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From Baxter Q-operators to local charges

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Abstract. We discuss how the shift operator and the Hamiltonian enter the hierarchy of Baxter Q-operators in the example of $\mathfrak{gl}(n)$ homogeneous spin-chains. Building on the construction that was recently carried out by the authors and their collaborators, we find that a reduced set of Q-operators can be used to obtain local charges. The mechanism relies on projection properties of the corresponding $\mathcal{R}$-operators on a highest/lowest weight state of the quantum space. It is intimately related to the ordering of the oscillators in the auxiliary space. Furthermore, we introduce a diagrammatic language that makes these properties manifest and the results transparent. Our approach circumvents the paradigm of constructing the transfer matrix with equal representations in quantum and auxiliary space and underlines the strength of the Q-operator construction.

Keywords: algebraic structures of integrable models, integrable spin chains (vertex models), quantum integrability (Bethe ansatz), solvable lattice models

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Contents

1. Introduction

In 1971 R J Baxter introduced the Q-operator along with the celebrated Baxter equation/TQ relation [1] in order to calculate exactly the partition function of the

J. Stat. Mech. (2013) P02019

eight-vertex model. The method of functional relations and commuting transfer matrices originated in this work. It plays a fundamental role in the theory of integrable quantum systems.

One of the most important developments in the field since then is the framework of the quantum inverse scattering method (QISM), see e.g. [2]. This approach builds on the existence of Lax operators which are solutions to the Yang–Baxter equation and can be seen as the generating objects of the model. It employs the idea of an auxiliary space to construct transfer matrices which define one parameter families of commuting operators, labeled by the spectral parameter z. Here the transfer matrix built out of Lax matrices with the same representation in quantum and auxiliary space is especially important. The Hamiltonian, along with all other local charges of the spin-chain, arises as a logarithmic derivative around the shift point of this particular transfer matrix [3]. As the local charges belong to the family of commuting operators their spectrum can be obtained from the algebraic Bethe ansatz [4].

The link between Q-operators and the Hamiltonian as well as higher local charges is rather indirect. From the point of view of the algebraic Bethe ansatz the link is established by applying the Hamiltonian to Bethe vectors. Following this procedure one obtains the eigenvalues of the Hamiltonian in terms of Bethe roots [3]. By identifying the Baxter Q-functions with the polynomial which vanishes at the Bethe roots one may write the eigenvalues of the local integrals of motion for the Heisenberg chain as [5]

$$I_k = \frac{\partial^k}{\partial z^k} \ln \frac{Q(z+\delta)}{Q(z-\delta)}\bigg|_{z=0}. \tag{1.1}$$

The first two charges I_0 and I_1 correspond to the momentum and the nearest-neighbor Hamiltonian, respectively. The label δ is the $\mathfrak{sl}(2)$ weight of the local vacuum in the algebraic Bethe ansatz. As mentioned in [5], where integrable spin-chains emerged in the context of high energy QCD [6], the relation between the integrals of motion and the Q-functions can be extended to the operatorial level. From the viewpoint of the analytic Bethe ansatz, expressing the relevant T-function in terms of Q-functions, see e.g. [7], in principle allows derivation of higher rank analogues of the expression (1.1) using functional relations and analyticity properties[5]. Here, we neither make an ansatz for the wave function nor use functional relations to obtain local charges. The goal of this article is to provide a more *direct* connection between Q-operators and local charges.

In a series of papers [7]–[10] Q-operators were constructed for $\mathfrak{gl}(n)$ homogeneous spin-chains from fundamental principles. The construction follows the quantum inverse scattering method employing degenerate solutions of the Yang–Baxter equation as generating objects. While the quantum space of these Lax operators is determined by the model, the auxiliary space is infinite dimensional. It is realized by a set of oscillator algebras as pioneered in [11]. All functional relations involving Q-operators and transfer matrices of the type mentioned above follow from remarkable fusion/factorization properties of the degenerate Lax operators. The hierarchy of Q-operators is most easy to construct. It is best illustrated using Hasse diagrams [12]. In [7] the nearest-neighbor Hamiltonian was obtained solving the Yang–Baxter equation for the R-matrix with equal

[5] To the best of our knowledge this procedure has never been carried out beyond the class of representations known as Kirillov–Reshetikhin modules [14]. Even for this class of representations, and importantly their infinite dimensional analogues considered in this paper, it is hard to find the general case in the literature.

J. Stat. Mech. (2013) P02019

representation in quantum and auxiliary space, see also [13], while the dispersion relation was derived for some finite dimensional representations from the functional relations of the Q-operators and generalized transfer matrices. Since $\mathcal{R}$-operators for Q-operators are conceptually simple solutions to the Yang–Baxter equation, it is desirable to obtain the nearest-neighbor Hamiltonian and also higher local charges from Q-operators avoiding this detour.

In this article we show how local charges can be extracted directly from the Q-operators built in [7]. No reference to the transfer matrix mentioned above is required. This avoids the notoriously complicated construction of the transfer matrix with equal representation in quantum and auxiliary space. In section 2 we give a brief review of the recent construction of Q-operators for $\mathfrak{gl}(n)$ homogeneous spin-chains and establish some notation. In section 3 we introduce an opposite product on the auxiliary space and obtain alternative presentations of the degenerate solutions used for the construction of Q-operators, hereafter referred to as $\mathcal{R}$-operators. Section 4 is dedicated to the extremely important projection properties of the degenerate Lax operators. Their alternative presentation obtained in section 3 is essential in order to fully exploit these properties. After developing these techniques we introduce a convenient diagrammatic language for $\mathcal{R}$-operators which extends to Q-operators in section 5. It further concerns the derivation of the shift operator and the Hamiltonian from Q-operators. Equation (5.22) is one of the main results of this paper. It defines the Hamiltonian density in terms of the novel $\mathcal{R}$-operators for Q-operators introduced in [7]. Section 6 offers a summary of our results and suggestions for further studies. Furthermore, we provide several appendices where specific examples are studied. They also contain definitions and properties of the operatorial shifted weights frequently used in this paper.

2. Review

In a series of papers [8, 9, 7] new solutions to the Yang–Baxter equation were derived. They allow construction of Baxter Q-operators for $\mathfrak{gl}(n)$ invariant spin-chains. These so called $\mathcal{R}$-operators are of remarkably compact form and can be written as[6]

$$\mathcal{R}_I(z) = \mathrm{e}^{\bar{\mathbf{a}}^{\dot{c}}_{c}\, J^{c}_{\dot{c}}} \cdot \mathcal{R}_{0,I}(z) \cdot \mathrm{e}^{-\mathbf{a}^{c}_{\dot{c}}\, J^{\dot{c}}_{c}}, \tag{2.1}$$

with

$$\mathcal{R}_{0,I}(z) = \rho_I(z) \prod_{k=1}^{|\bar{I}|} \Gamma\left(z - \frac{|\bar{I}|}{2} - \hat{\ell}^{\bar{I}}_k + 1\right). \tag{2.2}$$

These equations require some explanation. The letter I denotes a subset of the set $\{1, \dots, n\}$ of cardinality $|I|$. The undotted indices take values from the set I and the dotted ones from its conjugate $\bar{I}$,

$$a, b, c \in I, \qquad \dot{a}, \dot{b}, \dot{c} \in \bar{I}, \qquad A, B, C \in I \cup \bar{I}. \tag{2.3}$$

The $\mathcal{R}$-operators are composed out of $|I| \cdot |\bar{I}|$ families of oscillators

$$[\mathbf{a}^{a}_{b}, \bar{\mathbf{a}}^{\dot{c}}_{d}] = \delta^{a}_{d}\delta^{\dot{c}}_{b} \tag{2.4}$$

[6] For reasons that will become clear in section 3 we explicitly denote the product by '$\cdot$'.

J. Stat. Mech. (2013) P02019

and $\mathfrak{gl}(n)$ generators J_B^A with

$$[J_B^A, J_D^C] = \delta_B^C J_D^A - \delta_D^A J_B^C. \tag{2.5}$$

The choice of the set I naturally identifies a subalgebra $\mathfrak{gl}(I)$ of $\mathfrak{gl}(n)$, i.e. the subalgebra spanned by J_b^a, see (2.3). The precise definition of the operators $\hat{\ell}_k^{\bar{I}}$ entering (2.2) can be found in appendix A, they are operatorial shifted weights of the subalgebra $\mathfrak{gl}(\bar{I})$. The operators $\mathcal{R}_I$ are elements of a suitable extension of the product space $\mathcal{U}(\mathfrak{gl}(n)) \otimes \mathcal{H}^{(I,\bar{I})}$, in the following denoted by $\mathfrak{A}_I$. The normalization ρ_I is not determined by the Yang–Baxter relation and is discussed in the following sections.

The $\mathcal{R}$-operators above satisfy the Yang–Baxter equation

$$\mathcal{L}(z_1)\, \mathbf{L}_I(z_2) \cdot \mathcal{R}_I(z_2 - z_1) = \mathcal{R}_I(z_2 - z_1) \cdot \mathbf{L}_I(z_2)\, \mathcal{L}(z_1). \tag{2.6}$$

Here $\mathbf{L}_I$ denotes the operator $\mathcal{R}_I$ with fundamental representation in the $\mathfrak{gl}(n)$ part

$$\mathbf{L}_I(z) = \begin{pmatrix} z\delta_b^a + H_b^a & \bar{\mathbf{a}}_b^{\dot{a}} \\ -\mathbf{a}_{\dot{b}}^a & \delta_{\dot{b}}^{\dot{a}} \end{pmatrix} \qquad \text{for } I = \{1, \dots, |I|\}, \tag{2.7}$$

with[7] $H_b^a = -\bar{\mathbf{a}}_b^{\dot{c}} \mathbf{a}_{\dot{c}}^a - \frac{1}{2}\delta_{\dot{c}}^{\dot{c}} \delta_b^a$, where summation over the dotted indices is understood. The operator $\mathcal{L}$ denotes the well-known Lax matrix

$$\mathcal{L}(z) = \begin{pmatrix} z\delta_b^a + J_b^a & J_b^{\dot{a}} \\ J_{\dot{b}}^a & z\delta_{\dot{b}}^{\dot{a}} + J_{\dot{b}}^{\dot{a}} \end{pmatrix}. \tag{2.8}$$

