Mathematical properties of cosmological models with accelerated expansion

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Abstract

An introduction to solutions of the Einstein equations defining cosmological models with accelerated expansion is given. Connections between mathematical and physical issues are explored. Theorems which have been proved for solutions with positive cosmological constant or nonlinear scalar fields are reviewed. Some remarks are made on more exotic models such as the Chaplygin gas, tachyons and k-essence.

1   Introduction

Recent cosmological observations indicate that the expansion of the universe is accelerating and this has led to a great deal of theoretical activity. Models of accelerated cosmological expansion also raise a variety of interesting mathematical questions. The purpose of the following is to first give a pedagogical introduction to this subject suitable for the mathematically inclined reader and then to present an overview of some of the mathematical results which have been obtained up to now and the many challenges which remain.

The simplest class of cosmological models consists of those with the highest symmetry, i.e. those which are homogeneous and isotropic. The underlying spacetimes are the FLRW (Friedmann-Lemaitre-Robertson-Walker) models. A further simplification can be achieved by assuming that the metric

of the slices of constant time is flat. The spacetime metric can be written in the form:
$$- dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \tag{1}$$
for a suitable scale factor $a(t)$. These are the models most frequently used in the literature due both to their simplicity and the fact that spatially flat FLRW models appear to give a good description of our universe.

The physical interpretation of $a(t)$ is that if two typical galaxies are a distance $D(t)$ apart at time t then $D(t_2)/D(t_1) = a(t_2)/a(t_1)$ for any times t_1 and t_2. The statement that the universe is expanding corresponds to the condition that the time derivative $\dot{a}$ is positive. Accelerated expansion means that the second derivative $\ddot{a}$ is positive.

The function $a(t)$ should be determined by the field equations for gravity and in the following we always take the Einstein equations for this purpose. There are two choices to be made. The one concerns the cosmological constant Λ. The other concerns the description of the matter content of spacetime. This means choosing the variables which describe the matter, the equations of motion these are to satisfy and the definition of the energy-momentum tensor as a function of the matter fields and the spacetime metric. Under the assumption of FLRW symmetry this will lead to an evolution equation for $a(t)$. The easiest way to produce models with accelerated expansion is to choose a positive cosmological constant ($\Lambda > 0$). A more sophisticated alternative is to choose $\Lambda = 0$ but to include a suitable nonlinear scalar field among the matter fields.

The rest of this article is structured as follows. It starts with a brief introduction to some physical ideas relevant to accelerated cosmological expansion. Then mathematical theorems about spacetimes with positive cosmological constant motivated by the physics are described. After that these results are compared with the original physical motivation. Once the case of a positive cosmological constant has been described it is discussed why it might be good to replace the cosmological constant by a nonlinear scalar field and what changes when that is done. Finally, some future research directions involving more general models for cosmic acceleration are indicated. In particular, comments are made on the Chaplygin gas and the tachyon condensate.

2 Physical background

Accelerated expansion plays a role in cosmology in two different regimes. The first is the very early universe while the second is the period between the decoupling of the microwave background radiation and the present. Accelerated expansion in the early universe is associated with the name inflation which was introduced by Guth [10]. The paper [10] was extremely important in popularizing the concept of inflation. Related ideas had been considered previously by other authors.

One of the attractive features of inflation is that it is claimed to solve certain 'problems' in cosmology. It is justified to ask in which sense these can really be considered as problems but these philosophical questions will not be entered into in the following. Among these issues are

- homogeneity and isotropy

- flatness problem

- horizon problem

The first issue is that, after averaging on a suitable scale, our universe is homogeneous and isotropic. There are two basically different kinds of possible reason for this. One is that it was always homogeneous and isotropic. This possibility is perceived by many as unsatisfactory. The alternative is that the universe was originally anisotropic and inhomogeneous and that some dynamical mechanism later made it homogeneous and isotropic. In the second explanation this mechanism must be found. The second issue is that it appears that the curvature of space on cosmological scales is very small today and was even smaller at decoupling. It is often perceived that this smallness requires an explanation. The third issue is that the temperature of the microwave background is essentially the same at points such that there would have been no time to send a signal to both from some common point since the big bang in a standard Friedmann model (without accelerated expansion). Can this be explained? Inflation has something to say about all three issues, as will be shown later.

