

for some constant $C_0 > 0$. From (4) and (5) we deduce that if $F^i = 0$ then $E(u) = 0 \iff e_{ij}(u) = 0$. That is, in absence of external forces, the energy of a displacement u^i is zero if and only if u^i is a rigid-body translation or rotation, in accordance with physical intuition.

The pure traction problem of linear elasticity (i.e; where the forces at the boundary are prescribed) consists in finding a displacement $u^i \in H^1(\Omega)$ that minimize (4). Korn's inequality is used to prove that the functional (4) is coercive, existence of weak solutions then follows by the Lax-Milgram theorem.

We define the operator $l_{ij}(u)$ as the trace free part of $e_{ij}(u)$

$$l_{ij}(u) = e_{ij}(u) - \frac{1}{n} e(u) \delta_{ij}, \tag{6}$$

where

$$e(u) = \delta^{ij} e_{ij}(u) = \partial^i u_i. \tag{7}$$

The following is the main result of this article.

Theorem 1.1 *Let Ω be a Lipschitz domain in $\mathbb{R}^n$, with $n \geq 3$. Then, there exists a constant C, independent of u, such that*

$$||u||^2_{H^1(\Omega)} \leq C \int_\Omega (u^i u_i + l_{ij}(u) l^{ij}(u))\, d\mu, \quad \text{for all } u \in H^1(\Omega). \tag{8}$$

For a definition of Lipschitz domains see [1] and [12]. These domains include domains with corners like cubic domains.

Inequality (8) implies inequality (2) since we have

$$l_{ij}(u) l^{ij}(u) = e_{ij}(u) e^{ij}(u) - \frac{1}{n}(e(u))^2 \leq e_{ij}(u) e^{ij}(u). \tag{9}$$

The kernel $l_{ij}(u) = 0$ is given by the conformal Killing vectors, which include the Killing vectors and also the dilatations and special conformal transformations given by

$$ax^i, \quad k^j\,(2\,x_j\,x^i - \delta_j{}^i\,x_l\,x^l), \tag{10}$$

where k^i and a are arbitrary constants. If $n \geq 3$ then the dimension of the kernel is $(n+1)(n+2)/2$.

For vectors u^i which vanish at the boundary $\partial\Omega$ Korn's inequality (2) follows easily by integration by parts (this is known in the literature as Korn's inequality in the first case). The same argument applies to the inequality (8): if we assume that $u^i = 0$ on $\partial\Omega$, then for $n \geq 2$ we have

$$2\int_\Omega l_{ij}(u)l^{ij}(u)\,d\mu = \int_\Omega \partial_i u_j \partial^i u^j\,d\mu + \frac{(n-2)}{n}\int_\Omega (\partial^i u_i)^2\,d\mu \tag{11}$$

$$\geq \int_\Omega \partial_i u_j \partial^i u^j\,d\mu \tag{12}$$

Note that we can replace the operator l in (8) by

$$l'_{ij}(u) = e_{ij}(u) - \alpha\delta_{ij}e(u), \tag{13}$$

where α is an arbitrary real number, because we have the inequality

$$l_{ij}(u)l^{ij}(u) \leq l'_{ij}(u)l'^{ij}(u) = l_{ij}(u)l^{ij}(u) + \frac{(e(u)(1-n\alpha))^2}{n} \tag{14}$$

The case $n = 2$ is special. In this case, equations $l(u)_{ij} = 0$ are given by

$$\partial_1 u_1 - \partial_2 u_2 = 0, \quad \partial_1 u_2 - \partial_2 u_1 = 0. \tag{15}$$

These equations are the Cauchy-Riemann equations for the complex function $F = u_1 + iu_2$. That is, every analytical function F provides a solution of $l(u)_{ij} = 0$. Then, for $n = 2$, the kernel of the operator l is infinite dimensional. This implies that theorem 1.1 *does not hold* for $n = 2$ as we will see. Note that Korn's inequality (2) holds in this case and also inequality (12) for vectors u^i which vanish at the boundary.