Baxter Q-operators are constructed as regularized traces over the oscillator space of monodromies built from the operators $\mathcal{R}_I$. Following [7], they are given by

$$\mathbf{Q}_I(z) = \mathrm{e}^{\mathrm{i}z\,\phi_I}\, \hat{\mathrm{Tr}}_{\mathcal{H}^{(I,\bar{I})}} \{\mathcal{D}_I\, \mathcal{R}_I(z) \otimes \cdots \otimes \mathcal{R}_I(z)\} \qquad \text{with } \phi_I = \sum_{a \in I} \phi_a. \tag{2.9}$$

Here the quantum space consists of L sites and will be denoted by $\mathcal{V} = \mathcal{V}_L \otimes \cdots \otimes \mathcal{V}_1$. In the following each $\mathcal{V}_i$ corresponds to the same representation Λ of $\mathfrak{gl}(n)$. The regulator in (2.9) is defined as

$$\mathcal{D}_I = \exp\Big\{-\mathrm{i} \sum_{a,\dot{b}} \phi_{a\dot{b}}\, \mathbf{h}_{a\dot{b}}\Big\}, \tag{2.10}$$

where we introduced the twist parameters $\phi_{a\dot{b}} = \phi_a - \phi_{\dot{b}}$, the number operator $\mathbf{h}_{a\dot{b}} = \bar{\mathbf{a}}_a^{\dot{b}} \mathbf{a}_{\dot{b}}^a + \frac{1}{2}$ and the normalized trace

$$\hat{\mathrm{Tr}}_{\mathcal{H}}\{\mathrm{e}^{-\mathrm{i}\phi\mathbf{h}} \star\} = \frac{\mathrm{Tr}_{\mathcal{H}}\{\mathrm{e}^{-\mathrm{i}\phi\mathbf{h}} \star\}}{\mathrm{Tr}_{\mathcal{H}}\{\mathrm{e}^{-\mathrm{i}\phi\mathbf{h}}\}}. \tag{2.11}$$

The operators $\mathbf{Q}_I$ generate a large family of commuting operators. These operators are functionally dependent, they satisfy certain quadratic equations known as QQ-relations. These functional relations can be regarded as 'off-shell' Bethe equations. The hierarchy of the 2^n Q-operators can be graphically exposed in the Hasse diagram [12].

[7] Here we only consider the *minimal* case discussed in [7].

J. Stat. Mech. (2013) P02019

3. Alternative presentation of $\mathcal{R}$-operators: reordering oscillators

The solution (2.1) to the Yang–Baxter equation (2.6) is presented as a normal ordered expression in the oscillators $\{\bar{\mathbf{a}}^{\dot{c}}_{c}, \mathbf{a}^{c}_{\dot{c}}\}$. For reasons that will become clear in the next section we are also interested in its expression which is anti-normal ordered in the oscillators of the auxiliary space. The anti-normal ordered form of the $\mathcal{R}$-operators can be obtained either from the Yang–Baxter equation or directly by reordering the oscillators in (2.1). As we will see, the approach from the Yang–Baxter equation will be very powerful to obtain the desired expressions. However, it is not possible to fix the relative normalization by this method.

3.1. The Yang–Baxter approach

To derive the expression for the anti-normal ordered $\mathcal{R}$-operators directly from the Yang–Baxter equation it turns out to be convenient to introduce an opposite product on $\mathfrak{A}_I$. Let $\mathcal{O} \in \mathfrak{A}_I$ be written as

$$\mathcal{O} = \sum_k a(k) \otimes b(k), \tag{3.1}$$

with $a(k) \in \mathcal{U}(\mathfrak{gl}(n))$ and $b(k) \in \mathcal{H}^{(I,\bar{I})}$. Given two elements of $\mathfrak{A}_I$ the product used in (2.6) is defined as

$$\mathcal{O}_1 \cdot \mathcal{O}_2 = \sum_{k,l} a_1(k)\, a_2(l) \otimes b_1(k)\, b_2(l). \tag{3.2}$$

We now define the opposite product $\circ$ as

$$\mathcal{O}_1 \circ \mathcal{O}_2 = \sum_{k,l} a_1(k)\, a_2(l) \otimes b_2(l)\, b_1(k). \tag{3.3}$$

One can easily check that this product is associative. The Yang–Baxter equation (2.6) can then be written as

$$\mathcal{L}(z_1)\, \mathcal{R}_I(z_2 - z_1) \circ \mathbf{L}_I(z_2) = \mathbf{L}_I(z_2) \circ \mathcal{R}_I(z_2 - z_1)\, \mathcal{L}(z_1). \tag{3.4}$$

We would like to stress that this is exactly the same equation as (2.6) only rewritten in terms of the opposite product. Now, in analogy to [7] we substitute the ansatz

$$\mathcal{R}_I(z) = \mathrm{e}^{\bar{\mathbf{a}}^{\dot{c}}_{c}\, J^{c}_{\dot{c}}} \circ \tilde{\mathcal{R}}_{0,I}(z) \circ \mathrm{e}^{-\mathbf{a}^{c}_{\dot{c}}\, J^{\dot{c}}_{c}} \tag{3.5}$$

into (3.4) and as before obtain four sets of defining relations for $\tilde{\mathcal{R}}_{0,I}$. As there is some redundancy in these equations we only present one of them,

$$\tilde{\mathcal{R}}_{0,I}(z) \left(\left(z + \frac{\bar{I}}{2} \right) J^{\dot{a}}_{b} - J^{c}_{b} J^{\dot{a}}_{c} \right) = J^{\dot{a}}_{b}\, \tilde{\mathcal{R}}_{0,I}(z). \tag{3.6}$$

Comparing this equation to [7] it is easy to recognize that $\tilde{\mathcal{R}}_{0,I}(z)$ satisfies the same defining relation as $\mathcal{R}^{-1}_{0,\bar{I}}(z + n/2)$. We conclude that

$$\tilde{\mathcal{R}}_{0,I}(z) = \tilde{\rho}_I(z) \prod_{k=1}^{|I|} \frac{1}{\Gamma(z + |\bar{I}|/2 - \hat{\ell}^{I}_{k} + 1)}. \tag{3.7}$$

The ratio of ρ_I and $\tilde{\rho}_I$ entering (2.2) and (3.7) can be determined by requiring that (2.1) and (3.5) are the *same* operators. This is investigated in section 3.2.

3.2. The direct approach

The $\mathcal{R}$-operators (2.1) and (3.5) can be written as

$$\mathcal{R}_I(z) = \sum_{n,m=0}^{\infty} \frac{(-1)^m}{n!\,m!}\, \bar{\mathbf{a}}^{\dot{a}_1}_{a_1} \cdots \bar{\mathbf{a}}^{\dot{a}_n}_{a_n}\, \mathbf{a}^{b_1}_{\dot{b}_1} \cdots \mathbf{a}^{b_m}_{\dot{b}_m}\, J^{a_1}_{\dot{a}_1} \cdots J^{a_n}_{\dot{a}_n}\, \mathcal{R}_{0,I}(z)\, J^{\dot{b}_1}_{b_1} \cdots J^{\dot{b}_m}_{b_m}, \tag{3.8}$$

$$\mathcal{R}_I(z) = \sum_{n,m=0}^{\infty} \frac{(-1)^m}{n!\,m!}\, \mathbf{a}^{b_1}_{\dot{b}_1} \cdots \mathbf{a}^{b_m}_{\dot{b}_m}\, \bar{\mathbf{a}}^{\dot{a}_1}_{a_1} \cdots \bar{\mathbf{a}}^{\dot{a}_n}_{a_n}\, J^{a_1}_{\dot{a}_1} \cdots J^{a_n}_{\dot{a}_n}\, \tilde{\mathcal{R}}_{0,I}(z)\, J^{\dot{b}_1}_{b_1} \cdots J^{\dot{b}_m}_{b_m}. \tag{3.9}$$

Here we expanded the exponentials using the definition of the products in (3.2) and (3.3), respectively. To obtain the relation between $\mathcal{R}_{0,I}$ and $\tilde{\mathcal{R}}_{0,I}$ we have to reorder the oscillators in one of the two expressions. We find that for each pair $\bar{\mathbf{a}}^{\dot{c}}_{c}$, $\mathbf{a}^{c}_{\dot{c}}$ that is reordered in (3.8) $\mathcal{R}_{0,I}$ is 'conjugated' by the corresponding $\mathfrak{gl}(\{c,\dot{c}\})$ generators as

$$\mathcal{R}_{0,I} \longrightarrow \sum_{k=0}^{\infty} \frac{1}{k!}\, (J^{c}_{\dot{c}})^k\, \mathcal{R}_{0,I}(z)\, (J^{\dot{c}}_{c})^k. \tag{3.10}$$

This relation is obtained using (D.1) and does not rely on the precise form of $\mathcal{R}_{0,I}$. After subsequent conjugation of $\mathcal{R}_{0,I}$ with all $|I|\cdot|\bar{I}|$ $\mathfrak{gl}(\{c,\dot{c}\})$ subalgebra generators of $\mathfrak{gl}(n)$ one obtains an expression for $\tilde{\mathcal{R}}_{0,I}$,

$$\tilde{\mathcal{R}}_{0,I}(z) = \sum_{\{k_{c\dot{c}}\}=0}^{\infty} \prod_{c\in I,\dot{c}\in\bar{I}} \frac{1}{\sqrt{k_{c\dot{c}}!}}\, (J^{c}_{\dot{c}})^{k_{c\dot{c}}}\, \mathcal{R}_{0,I}(z) \prod_{c\in I,\dot{c}\in\bar{I}} \frac{1}{\sqrt{k_{c\dot{c}}!}} (J^{\dot{c}}_{c})^{k_{c\dot{c}}}. \tag{3.11}$$

This fixes the ratio of the prefactors ρ and $\tilde{\rho}$ appearing in (2.2) and (3.7). Naively, (3.11) appears rather different from (3.7). However, they must coincide since they satisfy the same defining relations. This is explicitly demonstrated for the case of $\mathfrak{gl}(2)$ in section C.1. See also section 4 for a clarifying discussion on the normalization.

