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Algebraic Curve for the SO(6) sector of AdS/CFT

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Abstract

We construct the general algebraic curve of degree four solving the classical sigma model on $\mathbb{R} \times S^5$. Up to two loops it coincides with the algebraic curve for the dual sector of scalar operators in $\mathcal{N} = 4$ SYM, also constructed here. We explicitly reproduce some particular solutions.

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1 Introduction

Since 't Hooft's discovery of planar limit in field theories [1], the idea that the planar non-abelian gauge theory could be exactly solvable, or integrable, always facinated string and field theorists. The analogy between planar graphs of the 4D YM theory and the dynamics of string world sheets of a fixed low genus (described by some unidentified 2D CFT) already pronounced in [1] lead to numerous attempts aimed at the precise formulation of the YM string.

This circle of ideas lead to a dual, matrix model formulation [2] of completely integrable toy models of the string theory and 2D quantum gravity, having also their dual description in the usual world sheet formalism [3]. Big planar graphs of the matrix models find their description in terms of the Liouville string theory proposed in [4]. However the problem of such dual description is still open in the original bosonic 4D YM theory.

Fortunately, these ideas are beginning to work in the 4D world thanks to supersymmetry. The work of Maldacena [5], inspired by some earlier ideas and observations [6], lead to a precise formulation of the string/gauge duality at least in case of IIB superstrings on $AdS_5 \times S^5$ and the conformal $\mathcal{N} = 4$ SYM theory. A lot of work has been done since then to find the AdS/CFT dictionary identifying the string analogs of physical operators and correlators in the $\mathcal{N} = 4$ SYM theory, see [7] for reviews.

The two most recent important advances in this field are the BMN correspondence [8] (see [9] for reviews) and the semiclassical spinning strings duality [10] (see [11] for reviews). These promise to enable quantitative comparisons between both theories even though the duality is of a strong/weak type. Let us only comment briefly on classical spinning strings on $AdS_5 \times S^5$ which were investigated in [12–15]. These were argued to be dual to long SYM operators first investigated in [16,17] where a remarkable agreement was found up to two-loops. Agreement for other particular examples [18,19] as well as at the level of the Hamiltonian [20–23] was obtained until discrepancies surfaced at three-loops, first in the (near) BMN correspondence [24], later also for spinning strings [25]. This problem, which might be the order-of-limits issue explained in [26], is not resolved at the moment. We will observe further evidence of two-loop agreement/three-loop discrepancy in this work.

Luckily, in the last few years the first signs of integrability were observed on both sides of the duality. The first, striking observation of integrability in $\mathcal{N} = 4$ SYM was made by Minahan and Zarembo [27]. They investigated the sector of single-trace local operators of $\mathcal{N} = 4$ supersymmetric gauge theory composed from scalars

$$\mathrm{Tr}\, \Phi_{m_1} \Phi_{m_2} \ldots \Phi_{m_L}. \tag{1.1}$$

It was found that the planar one-loop dilatation operator, which measures their anomalous scaling dimensions, is the Hamiltonian of an integrable spin chain. This chain has $\mathfrak{so}(6)$ symmetry and the spins transform in the vector representation. A basis of the spin chain Hilbert space is thus given by the states

$$|m_1, m_2, \ldots, m_L\rangle \tag{1.2}$$

which correspond, up to cyclic permutations, to single-trace local operators. It was subsequently shown that integrability not only extends to all local single-trace operators of $\mathcal{N}=4$ SYM [28,29] (extending earlier findings of integrability in gauge theory, c.f. [30] for a review), but more surprisingly also to higher loops (at least in some subsectors) [31, 32] (see also [33] for integrability in a related theory). The hypothesis of all-loop integrability together with input from the BMN conjecture [8] has allowed to make precise predictions of higher-loop scaling dimensions [31,34,25,26], which have just recently been verified by explicit computations [35], see also [36]. Integrability and the Bethe ansatz was also an essential tool in obtaining scaling dimensions for states dual to spinning strings. For reviews of gauge theory results and integrability, see [37,38].

Integrability in string theory on $AdS_5 \times S^5$, whose sigma model was explicitly formulated in [39], is based on a so-called *Lax pair*, a family of flat connections of the two-dimensional world sheet theory. Its existence is a common feature of sigma models on coset spaces and it can be used to construct Pohlmeyer charges [40]. These multi-local charges have been discussed in the context of classical bosonic string theory on $AdS_5 \times S^5$ in [41], while a family of flat connections for the corresponding superstring was identified in [42]. The Lax pair of the string sigma model was first put to use in [22] in the case of the restricted target space $\mathbb{R} \times S^3$, where $\mathbb{R}$ represents the time coordinate of AdS_5. There the analytic properties of the monodromy of the flat connections around the closed string were investigated and translated to integral equations similar to the ones encountered in the algebraic Bethe ansatz of gauge theory. This lead to the first rigorous proof of two-loop agreement of scaling dimensions for an entire sector of states. The possibility to quantize the sigma model by discretizing the continuous equations, by analogy with the finite chain Bethe ansatz for gauge theory, was suggested in [22]. A concrete proposal for the quantization of these equations was given in [43]. It reproduces the near BMN results of [24] and, even more remarkably, a generic $\sqrt[4]{\lambda}$ behavior at large coupling in agreement predicted in [44]. Curiously, this proposal appears to have a spin chain correspondence in the weak-coupling extrapolation [45], which however does not agree with gauge theory.

Integrability is usually closely related with the theory of algebraic curves. In most of cases, integrable models of 2D field theory or integrable matrix models are completely characterized by their algebraic curves. Often the algebraic curve unambiguously defines also the quantum version of the model. We also know many examples of algebraic curves characterizing the massive 4D $\mathcal{N}=2$ SYM theories, starting from the famous Seiberg-Witten curve [46], as well as for the $\mathcal{N}=1$ SYM theories [47]. However, the curves describe in that cases only particular BPS sectors of the gauge theories characterized by massive moduli. The $\mathcal{N}=4$ SYM CFT gives us the first hope for an entirely integrable 4D gauge theory, including the non-BPS states.

There is an increasing evidence that the full integrability on both sides of AdS/CFT duality might be governed by similar algebraic curves. On the SYM side, such a curve for the quasi-momentum can be built so far only for small 't Hooft coupling. Its mere existence is due to the perturbative integrability, confirmed up to three-loops [32,25] and hopefully existing for all-loops [26] and even non-perturbatively. On the string sigma model side, we already know the entire classical curve, also for the quasimomentum, for the $\mathbb{R} \times S^3$ [22] and $AdS_3 \times S^1$ [23] sectors of the theory, dual to $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ sectors of the gauge theory, respectively.

In this paper will make a new step in the direction of construction of the full algebraic curve of the classical $AdS_5 \times S^5$ sigma model and of its perturbative counterpart on the SYM side following from the one-loop integrability of the full SYM theory found in [28, 29]. We will accomplish this program for the $\mathfrak{so}(6)$ sector of SYM, which is closed at one-loop as well as in the thermodynamic limit [48], and its dual, the sigma model for the string on $\mathbb{R}_t \times S^5$. We will show in both theories that the projection of the algebraic curve onto the complex plane is a Riemann surface with four sheets corresponding to the four-dimensional chiral spinor representation of SU(4) $\sim$ SO(6). In the SYM case such a curve solves the classical limit of the corresponding Bethe equations. We will identify and fix all the parameters of this curve on both sides of AdS/CFT.

On the gauge side, the main tool of our analysis will be transfer matrices in the algebraic Bethe ansatz framework (see e.g. [49] for an introduction). These can be derived from a Lax-type formulation of the Bethe equations proposed in [50] for the $\mathfrak{su}(m)$ algebras. On the string side, we will construct the so called finite gap solution of the $\mathbb{R} \times S^5$ sigma model, also based on the Lax method [51].

This paper is organized as follows. In Sec. 2 we review the vector $\mathfrak{so}(6)$ spin chain which is dual to the one-loop planar dilatation operator of $\mathcal{N} = 4$ SYM in the sector of local operators composed from scalar fields. Special attention is paid to various transfer matrices and their analytic properties. In the thermodynamic limit will then construct the generic solution in terms of an algebraic curve and illustrate by means of two examples. All this is meant to serve as an introduction to the treatment of the string sigma model in the sections to follow. We start in Sec. 3 by investigating the properties of the monodromy of the Lax pair around the string. They are similar to the ones encountered for the spin chain, but there is an additional symmetry for the spectral parameter. These are used in Sec. 4 to reconstruct an algebraic curve associated to each solution of the equations of motion. We will show that the algebraic curve is uniquely defined by the analytic properties of the monodromy matrix. It thus turns out that solutions are completely characterized by their $\mathcal{B}$-periods (mode numbers) and filling fractions (excitation numbers). In Sec. 5 we construct equations similar to the Bethe equations of the spin chain. These are equivalent to the algebraic curve, but allow for a particle/scattering interpretation.[1] We conclude in Sec. 6. In the appendix we present results which are not immediately important for the AdS/CFT correspondence. Let us mention in particular Sec. F where we apply our formalism to the quite simple, yet interesting case of $\mathbb{R} \times S^2$ and compare to particular limits of known solutions.

2 The $\mathfrak{so}(6)$ Spin Chain

In this section we will review the integrable $\mathfrak{so}(6)$ spin chain with spins transforming in the vector representation of the symmetry algebra. Due to the isomorphism of the algebras $\mathfrak{so}(6)$ and $\mathfrak{su}(4)$ we can rely on a vast collection of results on integrable spin

[1] These equations were proposed independently by M. Staudacher and confirmed by comparing to explicit solutions of the string equations of motion [52].

chain with unitary symmetry algebra. We will use these firm facts to gain a better understanding of the Bethe ansatz in the thermodynamic limit for higher-rank symmetry groups. We identify some key properties of the resolvents which describe the distribution of Bethe roots. In the following chapters we will derive similar properties for the sigma model which later on will be used to (re)construct a similar Bethe ansatz for classical string theory.

2.1 Spin Chains Operators

Dilatation Operator. Let us consider single-trace local operators of $\mathcal{N} = 4$ SYM composed from L scalars without derivatives. These are isomorphic to states of a quantum $\mathfrak{so}(6)$ spin chain with spins transforming in the vector ($\mathbf{6}$) representation. The planar one-loop dilatation generator of $\mathcal{N} = 4$ SYM closes on these local operators, it can thus be written in terms of a spin chain Hamiltonian $\mathbf{H}$ as follows

$$\mathbf{D} = L + g^2\mathbf{H} + \cdots , \qquad g^2 = \frac{g_{\mathrm{YM}}^2 N}{8\pi^2} . \tag{2.1}$$

The Hamiltonian was derived in [27], it is given by the nearest-neighbor interaction $\mathcal{H}$ [2]

$$\mathbf{H} = \sum_{p=1}^{L} \mathcal{H}_{p,p+1}, \qquad \mathcal{H} = 2\mathcal{P}^{\mathbf{15}} + 3\mathcal{P}^{\mathbf{1}} = \mathcal{I} - \mathcal{S} + \tfrac{1}{2}\mathcal{K}^{\mathbf{6},\mathbf{6}}. \tag{2.2}$$