To prove that theorem 1.1 is not valid in two dimensions we argue by contradiction. Let us assume that inequality (8) holds. Then, following the same argument used in [3] to prove that Korn's inequality implies that the kernel of e is finite dimensional, we conclude that the kernel of l is finite dimensional. This provides the required contradiction for $n = 2$. For $n \geq 3$ this is an alternative proof of the above mentioned fact that the space of conformal Killing vectors is finite dimensional.

The operators $e_{ij}(u)$ and $l_{ij}(u)$ have a natural generalization for Riemannian manifolds. Let M be a Riemannian manifold with metric h_{ij} and covariant derivative D_i. Then, the operators e and l generalize to

$$\mathbf{e}_{ij}(u) = \frac{1}{2}(D_i u_j + D_j u_i) \tag{16}$$

and

$$\mathbf{l}_{ij}(u) = \mathbf{e}_{ij}(u) - \frac{1}{n}\mathbf{e}(u)h_{ij}, \tag{17}$$

where

$$\mathbf{e}(u) = h^{ij}\mathbf{e}_{ij} = D_i u^i, \tag{18}$$

and the indices are moved with the metric h_{ij} and its inverse h^{ij} (i.e; $u_i = h_{ij}u^j$).

The operator $\mathbf{l}$ is conformal invariant in the following sense. If $\tilde{h}_{ij} = e^{2f} h_{ij}$ is a metric conformal to h_{ij} (where f is an arbitrary function) and $\tilde{\mathbf{l}}$ is the corresponding operator, then we have the relation

$$\tilde{\mathbf{l}}(\tilde{u})_{ij} = e^{2f}\mathbf{l}(u)_{ij}, \tag{19}$$

where $\tilde{u}^i = u^i$ and $\tilde{u}_i = e^{2f} u_i$ (in Eq. (19), indices of quantities with tilde are moved with the metric $\tilde{h}_{ij}$ and its inverse).

In [3] a Riemannian version of Korn's inequality was proved. Using the same arguments and Theorem 1.1 the following result follows.

Corollary 1.2 *Let $\Omega \subset M$ be an open set with Lipschitz boundary and assume that the metric h_{ij} is in $C^1(M)$. Then, there is a positive constant C such that*

$$\int_\Omega D_i u_j u D^i u^j \, d\mu_h \leq C \int_\Omega u^i u_i + \mathbf{l}_{ij}(u)\mathbf{l}^{ij}(u) \, d\mu_h, \quad \textit{for all } u \in H^1(\Omega), \tag{20}$$

where $d\mu_h$ is the volume element of the metric h_{ij}.

Inequality (20) has applications in General Relativity. In the Cauchy formulation of the theory, the initial data have to satisfy the so called constraint equations on a Riemannian manifold (see the recent review article [2] and reference therein). Solutions of one of these equations (the "momentum constraint") can be obtained as a solution of a variational problem for the following energy under suitable boundary conditions for u^i

$$E'(u) = \int_\Omega (\mathbf{l}_{ij}(u)\mathbf{l}^{ij}(u) - Q^i u_i) \, d\mu_h, \tag{21}$$

where Q^i is a given vector. The energy (21) has similar form to the elastic energy (4), the difference is that the strain tensor $\mathbf{e}_{ij}$ is replaced by $\mathbf{l}_{ij}$. For black holes, the boundary conditions for u^i are analogous to the pure traction problem of linear elasticity. That is, the solution is the infimum of $E'(u)$ for all $u \in H^1(\Omega)$ (see [5, 6]). As in the case of elasticity, inequality (8) is used to prove that $E'(u)$ is coercive in $u \in H^1(\Omega)$, and then the existence of solution follows by the Lax-Milgram theorem (see [5, 6] for details).

In elasticity, the energy (21) has no direct physical meaning since in absence of external forces it is zero not only for rigid-body displacement but also for dilatations. That is, the "bulk modulus" coefficient of the material is equal to zero; no elastic material has this property. On the other hand, the constraint equations of General Relativity are conformal invariant (see [2]), this is why $\mathbf{l}_{ij}$ and not $\mathbf{e}_{ij}$ appears in (21).

2 Proof of Theorem 1.1

The strategy of the proof follows the proof of Korn's inequality given in [11] and [7]. The main tool is the following remarkably lemma proved in [11] (see also [10]).