4. Projection properties of $\mathcal{R}$-operators

The construction of local charges in the conventional QISM relies on the *fundamental* R-matrix $\mathbf{R}$ for which the auxiliary space is the same as the quantum space at each site. It requires the existence of a special point z_* where the R-matrix reduces to the permutation operator

$$\mathbf{R}(z_*) = \mathbf{P}, \tag{4.1}$$

see e.g. [2]. This property is often referred to as the regularity condition. The construction of local charges presented here bypasses the use of the fundamental R-matrix and is based on remarkable properties of the $\mathcal{R}$-operators in (2.1) for special values of the spectral parameter. For the example of the fundamental representation in the quantum space we find that the operator $\mathbf{L}_I$ given in (2.7) degenerates at *two* special points. Using *both*

J. Stat. Mech. (2013) P02019

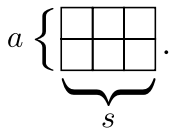
products introduced in section 2 we find that

$$\mathbf{L}_I\left(+\frac{|\bar{I}|}{2}\right) = \begin{pmatrix}\bar{\mathbf{a}}_b^{\dot{c}}\\ \delta_{\dot{b}}^{\dot{c}}\end{pmatrix}\cdot\begin{pmatrix}-\mathbf{a}_{\dot{c}}^{a} & \delta_{\dot{c}}^{\dot{a}}\end{pmatrix}, \qquad \mathbf{L}_I\left(-\frac{|\bar{I}|}{2}\right) = \begin{pmatrix}\bar{\mathbf{a}}_b^{\dot{c}}\\ \delta_{\dot{b}}^{\dot{c}}\end{pmatrix}\circ\begin{pmatrix}-\mathbf{a}_{\dot{c}}^{a} & \delta_{\dot{c}}^{\dot{a}}\end{pmatrix}. \tag{4.2}$$

Interestingly, properties similar to (4.2) appear quite naturally in the derivation of the Baxter equation, see e.g. [15]. This in turn is strictly connected to Baxter's original idea [1]. The degeneration can be understood from the spectral parameter dependent part of the $\mathcal{R}$-operators, compare to (2.1) and (3.5). For the fundamental representation, it originates from a reduction of the rank of $\mathcal{R}_{0,I}$ and $\tilde{\mathcal{R}}_{0,I}$ at the special points $\hat{z} = +|\bar{I}|/2$ and $\check{z} = -|\bar{I}|/2$, respectively. Of distinguished importance are $\mathbf{L}_I$-operators with $|\bar{I}| = 1$. In this case the rank of the oscillator independent part reduces to 1.

Relations of the type (4.2) hold for any highest/lowest weight representation of $\mathfrak{gl}(n)$. Their precise form can be obtained via a careful analysis of the spectrum of the shifted weight operators $\hat{\ell}_i^K$ entering $\mathcal{R}_I$. However, the analysis is technically involved. Details can be found in appendix A. In the following we restrict ourselves to representations corresponding to rectangular Young diagrams and their infinite dimensional generalization.

A rectangular Young diagram is labeled by two parameters (s, a) with $s, a \in \mathbb{N}$ according to



(4.3)

For representations of this type, also known as Kirillov–Reshetikhin modules [14], there exist two values of the spectral parameter such that $\mathcal{R}_{0,I}$ and $\tilde{\mathcal{R}}_{0,I}$ respectively are projectors on a highest weight state[8] for $|I| = n-a$. The number of highest/lowest weight states for such representations is $\binom{n}{a}$ and exactly coincides with the number of operators $\mathcal{R}_I$ with $|I| = n-a$. Each $\mathcal{R}_{0,I}$ and $\tilde{\mathcal{R}}_{0,I}$ of cardinality $n-a$ projects on a different highest weight state depending on the elements in I. With an appropriate normalization discussed in section 4.2 we find for $|I| = n-a$ that

$$\mathcal{R}_{0,I}(\hat{z}) = |hws\rangle\langle hws| \qquad \text{and} \qquad \tilde{\mathcal{R}}_{0,I}(\check{z}) = |hws\rangle\langle hws|. \tag{4.4}$$

As a direct consequence of (4.4) we obtain that the $\mathcal{R}$-operator at the special points $\hat{z}$ and $\check{z}$ can be written as

$$\mathcal{R}_I(\hat{z}) = \mathrm{e}^{\bar{\mathbf{a}}_c^{\dot{c}} J_{\dot{c}}^c}\cdot|hws\rangle\langle hws|\cdot\mathrm{e}^{-\mathbf{a}_{\dot{c}}^c J_c^{\dot{c}}} \qquad \text{and} \qquad \mathcal{R}_I(\check{z}) = \mathrm{e}^{\bar{\mathbf{a}}_c^{\dot{c}} J_{\dot{c}}^c}\circ|hws\rangle\langle hws|\circ\mathrm{e}^{-\mathbf{a}_{\dot{c}}^c J_c^{\dot{c}}}. \tag{4.5}$$

As we will see, these properties carry over to non-compact representations with highest weight that fulfil a *generalized rectangularity condition*, see section 4.1 and appendix A. However, not all $\mathcal{R}$-operators of a certain cardinality $|I|$ share the projection property, see the discussion in section C.3. It will become clear in section 5 that as a consequence of (4.5) the Q-operators at the special points $\hat{z}$ and $\check{z}$ are related by the shift operator, see (5.13).

[8] The notion of highest weight state depends on the choice of the raising generators. For rectangular representations $a!(n-a)!$ such choices correspond to the same highest weight state.

J. Stat. Mech. (2013) P02019

4.1. Reduction

It emerged in the discussion of the fundamental representation that at special values of the spectral parameter the operators $\mathcal{R}_{0,I}$ and $\tilde{\mathcal{R}}_{0,I}$ become projectors on a certain subspace. In the following we show when and how this happens. The analysis is clearly connected to the pattern of the decomposition of the $\mathfrak{gl}(n)$ representation at a site

$$\Lambda \to \oplus_\alpha m_\alpha \, (\Lambda^I_\alpha, \Lambda^{\bar{I}}_\alpha) \tag{4.6}$$

under the restriction $\mathfrak{gl}(n) \downarrow \mathfrak{gl}(I) \oplus \mathfrak{gl}(\bar{I})$. We are specifically interested in representations Λ and a set I such that $\hat{\ell}^{\bar{I}}_k$ are bounded from above and $\hat{\ell}^I_k$ are bounded from below[9]. The bound is saturated for all k by the subspace $(\Lambda^I_{\alpha_0}, \Lambda^{\bar{I}}_{\alpha_0})$ of Λ annihilated by the action of generators $J^{\dot{a}}_a$,

$$J^{\dot{a}}_a |\Lambda^I_{\alpha_0}, \Lambda^{\bar{I}}_{\alpha_0}\rangle = 0, \tag{4.7}$$

where the indices take values according to (2.3). The subspace $(\Lambda^I_{\alpha_0}, \Lambda^{\bar{I}}_{\alpha_0})$ is nothing but the $\mathfrak{gl}(I) \oplus \mathfrak{gl}(\bar{I})$ irreducible representation generated by the action of J^a_b and $J^{\dot{a}}_{\dot{b}}$ on the $\mathfrak{gl}(n)$ highest weight state[10] $|\Lambda\rangle$. Moreover, the eigenvalues of any fixed $\hat{\ell}^K_k$ are integer spaced. The fact that the operators $\mathcal{R}_{0,I}$ and $\tilde{\mathcal{R}}_{0,I}$ become projectors on this subspace for special values of the spectral parameter is an immediate consequence of the properties of the operators $\hat{\ell}^K_k$ together with the pole structure of the gamma function.

The class of representations considered at the end of the previous section (here referred to as generalized rectangular representations) have a number of remarkable features, see appendix A. In particular, there is at least one set I such that the subspace on which $\mathcal{R}_{0,I}$ and $\tilde{\mathcal{R}}_{0,I}$ project is one-dimensional. This fact is equivalent to the existence of a state such that

$$J^{\dot{a}}_a |\Lambda^I_0, \Lambda^{\bar{I}}_0\rangle = 0, \qquad J^a_b |\Lambda^I_0, \Lambda^{\bar{I}}_0\rangle = \lambda_I \, \delta^a_b |\Lambda^I_0, \Lambda^{\bar{I}}_0\rangle, \qquad J^{\dot{a}}_{\dot{b}} |\Lambda^I_0, \Lambda^{\bar{I}}_0\rangle = \bar{\lambda}_I \, \delta^{\dot{a}}_{\dot{b}} |\Lambda^I_0, \Lambda^{\bar{I}}_0\rangle \tag{4.8}$$

for a properly chosen set I and some $\bar{\lambda}_I, \lambda_I$, see appendix A for details. For convenience the state defined in (4.8) will be denoted as $|hws\rangle$.

4.2. On the normalization of $\mathcal{R}$-operators

Besides the ratio of ρ_I and $\tilde{\rho}_I$ which is fixed by (3.11) an overall normalization of the $\mathcal{R}$-operators has not yet been chosen. In our previous analysis we determined the one-dimensional subspace that saturates the bound of $\hat{\ell}^{\bar{I}}_k$ and $\hat{\ell}^I_k$. The action of the shifted weights on this subspace is given in appendix A. As already mentioned, for our purposes it is convenient to choose a normalization such that (4.5) holds, i.e.

$$\begin{aligned} \mathcal{R}_{0,I}(z) &= \kappa_I(z) \prod_{k=1}^{|\bar{I}|} \frac{\Gamma(z - |\bar{I}|/2 - \hat{\ell}^{\bar{I}}_k + 1)}{\Gamma(z - |\bar{I}|/2 + k - \bar{\lambda}_I)}, \\ \tilde{\mathcal{R}}_{0,I}(z) &= \tilde{\kappa}_I(z) \prod_{k=1}^{|I|} \frac{\Gamma(z + |\bar{I}|/2 + k - \lambda_I)}{\Gamma(z + |\bar{I}|/2 - \hat{\ell}^I_k + 1)}, \end{aligned} \tag{4.9}$$

[9] For any finite dimensional representation this is true for any set I.

[10] The raising generators that enter the $\mathfrak{gl}(n)$ highest weight condition for $|\Lambda\rangle$ are chosen to include $J^{\dot{a}}_a$.

J. Stat. Mech. (2013) P02019

compare to (2.2) and (3.7). Above, κ_I and $\tilde{\kappa}_I$ are periodic functions of $\hat{\ell}_k^{\bar{I}}$ and $\hat{\ell}_k^I$ of period one, respectively. Furthermore, they coincide on the highest weight $\kappa_I(z)|hws\rangle = \tilde{\kappa}_I(z)|hws\rangle = |hws\rangle$ and in analogy to ρ_I and $\tilde{\rho}_I$ are dependent by (3.11), see also section C.1 for the example of $\mathfrak{gl}(2)$. As discussed in section 3.1, from the study of the Yang–Baxter equation it seems to be rather natural to fix the overall normalization such that

$$\tilde{\mathcal{R}}_{0,\bar{I}}\left(z-\frac{|\bar{I}|}{2}\right)=\mathcal{R}_{0,I}^{-1}\left(z+\frac{|I|}{2}\right). \tag{4.10}$$

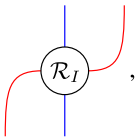
Interestingly, this relation implies the crossing equation

$$\left(\mathcal{R}_{\bar{I}}\left(z-\frac{|\bar{I}|}{2}\right)\right)^{*}=\mathcal{R}_{I}^{-1}\left(z+\frac{|I|}{2}\right) \tag{4.11}$$

with $(\bar{\mathbf{a}}_{\dot{a}_1}^{a_1}\cdots\bar{\mathbf{a}}_{\dot{a}_m}^{a_m}\,\mathbf{a}_{b_1}^{\dot{b}_1}\cdots\mathbf{a}_{b_n}^{\dot{b}_n})^{*}=\bar{\mathbf{a}}_{b_1}^{\dot{b}_1}\cdots\bar{\mathbf{a}}_{b_m}^{\dot{b}_m}\,\mathbf{a}_{\dot{a}_1}^{a_1}\cdots\mathbf{a}_{\dot{a}_n}^{a_n}$. However, an explicit study of these relations is left to the future.