Accelerated cosmological expansion at the present epoque is a relatively recent discovery, dating from the late 1990's. There is now very strong observational evidence, which continues to accumulate, that the velocity of recession of distant galaxies is accelerating. On the theoretical side this phenomenon is associated with the names dark energy and quintessence. The

latter term was introduced by Caldwell, Dave and Steinhardt [5]. There are a number of different lines of evidence for cosmic acceleration at times after decoupling which include

- supernovae of type Ia

- microwave background fluctuations

- gravitational lensing

- galaxy clustering

Here only the supernova data will be discussed. A supernova of type Ia is an exploding star which is bright enough to be visible at cosmological distances. The characteristics of an event of this type which can in principle be observed are the red shift, the light curve (observed brightness as a function of time) and the spectrum. In recent years it has become possible to observe these data in practice for a useful sample of objects. The light curve and spectrum provide the information needed to identify a supernova as being of type Ia. The advantage of this is that type Ia supernovae have universal properties which allow their intrinsic brightness to be determined. In a first approximation, all of these objects have the same intrinsic brightness at the maximum of their light curves. The number of objects of this type observed so far is just over 150. The projected space mission SNAP (Supernova Acceleration Probe) is planned to observe about 2000 per year. The way in which data can be compared with theoretical models will be outlined in Section 6.

3   Mathematical developments

It has been known for a long time that spacetimes with a positive cosmological constant have a tendency to isotropize at late times, a circumstance associated with the name 'cosmic no hair theorem'. In [22] Starobinsky wrote down formal expansions for the late-time behaviour of spacetimes with positive cosmological constant. He studied the case where the matter is described by a perfect fluid with linear equation of state $p = (\gamma - 1)\rho$, where γ is a constant belonging to the interval $[1, 2)$. He also discussed the vacuum case which, as it turns out, gives the leading order terms in the expansion of the geometry for the case with fluid as well. In a certain sense the solutions all

look like the de Sitter solution at late times. This will be made more precise below.

It will be convenient in the following to write the de Sitter solution in the form

$$- dt^2 + e^{2Ht}(dx^2 + dy^2 + dz^2) \tag{2}$$

where $H = \sqrt{\Lambda/3}$. These coordinates only cover half of de Sitter space but this is no disadvantage in the following where the subject of interest is the limit $t \to \infty$. The expansions of [22] are expressed in terms of Gauss coordinates. In other words $g_{00} = -1$ and $g_{0a} = 0$ where Latin indices are spatial indices. In the vacuum case the expansion of the spatial metric is

$$g_{ab}(t, x) = e^{2Ht}(g_{ab}^0(x) + g_{ab}^2(x)e^{-2Ht} + g_{ab}^3(x)e^{-3Ht} + \ldots) \tag{3}$$

The fact that the coefficient of e^{Ht} vanishes is a result of the analysis. Putting $g_{ab}^0 = \delta_{ab}$ and setting all the other coefficients to zero gives the de Sitter solution.

In [22] it is not specified how this infinite series is to be interpreted mathematically but it is natural to interpret it as a formal series. This means that there is no assertion that the series converges or even that it is asymptotic. Recall that a series as above is called asymptotic if for each positive integer M there is a positive constant C_M such that

$$\left| g_{ab} - \sum_0^M g_{ab}^m e^{(2-m)Ht} \right| \leq C_M e^{(1-M)Ht} \tag{4}$$

In other words, the sum of any finite truncation of the series differs from the quantity to which it is asymptotic by a remainder of order equal to the next term beyond the truncation. A convergent series is asymptotic but not necessarily conversely. At this point in the discussion it is not even claimed that the above series is asymptotic. It is just a formal expression which solves the Einstein equations in the sense that if we substitute it into the Einstein equations and manipulate the infinite series according to rather obvious rules all terms cancel.