The spin chain operators $\mathcal{P}^{\mathbf{20}'}, \mathcal{P}^{\mathbf{15}}, \mathcal{P}^{\mathbf{1}}$ project to the modules $\mathbf{1}, \mathbf{15}, \mathbf{20}'$ which appear in the tensor product of two spins, $\mathbf{6}\times\mathbf{6}$. These can be written using the operators $\mathcal{I}, \mathcal{S}, \mathcal{K}^{\mathbf{6},\mathbf{6}}$ which are the identity, the permutation of two spins and the trace $(\mathcal{K}^{\mathbf{6},\mathbf{6}})^{ij}{}_{kl} = \delta^{ij}\delta_{kl}$ of two $\mathfrak{so}(6)$ vectors, respectively

$$\mathcal{P}^{\mathbf{20}'} = \tfrac{1}{2}\mathcal{I} + \tfrac{1}{2}\mathcal{S} - \tfrac{1}{6}\mathcal{K}^{\mathbf{6},\mathbf{6}}, \qquad \mathcal{P}^{\mathbf{15}} = \tfrac{1}{2}\mathcal{I} - \tfrac{1}{2}\mathcal{S}, \qquad \mathcal{P}^{\mathbf{1}} = \tfrac{1}{6}\mathcal{K}^{\mathbf{6},\mathbf{6}}. \tag{2.3}$$

R-matrices. Minahan and Zarembo have found out that the Hamiltonian (2.2) obtained from $\mathcal{N} = 4$ SYM is integrable [27]: It coincides with the Hamiltonian of a standard $\mathfrak{so}(m)$ spin chain investigated by Reshetikhin [53]. In this section we focus on the case $m = 6$, but we will present generalizations of some expressions in App. A. Integrability for a standard quantum spin chain means that the (nearest-neighbor) Hamiltonian density $\mathcal{H}$ can be obtained via

$$\mathcal{R}(u) = \mathcal{S}\big(1 + iu\mathcal{H} + \mathcal{O}(u^2)\big) \tag{2.4}$$

from the expansion of an R-matrix $\mathcal{R}(u)$ (see e.g. [37] for an introduction in the context of gauge theory) which satisfies the Yang-Baxter relation

$$\mathcal{R}_{12}(u_1 - u_2)\mathcal{R}_{13}(u_1 - u_3)\mathcal{R}_{23}(u_2 - u_3) = \mathcal{R}_{23}(u_2 - u_3)\mathcal{R}_{13}(u_1 - u_3)\mathcal{R}_{12}(u_1 - u_2). \tag{2.5}$$

[2]We shall distinguish between global and local spin chain operators by boldface and curly letters, respectively. For their eigenvalues we shall use regular letters.

In addition to the YBE, we would like to demand the inversion formula

$$\mathcal{R}_{12}(u_1-u_2)\mathcal{R}_{21}(u_2-u_1)=\mathcal{I}. \tag{2.6}$$

The R-matrix for two vectors of $\mathfrak{so}(6)$ is given by [53][3]

$$\begin{aligned}\mathcal{R}^{\mathbf{6},\mathbf{6}}(u)&=\mathcal{P}^{\mathbf{20}'}+\frac{u-i}{u+i}\,\mathcal{P}^{\mathbf{15}}+\frac{(u-i)(u-2i)}{(u+i)(u+2i)}\,\mathcal{P}^{\mathbf{1}}\\&=\frac{i}{u+i}\,\mathcal{S}+\frac{u}{u+i}\,\mathcal{I}-\frac{iu}{(u+i)(u+2i)}\,\mathcal{K}^{\mathbf{6},\mathbf{6}}.\end{aligned} \tag{2.7}$$

It yields, via (2.4), the spin chain Hamiltonian (2.2). For completeness, we shall also state the R-matrices between a vector and a (anti)chiral spinor

$$\begin{aligned}\mathcal{R}^{\mathbf{4},\mathbf{6}}(u)&=\mathcal{P}^{\mathbf{20}}+\frac{u-\frac{3i}{2}}{u+\frac{3i}{2}}\,\mathcal{P}^{\bar{\mathbf{4}}}=\mathcal{I}-\frac{\frac{i}{2}}{u+\frac{3i}{2}}\,\mathcal{K}^{\mathbf{4},\mathbf{6}},\\\mathcal{R}^{\bar{\mathbf{4}},\mathbf{6}}(u)&=\mathcal{P}^{\overline{\mathbf{20}}}+\frac{u-\frac{3i}{2}}{u+\frac{3i}{2}}\,\mathcal{P}^{\mathbf{4}}=\mathcal{I}-\frac{\frac{i}{2}}{u+\frac{3i}{2}}\,\mathcal{K}^{\bar{\mathbf{4}},\mathbf{6}}.\end{aligned} \tag{2.8}$$

These also satisfy the Yang-Baxter equation (2.5) when we assign any of the three representations $\mathbf{6},\mathbf{4},\bar{\mathbf{4}}$ to the three spaces labeled by $1,2,3$ (the remaining R-matrices between $\mathbf{4}$ and $\bar{\mathbf{4}}$ can be found in App. B). Here we can again express the projectors in terms of identity $\mathcal{I}$ and spin-trace $\mathcal{K}^{\mathbf{4},\mathbf{6}},\mathcal{K}^{\bar{\mathbf{4}},\mathbf{6}}$ defined with Clifford gamma matrices by $(\mathcal{K}^{\mathbf{4},\mathbf{6}})^{\beta j}{}_{\alpha i}=(\gamma^j\gamma_i)^{\beta}{}_{\alpha}$ and $(\mathcal{K}^{\bar{\mathbf{4}},\mathbf{6}})^{\dot\beta j}{}_{\dot\alpha i}=(\gamma^j\gamma_i)^{\dot\beta}{}_{\dot\alpha}$

$$\mathcal{P}^{\mathbf{20}}=\mathcal{I}-\tfrac{1}{6}\mathcal{K}^{\mathbf{4},\mathbf{6}},\quad \mathcal{P}^{\bar{\mathbf{4}}}=\tfrac{1}{6}\mathcal{K}^{\mathbf{4},\mathbf{6}},\qquad \mathcal{P}^{\overline{\mathbf{20}}}=\mathcal{I}-\tfrac{1}{6}\mathcal{K}^{\bar{\mathbf{4}},\mathbf{6}},\quad \mathcal{P}^{\mathbf{4}}=\tfrac{1}{6}\mathcal{K}^{\bar{\mathbf{4}},\mathbf{6}}. \tag{2.9}$$

Transfer matrices. An R-matrix describes elastic scattering of two spins, it gives the phase shift for both spins at the same time. For a spin chain, it can also be viewed as a (quantum) SO(6) lattice link variable. If we chain up the link variables around the closed chain, we obtain a Wilson loop. The open Wilson loop is also known as the monodromy matrix $\mathbf{\Omega}_a(u)$, where a labels the auxiliary space of the Wilson line. The complex number u of the Wilson loop is the spectral parameter. In the $\mathbf{6}$ representation it is convenient to use the combination

$$\mathbf{\Omega}^{\mathbf{6}}_a(u)=\frac{(u+i)^L}{u^L}\,\mathcal{R}^{\mathbf{6},\mathbf{6}}_{a,1}(u-i)\mathcal{R}^{\mathbf{6},\mathbf{6}}_{a,2}(u-i)\cdots\mathcal{R}^{\mathbf{6},\mathbf{6}}_{a,L}(u-i). \tag{2.10}$$

The closed Wilson loop is also known as the transfer matrix

$$\mathbf{T}_{\mathbf{6}}(u)=\mathrm{Tr}_a\,\mathbf{\Omega}^{\mathbf{6}}_a(u). \tag{2.11}$$

We can also write the monodromy and transfer matrices in the spinor representations

$$\begin{aligned}\mathbf{\Omega}^{\mathbf{4}}_a(u)&=\frac{(u+\frac{i}{2})^L}{u^L}\,\mathcal{R}^{\mathbf{4},\mathbf{6}}_{a,1}(u-i)\cdots\mathcal{R}^{\mathbf{4},\mathbf{6}}_{a,L}(u-i),\qquad \mathbf{T}_{\mathbf{4}}(u)=\mathrm{Tr}_a\,\mathbf{\Omega}^{\mathbf{4}}_a(u),\\\mathbf{\Omega}^{\bar{\mathbf{4}}}_a(u)&=\frac{(u+\frac{i}{2})^L}{u^L}\,\mathcal{R}^{\bar{\mathbf{4}},\mathbf{6}}_{a,1}(u-i)\cdots\mathcal{R}^{\bar{\mathbf{4}},\mathbf{6}}_{a,L}(u-i),\qquad \mathbf{T}_{\bar{\mathbf{4}}}(u)=\mathrm{Tr}_a\,\mathbf{\Omega}^{\bar{\mathbf{4}}}_a(u).\end{aligned} \tag{2.12}$$

[3] This R-matrix coincides with the one given in [27,53] up to an overall factor and a redefinition of u. The redefinition is needed to comply with (2.6).

The transfer matrices commute due to the Yang-Baxter relation (2.5), as one can easily convince oneself by inserting the inversion relation (2.6) between both Wilson loops.

Local Charges. The transfer matrices give rise to commuting charges when expanded in powers of u. Local charges $\mathbf{Q}_r$ are obtained from $\mathbf{T}_{\mathbf{6}}(u)$, (the representation of the Wilson loop coincides with the spin representation) when expanded around $u = i$, i.e.

$$\frac{(u+i)^L}{(u+2i)^L}\,\mathbf{T}_{\mathbf{6}}(u+i) = \mathbf{U}\,\exp i\sum_{r=2}^{\infty} u^{r-1}\mathbf{Q}_r \tag{2.13}$$

The operator $\mathbf{U}$ is a global shift operator, it shifts all spins by one site. For gauge theory we are interested in the subspace of states with zero momentum, i.e. with eigenvalue $U = 1$ of $\mathbf{U}$. This is the physical state condition. The second charge $\mathbf{Q}_2$ is the Hamiltonian

$$\mathbf{Q}_2 = \mathbf{H}; \tag{2.14}$$

this fact can be derived from (2.4). Therefore all transfer matrices and charges are also conserved quantities. The third charge $\mathbf{Q}_3$ leads to pairing of states, a peculiar property of integrable spin chains [54, 31].

Global Charges. An interesting value of the spectral parameter is $u = \infty$ where one finds the generators of the symmetry algebra, in our case of $\mathfrak{so}(6) = \mathfrak{su}(4)$. Let us first note the symmetry generators for different representations

$$\mathcal{J}^{\mathbf{6},\mathbf{6}} = \mathcal{S} - \mathcal{K}^{\mathbf{6},\mathbf{6}}, \qquad \mathcal{J}^{\mathbf{4},\mathbf{6}} = \tfrac{1}{2}\mathcal{I} - \tfrac{1}{2}\mathcal{K}^{\mathbf{4},\mathbf{6}}, \qquad \mathcal{J}^{\bar{\mathbf{4}},\mathbf{6}} = \tfrac{1}{2}\mathcal{I} - \tfrac{1}{2}\mathcal{K}^{\bar{\mathbf{4}},\mathbf{6}}. \tag{2.15}$$

The two vector spaces on which these operators act are interpreted as follows: The first ($\mathbf{6}, \mathbf{4}, \bar{\mathbf{4}}$) specifies the parameters for the rotation which acts on the second space ($\mathbf{6}$ in all cases). To be more precise, consider the operator $\mathcal{J}^{\alpha i}{}_{\beta j}$ where α, β belong to $\mathbf{6}, \mathbf{4}, \bar{\mathbf{4}}$ while i, j belong to the second $\mathbf{6}$. In all three cases, the indices α, β can be combined into an index of the adjoint representation which determines the parameters of the rotation. Note that the expressions in (2.15) respect the symmetry properties of the adjoint representation in the tensor products $\mathbf{6} \times \mathbf{6}$, $\mathbf{4} \times \bar{\mathbf{4}}$ and $\bar{\mathbf{4}} \times \mathbf{4}$.