5. Diagrammatics and local charges

As is customary, we denote R-matrices by two crossing lines. In the construction of generalized transfer matrices each vertical line corresponds to the quantum space associated to a spin-chain site. Likewise, horizontal lines represent the auxiliary space. In the following $\mathcal{R}$-operators generating Q-operators are depicted as

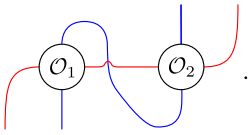
$\mathcal{R}_I(z) =$ 

, (5.1)

compare to (2.9). We will now develop a pictorial language for the $\mathcal{R}$-operators, which incorporates all the aforementioned properties, see sections 3 and 4. One of its main advantages is that the opposite product (3.3), which might look unfamiliar in the equations, is translated to a rather natural composition rule. It is a key ingredient to reveal the emergence of local charges from Q-operators.

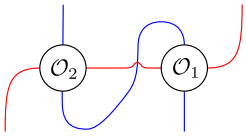
5.1. Two multiplication rules

As discussed in section 3, it is natural to introduce two different multiplication rules. Diagrammatically the product $\cdot$ can then be understood as

$\mathcal{O}_1\cdot\mathcal{O}_2 =$ 

. (5.2)

Here the oscillator (red) and $\mathcal{U}(\mathfrak{gl}(n))$ (blue) components of $\mathcal{O}_1$ are both multiplied from the left to $\mathcal{O}_2$. On the other hand, the product $\circ$ is denoted by

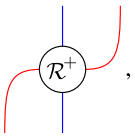
$\mathcal{O}_1 \circ \mathcal{O}_2 =$ 

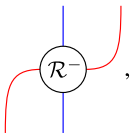
. (5.3)

Here the $\mathcal{U}(\mathfrak{gl}(n))$ part of $\mathcal{O}_1$ is also multiplied from the left to $\mathcal{O}_2$, but the oscillator part is multiplied to the right. In summary, once the operators $\mathcal{O}_i$ are written as (3.1) the order of the factors (from left to right) in (5.2) and (5.3) is obtained by following the lines from bottom to top.

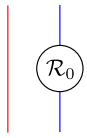
5.2. $\mathcal{R}$-operators

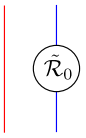
We will now develop a diagrammatic expression for $\mathcal{R}_I$ for both (2.1) and (3.5). To be pedagogical we proceed slowly. It is clear that $\mathcal{R}_I$ can be regarded as a composite object of four parts, namely

$e^{\bar{\mathbf{a}}^{\dot{c}}_{\dot{c}} J^{c}_{\dot{c}}} =$ 

, $e^{-\mathbf{a}^{c}_{\dot{c}} J^{\dot{c}}_{c}} =$ 

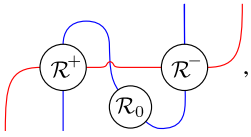
, (5.4)

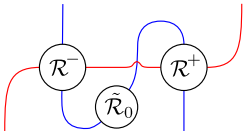
$\mathcal{R}_{0,I} =$ 

, $\tilde{\mathcal{R}}_{0,I} =$ 

. (5.5)

$\mathcal{R}_{I,0}$ and $\tilde{\mathcal{R}}_{I,0}$ act trivially in the auxiliary space, this is depicted by the straight line in (5.5). The label I is suppressed in the pictures. Let us now construct the two expressions of $\mathcal{R}_I$ given in (2.1) and (3.5). Using the ingredients above and the multiplication rules (5.2) and (5.3) one finds

$\mathcal{R}_I =$ 

, $\mathcal{R}_I =$ 

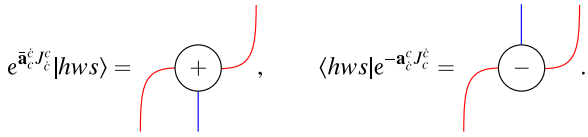
. (5.6)

When reading the diagrams from bottom to top it becomes clear that the expression on the left-hand side is normal ordered, while the expression on the right-hand side is anti-normal ordered in the oscillator space.

5.3. Projection properties

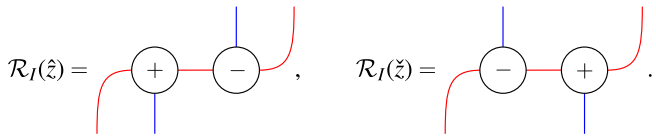
In section 4 we argued that at the special points $\hat{z}$ and $\check{z}$ some $\mathcal{R}$-operators of certain cardinality $|I|$ decompose into an outer product, see (4.5). This fact is a consequence of the degeneration (to rank 1) of $\mathcal{R}_{0,I}$ and $\tilde{\mathcal{R}}_{0,I}$ for generalized rectangular representations.

In the diagrammatics introduced, the building blocks of (4.5) are denoted by



(5.7)

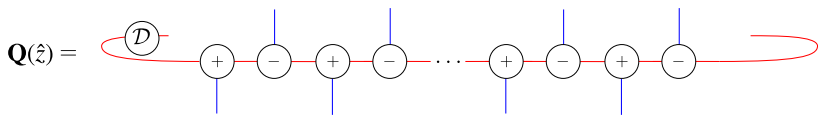
As the above expressions are 'vector' and 'covector' in the quantum space, a quantum space operator acts on them by left and right multiplication, respectively. This is indicated by one missing ingoing/outgoing vertical line. Using the notions of the two defined products we see that according to (4.5) at the special points (5.6) simplifies to



(5.8)

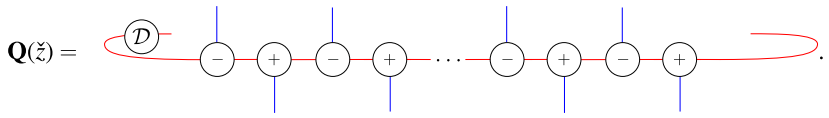
5.4. Baxter Q-operators

Baxter Q-operators can be built from the monodromy of the $\mathcal{R}$-operators as reviewed in section 2. Here, we will concentrate on the Q-operators constructed out of the $\mathcal{R}$-operators that satisfy condition (4.5), see section 4 for more details. Hereafter, the index I of the chosen Q-operator will be omitted. The diagrammatics for these $\mathcal{R}$-operators was developed above. From (5.8) it is clear that the corresponding Q-operators at the special points are given by



(5.9)

and



(5.10)

Here $\mathcal{D}$ denotes the regulator in (2.10). For convenience we recall that in (5.9) and (5.10) there is one ingoing and one outgoing vertical line for each spin-chain site. As indicated in the picture, the auxiliary space is closed by the trace, see (2.9).

5.5. Shift mechanism

The homogeneous spin-chain has the property of being translationally invariant; the shift operator defined as

$$\mathbf{U}\, X_n\, \mathbf{U}^{-1} = X_{n+1}, \qquad \mathbf{U}\, X_L\, \mathbf{U}^{-1} = f_1(\phi)\, X_1\, f_1^{-1}(\phi) \tag{5.11}$$

commutes with the Hamiltonian and all generalized transfer matrices. The operator $f_1(\phi)$ arises from the twisted boundary conditions and is explicitly given below.

The shift operator can be written as (see e.g. [16, 2])

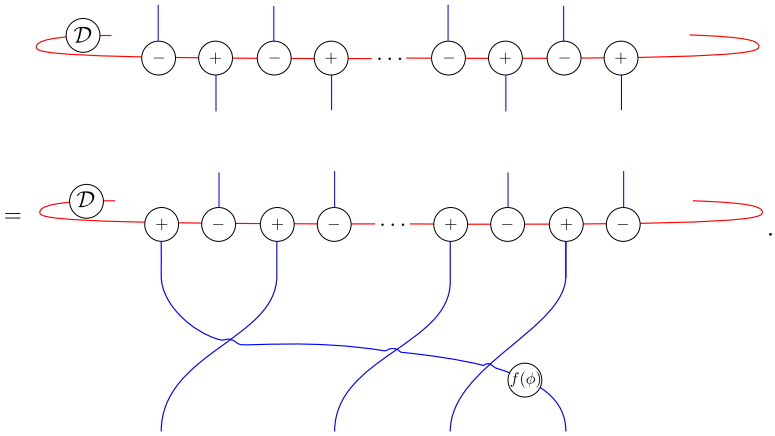
$$\boldsymbol{U} = f_1(\phi)\, \mathbf{P}_{1,2}\mathbf{P}_{2,3}\cdots\mathbf{P}_{L-1,L}, \tag{5.12}$$

J. Stat. Mech. (2013) P02019

where $\mathbf{P}_{i,i+1}$ acts as a permutation on sites i and $i+1$ in the quantum space. The main result of this subsection is to show the relation

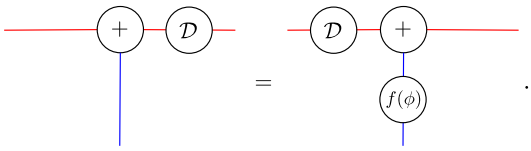
$$\mathbf{Q}(\check{z}) = \mathbf{U}\,\mathbf{Q}(\hat{z}). \tag{5.13}$$

The label I has been omitted following the logic as in sections 5.3 and 5.4. This equation is immediately proven once it is rewritten in the diagrammatic language developed previously,



(5.14)

The only non-trivial step in the proof is to move the last term $\oplus$ in the left-hand side of (5.14) through the regulator $\textcircled{\mathcal{D}}$. This is carried out using the relation



(5.15)

A direct computation shows that

$$f(\phi) = \exp\left\{ \mathrm{i} \sum_{c\in I} \phi_c (J_c^c - \lambda_I) + \mathrm{i} \sum_{\dot{c}\in \bar{I}} \phi_{\dot{c}} (J_{\dot{c}}^{\dot{c}} - \bar{\lambda}_I) \right\}. \tag{5.16}$$

This proves relation (5.13) for the large class of generalized rectangular representations. Using (5.13) and the form of the Q-operator eigenvalues[11] in terms of Bethe roots $\{z_i\}_{i=1}^M$

$$Q(z) = \mathrm{e}^{\mathrm{i}z\phi_I} \prod_{i=1}^{M} (z - z_i). \tag{5.17}$$

The eigenvalues of the shift operators are written as

$$U(\{z_i\}) = \mathrm{e}^{\mathrm{i}(\check{z}-\hat{z})\phi_I} \prod_{i=1}^{M} \frac{\check{z} - z_i}{\hat{z} - z_i}. \tag{5.18}$$

[11] Criteria concerning the diagonalizability of Q-operators can be obtained following the arguments in [19]. One finds sufficient conditions on $\mathcal{R}$-operators such that the corresponding transfer matrix is a normal operator. This analysis requires a case-by-case study as it is a property connected to the choice of quantum space. One can show that the Q-operators considered in this section are diagonalizable.