In [19] a theorem was proved concerning the above formal series. To formulate it, let A_{ab} be a three-dimensional Riemannian metric and B_{ab} a symmetric tensor which is transverse traceless with respect to A_{ab}. This means that $A^{ab}B_{ab} = 0$ and $\nabla^a B_{ab} = 0$ where the covariant derivative is that associated to the metric A_{ab}. Given A_{ab} and B_{ab} of this form which are

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smooth (C^∞) there exists a unique series of the above form satisfying the vacuum Einstein equations with $\Lambda > 0$ with smooth coefficients g_{ab}^m which satisfies the conditions $g_{ab}^0 = A_{ab}$ and $g_{ab}^3 = B_{ab}$. Notice that on the basis of function counting these solutions are as general as the general solution of the vacuum Einstein equations. For the general solution can be specified by giving the induced metric and the second fundamental form on a space-like hypersurface and these must satisfy one scalar and one vector equation. Thus in both cases we have the same type of data and the same number of constraints which they have to satisfy. In the present case the constraints are simpler than in the ordinary Cauchy problem. In [19] a corresponding theorem was also proved for the case of the Einstein equations coupled to a perfect fluid with a linear equation of state. The most difficult part of the proof is to show that the Einstein constraint equations are satisfied as a consequence of the 'constraints at infinity', i.e. the tranverse traceless nature of B_{ab} with respect to A_{ab}.

It is desirable to extend the above results about formal power series and function counting to show that there exists a large class of solutions which have asymptotic expansions of the above form and that these are general in the sense that they include all solutions arising from a non-empty open set of initial data on a Cauchy surface. One place to look for such an open set is as an open neighbourhood of standard data for the de Sitter solution on a hypersurface $t =$const. In the vacuum case a result of this kind was proved in [19] using results of Friedrich [6], [7] on the stability of de Sitter space. The corresponding result with a perfect fluid, which is what would be desirable for cosmology, is still open. The proofs in the vacuum case use the conformal field equations. The results can be extended in some cases to conformally invariant matter fields but for other matter fields, including most fluids, it is not at all clear that the method could work. If the metric is conformally rescaled as in the conformal method either the rescaled metric or the conformal factor will for most fluids be non-smooth, involving non-integral powers of the time coordinate.

Another possible direction in which the existing results could be extended is to other spacetime dimensions. In the context of formal power series of the vacuum Einstein equations with $\Lambda > 0$ this has been done in [19]. The result is the series:

$$g_{ab} = e^{2Ht}(g_{ab}^0 + \sum_{m=1}^{\infty} \sum_{l=0}^{L_m} (g_{ab})_{m-2,l} t^l e^{-mHt}) \tag{5}$$

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where $H = \sqrt{2\Lambda/n(n-1)}$ in spatial dimension n. For each m the quantity L_m is a finite integer. The terms with $l > 0$ will be refererred to as 'logarithmic terms' since t is logarithmic in the expansion parameter e^{Ht}. Again it is possible to prescribe two quantities A_{ab} and B_{ab} which this time have to satisfy an inhomogeneous version of the transverse traceless condition in general. The inhomogeneity is determined by A_{ab}. The prescribed coefficients are $g_{ab}^0 = A_{ab}$ and $(g_{ab})_{n-2,0} = B_{ab}$. In general logarithmic terms are required to get a consistent formal expansion. They can only be avoided if n is odd, $n = 2$ or A_{ab} satisfies some strong restrictions.

At the present time the results on formal asymptotic expansions for higher dimensional vacuum spacetimes have not been extended to existence theorems for all solutions corresponding to a non-empty open set of initial data on a regular Cauchy surface. It has, however, been proved that there exists a very large class of solutions of the Einstein equations with asymptotic expansions as above. Tensors A_{ab} and B_{ab} satisfying the constraints at infinity can be prescribed arbitrarily under the assumption that they are analytic (C^ω). This was proved in [19] using Fuchsian techniques. The generality of the solutions is judged using function counting. These results can probably be extended to fluids with linear equation of state in $3 + 1$ dimensions but this has not been worked out.