We now express the R-matrices in terms of these symmetry generators and find

$$\begin{aligned}
\mathcal{R}^{\mathbf{6},\mathbf{6}}(u) &= \frac{u}{u+i}\,\mathcal{I} + \frac{iu}{(u+i)(u+2i)}\,\mathcal{J}^{\mathbf{6},\mathbf{6}} - \frac{2}{(u+i)(u+2i)}\,\mathcal{S},\\
\mathcal{R}^{\mathbf{4},\mathbf{6}}(u) &= \frac{u+i}{u+\frac{3i}{2}}\,\mathcal{I} + \frac{i}{u+\frac{3i}{2}}\,\mathcal{J}^{\mathbf{4},\mathbf{6}},\\
\mathcal{R}^{\bar{\mathbf{4}},\mathbf{6}}(u) &= \frac{u+i}{u+\frac{3i}{2}}\,\mathcal{I} + \frac{i}{u+\frac{3i}{2}}\,\mathcal{J}^{\bar{\mathbf{4}},\mathbf{6}}.
\end{aligned} \tag{2.16}$$

One then finds that the expansion at infinity

$$\begin{aligned}
\frac{u+i}{u}\,\mathcal{R}^{\mathbf{6},\mathbf{6}}(u-i) &= \mathcal{I} + \frac{i}{u}\mathcal{J}^{\mathbf{6},\mathbf{6}} + \mathcal{O}(1/u^2),\\
\frac{u+\frac{i}{2}}{u}\,\mathcal{R}^{\mathbf{4},\mathbf{6}}(u-i) &= \mathcal{I} + \frac{i}{u}\mathcal{J}^{\mathbf{4},\mathbf{6}} + \mathcal{O}(1/u^2),\\
\frac{u+\frac{i}{2}}{u}\,\mathcal{R}^{\bar{\mathbf{4}},\mathbf{6}}(u-i) &= \mathcal{I} + \frac{i}{u}\mathcal{J}^{\bar{\mathbf{4}},\mathbf{6}} + \mathcal{O}(1/u^2).
\end{aligned} \tag{2.17}$$

Here we have used the same shifts and prefactors as in the construction of the monodromy matrices (2.10,2.12). In all three cases, the monodromy matrix therefore has the expansion

$$\boldsymbol{\Omega}^{\mathbf{R}}_a(u) = \mathbf{I}_a + \frac{i}{u}\,\mathbf{J}^{\mathbf{R}}_a + \mathcal{O}(1/u^2) \tag{2.18}$$

at $u = \infty$ with the global rotation operators

$$\mathbf{J}^{\mathbf{R}}_a = \sum_{p=1}^{L} \mathcal{J}^{\mathbf{R},\mathbf{6}}_{a,p}. \tag{2.19}$$

If we expand further around $u = \infty$ we will find multi-local operators along the spin chain. These are the generators of the Yangian, see e.g. [55] and [56] in the context of $\mathcal{N} = 4$ SYM.

2.2 Bethe ansatz

States. Consider a spin chain state

$$\big|\{u_{j,k}\}, L\big\rangle \sim \left(\prod_{j=1}^{3}\prod_{k=1}^{K_j} \mathbf{B}_j(u_{j,k})\right)|0, L\rangle. \tag{2.20}$$

The vacuum state $|0, L\rangle$ is the tensor product of L spins in a highest weight configuration of the $\mathbf{6}$. In other words, $|0, L\rangle$ is the ferromagnetic vacuum with all spins aligned to give a maximum total spin. The operator $\mathbf{B}_j(u)$, $u \in \mathbb{C}$, $j = 1, 2, 3$, creates an excitation with rapidity u and quantum numbers of the j-th simple root of $\mathfrak{su}(4)$. A state with a given weight $[r_1, r_2, r_3]$ (Dynkin labels) of $\mathfrak{su}(4)$ has excitation numbers K_j given by (c.f. [29]):

$$\begin{aligned} K_1 &= \tfrac{1}{2}L - \tfrac{3}{4}r_1 - \tfrac{1}{2}r_2 - \tfrac{1}{4}r_3, \\ K_2 &= \phantom{\tfrac{1}{2}}L - \tfrac{1}{2}r_1 - \phantom{\tfrac{1}{2}}r_2 - \tfrac{1}{2}r_3, \\ K_3 &= \tfrac{1}{2}L - \tfrac{1}{4}r_1 - \tfrac{1}{2}r_2 - \tfrac{3}{4}r_3. \end{aligned} \tag{2.21}$$

Transfer matrices. Now let us assume that the state $|\{u_{j,k}\}, L\rangle$ is an eigenstate of all transfer matrices $\mathbf{T}_{\mathbf{R}}(u)$ for all values of the spectral parameter u. Then it can be shown that the eigenvalue of the transfer matrix in the $\mathbf{6}$ representation is given by (see

App. C for a derivation)

$$
\begin{aligned}
T_{\mathbf{6}}(u) = \; & \frac{R_2(u-\frac{3i}{2})}{R_2(u-\frac{i}{2})} \frac{V(u+i)}{V(u)} \\
+ & \frac{R_1(u-i)}{R_1(u)} \frac{R_2(u+\frac{i}{2})}{R_2(u-\frac{i}{2})} \frac{R_3(u-i)}{R_3(u)} \frac{V(u-i)}{V(u)} \frac{V(u+i)}{V(u)} \\
+ & \frac{R_1(u+i)}{R_1(u)} \frac{R_3(u-i)}{R_3(u)} \frac{V(u-i)}{V(u)} \frac{V(u+i)}{V(u)} \\
+ & \frac{R_1(u-i)}{R_1(u)} \frac{R_3(u+i)}{R_3(u)} \frac{V(u-i)}{V(u)} \frac{V(u+i)}{V(u)} \\
+ & \frac{R_1(u+i)}{R_1(u)} \frac{R_2(u-\frac{i}{2})}{R_2(u+\frac{i}{2})} \frac{R_3(u+i)}{R_3(u)} \frac{V(u-i)}{V(u)} \frac{V(u+i)}{V(u)} \\
+ & \frac{R_2(u+\frac{3i}{2})}{R_2(u+\frac{i}{2})} \frac{V(u-i)}{V(u)} \,.
\end{aligned} \tag{2.22}
$$

Note that the monodromy matrix $\mathbf{\Omega}_a^{\mathbf{6}}(u)$ is a $\mathbf{6}\times\mathbf{6}$ matrix in the auxiliary space labelled by a. This explains why the transfer matrix as its trace consists of six terms. For convenience, we have defined the functions $R_j(u), V(u)$

$$
R_j(u) = \prod_{k=1}^{K_j} (u - u_{j,k}), \qquad V(u) = u^L, \tag{2.23}
$$

which describe two and one-particle scattering, respectively. Let us also state the eigenvalues of the transfer matrices in the spinor representations

$$
\begin{aligned}
T_{\mathbf{4}}(u) = \; & \frac{R_1(u-\frac{3i}{2})}{R_1(u-\frac{i}{2})} \frac{V(u+\frac{i}{2})}{V(u)} \\
+ & \frac{R_1(u+\frac{i}{2})}{R_1(u-\frac{i}{2})} \frac{R_2(u-i)}{R_2(u)} \frac{V(u+\frac{i}{2})}{V(u)} \\
+ & \frac{R_2(u+i)}{R_2(u)} \frac{R_3(u-\frac{i}{2})}{R_3(u+\frac{i}{2})} \frac{V(u-\frac{i}{2})}{V(u)} \\
+ & \frac{R_3(u+\frac{3i}{2})}{R_3(u+\frac{i}{2})} \frac{V(u-\frac{i}{2})}{V(u)}
\end{aligned} \tag{2.24}
$$

and its conjugate

$$
\begin{aligned}
T_{\bar{\mathbf{4}}}(u) = {} & \qquad\qquad\qquad\qquad\qquad\qquad \frac{R_3(u-\frac{3i}{2})}{R_3(u-\frac{i}{2})}\,\frac{V(u+\frac{i}{2})}{V(u)} \\
+ {} & \qquad\qquad\quad \frac{R_2(u-i)}{R_2(u)}\,\frac{R_3(u+\frac{i}{2})}{R_3(u-\frac{i}{2})}\,\frac{V(u+\frac{i}{2})}{V(u)} \\
+ {} & \frac{R_1(u-\frac{i}{2})}{R_1(u+\frac{i}{2})}\,\frac{R_2(u+i)}{R_2(u)} \qquad\qquad\quad \frac{V(u-\frac{i}{2})}{V(u)} \\
+ {} & \frac{R_1(u+\frac{3i}{2})}{R_1(u+\frac{i}{2})} \qquad\qquad\qquad\qquad\quad \frac{V(u-\frac{i}{2})}{V(u)}.
\end{aligned}
\tag{2.25}
$$

Bethe equations. As they stand, the above expressions for $T_{\mathbf{R}}(u)$ are rational functions of u. From the definition of $\mathbf{T}_{\mathbf{R}}(u)$ in (2.10,2.11,2.12) and $\mathcal{R}^{\mathbf{R},\mathbf{6}}(u)$ in (2.7) it follows that $T_{\mathbf{R}}(u)$ is a polynomial in $1/u$ [4] of degree at most $2L$ (for $\mathbf{R}=\mathbf{6}$; for $\mathbf{R}=\mathbf{4}$ or $\mathbf{R}=\bar{\mathbf{4}}$ the maximum degree is L). This means that a state $|\{u_{j,k}\},L\rangle$ cannot be an eigenstate of the transfer matrices if $T_{\mathbf{R}}(u)$ has poles anywhere in the complex plane except the obvious singularity at $u=0$ from the definition of $\mathbf{\Omega}^{\mathbf{R}}_a(u)$. From the cancellation of poles for all $1/u\in\mathbb{C}$ one can derive a set of equations which in effect allowed rapidities $\{u_{j,k}\}$ to make up an eigenstate. These are precisely the Bethe equations [53, 57]

$$
\begin{aligned}
\frac{R_1(u_{1,k}+i)}{R_1(u_{1,k}-i)}\,\frac{R_2(u_{1,k}-\frac{i}{2})}{R_2(u_{1,k}+\frac{i}{2})} \qquad\qquad\qquad &= -1, \\
\frac{R_1(u_{2,k}-\frac{i}{2})}{R_1(u_{2,k}+\frac{i}{2})}\,\frac{R_2(u_{2,k}+i)}{R_2(u_{2,k}-i)}\,\frac{R_3(u_{2,k}-\frac{i}{2})}{R_3(u_{2,k}+\frac{i}{2})} &= -\frac{V(u_{2,k}+\frac{i}{2})}{V(u_{2,k}-\frac{i}{2})}, \\
\frac{R_2(u_{3,k}-\frac{i}{2})}{R_2(u_{3,k}+\frac{i}{2})}\,\frac{R_3(u_{3,k}+i)}{R_3(u_{3,k}-i)} &= -1.
\end{aligned}
\tag{2.26}
$$

Effectively, they ensure that $T_{\mathbf{6}}(u), T_{\mathbf{4}}(u)$ and $T_{\bar{\mathbf{4}}}(u)$ are all analytic for $1/u\in\mathbb{C}$.

Local Charges. In $T_{\mathbf{6}}(u)$ all terms but one are proportional to $V(u-i)=(u-i)^L$. Thus the first L terms in the expansion in u around i are determined by this one term alone (unless there are singular roots at $u=+\frac{i}{2}$ which would lower the bound)

$$
\frac{V(u+i)}{V(u+2i)}\,T_{\mathbf{6}}(u+i) = \frac{R_2(u-\frac{i}{2})}{R_2(u+\frac{i}{2})} + \mathcal{O}(u^L).
\tag{2.27}
$$

According to (2.13) this is precisely the combination for the expansion in terms of local charges. Comparing (2.27) to (2.13) we obtain for the global shift and local charge eigenvalues U, Q_r

$$
U = \prod_{k=1}^{K_2} \frac{u_{2,k}-\frac{i}{2}}{u_{2,k}+\frac{i}{2}}, \qquad Q_r = \frac{i}{r-1}\sum_{k=1}^{K_2}\left(\frac{1}{(u_{2,k}+\frac{i}{2})^{r-1}} - \frac{1}{(u_{2,k}-\frac{i}{2})^{r-1}}\right).
\tag{2.28}
$$

[4] The expansion in $1/u$ instead of the more common one in u is due to our definitions.