J. Stat. Mech. (2013) P02019

The identification of the special points $\hat{z}$ and $\check{z}$ is particularly important as it reveals how higher local charges may be extracted from Q-operators. In the next subsection this is elucidated for the case of the nearest-neighbor Hamiltonian.

5.6. The nearest-neighbor Hamiltonian and its action

We identified two special points $\hat{z}$ and $\check{z}$ at which the Q-operators are related by the shift operator, see (5.13). This enables us to give a direct operatorial derivation of (1.1). The main result of this section is that

$$\mathbf{H} = \frac{\mathbf{Q}'(\check{z})}{\mathbf{Q}(\check{z})} - \frac{\mathbf{Q}'(\hat{z})}{\mathbf{Q}(\hat{z})} \tag{5.19}$$

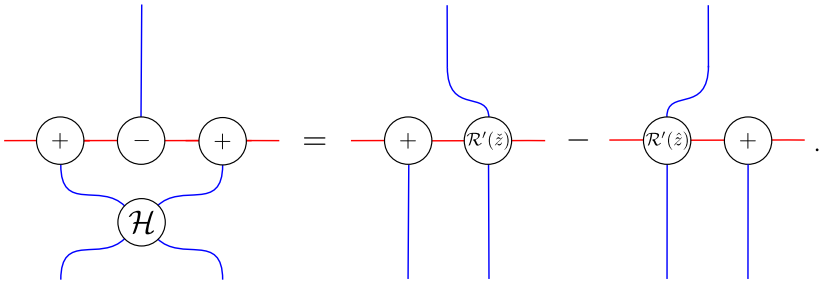
is a nearest-neighbor Hamiltonian, i.e.

$$\mathbf{H} = \sum_{i=1}^{L} \mathcal{H}_{i,i+1}. \tag{5.20}$$

It is of prime importance as it yields the total energy of the spin-chain and determines the time-evolution of the system. An important step in the derivation of the Hamiltonian (5.20) is to rewrite (5.19) as

$$\mathbf{H}\,\mathbf{Q}(\check{z}) = \mathbf{Q}'(\check{z}) - \mathbf{U}\,\mathbf{Q}'(\hat{z}). \tag{5.21}$$

Here we used (5.13). The derivation of (5.19) is a direct consequence of the truly remarkable identity



(5.22)

The significance of this equation is twofold. Firstly, it contains the non-trivial statement that the right-hand side of (5.22) can be written as the left-hand side for 'some' $\mathcal{H}$ that, as encoded in the picture, acts non-trivially only in the quantum space. This is proven in section B.1 using the so called Sutherland equation, originally introduced to provide a criterion for a local Hamiltonian to commute with a given transfer matrix [17], and the special properties of the $\mathcal{R}$-operator. Secondly, (5.22) defines $\mathcal{H}$ uniquely in terms of the $\mathcal{R}$-operators for Q-operators. The fact that this way of defining $\mathcal{H}$ can be particularly convenient for practical purposes is supported by the non-trivial example of $\mathfrak{sl}(2)$ spin $-\frac{1}{2}$ in section C.2.

Using (5.22), (5.21) can be shown quickly. The derivative of the Q-operators follows immediately from the definition in (2.9) for any set I

$$\mathbf{Q}'_I(z) = \mathrm{i}\,\phi_I\,\mathbf{Q}_I(z) + \mathrm{e}^{\mathrm{i}z\,\phi_I} \sum_{k=1}^{L} \hat{\mathrm{Tr}}_{\mathcal{H}^{(I,\bar{I})}} \left\{ \mathcal{D}_I\,\mathcal{R}_I(z) \otimes \cdots \otimes \underbrace{\mathcal{R}'_I(z)}_{k\text{th side}} \otimes \cdots \otimes \mathcal{R}_I(z) \right\}. \tag{5.23}$$

J. Stat. Mech. (2013) P02019

Taking a closer look at (5.21), one finds that the right-hand side can be rearranged as a sum of local contributions corresponding to the Hamiltonian density $\mathcal{H}$. This is carried out by pairing terms according to (5.22). Furthermore, from (5.21) it follows that

$$\mathcal{H}_{L,L+1} = f_1(\phi)\mathcal{H}_{L,1}f_1^{-1}(\phi). \tag{5.24}$$

Thus, we have shown (5.21). It is worth mentioning that an analogous and equivalent relation to (5.22) holds for the action of $\mathcal{H}$ from the right, see also (B.4). On the level of eigenvalues, upon using (5.17), (5.19) gives the famous energy formula

$$E(\{z_i\}) = \sum_{i=1}^{M}\left(\frac{1}{\check{z}-z_i} - \frac{1}{\hat{z}-z_i}\right). \tag{5.25}$$

This coincides with (1.1) for $k = 1$. We would like to stress again that the auxiliary and quantum spaces of the $\mathcal{R}$-operators are of different nature. The mechanism by which the Hamiltonian density can be extracted from the $\mathcal{R}$-operators is encoded in (5.22). The explicit expression for $\mathcal{H}$ for generalized rectangular representations in the quantum space is provided in section B.2. If we further restrict ourselves to certain representations one obtains rather convenient expressions for the Hamiltonian density. This is carried out for the fundamental representation and the $\mathfrak{sl}(2)$ spin $-\frac{1}{2}$ case.

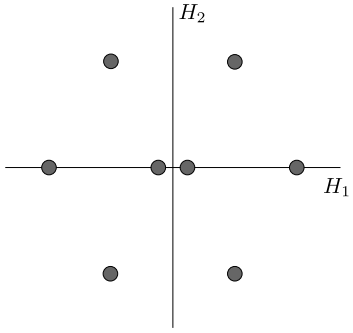
6. Concluding remarks

In this paper we studied how the local charges directly enter the hierarchy of Q-operators. Our studies provide a transparent derivation of the operatorial version of the eigenvalue formula for the first two local charges in (1.1) by employing the recent construction of Q-operators [7]. The method states clearly which Q-operators[12] in the hierarchy can be used to extract local charges, given a rectangular representation and their non-compact generalizations in the quantum space Λ. It further implies the validity of the dispersion relation for these representations.

We found that the Q-operators constructed in [7] provide an intuitive way to extract local charges. The mechanism relies on special features of the novel $\mathcal{R}$-operators used as generating objects for the Q-operators. Each $\mathcal{R}$-operator admits two alternative presentations corresponding to a normal or anti-normal ordered form in the auxiliary oscillator part. Some properties of the $\mathcal{R}$-operators that are manifest in one presentation are hidden in the other and vice versa. Following this paradigm, we identified *two* special points at which the $\mathcal{R}$-operators degenerate under the two products defined in section 3.1. This fact can be traced back to the reduction of the oscillator independent part $\mathcal{R}_{0,I}$ and $\tilde{\mathcal{R}}_{0,I}$ to rank 1, respectively. They become projectors on a highest weight state. The diagrammatics developed incorporates this reduction. On the level of Q-operators this leads to the shift operator which relates the Q-operator at these two points, see (5.13). Furthermore, it yields to the identification of the Hamiltonian density for the nearest-neighbor Hamiltonian. Here, we would like to stress that in this method, the quantum and auxiliary spaces are in general of different nature. An explicit formula of the Hamiltonian density and its action in terms of the quantum space generators at the corresponding two

[12] The zeros of the eigenvalues of these distinguished Q-operators are usually referred to as momentum carrying Bethe roots.

J. Stat. Mech. (2013) P02019



(H_1, H_2)	$\mathcal{R}_{0,\{3\}}(z)$	$\mathcal{R}_{0,\{23\}}(z)$
$(1, 0)$	1	$z + \frac{1}{2}$
$(\frac{1}{2}, \frac{\sqrt{3}}{2})$	z	$z + \frac{1}{2}$
$(\frac{1}{2}, -\frac{\sqrt{3}}{2})$	1	$(z + \frac{1}{2})(z - \frac{3}{2})$
$(0, 0)$	z	$(z + \frac{1}{2})(z - \frac{3}{2})$
$(0, 0)$	$z - 2$	$(z + \frac{1}{2})(z - \frac{3}{2})$
$(-\frac{1}{2}, \frac{\sqrt{3}}{2})$	$z(z - 2)$	$(z + \frac{1}{2})(z - \frac{3}{2})$
$(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$	z	$(z + \frac{1}{2})(z - \frac{3}{2})(z - \frac{1}{2})$
$(-1, 0)$	$z(z - 2)$	$(z + \frac{1}{2})(z - \frac{3}{2})(z - \frac{1}{2})$

Figure 1. The polynomial structure of the $\mathcal{R}$-operators using the normalization in [7].

sites was obtained. In section C.2 we show that this formulation is particularly convenient for the non-compact $\mathfrak{sl}(2)$ spin-chain of representation $s = -1/2$. Interestingly, spin-chains of this type emerge in the study of certain four dimensional gauge theories, see e.g. [18].

The generalization to non-rectangular representations remains open. It is known that the nearest-neighbor Hamiltonian is non-Hermitian and the transfer matrices are non-normal operators for these representations [19]. In the example of the adjoint representation of $\mathfrak{gl}(3)$ it is easy to show that the oscillator independent part of the $\mathcal{R}$-operator does not reduce to rank 1 at any point of the spectral parameter z. We have collected their polynomial structure in figure 1. It would be interesting to study this problem more carefully to define local charges using the developed method as a guiding principle. This might also shed some light on representations without highest- or lowest weight states. Besides the $\mathfrak{gl}(n)$ homogeneous spin-chains there exists a large zoo of quantum integrable models. Clearly, it would be very interesting to study the presented mechanism in these models. In particular, it is certainly interesting to apply the method to the case of $\mathfrak{gl}(n|m)$.