The above results require no symmetry assumptions. Under the assumption of spatial homogeneity much more is known. A theorem of Wald [26] shows that for spacetimes of Bianchi types I-VIII with positive cosmological constant and matter satisfying the dominant and strong energy conditions solutions which exist globally in the future have certain asymptotic properties as $t \to \infty$. This implies that the asymptotics of these spacetimes have some of the properties which follow from the asymptotic expansions discussed above. To go further the matter model must be specified. For matter described by the Vlasov equation global existence and more refined asymptotics have been proved by Lee [15]. When the matter model is a perfect fluid with linear equation of state similar results have been proved in [20]. These results confirm many of the features expected from the formal asymptotic expansions. There is also a class of highly symmetric inhomogeneous spacetimes with $\Lambda > 0$ for which global existence and asymptotic properties has been proved for large initial data. These are solutions of the Einstein-Vlasov system with plane or hyperbolic symmetry [23], [24].

4 Mathematics and physics compared

In all the classes of spacetimes with a positive cosmological constant which expand forever the available mathematical results all indicate isotropization at late times. To see the reason for this, introduce the second fundamental form of the hypersurfaces $t =$const., which in Gauss coordinates is given by $k_{ab} = -(1/2)\partial_t g_{ab}$. It turns out that the tracefree part of k_{ab} becomes negligible in comparison with its trace $\mathrm{tr}k$, which is the mean curvature. Equivalently each eigenvalue of the second fundamental form divided by the mean curvature tends to $1/3$ as $t \to \infty$. In the FLRW models these values are exactly equal to $1/3$. In the terminology more common in general relativity the ratio of shear to expansion tends to zero. This is the meaning of isotropization.

At first sight it seems that the spacetime does not become homogeneous at late times, since the coefficient g_{ab}^0 of the leading term in the expansion is not homogeneous. There is, however, a more subtle sense in which it does become homogeneous. Globally in space there is certainly no uniform convergence to a homogeneous metric. This is also the case for spacetime regions of constant coordinate size in the Gaussian coordinates which have been used. On a spatial region of fixed physical size, however, things look different. A region of this kind has a coordinate size which goes to zero exponentially. Since any metric can be approximated increasingly well by a flat metric on a region of ever decreasing size it follows that on a region of fixed physical size the metric converges uniformly and exponentially to the de Sitter metric. In this sense the spacetime does become homogeneous.

Consider next the flatness problem. If the metric has an asymptotic expansion of the form given in the last section then it can be computed directly that the scalar curvature of the spatial metric converges to zero exponentially as $t \to \infty$ and this is what we want to solve the flatness problem. In fact even more can be said. The curvature invariants $R_{ab}R^{ab}$ and $R_{abcd}R^{abcd}$ associated with the three-dimensional metric also decay exponentially. Thus it is not just the scalar curvature which decays; the entire curvature of the spatial metric decays just as fast. It should be noted that although the results of [23] and [24] give a lot of information on the spacetimes to which they are applicable, they are apparently not strong enough to give curvature decay.

It is not so easy to address the horizon problem by a simple and precise mathematical statement. What can be said is the following. A positive cosmological constant leads to solutions of the Einstein equations which look

like de Sitter space on a long time interval and a long time interval in de Sitter space does not suffer from the horizon problem.

5   Scalar fields

As already mentioned in the introduction, an alternative to a positive cosmological constant as a mechanism for producing solutions of the Einstein equations with accelerated expansion is a suitable nonlinear scalar field. Consider a minimally coupled scalar field in a spacetime with vanishing cosmological constant. The energy-momentum tensor of the scalar field is of the form

$$T_{\alpha\beta} = \nabla_\alpha \phi \nabla_\beta \phi - \left[\frac{1}{2}\nabla^\gamma \phi \nabla_\gamma \phi + V(\phi)\right] g_{\alpha\beta} \tag{6}$$