Note that the eigenvalue of the second charge Q_2 is the energy $E = Q_2$, eigenvalue of the Hamiltonian, see (2.14).

The expansion in terms of local charges is a distinctive feature of $T_{\mathbf{6}}(u)$, which is in the same representation as the spins. For $T_{\mathbf{6}}(u)$ we can expand around $u = \pm i$ and only one of the six terms does contribute in the leading few powers as in (2.27). In contradistinction, at least two terms contribute to the expansion of $T_{\mathbf{4}}(u)$ and $T_{\bar{\mathbf{4}}}(u)$ at every point u. Therefore, neither $T_{\mathbf{4}}(u)$ nor $T_{\bar{\mathbf{4}}}(u)$ can be used to yield *local* charges, which are the sums of the magnon charges as in (2.28).

2.3 Thermodynamic Limit

In the thermodynamic limit the length L of the spin chain as well as the number of excitations K_j approach infinity while focusing on the low-energy spectrum [58, 16]. Let us now rescale the parameters

$$\{K_j, u, r_j, D, g\} \mapsto L\{K_j, u, r_j, D, g\} \tag{2.29}$$

while $E \mapsto E/L$. The Bethe roots $u_{j,k}$ condense on (not necessarily connected) curves $\mathcal{C}_j$ in the complex plane with a density function $\rho_j(u)$, i.e.

$$\sum_{k=1}^{K_j} \ldots \to L \int_{\mathcal{C}_j} du\, \rho_j(u)\, \ldots \tag{2.30}$$

This fixes the normalization of the densities to

$$\int_{\mathcal{C}_j} du\, \rho_j(u) = K_j. \tag{2.31}$$

It is useful to note the following limits of fractions involving R and V

$$\frac{R_j(u+a)}{R_j(u+b)} \to \exp\big((b-a)G_j(u)\big), \qquad \frac{V(u+a)}{V(u+b)} \to \exp\left(\frac{a-b}{u}\right) \tag{2.32}$$

with the resolvent

$$G_j(u) = \int_{\mathcal{C}_j} \frac{dv\, \rho_j(v)}{v-u}\,. \tag{2.33}$$

We can now determine the limit of the transfer matrices. Let us start with the fundamental representation (2.24), we obtain

$$T_{\mathbf{4}}(u) \to \exp\big(ip_1(u)\big) + \exp\big(ip_2(u)\big) + \exp\big(ip_3(u)\big) + \exp\big(ip_4(u)\big). \tag{2.34}$$

The four exponents $p_{1,2,3,4}(u)$ read

$$\begin{aligned}
p_1(u) &= \tilde{G}_1(u), \\
p_2(u) &= \tilde{G}_2(u) - \tilde{G}_1(u), \\
p_3(u) &= \tilde{G}_3(u) - \tilde{G}_2(u), \\
p_4(u) &= \qquad\quad\; - \tilde{G}_3(u),
\end{aligned} \tag{2.35}$$

where we have defined the singular resolvents $\tilde{G}_j(u)$ as

$$\begin{aligned}\tilde{G}_1(u) &= G_1(u) + 1/2u,\\ \tilde{G}_2(u) &= G_2(u) + 1/u,\\ \tilde{G}_3(u) &= G_3(u) + 1/2u.\end{aligned} \tag{2.36}$$

Note that the exponents add up to zero

$$p_1(u) + p_2(u) + p_3(u) + p_4(u) = 0. \tag{2.37}$$

The limit of a transfer matrix in an arbitrary representation $\mathbf{R}$ now reads simply

$$T_{\mathbf{R}}(u) \to \sum_{k=1}^{R} \exp\bigl(ip_k^{\mathbf{R}}(u)\bigr). \tag{2.38}$$

The functions $p^{\mathbf{4}}(u) = p(u)$ are related to the transfer matrix in the fundamental representation. From (2.22,2.25) we can derive up similar functions $p^{\mathbf{6}}(u), p^{\bar{\mathbf{4}}}(u)$ for the vector and conjugate fundamental representation. For each component of the multiplet there is an exponent $p_k^{\mathbf{R}}(u)$

$$\begin{aligned}p^{\mathbf{4}} &= (p_1, p_2, p_3, p_4),\\ p^{\mathbf{6}} &= (p_1 + p_2,\ p_1 + p_3,\ p_1 + p_4,\ p_2 + p_3,\ p_2 + p_4,\ p_3 + p_4),\\ &= (p_1 + p_2,\ p_1 + p_3,\ p_1 + p_4,\ -p_1 - p_4,\ -p_1 - p_3,\ -p_1 - p_2),\\ p^{\bar{\mathbf{4}}} &= (p_1 + p_2 + p_3,\ p_1 + p_2 + p_4,\ p_1 + p_3 + p_4,\ p_2 + p_3 + p_4)\\ &= (-p_4, -p_3, -p_2, -p_1).\end{aligned} \tag{2.39}$$

2.4 Properties of the Resolvents

Bethe Equations and Sheets. We know that $T_{\mathbf{4}}(u)$ is a polynomial in $1/u$. It therefore has no singularities except at $u = 0$ and it should remain analytic in the thermodynamic limit. This is ensured by the Bethe equations (2.26) whose limit reads

$$\begin{aligned}2\tilde{G\!\!\!/}_1(u) - \tilde{G}_2(u) &= p\!\!\!/_1(u) - p\!\!\!/_2(u) = 2\pi n_{1,a}, && u \in \mathcal{C}_{1,a},\\ 2\tilde{G\!\!\!/}_2(u) - \tilde{G}_1(u) - \tilde{G}_3(u) &= p\!\!\!/_2(u) - p\!\!\!/_3(u) = 2\pi n_{2,a}, && u \in \mathcal{C}_{2,a},\\ 2\tilde{G\!\!\!/}_3(u) - \tilde{G}_2(u) &= p\!\!\!/_3(u) - p\!\!\!/_4(u) = 2\pi n_{3,a}, && u \in \mathcal{C}_{3,a}.\end{aligned} \tag{2.40}$$

Here we have split up the curves $\mathcal{C}_j$ into their connected components $\mathcal{C}_{j,a}$ with

$$\mathcal{C}_j = \mathcal{C}_{j,1} \cup \cdots \cup \mathcal{C}_{j,A_j} \tag{2.41}$$

and introduced a mode number $n_{j,a}$ for each curve to select the branch of the logarithm that was used to bring the equations (2.26) into the form (2.40). Furthermore $\tilde{G\!\!\!/}$ and $p\!\!\!/$ are the principal values of $\tilde{G}$ and p, respectively, at a cut, e.g.

$$\tilde{G\!\!\!/}_j(u) = \tfrac{1}{2}\tilde{G}_j(u - \epsilon) + \tfrac{1}{2}\tilde{G}_j(u + \epsilon). \tag{2.42}$$

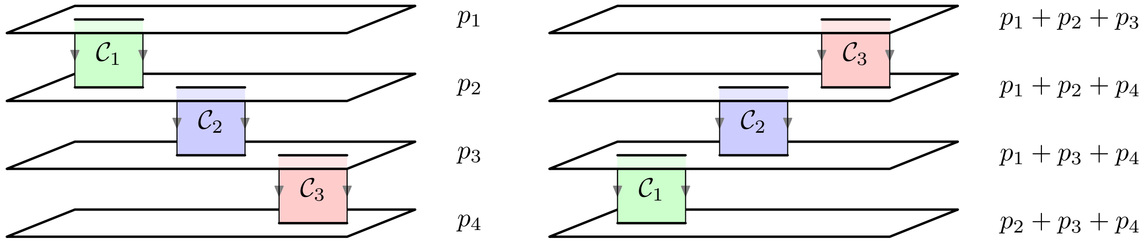

Figure 1: Transfer matrix in **4** and $\bar{\mathbf{4}}$ representation.

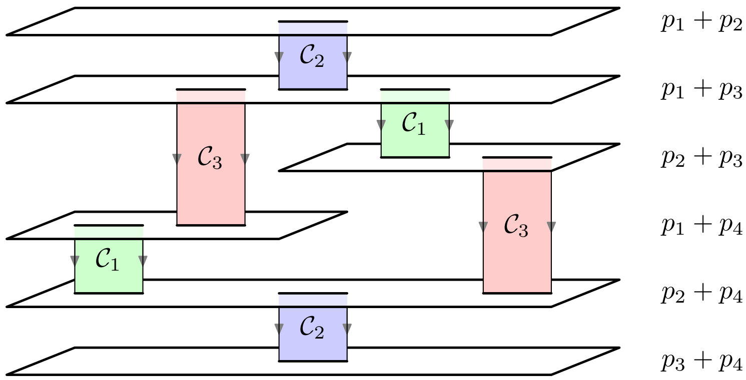

Figure 2: Transfer matrix in **6** representation.

Let us explain the meaning of the Bethe equations in words. The first one implies that a cut in $p_1(u)$ or $p_2(u)$ at $\mathcal{C}_{1,a}$ can be analytically continued by the function $p_2(u)$ and $p_1(u)$, respectively (up to a shift by $\pm 2\pi n_{1,a}$). For the transfer matrices in (2.38) neither the interchange between p_1 and p_2 nor a shift by an integer multiple of 2π has any effect. Similarly, p_2 and p_3 or p_3 and p_4 are connected by cuts along $\mathcal{C}_2$ or $\mathcal{C}_3$ as depicted in Fig. 1. Therefore the transfer matrices are analytic except at $u = 0$. In total, the functions $p_{1,2,3,4}(u)$ (modulo 2π) make up four sheets of a Riemann surface, an algebraic curve of degree four. The function $p = (p_1, p_2, p_3, p_4)$ is not single valued due to the ambiguities by multiples of 2π. In dp the (constant) ambiguities drop out. The differential dp therefore is a holomorphic function on the algebraic curve except at the singular points $u = 0$ on each sheet, see (2.36,2.39).

The configurations of sheets and their connections are displayed in Fig. 1,2. The sheet function $p_k^{\mathbf{R}}(u)$ is obtained by summing up the outgoing singular resolvents $\tilde{G}_j(u)$ and subtracting the incoming ones, c.f. (2.35,2.39).

Let us note here that the equations (2.40) are reminiscent of the saddle point equations of [59] for the RSOS type multi-matrix models. However the potential part is different and, most importantly, the right hand sides of (2.40) would be zero for RSOS models.