The $\mathcal{R}_0$-operators at special values of the spectral parameter discussed in section 4 share striking similarities with so called extremal-projectors, see [20]. This connection deserves further investigation. Besides, there is more territory to be explored in this direction. In particular, it would be interesting to recover the $\mathcal{R}$-operators used in this paper from the formula of Khoroshkin and Tolstoy for the universal R-matrix for the Yangian double [21], see also [22, 23] for recent applications in the case of quantum affine algebras. This exercise, conceptually interesting on its own, would allow one to 'dress' the $\mathcal{R}$-operators used in this paper with their preferred normalization factors. The relation between such factors and crossing symmetry has been briefly discussed in section 4.2. Remarkably, it appears that proper normalizations are necessary in order to exploit the symmetry between the two presentations of $\mathcal{R}$-operators used in this work.

The program of developing a systematic approach to the theory of quantum integrable models based on the Q-operator method has received increasing attention in recent years, see e.g. [22, 24] and references therein. Despite constant progress a number of questions remain open. We believe that the properties of the $\mathcal{R}$-operators emphasized in this work play an important role in this program. Furthermore, the calculation of correlation functions in quantum integrable models remains an outstanding problem. It is believed that the Q-operator method plays a prominent role in this investigation [25]. Also, it is

J. Stat. Mech. (2013) P02019

worth mentioning potential applications of such integrability techniques in the study of structure constants of $\mathcal{N}=4$ super Yang–Mills theory (SYM) along the lines presented in [26] and references therein.

Recently, an intriguing connection has been observed between tree level scattering amplitudes in $\mathcal{N}=4$ SYM and certain contributions to the $\mathfrak{psu}(2,2|4)$ integrable Hamiltonian corresponding to the dilatation operator of the theory [27]. It would be interesting to bring together the new point of view on integrable Hamiltonians (5.22) presented in this paper with this remarkable connection. This might shed some light on the role the degenerate representations of the Yangian algebra, crucial in the Q-operator construction, play in the surprisingly rich structure [28] behind scattering amplitudes of $\mathcal{N}=4$ SYM.

Acknowledgments

We would like to thank Matthias Staudacher for proposing this interesting problem. Furthermore, we thank Yuri Aisaka, Nils Kanning, Tomasz Łukowski and Matthias Staudacher for useful discussions. We would also like to thank Tomasz Łukowski for providing `Mathematica` code for the construction of $\mathcal{R}$-matrices for compact representations. The graphics were made using `Inkscape`. RF thanks the IAS Jerusalem for hospitality during the course of this work. CM is partially supported by a DFG grant in the framework of the SFB 676 '*Particles, Strings and the Early Universe*'.

Appendix A. Shifted weights

The quantities $\hat{\ell}_k^K$ are important building blocks for the $\mathcal{R}$-operators used in this paper. Spelling out their characteristics is an essential step in the study of the properties of Q-operators. The labels of $\hat{\ell}_k^K$ correspond to a subset K of $\{1,2,\ldots,n\}$ and an index $k=1,2,\ldots,|K|$. For $\mathfrak{gl}(n)$ there are $n\cdot 2^{n-1}$ such $\hat{\ell}_k^K$. The set K identifies a natural embedding of $\mathfrak{gl}(K)$ in $\mathfrak{gl}(n)$. The Casimirs of $\mathfrak{gl}(K)$ defined as

$$C_i^{(K)} = J_{a_i}^{a_1} J_{a_1}^{a_2} \cdots J_{a_{i-1}}^{a_i} \qquad \text{with } a_j \in K \tag{A.1}$$

are symmetric polynomials of $\hat{\ell}_k^K$ via the following formula[13]:

$$C_i^{(K)} = \sum_{k\in K} \prod_{j\neq k} \left(1 + \frac{1}{\hat{\ell}_k^K - \hat{\ell}_j^K}\right) (\hat{\ell}_k^K)^i. \tag{A.2}$$

In general, not all $\hat{\ell}_k^K$ do commute among themselves. For a chosen path in the Hasse diagram, i.e. a sequence of sets $\mathcal{P} \equiv \emptyset \subset \{a\} \subset \{a,b\} \subset \ldots \subset \{1,2,\ldots,n\}$ ordered by inclusion, all the $n(n+1)/2$ corresponding $\hat{\ell}_k^K$ commute among themselves. In particular, for a given irreducible representation of $\mathfrak{gl}(n)$ there exists a basis such that all $\hat{\ell}_k^K$ corresponding to the chosen path $\mathcal{P}$ act diagonally[14]. This basis coincides with the famous Gelfand–Tsetlin basis (see e.g. [30] for a nice review and collection of references). The

[13] We refer to the previous paper [7].

[14] An interesting class of infinite dimensional representations of $\mathfrak{gl}(n)$ for which the $\hat{\ell}_k^K$ act as multiplication operators was introduced in [29] in connection with the method of separation of variables.

algebra $\mathfrak{gl}(n)$ admits a large zoo of representations. In the following we consider some specific examples in more detail. In the previous paper [7] we defined the generalized rectangularity condition as $J^A_C J^C_B = \alpha J^A_B + \beta \delta^A_B$. In this case, using (A.1) and (A.2) for the full set $K = \{1, \dots, n\}$, one can show that

$$\ell_i = \begin{cases} (\bar{\lambda}_I - i + 1) & i \le a, \\ (\lambda_I - i + 1) & i > a, \end{cases} \tag{A.3}$$

where a is an integer with $0 \le a \le n$, compare to (4.3), and $\lambda_I, \bar{\lambda}_I$ are in general complex numbers related to α and β via $\alpha = \lambda_I + \bar{\lambda}_I + |I|$, $\beta = -\lambda_I(\bar{\lambda}_I + |I|)$. The label I is introduced for consistency with section 4.1, where $|I| = n - a$. Moreover, (A.3) should be understood up to permutation of ℓ_i. Generalized rectangular representations have a number of remarkable features. Among others, the tensor product of such representations is multiplicity free, the weight diagram is multiplicity free and the corresponding $\mathcal{L}$-operator (2.8) satisfies $\mathcal{L}(z)\mathcal{L}(\alpha - z) = z(\alpha - z) + \beta$. If we further restrict ourselves to generalized rectangular representations with highest weight we have

$$\hat{\ell}^{\bar{I}}_i |\Lambda^I_0, \Lambda^{\bar{I}}_0\rangle = \ell_i |\Lambda^I_0, \Lambda^{\bar{I}}_0\rangle = (\bar{\lambda}_I - i + 1)|\Lambda^I_0, \Lambda^{\bar{I}}_0\rangle, \tag{A.4}$$

$$\hat{\ell}^{I}_i |\Lambda^I_0, \Lambda^{\bar{I}}_0\rangle = (\ell_{i+|\bar{I}|} + |\bar{I}|)|\Lambda^I_0, \Lambda^{\bar{I}}_0\rangle = (\lambda_I - i + 1)|\Lambda^I_0, \Lambda^{\bar{I}}_0\rangle, \tag{A.5}$$

where the state $|\Lambda^I_0, \Lambda^{\bar{I}}_0\rangle$ was defined in (4.8). It is worth stressing that for generalized rectangular representations, for any set K, the shifted weight operators $\hat{\ell}^K_k, \hat{\ell}^{\bar{K}}_k$ contain the same information. This is no longer the case for more general representations.

Appendix B. Hamiltonian density

B.1. The action of the Hamiltonian density

In this appendix we explain the origin of (5.22). The starting point is the equation

$$[\mathcal{H}_{1,2}, \mathcal{R}_{I,1}(z)\mathcal{R}_{I,2}(z)] = \mathcal{R}_{I,1}(z)\mathcal{R}'_{I,2}(z) - \mathcal{R}'_{I,1}(z)\mathcal{R}_{I,2}(z). \tag{B.1}$$

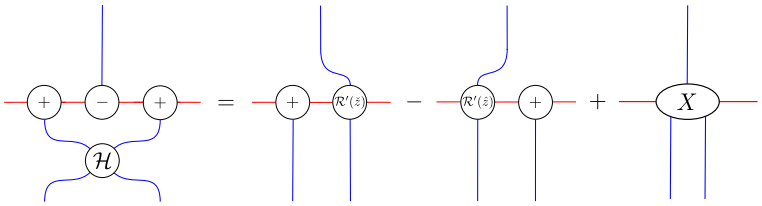
It is a special case of Sutherland's equation [17, 16] and ensures the commutativity of $\mathbf{H}$ with $\mathbf{Q}_I$ and the complete hierarchy of commuting operators. This equation plays a crucial role in the construction of higher charges using the boost operator approach [31]. See also [32] where it was applied in a systematic study of integrable long-range spin-chains. Equation (B.1) follows from the Yang–Baxter equation

$$\mathbf{R}_{1,2}(z_1 - z_2)\mathcal{R}_{I,1}(z_1)\mathcal{R}_{I,2}(z_2) = \mathcal{R}_{I,2}(z_2)\mathcal{R}_{I,1}(z_1)\mathbf{R}_{1,2}(z_1 - z_2), \tag{B.2}$$

where $\mathbf{R}_{1,2}$ denotes the fundamental R-matrix entering (4.1) with equal representation Λ of $\mathfrak{gl}(n)$ in 1 and 2. $\mathcal{R}_{I,1}$ and $\mathcal{R}_{I,2}$ are the $\mathcal{R}$-operators defined via (2.6). Expanding (B.2) around $z_1 = z_2 = z$ and using the regularity condition (4.1) together with $\mathcal{H}_{1,2} = \mathbf{P}_{1,2}\mathbf{R}'_{1,2}(0)$ one obtains (B.1). Equation (B.1) contains the free parameter z. To prove (5.22) we will focus on the values $\hat{z}$ and $\check{z}$. Instead of writing formulas we rely on the