Lemma 2.1 *Let* Ω *be a Lipschitz domain and let* u *be a distribution on* Ω *such that* $u \in H^{-p-q}(\Omega)$ *and* $\partial^\alpha u \in H^{-p-q}(\Omega)$, $|\alpha| \leq q$, *for some integers* $p \geq 0$ *and* $q \geq 1$. *Then* $u \in H^{-p}(\Omega)$.

For the definition of Sobolev spaces with negative exponents see [9]. We will use the standard notation $H^0(\Omega) = L^2(\Omega)$.

This lemma is a generalization of Theorem 3.2, Chapter III, page 111, in [7] (see also Remark 3.1 on page 112 and the Comments (Sect. 8) on page 196 in the same chapter) where the case $q = 0$, $p = 1$ is proved. This particular case is enough for proving Korn's inequality. However, for the inequality (8) we need to take one more derivative, and hence we will use Lemma 2.1 for $q = p = 1$.

Proof Following [7], we divide the proof in two steps.

Step 1 Using Lemma 2.1, we will prove that $u_i, l_{ij}(u) \in L^2(\Omega)$ implies $u_i \in H^1(\Omega)$.

We have the following identity

$$\partial_k \partial_j u_i = \partial_j e_{ik}(u) + \partial_k e_{ij}(u) - \partial_i e_{jk}(u). \tag{22}$$

From this we deduce

$$\partial_k \partial_j u_i = \partial_j l_{ik}(u) + \partial_k l_{ij}(u) - \partial_i l_{jk}(u) + \frac{1}{n}(-\partial_j e(u)\delta_{ik} - \partial_k e(u)\delta_{ij} + \partial_i e(u)\delta_{jk}). \tag{23}$$

Taking a derivative ∂^k of Eq. (23) we obtain

$$\partial_j \Delta u_i = \partial_j \partial^k l_{ik} + \Delta l_{ij} - \partial_i \partial^k l_{jk} - \frac{1}{n-1}\delta_{ji}\partial^k \partial^f l_{kf}, \tag{24}$$

where we have used

$$\partial^i \partial^j l_{ij}(u) = \frac{(n-1)}{n}\Delta e(u). \tag{25}$$

By hypothesis we have $l_{ij}(u) \in L^2(\Omega)$, then the right hand side of Eq. (24) is in $H^{-2}(\Omega)$ and hence $\partial_j \Delta u_i \in H^{-2}(\Omega)$. We use Lemma 2.1 for the functions Δu_i with $p = q = 1$ to conclude that $\Delta u_i \in H^{-1}(\Omega)$. Then, by the identity

$$\partial^i l_{ij}(u) = \frac{1}{2}\Delta u_j + \left(\frac{1}{2} - \frac{1}{n}\right)\partial_j e(u) \tag{26}$$

we conclude that $\partial_j e(u) \in H^{-1}(\Omega)$.

Going back to Eq. (23) and using $\partial_j e(u) \in H^{-1}(\Omega)$ we conclude that $\partial_k \partial_j u_i \in H^{-1}(\Omega)$. We apply again Lemma 2.1 for the function $\partial_j u_i$ with $p = 0$ and $q = 1$ and we obtain $\partial_j u_i \in L^2(\Omega)$, that is, $u_i \in H^1(\Omega)$.

Step 2. Let H be the space of $u^i \in L^2(\Omega)$ such that $l_{ij}(u) \in L^2(\Omega)$. H is a Hilbert space for the norm

$$\int_\Omega (u^i u_i + l_{ij}(u) l^{ij}(u)) dv. \tag{27}$$

In Step 1 we have proved that $u \in H^1(\Omega) \iff u \in H$. We apply the closed graph theorem to the identity mapping from $H^1(\Omega)$ into H to obtain inequality (8). □

The proof fails for $n = 2$ because in this case we can not use Eq. (26) to conclude that $\partial_j e(u) \in H^{-1}(\Omega)$.

As it was mentioned in [6], an alternative proof of this theorem, under stronger assumptions on the regularity of the boundary, can be obtained using Proposition 12.1 of [13].

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