Local Charges. The expansion of $T_{\mathbf{6}}(u)$ at $u = i$ gives the local charges. In the thermodynamic limit, this point is scaled to $u = 0$ and from (2.27,2.32) we find [60]

$$\tilde{G}_2(u) = p_1(u) + p_2(u) = \frac{1}{u} + \sum_{r=1}^{\infty} u^{r-1} Q_r, \qquad (2.43)$$

where Q_r has been rescaled by L^{r-1}. The first charge Q_1 is the total momentum around the spin chain which should equal

$$Q_1 = 2\pi n_0 \qquad (2.44)$$

for gauge theory states. The second charge

$$Q_2 = E = (D-1)/g^2. \qquad (2.45)$$

is the energy eigenvalue of the Hamiltonian. The other two resolvents are non-singular

$$\tilde{G}_1(u), \tilde{G}_3(u) = \mathcal{O}(u^0) \qquad (2.46)$$

and their expansion (thus) does not correspond to local quantities. For convenience, we also display the expansion of the sheet functions at zero

$$+p_1(u), +p_2(u), -p_3(u), -p_4(u) = \frac{1}{2u} + \mathcal{O}(u^0), \qquad (2.47)$$

Global Charges. The charges of the symmetry algebra are obtained from the monodromy matrix at $u = \infty$, see Sec. 2.1. When we expand the resolvents $G_j(u)$ at infinity

$$G_j(u) = -\frac{1}{u}\int_{\mathcal{C}_j} dv\, \rho_j(v) + \mathcal{O}(1/u^2) = -\frac{K_j}{u} + \mathcal{O}(1/u^2). \qquad (2.48)$$

we find the fillings of the cuts (2.31), which are related to the representation $[r_1, r_2, r_3]$ of the state via (2.21). The singular resolvents directly relate to the Dynkin labels as follows ($L = 1$ after rescaling)

$$\begin{aligned}
\tilde{G}_1(u) &= \frac{1}{u}\left(\tfrac{1}{2} - K_1\right) + \mathcal{O}(1/u^2) = \frac{1}{u}\left(\tfrac{3}{4}r_1 + \tfrac{1}{2}r_2 + \tfrac{1}{4}r_3\right) + \mathcal{O}(1/u^2),\\
\tilde{G}_2(u) &= \frac{1}{u}\left(1 - K_2\right) + \mathcal{O}(1/u^2) = \frac{1}{u}\left(\tfrac{1}{2}r_1 + r_2 + \tfrac{1}{2}r_3\right) + \mathcal{O}(1/u^2),\\
\tilde{G}_3(u) &= \frac{1}{u}\left(\tfrac{1}{2} - K_3\right) + \mathcal{O}(1/u^2) = \frac{1}{u}\left(\tfrac{1}{4}r_1 + \tfrac{1}{2}r_2 + \tfrac{3}{4}r_3\right) + \mathcal{O}(1/u^2).
\end{aligned} \qquad (2.49)$$

For convenience, we also display the expansion of the sheet functions at infinity

$$\begin{aligned}
p_1(u) &= \frac{1}{u}\left(+\tfrac{3}{4}r_1 + \tfrac{1}{2}r_2 + \tfrac{1}{4}r_3\right) + \mathcal{O}(1/u^2),\\
p_2(u) &= \frac{1}{u}\left(-\tfrac{1}{4}r_1 + \tfrac{1}{2}r_2 + \tfrac{1}{4}r_3\right) + \mathcal{O}(1/u^2),\\
p_3(u) &= \frac{1}{u}\left(-\tfrac{1}{4}r_1 - \tfrac{1}{2}r_2 + \tfrac{1}{4}r_3\right) + \mathcal{O}(1/u^2),\\
p_4(u) &= \frac{1}{u}\left(-\tfrac{1}{4}r_1 - \tfrac{1}{2}r_2 - \tfrac{3}{4}r_3\right) + \mathcal{O}(1/u^2).
\end{aligned} \qquad (2.50)$$

2.5 Algebraic curve

Let us now try to restore the function $p(x)$ from the information derived in the previous subsection,[5] namely, from the Riemann-Hilbert equations (2.40)

$$\not{p}_k(x) - \not{p}_{k+1}(x) = 2\pi n_a \qquad \text{for } x \in \mathcal{C}_a, \tag{2.51}$$

where $\mathcal{C}_a$ connects sheets k and $k+1$ and from the behavior at the various sheets (2.35) at $x \to \infty$ (2.50)

$$p_k \sim \frac{1}{u} + \mathcal{O}(1/u^2) \tag{2.52}$$

as well as $x \to 0$ (2.47)

$$+p_1, +p_2, -p_3, -p_4 = \frac{1}{2u} + 0 \log u + \mathcal{O}(u^0) \tag{2.53}$$

From the discussion in Sec. 2.4 we know that $\exp(ip)$ is a single valued holomorphic function on the Riemann surface with four sheets except at the points 0 and ∞. It is however not an algebraic curve because it has an essential singularity of the type $\exp(i/u)$ at $u = 0$. While p only has pole-singularities, it is defined only modulo 2π. This problem is overcome in the derivative p' which is has a double pole $1/u^2$ at $u = 0$, but no single pole $1/u$, neither at $u = 0$ nor at $u = \infty$.

All this suggests that there exists a function $p(u)$, the quasi-momentum, such that its derivative

$$y(u) = u^2 \frac{dp}{du}(u). \tag{2.54}$$

satisfies a quartic algebraic equation

$$F(y,u) = P_4(u)\, y^4 + P_2(u)\, y^2 + P_1(u)\, y + P_0(u) = P_4(u) \prod_{k=1}^{4} \big(y - y_k(u)\big) = 0. \tag{2.55}$$

For a solution with finitely many cuts we may assume the coefficients $P_k(u)$ to be polynomials in u. The term y^3 is absent because $p_1 + p_2 + p_3 + p_4 = 0$. We have adjusted y to approach a constant limiting value at $x = 0$ as well as at $x = \infty$. It follows that all the polynomials $P_k(u)$ have the same order $2A$ and a non-vanishing constant coefficient. Altogether the function $F(y,u)$ which determines the curve is parameterized by $8A+4$ coefficients minus one overall normalization.

Let us now investigate the analytic structure of the solution of $F(y,u) = 0$ and compare it to the structure of p. In general we can expect that p behaves like $\sqrt{u-u^*}$ at a branch point u^*, consequently $y \sim 1/\sqrt{u-u^*}$. To satisfy the equation $F(y,u) = 0$ at $y = \infty$ we should look for zeros of $P_4(u)/P_2(u)$. Incidentally, we find precisely the correct behavior for y due to the missing of the y^3 term.[6] For a generic $P_2(u)$, the branch

[5] In the $\mathfrak{su}(2)$ case the corresponding hyperelliptic curve was constructed in [22] using the method proposed in [61].

[6] A pole on a single sheet could never be cancelled in $p_1 + p_2 + p_3 + p_4 = 0$. In contrast, a branch singularity $+\alpha/\sqrt{u-u^*}$ will be cancelled by an accompanying singularity $-\alpha/\sqrt{u-u^*}$ on the sheet which is connected along the branch cut.

points are thus the roots of $P_4(u)$

$$P_4(u) = \prod_{a=1}^{A} (u - a_a)(u - b_a). \tag{2.56}$$

Therefore, A is the number of cuts and a_a, b_a are the branch points. The algebraic equation (2.55) potentially has further cuts. The associated singularities are of the undesired form $y \sim (u - u^*)^{r+1/2}$ or $p \sim (u - u^*)^{r+3/2}$ and we have to ensure their absence. Their positions can be obtained as roots of the discriminant of the quartic equation

$$\begin{aligned} R &= -4P_1^2P_2^3 + 16P_0P_2^4 - 27P_1^4P_4 + 144P_0P_1^2P_2P_4 - 128P_0^2P_2^2P_4 + 256P_0^3P_4^2 \\ &= P_4^5\,(y_1 - y_2)^2(y_1 - y_3)^2(y_1 - y_4)^2(y_2 - y_3)^2(y_2 - y_4)^2(y_3 - y_4)^2. \end{aligned} \tag{2.57}$$

All solutions of $R(u) = 0$ with odd multiplicity give rise to undesired branch cuts, in other words we have to demand that the discriminant is a perfect square

$$R(u) = Q(u)^2 \tag{2.58}$$

with a polynomial $Q(u)$.[7] This fixes $5A$ coefficients and we remain with only $3A+3$ free coefficients.

First of all we can fix the coefficients of the double pole in p' at $u = 0$ according to (2.53). This fixes three coefficients, $P_4(0) = -8P_2(0) = 16P_0(0)$ and $P_1(0) = 0$.

The function $p(u)$ has to be single-valued (modulo 2π) on the curve. We can put the $\mathcal{A}$-cycles to zero[8]

$$\oint_{\mathcal{A}_a} dp = 0. \tag{2.59}$$

The cycle $\mathcal{A}_a$ surrounds the cut $\mathcal{C}_a$. Note that there are only $A-3$ independent $\mathcal{A}$-cycles in agreement with the genus of the algebraic curve, $A-3$. The sum of all $\mathcal{A}$-cycles on each of the three independent sheets can be joined to a cycle around the punctures at $u = 0$. Here we expect a double pole, but not a single pole, (2.53)

$$\oint_0 dp_k = 0. \tag{2.60}$$

The $\mathcal{A}$-cycles together with the absence of single poles at $u = 0$ yield A constraints.

Next we consider the $\mathcal{B}$-periods. The cycle $\mathcal{B}_a$ connects the points $u = \infty$ of two sheets $k, k+1$ going through a cut $\mathcal{C}_a$ which connects these sheets.[9] We now rewrite the Bethe equations (2.51) as A integer $\mathcal{B}$-periods

$$\int_{\mathcal{B}_a} dp = 2\pi n_a \tag{2.61}$$

[7] An equivalent condition is: All solutions to the equations $dF(y,u) = 0$ and $P_4(u) \neq 0$ lie on the curve $F(y,u) = 0$. The condition $dF(y,u) = 0 \Rightarrow F(y,u) = 0$ eliminates branch points and $P_4(u) \neq 0$ preserves the desired ones.

[8] Even though we should assume multiples of 2π as the periods, the cuts can be chosen in such a way as to yield single-valued functions p_k [22].

[9] Here the property $p' \sim 1/x^2$, (2.52), is useful to mark the points $u = \infty$.

where n_a is the mode number associated to the cut $\mathcal{C}_a$.

We can now integrate $p'(u)$ and obtain $p(u)$. The integration constants are determined by the value at $u = \infty$, (2.52). At this point we are left with precisely A undefined coefficients. These can be identified with the filling fractions

$$K_a = -\frac{1}{2\pi i} \int_{\mathcal{A}_a} p(u)\, du. \tag{2.62}$$

In the integral representation these correspond to the quantities

$$K_a = \int_{\mathcal{C}_a} \rho(u)\, du. \tag{2.63}$$

When all filling fractions K_a and integer mode numbers n_a are fixed, we can calculate in principle any function of physical interest. In particular, $\tilde{G}_2 = p_1 + p_2 = -p_3 - p_4$ gives an infinite set of local charges (2.43), including the anomalous dimension.

2.6 Examples

Here we will discuss the algebraic solutions of the $\mathfrak{so}(6)$ integrable spin chain found in [18]. The filling fractions (K_1, K_2, K_3) of the three solutions are given by $(\frac{1}{2}\alpha, \alpha, \frac{1}{2}\alpha)$, $(\frac{1}{2}\alpha, \alpha, 0)$ and $(1-\alpha, 1-\frac{1}{2}\alpha, 0)$. For the first two cases (i, ii), there are two symmetric cuts $\mathcal{C}_\pm$ for G_2 stretching from points $\pm a$ to $\pm b$, while the cut for G_1 (and G_3) stretches all along the imaginary axis. The latter cut is however not a genuine branch cut, because both branch points coincide at $u = \infty$. It merely permutes the sheets p_1 and p_2 (or p_3 and p_4) when crossing the imaginary axis. Therefore this cut effectively screens all the charges behind the imaginary axis (looking from either side of it). In our treatment we shall lift this unessential branch cut, the structure of cuts and sheets will thus be different as we will describe below. The solution for case (iii) was found to be equivalent to the case (ii) upon analytic continuation. For us this means that the algebraic curve underlying the solution is actually the same, only the labelling of its sheets is modified.