where V is a smooth non-negative function, the potential. To see the connection with a cosmological constant, consider the spatially homogeneous case. Then the energy density is given by $\rho = T_{00} = \dot{\phi}^2/2 + V(\phi)$ while the pressure is given by $p = T_{11} = \dot{\phi}^2/2 - V(\phi)$. There are now different possible regimes. If the kinetic energy is much larger than the potential energy on a certain time interval then on that interval the energy density is approximately equal to the pressure. Thus in a certain loose sense the matter can be approximated by a stiff fluid, which satisfies $p = \rho$. If the kinetic and potential energies are approximately equal on a certain time interval then the pressure is approximately zero there. On that interval the matter can be approximated by dust, which satisfies $p = 0$. Finally, if the potential energy is much larger than the kinetic energy then the pressure is approximately equal to minus the energy density. It is the third case which is related to a cosmological constant. If we think of the cosmological constant as a matter field whose energy-momentum tensor is proportional to the metric then this fictitious matter satisfies $p = -\rho$. In particular, the pressure is negative and comparable in size to the energy density and this is what gives rise to accelerated expansion.

The nature of the dynamics with a nonlinear scalar field depends crucially on the potential V. A useful intuitive picture for guessing what happens with a given potential is the 'rolling' picture. In any spatially homogeneous spacetime the equation of motion for the scalar field is

$$\ddot{\phi} - (\mathrm{tr}k)\dot{\phi} + V'(\phi) = 0 \tag{7}$$

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This is similar to the equation of motion of a ball which rolls on the graph of the function V with variable friction determined by the mean curvature tr$k$. Physical intuition then suggests that the ball should roll down the slope and settle down in a local minimum of the potential. It turns out that accelerated expansion eventually stops if the minimum value of the potential is zero and that for that reason the case of a strictly positive minimum is mathematically more tractable.

Depending on how the acceleration of the universe varies with time it may or may not be consistent with the simplest model where there is a positive cosmological constant and any other matter present satisfies the strong energy condition and so cannot by itself cause acceleration. If the obervations are not consistent with acceleration caused only by a cosmological constant then the next simplest possibility is the nonlinear scalar field. Whether a cosmological constant is enough to explain the observations does not yet seem to be settled although there is some work on the problem in the literature. (See e.g. [1].)

There have been many suggestions for the form of the potential V in the context of inflation or quintessence but there is no clear winner at the moment. If there is a scalar field causing cosmological expansion then we do not know what it is. In these circumstances it makes sense to study the properties of large classes of potentials. In [20] the case of a potential with a strictly positive minimum was discussed. For spacetimes containing a scalar field of this type and ordinary matter satisfying the dominant and strong energy conditions it was shown that there are rather direct generalizations of Wald's theorem [26] and the results of [15].

More specifically, it can be shown under weak assumptions that if the potential is bounded below by a positive constant then $V'(\phi)$ tends to zero as $t \to \infty$. Either ϕ converges to a finite value which is a critical point of V or ϕ tends to plus or minus infinity. If ϕ converges to a finite value and if the corresponding critical point of V is a non-degenerate local minimum ϕ_1 then the solution has asymptotics like that in Wald's theorem, with $V(\phi_1)$ playing the role of an effective cosmological constant. The mean curvature tr$k$ converges to a constant $-3H_1$.

In [20] the statement was made that when the potential has a non-degenerate positive minimum a solution for which the scalar field converges to this minimum has no oscillations. This is misleading and should be replaced by the statement that the deviation of the scalar field from the point where the potential attains its minimum and the modulus of $\dot{\phi}$ decay exponentially

as $t \to \infty$. This implies in particular that $\dot{\phi}$ is absolutely integrable. The equation for $u = (\phi, \dot{\phi})$ can be written in the form $\dot{u} = Au + R(t)u$ where A is a constant matrix and $R(t)$ is a matrix-valued function which decays exponentially. If $\beta > 9H_1^2/4$ where $\beta = V''(\phi_1)$ then the eigenvalues of A are not real. For a generic solution there is an oscillation modulating the leading order exponential decay of the scalar field.

A natural next step is to look at potentials which are strictly positive but which are allowed to go to zero at infinity. The best-studied case is that of power-law inflation. Analogues of Wald's theorem for this case were obtained in [13] and extended in [16]. The potential is of the form $V = V_0 e^{-\kappa \lambda \phi}$ for a positive constant λ. Here κ is a constant which in geometrical units ($G = c = 1$) satisfies $\kappa^2 = 8\pi$. Accelerated expansion at late times is obtained if $\lambda < \sqrt{2}$. If λ is greater than $\sqrt{2}$ then the expansion is decelerated at late times. In the accelerated case the scale factor behaves like a power of t greater than one at late times. When $\lambda > \sqrt{2}$ there are exact FLRW models where the scale factor is proportional to a power of t which is less than one [11].