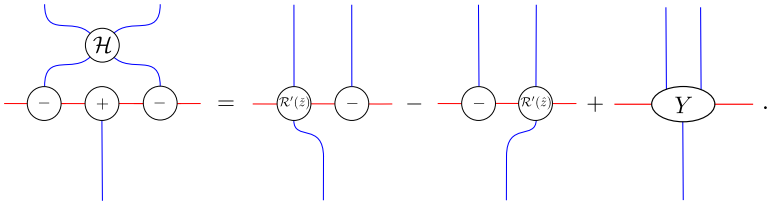
J. Stat. Mech. (2013) P02019

developed diagrammatics. Without loss of generality we may write



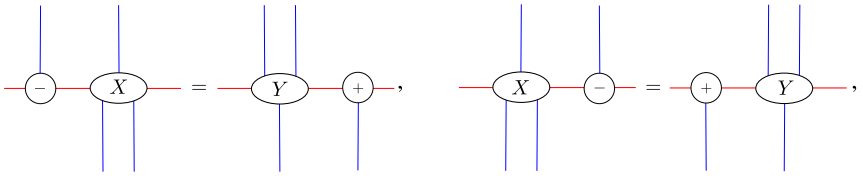
(B.3)

and



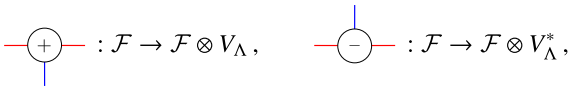
(B.4)

Inserting these equations in (B.1) specified to the values $\hat{z}$ and $\check{z}$ one obtains



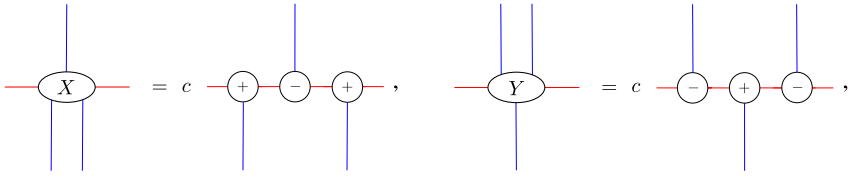
(B.5)

respectively. Observe that the operators



(B.6)

have no kernel. According to (5.7), $\mathcal{F}$ is associated to the oscillator horizontal line while V_Λ and V_Λ^* correspond to the quantum space at a site and its dual. Then, (B.5) immediately implies



(B.7)

where c is an arbitrary constant and can be reabsorbed into the definition of $\mathbf{H}$ in (B.3) and (B.4). This concludes the proof of (5.22).

B.2. A plug-in formula for the Hamiltonian density

For practical purposes we give a plug-in formula for the Hamiltonian density in this appendix. By multiplying (5.22) in the auxiliary space with  from the right one finds that

$$\mathcal{H}_{i,i+1}\mathcal{R}_{I,i}(\check{z})\,\mathcal{R}_{I,i+1}(\check{z}) = \mathcal{R}_{I,i}(\check{z})\,\mathcal{R}'_{I,i+1}(\check{z}) - \mathbf{P}_{i,i+1}\mathrm{e}^{\bar{\mathbf{a}}^{\dot{c}}_{c}J(i)^{c}_{\dot{c}}} \circ |hws\rangle_i \mathcal{R}'_{I,i+1}(\hat{z})\langle hws|_i \circ \mathrm{e}^{-\mathbf{a}^{c}_{\dot{c}}J(i)^{\dot{c}}_{c}}. \tag{B.8}$$

Interestingly, $\mathcal{R}_{I,i}(\check{z})\,\mathcal{R}_{I,i+1}(\check{z})$ can be inverted under $\cdot$ to obtain $\mathcal{H}_{i,i+1}$. As $\mathcal{H}_{i,i+1}$ does not depend on the auxiliary space, all oscillators can be removed in a consistent way. In this way one can write

$$\begin{aligned}\mathcal{H}_{i,i+1}\mathcal{R}_{0,i}(\check{z})\mathrm{e}^{-J(i)^{\dot{a}}_{a}J(i+1)^{a}_{\dot{a}}}\mathcal{R}_{0,i+1}(\check{z}) &= \mathcal{R}_{0,i}(\check{z})\mathrm{e}^{-J(i)^{\dot{a}}_{a}J(i+1)^{a}_{\dot{a}}}\mathcal{R}'_{0,i+1}(\check{z})\\ &- \mathbf{P}_{i,i+1}\sum_{\{k_{c\dot{c}}\},\{m_{c\dot{c}}\}=0}^{\infty}\prod_{c\in I,\dot{c}\in\bar{I}}\frac{1}{k_{c\dot{c}}!m_{c\dot{c}}!}(J^{c}_{\dot{c}}(i)+J^{c}_{\dot{c}}(i+1))^{k_{\dot{c}c}+m_{c\dot{c}}}\\ &\times\mathcal{R}_{0,I}(\hat{z})\,\tilde{\mathcal{R}}'_{0,i+1}(\hat{z})\prod_{c\in I,\dot{c}\in\bar{I}}(J^{\dot{c}}_{c}(i))^{k_{c\dot{c}}}(J^{\dot{c}}_{c}(i+1))^{m_{c\dot{c}}}.\end{aligned}\tag{B.9}$$

In analogy to (B.8) this yields the Hamiltonian density.

Appendix C. Examples

C.1. The $\mathfrak{gl}(2)$ case: reordering and projection in full detail

In this section we exploit the properties mentioned in the previous sections for the example of $\mathfrak{gl}(2)$ with $|I|=\{\dot{c}\}$, $I=\{c\}$ and $c,\dot{c}=1,2$. In this case $\mathcal{R}_{0,\{c\}}$ and $\tilde{\mathcal{R}}_{0,\{c\}}$ are given by

$$\mathcal{R}_{0,\{c\}}(z)=\kappa_{\{c\}}(z)\,\frac{\Gamma(z+1/2-\ell_1^{\{\dot{c}\}})}{\Gamma(z+1/2-\bar{\lambda}_c)},\qquad \tilde{\mathcal{R}}_{0,\{c\}}(z)=\tilde{\kappa}_{\{c\}}(z)\,\frac{\Gamma(z+3/2-\lambda_c)}{\Gamma(z+3/2-\ell_1^{\{c\}})};\tag{C.1}$$

see (2.1) and (3.5), respectively. We will now determine the explicit relation between ρ and $\tilde{\rho}$ as discussed in section 3.2. For the $\mathfrak{gl}(2)$ case $\mathcal{R}_I$ contains only one pair of oscillators. From this it follows that the conjugation in (3.11) has to be performed only once,

$$\tilde{\mathcal{R}}_{0,\{c\}}(z)=\sum_{n=0}^{\infty}\frac{1}{n!}(J^{c}_{\dot{c}})^{n}\mathcal{R}_{0,\{c\}}(z)(J^{\dot{c}}_{c})^{n}.\tag{C.2}$$

We can sum up this expression using the relation

$$(J^{a}_{\dot{a}})^{k}\,(J^{\dot{a}}_{a})^{k}=(-1)^{k}\frac{\Gamma(J^{\dot{a}}_{\dot{a}}-\ell_1+k)\Gamma(J^{\dot{a}}_{\dot{a}}-\ell_2+k)}{\Gamma(J^{\dot{a}}_{\dot{a}}-\ell_1)\Gamma(J^{\dot{a}}_{\dot{a}}-\ell_2)},\tag{C.3}$$

where $\ell_i=\ell_i^{\{a,\dot{a}\}}=\ell_i^{\{1,2\}}$. Applying the reflection formula for Gamma functions

$$\Gamma(1-z)\Gamma(z)=\frac{\pi}{\sin\pi z}\tag{C.4}$$

one finds that

$$\begin{aligned}\tilde{\mathcal{R}}_{0,\{c\}}(z)=-\kappa_{\{c\}}(z)\,&\frac{\sin\pi(z+1/2-\ell_1)\sin\pi(z+1/2-\ell_2)}{\sin\pi(z+1/2-\ell_1^{\{\dot{c}\}})\sin\pi(z+1/2-\ell_1^{\{c\}})}\\ &\times\frac{\Gamma(z+3/2-\lambda_c)}{\Gamma(z+3/2-\ell_1^{\{c\}})},\end{aligned}\tag{C.5}$$

using $C_1=J^{c}_{c}+J^{\dot{c}}_{\dot{c}}=1+\ell_1+\ell_2$ and that up to permutation of ℓ_1 and ℓ_2 it holds that $\ell_1=\bar{\lambda}_c$, $\ell_2=\lambda_c-1$, see (A.3). This is exactly what we expected from the analysis of the

J. Stat. Mech. (2013) P02019

Yang–Baxter equation, compare (C.1). Furthermore, it fixes the relative normalization

$$\tilde{\kappa}_{\{c\}}(z) = -\frac{\sin\pi(z+1/2-\ell_1)\sin\pi(z+1/2-\ell_2)}{\sin\pi(z+1/2-\ell_1^{\{\dot{c}\}})\sin\pi(z+1/2-\ell_1^{\{c\}})}\kappa_{\{c\}}(z). \tag{C.6}$$

Let us now look for the projection point as discussed section 4. For $\hat{z} = \lambda_{\dot{c}} - \frac{1}{2}$ one obtains[15]

$$\mathcal{R}_{0,\{c\}}(\hat{z}) = |hws\rangle\langle hws|. \tag{C.7}$$

On the other hand, at $\check{z} = \lambda_c - \frac{3}{2}$ we find

$$\tilde{\mathcal{R}}_{0,\{c\}}(\check{z}) = |hws\rangle\langle hws|. \tag{C.8}$$

The total trigonometric prefactor reduces to 1. Note that this is the case for arbitrary z on any state if the spectrum of $\ell^{\{\dot{c}\}}$ and $\ell^{\{c\}}$ is integer spaced.