Frolov-Tseytlin String. Let us start with case (ii/iii) for which there are no excitations of type 3, i.e. $G_3 = 0$. The string analog of this solution was originally studied in [10]. Therefore, the sheet $p_4 = -1/2u$ is detached from the other sheets and the curve factorizes as follows

$$\big(y(u) - 1/2u^2\big)\Big(\tilde{P}_3(u)z(u)^3 + \tilde{P}_1(u)z(u) + \tilde{P}_0(u)\Big) = 0. \tag{2.64}$$

Here we have introduced the shifted variable $z_k = y_k + 1/6$ which ensures that $z_1 + z_2 + z_3 = 0$ due to $y_1 + y_2 + y_3 + y_4 = 0$ and $y_4 = 1/2$. Now the cubic equation for z corresponds to a spin chain with $\mathfrak{su}(3)$ symmetry and spins in the fundamental representation. It can be solved analogously to the $\mathfrak{su}(4)$ case.

The simplest solution which does not reduce further to $\mathfrak{su}(2)$ requires two cuts which do not connect the same two sheets. The corresponding curve obviously has degree three and genus zero, i.e. it is algebraic. This agrees precisely with the solution of case (ii/iii)

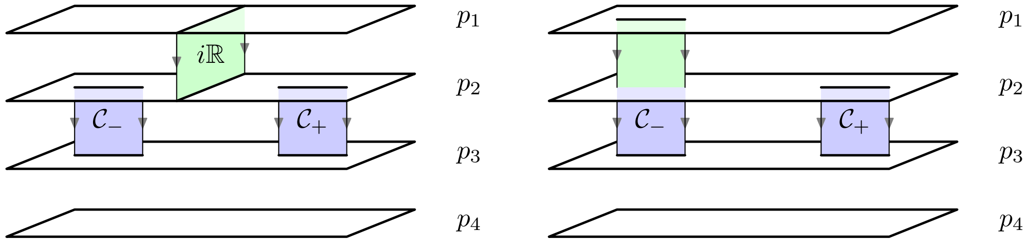

Figure 3: Frolov-Tseytlin string. The diagram on the left depicts the cuts as described in [18]. The cut along the imaginary line ($i\mathbb{R}$) interchanges the two involved sheets (p_1, p_2). When we remove this inessential branch cut, we obtain the diagram on the right. Here the cut $\mathcal{C}_-$ goes directly from p_1 to p_3 right through p_2 effectively screening half of it from $\mathcal{C}_+$.

in [18], which however appears to have three cuts, see the diagram in Fig. 3 on the left. As emphasized above, the cut between sheets p_1 and p_2 can be lifted by interchanging the two when crossing the imaginary axis. Here we have to make the choice on which side we should flip p_1 and p_2, we choose the left one. Now the cut $\mathcal{C}_-$ extends directly from p_3 to p_1, see the diagram in Fig. 3 on the right. Alternatively, one could choose $\mathcal{C}_+$ to extend between p_3 and p_1 while $\mathcal{C}_-$ remains between p_3 and p_2. The solution was originally found in [18], here we will demonstrate the properties discussed in Sec. 2.5. The coefficients describing the algebraic curve in (2.64) are given by

$$\begin{aligned}\tilde{P}_3(u) &= 27\left((1-\alpha)+(-8+36\alpha-27\alpha^2)(\pi n u)^2+16(\pi n u)^4\right), \\ \tilde{P}_1(u) &= -9\left((1-\alpha)+(-8+24\alpha-15\alpha^2)(\pi n u)^2+(16-48\alpha+36\alpha^2)(\pi n u)^4\right), \\ \tilde{P}_0(u) &= -2\left((1-\alpha)+(-8+18\alpha-9\alpha^2)(\pi n u)^2+(16-72\alpha+108\alpha^2-54\alpha^3)(\pi n u)^4\right).\end{aligned} \tag{2.65}$$

The discriminant (modulo factors of $\tilde{P}_0$) is indeed a perfect square

$$\begin{aligned}\tilde{R}(u) &= 4\tilde{P}_1(u)^3+27\tilde{P}_0(u)^2\tilde{P}_3(u) \\ &= -8503056\alpha^2(\pi n u)^6\left((4-9\alpha+5\alpha^2)+(-16+60\alpha-72\alpha^2+27\alpha^3)(\pi n u)^2\right)^2.\end{aligned} \tag{2.66}$$

For the solution we find the expansion around $u=0$

$$z_{1,2} = -\tfrac{1}{3}+2\alpha(\pi n u)^2+\mathcal{O}(u^3), \qquad z_3 = \tfrac{2}{3}-4\alpha(\pi n u)^2+\mathcal{O}(u^4). \tag{2.67}$$

and around $u=\infty$

$$z_{1,2} = -\tfrac{1}{3}+\tfrac{1}{2}\alpha+\mathcal{O}(1/u), \qquad z_3 = \tfrac{2}{3}-\alpha+\mathcal{O}(1/u^2). \tag{2.68}$$

After integrating to the function p we get the expansions

$$p_{1,2} = \frac{1}{2u}\pm f(\alpha,n)+2\alpha(\pi n)^2 u+\mathcal{O}(u^2), \qquad p_3 = -\frac{1}{2u}-4\alpha(\pi n)^2 u+\mathcal{O}(u^3). \tag{2.69}$$

and

$$p_{1,2} = \frac{1-\alpha}{2u}+\mathcal{O}(1/u^2), \qquad p_3 = \frac{-1+2\alpha}{2u}+\mathcal{O}(1/u^3). \tag{2.70}$$

Now $p_3+p_4 = -G_2$ is a physical sheets and p_4 is trivially $p_4 = -1/2u$. Therefore we can read off the energy as the negative coefficient of u in the expansion of p_3 around $u=0$.

From here it is not obvious to see that n must be integer. This fact can be derived from demanding that the $\mathcal{B}$-periods are integers.

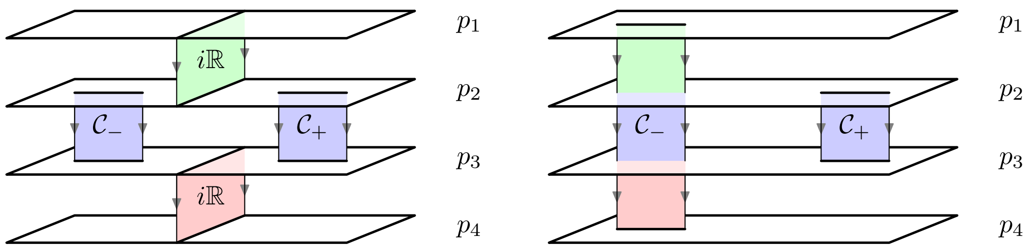

Figure 4: Pulsating string solution. The diagram on the left depicts the cuts as described in [18]. The cuts along the imaginary line ($i\mathbb{R}$) interchange the two involved sheets (p_1, p_2) and (p_3, p_4). When we remove these inessential branch cuts, we obtain the diagram on the right. Here the cut $\mathcal{C}_-$ goes directly from p_1 to p_4 right through p_2 and p_3 effectively screening it from $\mathcal{C}_+$.

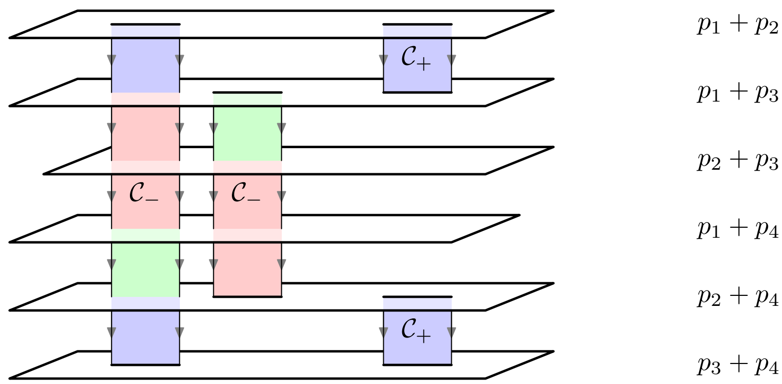

Figure 5: Six-sheeted version of the pulsating string. Note that on the outer two sheets $p_1 + p_2$ and $p_3 + p_4$, the physical sheets, both cuts $\mathcal{C}_+$ and $\mathcal{C}_-$ can be seen. In particular, these sheets do not change if we "expand" $\mathcal{C}_+$ instead of $\mathcal{C}_-$ in Fig. 4. Here the screening works because on the two sheets which $\mathcal{C}_+$ connects, a cut $\mathcal{C}_-$ starts in the same direction, this effectively cancels the forces on $\mathcal{C}_+$. The middle two sheets are free.

Pulsating String. The case (i) is analogous to the case (ii/iii). The string analog of this solution was originally found in [14]. In [18] the solution consisted of four cuts, two of which can however be removed, see Fig. 4. We see that the cut $\mathcal{C}_+$ connects p_2 with p_3, while $\mathcal{C}_-$ connects p_1 with p_4 directly. For the above reasons the two cuts are completely independent. The only connection is due to the momentum constraint, energy and higher local charges which are measured on $p_1 + p_2$. It is also interesting to look at the structure of cuts in the vector representation, see Fig. 5. Here, two sheets are decoupled. For the above reasons, the curve (2.55) factorizes in two

$$\big(\tilde{P}_2^+(u)\, y(u)^2 - \tilde{P}_0^+(u)\big)\big(\tilde{P}_2^-(u)\, y(u)^2 - \tilde{P}_0^-(u)\big) = 0. \tag{2.71}$$

From the solution in [18] we can deduce the coefficients

$$\begin{aligned} \tilde{P}_2^\pm(u) &= 4\big(1 \pm 4(1-\alpha)\pi n u + 4(\pi n u)^2\big), \\ \tilde{P}_0^\pm(u) &= -\big(1 \pm 2(1-\alpha)\pi n u\big)^2. \end{aligned} \tag{2.72}$$

This defines the algebraic curve corresponding to the pulsating string at one-loop.

2.7 Higher Loops

Before we turn to string theory, we make a digression towards higher loops. In [48] Minahan considered the $\mathfrak{so}(6)$ sector at higher loops. While this sector is actually not closed at higher loops, he argued that it closes in the thermodynamic limit. He then combined the one-loop Bethe equations for the $\mathfrak{su}(6)$ sector (2.40) with the higher-loop Bethe equations for the $\mathfrak{su}(2)$ subsector [25, 26]

$$\begin{aligned} 2\bar{\not{G}}_1(u) - \bar{G}_2(u) &= 2\pi n_{1,a}, \qquad u \in \bar{\mathcal{C}}_{1,a}, \\ 2\bar{\not{G}}_2(u) - \bar{G}_1(u) - \bar{G}_3(x) + F_{\text{gauge}}(u) &= 2\pi n_{2,a}, \qquad u \in \bar{\mathcal{C}}_{2,a}, \\ 2\bar{\not{G}}_3(u) - \bar{G}_2(u) &= 2\pi n_{3,a}, \qquad u \in \bar{\mathcal{C}}_{3,a}. \end{aligned} \tag{2.73}$$

with

$$F_{\text{gauge}}(u) = \frac{1}{\sqrt{u^2 - 2g^2}}\,. \tag{2.74}$$

At this point, we have added a bar to the resolvent, because there will be another (unbarred) resolvent $G_j(u)$ which is more closely related to physical quantities. The proposed generalization amounts to a modification of the singular term in $\tilde{G}_j(u)$, c.f. (2.36)[10]

$$\tilde{\bar{G}}_j(u) = \bar{G}_j(u) + M_{2j}^{-1}\,\frac{1}{\sqrt{u^2 - 2g^2}}\,, \tag{2.75}$$

The Bethe equations for $\tilde{\bar{G}}_1(u)$ are just the same as the ones at one-loop (2.40). Note that the pole at $u = 0$ in (2.36) has turned into a cut, but at $g = 0$ we recover the one-loop expressions. To understand the structure of the equations better, we should unfold the cut by a suitable coordinate transformation. This is achieved by the map $u \to x$ given in [26]

$$x(u) = \tfrac{1}{2}u + \tfrac{1}{2}\sqrt{u^2 - 2g^2}\,, \qquad u(x) = x + \frac{g^2}{2x}\,. \tag{2.76}$$