For inhomogeneous models there is just one interesting result. In [18] formal series expansions for spacetimes with power-law inflation and matter content given by a scalar field alone were written down. It would be interesting to extend the results of [19] for a cosmological constant to this case. The formal expansions are more complicated since they can include powers which are any linear combination with integer coefficients of one and λ. This is similar to the case of a perfect fluid where integer linear combinations of one and γ occur. Note that there is at present no analogue of the results of [6] and [7] known for the case of power-law inflation. It would also be interesting to extend the results of [24] to the case of a nonlinear scalar field. A first step in this direction is a local existence theorem for solutions of the Einstein equations coupled to the Vlasov equation and a linear scalar field which was obtained in [25].

If the potential is zero somewhere the dynamical behaviour becomes more complicated. This is what happens in chaotic inflation. The model case is that of a massive linear scalar field. There is accelerated expansion on some finite time interval but it eventually stops, a process known as reheating. After this the scalar field behaves like dust. At late times $\dot{\phi}$ does not decay faster than t^{-1} and so is not absolutely integrable. These conclusions are based on heuristic arguments [3].

6 Relations between perfect fluids and scalar fields

A type of matter model frequently used to produce accelerated expansion is a perfect fluid which violates the strong energy condition. The equation of state $p = f(\rho)$ satisfies $\rho + 3p < 0$. In the simplest case of a linear equation of state $p = (\gamma - 1)\rho$ this corresponds to choosing $\gamma < 2/3$. Unfortunately $\gamma < 1$ means that $dp/d\rho < 0$ and so the speed of sound becomes imaginary. As has been argued in [8] this suggests that for inhomogenous solutions the initial value problem is ill-posed. The case of homogeneous spacetimes should be thought of as a simple and important special case of the problem without symmetry and if the model makes no mathematical sense without symmetry it is suspect.

A solution to this difficulty is the observation that there is a certain equivalence between a perfect fluid and a scalar field and that the scalar field defines a model which is well-posed without symmetry restrictions. Consider first the case of a linear equation of state with $0 < \gamma < 2/3$ and no other matter fields. Suppose that a spatially flat FLRW solution is given for a fluid with this equation of state. We look for a potential such that the corresponding nonlinear scalar field can reproduce the fluid solution. Using the equation of state gives the relation $\dot{\phi}^2 = \frac{2\gamma}{2-\gamma}V$. Differentiating this with respect to time gives an equation relating $\ddot{\phi}$ and $V'(\phi)$. All terms in this equation have a common factor $\dot{\phi}$. Because $p \neq \rho$ it follows that $\dot{\phi} \neq 0$ and this factor can be cancelled. It follows that $\ddot{\phi} = \frac{\gamma}{2-\gamma}V'(\phi)$. The Hamiltonian constraint implies that $\mathrm{tr}k = -\sqrt{\frac{48\pi V}{(2-\gamma)^2}}$. Putting all this information into the equation of motion for the scalar field gives the equation $V' = -\sqrt{24\pi\gamma}V$. Solving this equation shows that $V = V_0 e^{-\sqrt{24\pi\gamma}\phi}$. Thus the only kind of potential which can work is the one we have already seen for power-law inflation, with $\lambda = \sqrt{3\gamma}$. The range of values of λ which occurs is exactly that which we already saw. It can be shown that this potential really does reproduce the fluid solution. To see this, notice that the initial data which must be chosen for the scalar field are uniquely determined by the data for the fluid. The quantities p and ρ defined from this scalar field satisfy the Euler equations since the energy-momentum tensor of the scalar field is divergence-free. Hence they agree with the fluid density and pressure everywhere. Note that this procedure does not extend to models which are homogeneous but

not isotropic.