C.2. The Hamiltonian action for the non-compact spin $s = -\frac{1}{2}$ chain

In this appendix we study how the Hamiltonian density for the non-compact spin $-\frac{1}{2}$ spin-chain emerges in the presented formalism. This spin-chain has received special interest in the context of the AdS_5/CFT_4 correspondence [33]. The $\mathcal{R}$-operators for $\mathfrak{gl}(2)$ were discussed in section C.1 in great detail. Restricting to $\mathfrak{sl}(2)$ one finds that one of the two $\mathcal{R}$-operators can be written as

$$\mathcal{R}_+(z) = \mathrm{e}^{\bar{\mathbf{a}}\,J_+}\,\mathcal{R}_{0,+}(z)\,\mathrm{e}^{\mathbf{a}\,J_-} \qquad \text{with } \mathcal{R}_{0,+}(z) = \frac{\Gamma(z+J_0)}{\Gamma(z+\frac{1}{2})}, \tag{C.9}$$

compare section C.1. The usual $\mathfrak{sl}(2)$ commutation relations are

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = -2J_0.$$

Furthermore, we define the action on the module via the common relations

$$J_+|m\rangle = (m+1)|m+1\rangle, \qquad J_-|m\rangle = m|m-1\rangle, \qquad J_0|m\rangle = (m+\tfrac{1}{2})|m\rangle. \tag{C.10}$$

It follows that

$$\mathcal{R}_{0,+}(z) = \sum_{m=0}^{\infty}\frac{\Gamma(z+1/2+m)}{\Gamma(z+1/2)}\,|m\rangle\langle m|, \qquad \mathcal{R}'_{0,+}(\hat{z}) = \sum_{m=1}^{\infty}\Gamma(m)\,|m\rangle\langle m|, \tag{C.11}$$

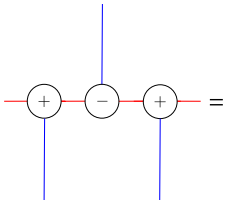
$$\tilde{\mathcal{R}}_{0,+}(z) = \sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(-z+1/2+m)}{\Gamma(-z+1/2)}\,|m\rangle\langle m|,$$

$$\tilde{\mathcal{R}}'_{0,+}(\check{z}) = \sum_{m=1}^{\infty}(-1)^{m+1}\Gamma(m)\,|m\rangle\langle m|, \tag{C.12}$$

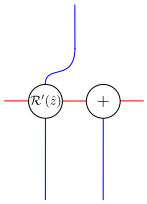
[15] Here we take $\kappa_{\{c\}}(z)|hws\rangle = |hws\rangle$.

J. Stat. Mech. (2013) P02019

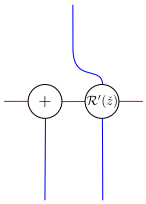
with $\hat{z} = -\frac{1}{2}$ and $\check{z} = \frac{1}{2}$. The relevant terms in (5.22) are then given by



$$= \sum_{m_1,m_2=0}^{\infty} \bar{\mathbf{a}}^{m_1} \left(|m_1, m_2\rangle\right) \langle 0|e^{\mathbf{a}J_2^-} \bar{\mathbf{a}}^{m_2}\,, \tag{C.13}$$



$$= -\sum_{m_1,m_2=0}^{\infty} \bar{\mathbf{a}}^{m_1} \left(h(m_1)\,|m_1, m_2\rangle - \sum_{\ell=1}^{m_2} \frac{1}{\ell} |m_1 + \ell, m_2 - \ell\rangle \right) \langle 0|e^{\mathbf{a}J_2^-} \bar{\mathbf{a}}^{m_2}\,, \tag{C.14}$$



$$= \sum_{m_1,m_2=0}^{\infty} \bar{\mathbf{a}}^{m_1} \left(h(m_2)\,|m_1, m_2\rangle - \sum_{\ell=1}^{m_1} \frac{1}{\ell} |m_1 - \ell, m_2 + \ell\rangle \right) \langle 0|e^{\mathbf{a}J_2^-} \bar{\mathbf{a}}^{m_2}\,. \tag{C.15}$$

From this we find that

$$\begin{aligned} \mathcal{H}|m_1 m_2\rangle = {} & \left(h(m_1) + h(m_2)\right)|m_1, m_2\rangle - \sum_{\ell=1}^{m_1} \frac{1}{\ell} |m_1 - \ell, m_2 + \ell\rangle \\ & - \sum_{\ell=1}^{m_2} \frac{1}{\ell} |m_1 + \ell, m_2 - \ell\rangle. \end{aligned} \tag{C.16}$$

Note that the constant discussed in section B.1 is fixed to be $c = 0$.

C.3. Projection properties of $\mathcal{R}$-operators for $\mathfrak{su}(2,2)$

It is instructive to present the structure of $\hat{\ell}_k^K$ in full detail for the interesting class of representations usually referred to as oscillator representations. These representations represent a subfamily of the generalized rectangular representations. The $\mathfrak{gl}(n)$ generators take the simple form

$$J_B^A = \bar{b}^A\, b_B + \gamma\, \delta_B^A, \qquad [b_B, \bar{b}^A] = \delta_B^A, \tag{C.17}$$

where γ commutes with all the generators. Using purely algebraic manipulations, one can show that for any fixed set $K \subseteq \{1, \ldots, n\}$ the corresponding set of $\hat{\ell}_k^K$ is given by

$$\{\gamma + \hat{N}^K, \gamma - 1, \gamma - 2, \ldots, \gamma - |K| + 1\}, \qquad \hat{N}^K \equiv \sum_{c \in K} \bar{b}^c\, b_c. \tag{C.18}$$

Notice that for each set K only one $\hat{\ell}_k^K$ is a non-trivial operator. The spectrum of $\hat{\ell}_k^K$ thus follows from the spectrum of $\widehat{N}^K$. The spectrum of $\widehat{N}^K$ in turn depends on the choice of

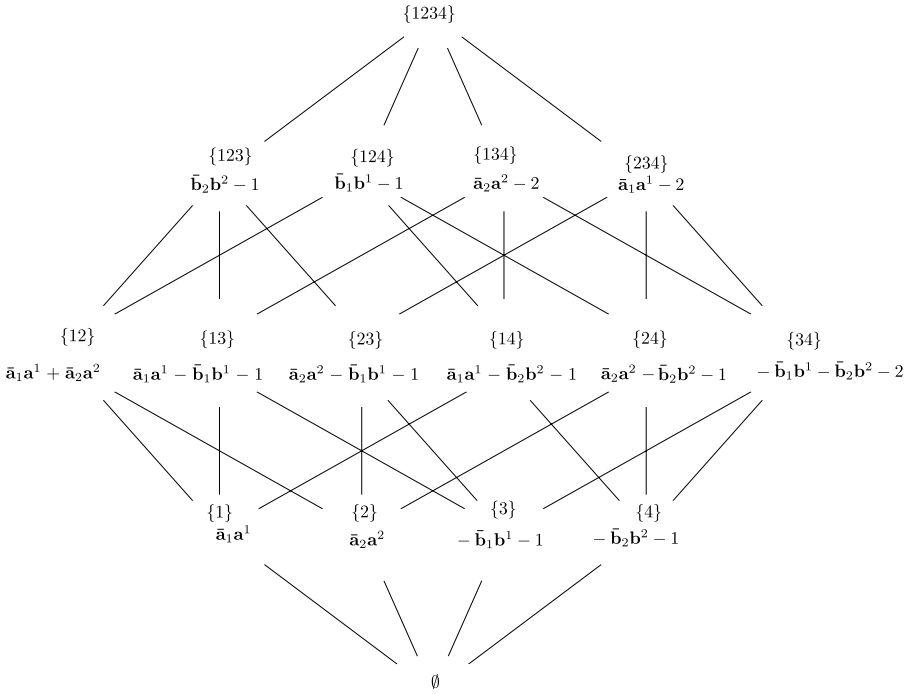


Figure C.1. Hasse diagram for $\mathfrak{su}(2,2)$ including the shifted weights of the subalgebra $\hat{\ell}_1^I - \gamma$ according to (C.18).

the vacuum for the oscillator algebra. After renaming the oscillators according to

$$\bar{b}^A = (\bar{a}^\alpha, b^{\dot{\alpha}}), \qquad b_A = (a_\alpha, -\bar{b}_{\dot{\alpha}}), \tag{C.19}$$

where $\alpha = 1, \ldots, p$ and $\dot{\alpha} = p+1, \ldots, n$, the vacuum is defined by

$$a_\alpha |0\rangle = 0 = b^{\dot{\alpha}} |0\rangle. \tag{C.20}$$

The representations obtained in this way are not irreducible, the operator $\widehat{N}^{\{1,\ldots,n\}}$ is central and its eigenvalues label an infinite family of unitary irreducible representations of $\mathfrak{su}(p, n-p)$. The case of $\mathfrak{su}(2,2)$ for $\widehat{N}^{\{1,\ldots,4\}} = 0$ is given in figure C.1. We conclude that only the Q-operator corresponding to the set $I = \{3,4\}$ fulfils the criteria given in section 4.1. The algebra $\mathfrak{su}(2,2)$ is the conformal algebra in four dimensions and the representation chosen in the example corresponds to the so called massless scalar field.

C.4. The Hamiltonian for the fundamental representation

For the fundamental representation one has $a = 1$, see (4.3). Therefore the $\mathcal{R}$-operators of cardinality $|I| = n - 1$ carry the information about the Hamiltonian. In this case the special points are located at $\hat{z} = +\frac{1}{2}$ and $\check{z} = -\frac{1}{2}$, compare (4.2). The derivative $\mathbf{L}'_I$ does not depend on the spectral parameter z and does not contain oscillators. It follows that equation (B.8) simplifies to

$$\mathcal{H}_{i,i+1} \mathbf{L}_{I,i}(\check{z})\, \mathbf{L}_{I,i+1}(\check{z}) = (\mathbf{1} - \mathbf{P})_{i,i+1} \mathbf{L}_{I,i}(\check{z}) \mathbf{L}'_{I,i+1}. \tag{C.21}$$

J. Stat. Mech. (2013) P02019

Furthermore, $\mathbf{L}'_I$ can be written as

$$\mathbf{L}'_I = \mathbf{L}_I(\hat{z}) - \mathbf{L}_I(\check{z}). \tag{C.22}$$

The well-known expression for the Hamiltonian density

$$\mathcal{H}_{i,i+1} = (\mathbf{P} - \mathbf{1})_{i,i+1} \tag{C.23}$$

follows, noting that

$$(\mathbf{1} - \mathbf{P})_{i,i+1}\mathbf{L}_{I,i}(\check{z})\mathbf{L}_{I,i+1}(\hat{z}) = 0. \tag{C.24}$$

Interestingly, as a consequence of (5.8), identity (C.24) holds true for any generalized rectangular representation.

Appendix D. The reordering formula

The reordering of the oscillators in the auxiliary space we are interested in is of the form

$$\mathrm{e}^{\bar{\mathbf{a}}A} \cdot B \cdot \mathrm{e}^{\mathbf{a}C} = \mathrm{e}^{\bar{\mathbf{a}}A} \circ \tilde{B} \circ \mathrm{e}^{\mathbf{a}C}. \tag{D.1}$$

Using $\mathrm{e}^{-\bar{\mathbf{a}}A}\mathbf{a}^n\mathrm{e}^{\bar{\mathbf{a}}A} = (\mathbf{a} + A)^n$ we find that

$$\tilde{B} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} A^n\, B\, C^n, \qquad B = \sum_{n=0}^{\infty} \frac{1}{n!} A^n\, \tilde{B}\, C^n. \tag{D.2}$$

Here we did not specify any commutation relations among A, B, C.

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J. Stat. Mech. (2013) P02019