To read off the physical information from the solution of the Bethe equations we will introduce a new resolvent in the x-plane

$$G_j(x) = \int_{\mathcal{C}_j} \frac{dy\,\rho_j(y)}{1 - g^2/2y^2}\,\frac{1}{y - x}\,. \tag{2.77}$$

where the densities are related as $du\,\rho_j(u) = dx\,\rho_j(x)$. The total momentum $U = 1$ and the anomalous dimension $E = (D - L)/g^2$ are read off from $G_2(x)$ as

$$U = \exp\bigl(iG_2(0)\bigr), \qquad E = G_2'(0). \tag{2.78}$$

The relation to the u-resolvent $\bar{G}_j$ is given by

$$\bar{G}_j(x + g^2/2x) = G_j(x) + G_j(g^2/2x) - G_j(0). \tag{2.79}$$

[10] We have used the inverse of the Cartan metric for $\mathfrak{su}(4)$, see (E.6), to specify the coefficient of the singular term, i.e. $M_{2j}^{-1} = (\frac{1}{2}, 1, \frac{1}{2})$

Let us now transform the singular resolvent to the x-plane as $\tilde{G}_j(x) = \bar{\tilde{G}}_j(u(x))$, we obtain

$$\tilde{G}_j(x) = G_j(x) + G_j(g^2/2x) - G_j(0) + M_{2j}^{-1}\,\frac{x}{x^2 - \frac{1}{2}g^2}\,. \qquad (2.80)$$

Note that now the singular term changes sign under the transformation $x \to g^2/2x$ while rest remains invariant, i.e. we have the transformation law

$$\tilde{G}_j(g^2/2x) = \tilde{G}_j(x) - 2M_{2j}^{-1}\frac{x}{x^2 - \frac{1}{2}g^2}\,. \qquad (2.81)$$

The Bethe equations

$$\begin{aligned} 2\tilde{\not{G}}_1(x) - \tilde{G}_2(x) &= \not{p}_1(x) - \not{p}_2(x) = 2\pi n_{1,a}, \qquad & x \in \mathcal{C}_{1,a}, \\ 2\tilde{\not{G}}_2(x) - \tilde{G}_1(x) - \tilde{G}_3(x) &= \not{p}_2(x) - \not{p}_3(x) = 2\pi n_{2,a}, \qquad & x \in \mathcal{C}_{2,a}, \\ 2\tilde{\not{G}}_3(x) - \tilde{G}_2(x) &= \not{p}_3(x) - \not{p}_4(x) = 2\pi n_{3,a}, \qquad & x \in \mathcal{C}_{3,a}. \end{aligned} \qquad (2.82)$$

now imply that $p(x)$ corresponds to an algebraic curve of degree four as explained in Sec. 2.5. This curve has slightly different properties (2.80,2.81) as compared to the one-loop case.

3 Classical Sigma-Model on $\mathbb{R} \times S^{m-1}$

In this section we will investigate the analytic properties of the monodromy of the Lax pair around the closed string. The string is the two-dimensional non-linear sigma model on $\mathbb{R} \times S^{m-1}$ supplemented by the Virasoro constraints. This is an interesting model, because (classically) it is a consistent truncation of the superstring on $AdS_5 \times S^5$.

3.1 The Sigma-Model

Consider the two-dimensional sigma model on $\mathbb{R} \times S^{m-1}$. Let $\{X_i\}_{i=0}^m$ denote the target space coordinates. While X_0 can take any value on $\mathbb{R}$, the other coordinates $\{X_i\}_{i=1}^m$ satisfy a constraint

$$X_1{}^2 + \cdots + X_m{}^2 = 1. \qquad (3.1)$$

Let us also introduce the following vector notation

$$\vec{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix}, \qquad \vec{X}^2 = 1 \qquad (3.2)$$

and a matrix $h_{\mathbf{V}}$ associated with $\vec{X}$ by

$$h_{\mathbf{V}} = 1 - 2\vec{X}\vec{X}^{\mathsf{T}} \qquad (\text{i.e.}\quad h_{\mathbf{V}ij} = \delta_{ij} - 2X_iX_j). \qquad (3.3)$$

The matrix describes a reflection along the vector $\vec{X}$, $\det h_{\mathbf{V}} = -1$, it satisfies $h_{\mathbf{V}}h_{\mathbf{V}}^{\mathsf{T}} = 1$, i.e. it is orthogonal, $h_{\mathbf{V}} \in \mathrm{O}(m)$. Furthermore it is symmetric, $h_{\mathbf{V}} = h_{\mathbf{V}}^{\mathsf{T}}$, and therefore it equals its own inverse

$$h_{\mathbf{V}}^{-1} = h_{\mathbf{V}}. \qquad (3.4)$$

The action of a bosonic string rotating on the S^5 sphere and restricted to a time-like geodesic $\mathbb{R}$ of AdS_5 is given by

$$S_\sigma = -\frac{\sqrt{\lambda}}{4\pi}\int d\sigma\, d\tau\, \Big(\partial_a \vec{X}\cdot\partial^a\vec{X} - \partial_a X_0\,\partial^a X_0 + \Lambda\big(\vec{X}^2-1\big)\Big), \tag{3.5}$$

where Λ is a Lagrange multiplier that constrains $\vec{X}$ to the unit sphere (3.1). Here it is useful to introduce light-cone coordinates

$$\sigma_\pm = \tfrac{1}{2}(\tau\pm\sigma), \qquad \partial_\pm = \partial_\tau \pm \partial_\sigma. \tag{3.6}$$

Then the equations of motion read

$$\partial_+\partial_-\vec{X} + (\partial_+\vec{X}\cdot\partial_-\vec{X})\vec{X} = 0, \qquad \partial_+\partial_- X_0 = 0. \tag{3.7}$$

A solution for the time coordinate which we use to fix the residual gauge of the string is

$$X_0(\tau,\sigma) = \frac{D\,\tau}{\sqrt{\lambda}} = \frac{D(\sigma_+ + \sigma_-)}{\sqrt{\lambda}}\,, \tag{3.8}$$

where D is the dimension in the AdS/CFT interpretation. In addition to the action, the string must satisfy the Virasoro constraints

$$(\partial_\pm\vec{X})^2 = (\partial_\pm X_0)^2, \tag{3.9}$$

which read

$$(\partial_\pm\vec{X})^2 = \frac{D^2}{\lambda} \tag{3.10}$$

in the gauge (3.8).

3.2 Flat and Conserved Currents

Let us next define the right current j_{V} and the left current ℓ_{V} by

$$j_{\mathrm{V}} := h_{\mathrm{V}}^{-1} dh_{\mathrm{V}}, \qquad \ell_{\mathrm{V}} := -dh_{\mathrm{V}} h_{\mathrm{V}}^{-1}. \tag{3.11}$$

Due to the special property (3.4) these currents coincide. In this case they are simply expressed in terms of $\vec{X}$ as

$$j_{\mathrm{V}} = \ell_{\mathrm{V}} = 2(\vec{X}d\vec{X}^{\mathsf{T}} - d\vec{X}\vec{X}^{\mathsf{T}}), \tag{3.12}$$

or in terms of their components,

$$(j_{\mathrm{V}})_{a,ij} = (\ell_{\mathrm{V}})_{a,ij} = 2\big(X_i\partial_a X_j - X_j\partial_a X_i\big). \tag{3.13}$$

From (3.11) it is clear that j_{V} satisfies the flatness condition

$$dj_{\mathrm{V}} + j_{\mathrm{V}}\wedge j_{\mathrm{V}} = 0. \tag{3.14}$$

The kinetic term of the Lagrangian of the sigma model on S^{m-1} is given by $\mathrm{Tr}\, j_{\mathbf{V}} \wedge * j_{\mathbf{V}}$. It is invariant under global right (and left) multiplication to $h_{\mathbf{V}}$ and $j_{\mathbf{V}}$ is the associated conserved current

$$d(*j_{\mathbf{V}}) = 0 \qquad (\text{i.e. } \partial_a j^a_{\mathbf{V}} = 0). \tag{3.15}$$

This relation can be verified using the equations of motion (3.7), in fact it is equivalent to them.

The current $j_{\mathbf{V}}$ is an element of $\mathfrak{so}(m)$ in the vector representation. Similarly, we can write a current in the spinor representation. Let $\{\gamma_i\}$ form the basis of the Clifford algebra of SO(m), i.e. they satisfy

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}. \tag{3.16}$$

Now we introduce the matrix

$$h_{\mathbf{S}} = \vec{\gamma} \cdot \vec{X} \tag{3.17}$$

in the spinor representation. It satisfies

$$h_{\mathbf{S}} = h_{\mathbf{S}}^{-1}, \tag{3.18}$$

therefore it can be regarded as a spinor equivalent of $h_{\mathbf{V}}$. As above this gives rise to equal right and left currents $j_{\mathbf{S}} = \ell_{\mathbf{S}}$. These are in fact equivalent to $j_{\mathbf{V}}$ through $j_{\mathbf{S}} \sim \gamma_i \gamma_j (j_{\mathbf{V}})_{ij}$.

3.3 Lax Pair and Monodromy Matrix

Having a flat and conserved current j, one can construct a family of flat currents

$$a(x) = \frac{1}{1-x^2}\, j + \frac{x}{1-x^2} *j \tag{3.19}$$

parameterized by the spectral parameter x. These give rise to a pair of Lax operators $(\mathcal{M}, \mathcal{L}) = d + a$

$$\mathcal{L}(x) = \partial_\sigma + a_\sigma(x) = \partial_\sigma + \frac{1}{2}\left(\frac{j_+}{1-x} - \frac{j_-}{1+x}\right),$$
$$\mathcal{M}(x) = \partial_\tau + a_\tau(x) = \partial_\tau + \frac{1}{2}\left(\frac{j_+}{1-x} + \frac{j_-}{1+x}\right), \tag{3.20}$$

where we make use of $*(j_\tau, j_\sigma) = (j_\sigma, j_\tau)$ and define $j_\pm = j_\tau \pm j_\sigma$. The conservation and flatness conditions for j are interpreted as the flatness condition for $a(x)$ for all values of x

$$da(x) + a(x) \wedge a(x) = 0 \qquad \text{or} \qquad \big[\mathcal{L}(x), \mathcal{M}(x)\big] = 0. \tag{3.21}$$

Here we have made use of the relations $*b \wedge c = -b \wedge *c$ and $*b \wedge *c = -b \wedge c$ for one-forms b, c in two dimensions.

We can now compute the monodromy of the operator $d + a(x)$ around the closed string. This is the Wilson line along the curve $\gamma(\sigma, \tau)$ which winds once around the string and which is starting and ending at the point (τ, σ)

$$\Omega(x, \tau, \sigma) = \mathrm{P} \exp \int_{\gamma(\tau,\sigma)} \big(-a(x)\big). \tag{3.22}$$

As the current $a(x)$ is flat, the actual shape of the curve γ is irrelevant. The monodromy $\Omega(x)$ depends on the starting point (τ, σ) through the defining equations of the Wilson line,

$$d\Omega(x) + \big[a(x), \Omega(x)\big] = 0. \tag{3.23}$$

which generates a similarity transformation. Physical information should be invariant under the choice of specific points on the world sheet. Therefore, the monodromy $\Omega(x)$ is not physical, but only its conjugacy class, i.e. the set of its eigenvalues. For our purposes this means that neither the curve nor its starting point is relevant. We can thus choose the curve γ to be given by $\tau = 0$, $\sigma \in [0, 2\pi]$. The monodromy matrix becomes

$$\Omega(x) = \mathrm{P}\exp \int_0^{2\pi} d\sigma\, \frac{1}{2}\left(\frac{j_+}{x-1} + \frac{j_-}{x+1}\right). \tag{3.24}$$

where the path ordering symbol P puts the values of σ in decreasing order from left to right.