The above analysis can be generalized to other equations of state. Consider again the case of a perfect fluid and no other matter fields. Some general assumptions will be made on the equation of state to make a smooth and complete discussion possible. It should, however, be noted that the considerations which follow may be usefully applied in more general situations. Here it is assumed that $dp/d\rho < C_1 < 1$ for a constant C_1 and $p/\rho > C_2 > -1$ for a constant C_2. Note that for any nonlinear scalar field $|p/\rho| \leq 1$. For a general equation of state the relation

$$\frac{1}{2}\dot{\phi}^2 - V(\phi) = f\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{8}$$

must be analysed. This can be rewritten in the form $F(\frac{1}{2}\dot{\phi}^2, V(\phi)) = 0$. Suppose that we have one solution of this equation. The implicit function theorem gives the existence of a function g which satisfies $F(g(V), V) = 0$ for V close to its value in the original solution. This is because the partial derivative of F with respect to its first argument is non-zero. The function g satisfies the relation

$$g'(V) = \frac{1 + f'(g(V) + V)}{1 - f'(g(V) + V)} \tag{9}$$

As a consequence the derivative of the locally defined function g remains bounded on its domain of definition and g can be extended to a longer interval provided it does not tend to zero at the endpoint of the interval. If g tended to zero then this would imply that $p/\rho \to -1$, in contradiction to what has been assumed concerning the equation of state. It follows that the relation (8) can be inverted globally to give $\dot{\phi}^2 = 2g(V)$. Following the same steps as in the case of a linear equation of state gives the equation

$$V'(\phi) = -\frac{1}{1 + g'(V)}\sqrt{48\pi g(V)(g(V) + V)} \tag{10}$$

An exotic fluid model for accelerated cosmological expansion is the Chaplygin gas [12] with equation of state $p = -A/\rho$ for a positive constant A. It satisfies $dp/d\rho > 0$ but violates the dominant energy condition for $\rho < A$, since in that case $p/\rho < -1$. It is ruled out by the assumptions made above. It turns out, however that there are cosmological models with this equation of state where $\rho > A$ everywhere so that this difficulty is avoided. The

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calculations as above can be done for the Chaplygin gas assuming the inequality $\rho > A$. The result is surprisingly simple. The potential is given by $V(\phi) = \frac{1}{2}\sqrt{A}(\cosh\sqrt{24\pi}\phi + \frac{1}{\cosh\sqrt{24\pi}\phi})$. Thus a potential is obtained which has a strictly positive lower bound and it satisfies the hypotheses of the theorems of [20]. Thus for the scalar field corresponding to the Chaplygin gas detailed information is available about late time asymptotics. Unfortunately, because of the fact that the transformation to the scalar field picture is not globally defined, it is not possible to immediately deduce full information on the late-time dynamics for the Chaplygin gas. It should also be remembered that the correspondence with a scalar field does not apply to solutions of the Einstein equations with a Chaplygin gas which are homogeneous but not isotropic.

Sometimes it is desirable to parametrize the degrees of freedom in a cosmological model with fluid in a way which is different from that using the equation of state. An important example of this is the machinery required to compare supernova observations with theoretical models. This will now be sketched. If we know both the apparent and intrinsic brightness of a source then we can compute its distance. (Technically, what can be computed is the so-called luminosity distance.) Consider a spatially flat FLRW model. Then the redshift z of an object is given in terms of the scale factor by $1 + z = a(t_o)/a(t_e)$, where t_o is the time at which the light from the object is observed and t_e the time at which it is emitted. In a model of this kind the luminosity distance can be computed to be $D_L = ra(t_o)(1 + z)$, where r is the spatial separation between the wordlines of observer and emitter as measured in standard coordinates. Let $H = -\mathrm{tr}k/3$, the Hubble parameter. If the luminosity distance and Hubble parameter are expressed in terms of redshift then the following relation results [21]:

$$H(z) = \left[\frac{d}{dz}\frac{D_L(z)}{1 + z} \right]^{-1} \tag{11}$$

Supernova data provides points on the curve $D_L(z)$ and the equation (11) in principle then determines $H(z)$.

Let us ignore complications due to having only discrete data and suppose we know the function $D_L(z)$ exactly. It will now be shown how the scale factor $a(t)$ can be reconstructed. Firstly, $H(z)$ can be computed using (11). An elementary computation shows that $dt/dz = -H(z)(1+z)^{-1}$. Integrating this gives t as a function of z and inverting this gives z as a function of t. Thus

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$H(t)$ can be determined. Integrating once more gives $a(t)$. In practise, in order to distinguish between different theoretical models, an ansatz is made for $H(z)$ containing some parameters and a best fit analysis of the data is carried out to obtain values for these parameters.

7 Tachyons and phantom fields

The ordinary scalar field we have considered up to now can be derived from a Lagrangian with density $\nabla_\alpha \phi \nabla^\alpha \phi + V(\phi)$. Recently dark energy models have been considered where the Lagrangian density is a more general nonlinear function $p(\nabla_\alpha \phi \nabla^\alpha \phi, \phi)$. This is known as k-essence [2]. A great advantage of the ordinary nonlinear scalar field is that it is guaranteed to have well-behaved dynamics in the full inhomogeneous case. The Cauchy problem is always well-posed. (This is even true if the potential is allowed to be negative.) In contrast, k-essence models need not have a well-posed local Cauchy problem. The equation of motion of the scalar field need not be hyperbolic. An additional complication is that since the equations are in general quasilinear rather than semilinear, the scalar field may develop shocks. In this case there is an additional source of singularities supplementing the familiar ones in general relativity. A useful discussion of the some of these points, and the question of which energy conditions are satisfied by k-essence models, can be found in [9]. The models which violate the dominant energy condition are called phantom or ghost models.

An interesting example is given by the case where the function p is given by $V(\phi)\sqrt{1 + \nabla_\alpha \phi \nabla^\alpha \phi}$, which is known as the tachyon or tachyon condensate. Note that although the word 'tachyon' originally denoted a particle which travels faster than light, the tachyon field considered here has no superluminal propagation. All characteristics of the equation lie inside the light cone. The tachyon condensate corresponds to an effective field theory for a large collection of tachyons. Consider now the special case where $V(\phi)$ is identically one. Then provided the gradient of ϕ is timelike this model is equivalent to a special case of the Chaplygin gas. To see this it suffices to define the four-velocity of the fluid by

$$u^\mu = \frac{\nabla^\mu \phi}{\sqrt{-\nabla_\alpha \phi \nabla^\alpha \phi}}. \tag{12}$$

This velocity field is irrotational. The equation for a Chaplygin gas in four-dimensional Minkowski space also describes a timelike hypersurface of zero

mean curvature (a membrane) in five-dimensional Minkowski space. Questions of global existence for these equations have been studied in [14].

8 Closing remarks

This paper gives a general introduction to the subject of cosmological models with accelerated expansion, taking a mathematical point of view. After some basic concepts have been introduced, the relevant physical background on inflation and quintessence is outlined. After this, various existing mathematical results in the case of a positive cosmological constant are presented. They are then confronted with the physical motivation. The exposition continues with a review of results in the case where the cosmological constant is replaced by a nonlinear scalar field. Some interesting open problems are mentioned. There are close relations between models with scalar fields and models with perfect fluids whose equation of state is more or less exotic. Some of these connections are explained. Following this it is explained how scalar fields defined by Lagrangians which are non-linear in the first derivatives give rise to models (known as k-essence) which various connections to both more conventional scalar fields (which are linear in derivatives) and perfect fluids.

At this moment new observations on cosmic acceleration are stimulating a vigorous model-building activity. One aspect of this is that if string theory is a theory of everything then it should, in particular, be able to explain dark energy. It is thus natural that string theory should be one of the main sources of new models. There are many models which are not touched on at all in this paper, in particular those coming from brane-world scenarios [17] or loop quantum cosmology [4]. We have taken a conservative strategy which covers some of the models which are easier to understand mathematically. Even with these limitations we could only treat a few aspects of the subject. A useful task for mathematical relativity is to establish clear definitions of the various models and to identify interesting dynamical issues concerning the solutions. Another task is to systematize the web of relations which exists relating different models and to determine which of them are (in an appropriate sense) really different. Apart from its pedagogical aspects this paper is intended to be a step towards meeting these challenges.

9   Acknowledgements

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