3.4 Eigenvalues of the Monodromy Matrix

Let us now choose the current in the vector representation $j = j_{\mathrm{V}}$. Since $j_{\mathrm{V}}^{\mathsf{T}} = -j_{\mathrm{V}}$ and $x \in \mathbb{C}$, Ω^{V} is a complex orthogonal matrix[11]

$$\Omega^{\mathrm{V}} \Omega^{\mathrm{V}\mathsf{T}} = 1. \tag{3.25}$$

Only the conjugacy class of $\Omega^{\mathrm{V}}(x)$, characterized by its eigenvalues, corresponds to physical observables. $\Omega^{\mathrm{V}} \in \mathrm{SO}(m, \mathbb{C})$ is diagonalized into the following general form

$$\Omega^{\mathrm{V}}(x) \simeq \begin{cases} \mathrm{diag}\left(e^{iq_1(x)}, e^{-iq_1(x)}, e^{iq_2(x)}, e^{-iq_2(x)}, \ldots, e^{iq_{[m/2]}(x)}, e^{-iq_{[m/2]}(x)}\right) & \text{for } m \text{ even}, \\ \mathrm{diag}\left(e^{iq_1(x)}, e^{-iq_1(x)}, e^{iq_2(x)}, e^{-iq_2(x)}, \ldots, e^{iq_{[m/2]}(x)}, e^{-iq_{[m/2]}(x)}, 1\right) & \text{for } m \text{ odd}, \end{cases} \tag{3.26}$$

where we express the eigenvalues in terms of quasi-momenta $\{q_k(x)\}_{k=1}^{[m/2]}$. However, there still remains the freedom of permutation of the eigenvalues, switching the sign and adding integer multiples of $2\pi i$. A more convenient quantity is the characteristic polynomial

$$\begin{aligned} \Psi^{\mathrm{V}}(\alpha) &= \det(\alpha - \Omega^{\mathrm{V}}) \\ &= \begin{cases} \prod_{k=1}^{[m/2]} (\alpha - e^{iq_k})(\alpha - e^{-iq_k}), & \text{for } m \text{ even} \\ (\alpha - 1) \prod_{k=1}^{[m/2]} (\alpha - e^{iq_k})(\alpha - e^{-iq_k}), & \text{for } m \text{ odd} \end{cases} \\ &\equiv \sum_{k=0}^{m} (-1)^{m-k}\, \alpha^k\, T_{\mathrm{V}[k]}. \end{aligned} \tag{3.27}$$

It is an m-th order polynomial and each coefficient $T_{\mathrm{V}[k]}$ is a symmetric polynomial of the eigenvalues. $T_{\mathrm{V}[k]}$ is the trace of the monodromy matrix in the k-th antisymmetric tensor product of the vector representation. Note that

$$T_{\mathrm{V}[m]} = T_{\mathrm{V}[0]} = 1, \quad T_{\mathrm{V}[m-k]} = T_{\mathrm{V}[k]} \tag{3.28}$$

[11] In fact it satisfies the reality condition $(\Omega^{\mathrm{V}}(x^*))^* = \Omega^{\mathrm{V}}(x)$, i.e. the complex values in $\Omega^{\mathrm{V}}(x)$ are introduced only through a complex x.

in agreement with representation theory.

For the monodromy matrix in the spinor representation we find

$$\Omega^{\mathrm{s}} \simeq \mathrm{diag}\big(\exp(\pm\tfrac{i}{2}q_1 \pm \tfrac{i}{2}q_2 \cdots \pm \tfrac{i}{2}q_{[m/2]})\big). \tag{3.29}$$

with all $2^{[m/2]}$ choices of signs. When n is even, we can reduce Ω^{s} further into its chiral and antichiral parts $\Omega^{\mathrm{s}\pm}$

$$\Omega^{\mathrm{s}} = \begin{pmatrix} \Omega^{\mathrm{s}+} & 0 \\ 0 & \Omega^{\mathrm{s}-} \end{pmatrix}. \tag{3.30}$$

For $\Omega^{\mathrm{s}+}$ and $\Omega^{\mathrm{s}-}$ we should only take those eigenvalues with an even or odd number of plus signs in (3.29), respectively.

3.5 Analyticity

The set of eigenvalues of $\Omega(x)$ depends analytically of x except at some special singular points, see below. This, however, does not directly imply the same for q_k as well. For one, q_k is defined only modulo 2π. Furthermore, the labelling of q_k's is defined at our will. Although the q_k's could fluctuate randomly from one point to the next according to these two ambiguities, we shall assume q_k to be analytic except at singular points and at a (finite) number of branch cuts $\mathcal{C}_a$. At the cuts, the q_k's can be permuted and shifted by multiples of 2π. Such a transformation is captured by the equations

$$\not{q}_k(x) \mp \not{q}_l(x) = 2\pi n_a \qquad \text{for } x \in \mathcal{C}_a. \tag{3.31}$$

Here $\not{q}(x)$ means the principal part of $q_k(x)$ across the cut

$$\not{q}_k(x) = \tfrac{1}{2}q_k(x+i\epsilon) + \tfrac{1}{2}q_k(x-i\epsilon). \tag{3.32}$$

The integer n_a is called the mode number of $\mathcal{C}_a$ and it is assumed to be constant along the cut. Without loss of generality we can restrict ourselves to a subset of allowed permutations, q_k with q_{k+1}, i.e.

$$\not{q}_k(x) - \not{q}_{k+1}(x) = 2\pi n_{k,a} \qquad \text{for } x \in \mathcal{C}_{k,a}. \tag{3.33}$$

These must be supplemented by (for m even)

$$\not{q}_{[m/2]-1}(x) + \not{q}_{[m/2]}(x) = 2\pi n_{[m/2],a} \qquad \text{for } x \in \mathcal{C}_{[m/2],a} \tag{3.34}$$

or (for m odd)

$$2\not{q}_{[m/2]}(x) = 2\pi n_{[m/2],a} \qquad \text{for } x \in \mathcal{C}_{[m/2],a}. \tag{3.35}$$

3.6 Asymptotics

Let us investigate the expansion at $x = \infty$. In leading order, the family of flat connections

$$a(x) = -\frac{1}{x} *j + \mathcal{O}(1/x^2) \tag{3.36}$$

yields the dual of the conserved current j. The expansion of the monodromy matrix $\Omega(x)$ at $x = \infty$ thus yields

$$\Omega(x) = I + \frac{1}{x}\int_0^{2\pi} d\sigma\, j_\tau + \mathcal{O}(1/x^2) = I + \frac{1}{x}\frac{4\pi J}{\sqrt{\lambda}} + \mathcal{O}(1/x^2). \tag{3.37}$$

Here, the conserved charges J of the sigma-model

$$J = \frac{\sqrt{\lambda}}{4\pi}\int_0^{2\pi} d\sigma\, j_\tau, \tag{3.38}$$

appear as the first order in the expansion in $1/x$. The eigenvalues of J^{V} in the vector representation are given by

$$J^{\mathrm{V}} \simeq \begin{cases} \mathrm{diag}\left(iJ_1, -iJ_1, iJ_2, -iJ_2, \ldots, iJ_{[m/2]}, -iJ_{[m/2]}\right) & \text{for } m \text{ even}, \\ \mathrm{diag}\left(iJ_1, -iJ_1, iJ_2, -iJ_2, \ldots, iJ_{[m/2]}, -iJ_{[m/2]}, 0\right) & \text{for } m \text{ odd}. \end{cases} \tag{3.39}$$

The charge eigenvalues J_k are related to the Dynkin labels $[s_1, s_2, \ldots]$ for even m by

$$J_k = \sum_{j=k}^{[m/2]} s_j - \tfrac{1}{2}s_{[m/2]-1} - \tfrac{1}{2}s_{[m/2]}, \tag{3.40}$$

and for odd m by

$$J_k = \sum_{j=k}^{[m/2]} s_j - \tfrac{1}{2}s_{[m/2]}. \tag{3.41}$$

When we compare (3.37,3.26,3.39) we find the expansion of the q_k's at $x = \infty$. Since q_k's are defined as in (3.26), there is a freedom of choosing signs and the ordering of q_k's as well as the branch of the logarithm. We fix them so that q_k has asymptotic behavior

$$q_k(x) = \frac{1}{x}\frac{4\pi J_k}{\sqrt{\lambda}} + \mathcal{O}(1/x^2). \tag{3.42}$$

In particular we fix the branch of all the logarithms such that all $q_k(x)$ vanish at $x = \infty$.

3.7 Singularities

To study the asymptotics of the monodromy matrix (3.24) at $x \to \pm 1$ we assume that at all values of σ the unitary matrix $u_\pm(\sigma)$ diagonalizes $j_\pm(\sigma)$ [12]

$$u_\pm(\sigma)\, j_\pm(\sigma)\, u_\pm^{-1}(\sigma) = j_\pm^{\mathrm{diag}}(\sigma). \tag{3.43}$$

We furthermore assume that the function $u_\pm(x,\sigma)$ is some analytic continuation of $u_\pm(\sigma)$ at $x = \pm 1$, i.e.

$$u_\pm(x,\sigma) = u_\pm(\sigma) + \mathcal{O}(x \mp 1). \tag{3.44}$$

[12] We thank Gleb Arutyunov for explaining to us the expansion.

When we now do a gauge transformation using $u_\pm(x,\sigma)$ we find that

$$u_\pm(x)\,\mathcal{L}(x)\,u_\pm(x)^{-1} = \partial_\sigma - \frac{1}{2}\,\frac{j_\pm^{\mathrm{diag}}}{x\mp 1} + \mathcal{O}\big((x\mp 1)^0\big) \tag{3.45}$$

where the gauge term $\partial_\sigma u_\pm\, u_\pm^{-1}$ is of order $\mathcal{O}((x\mp 1)^0)$. The higher-order off-diagonal terms in $\mathcal{L}(x)$ can be removed order by order by adding the appropriate terms to $u_\pm(x)$. Therefore $u_\pm(x)$ can be used to completely diagonalize $\mathcal{L}(x)$ for all σ. Thus we can drop the path ordering and write

$$u_\pm(x,2\pi)\,\Omega^{\mathrm{V}}(x)\,u_\pm^{-1}(x,0) = \exp\left(\frac{1}{2}\int_0^{2\pi} d\sigma\,\frac{j_\pm^{\mathrm{diag}}}{x\mp 1} + \mathcal{O}\big((x\mp 1)^0\big)\right) \tag{3.46}$$

In other words, to compute the leading singular behavior of the eigenvalues of $\Omega^{\mathrm{V}}(x)$, it suffices to integrate the eigenvalues of $j_\pm$. In App. D we will compute the next few terms in this series and thus find some local commuting charges of the sigma model.

From (3.12) we infer that the current $j_{\mathbf{V}}$ in the vector representation has only two non-zero eigenvalues. They are imaginary, have equal absolute value but opposite signs. The absolute value is determined through the Virasoro constraint (3.9), $(\partial_\pm\vec{X})^2 = (\partial_\pm X_0)^2$. We then find using (3.12)

$$-\tfrac{1}{8}\,\mathrm{Tr}\,(j_{\mathbf{V},\pm})^2 = \big(\partial_\pm\vec{X}\big)^2 = \big(\partial_\pm X_0\big)^2 = \frac{D^2}{\lambda}\,(\partial_\pm\tau)^2 = \frac{D^2}{\lambda}\,. \tag{3.47}$$

The diagonalized matrix $j_{\mathbf{V}\pm}^{\mathrm{diag}}$ thus takes the form

$$j_{\mathbf{V},\pm}^{\mathrm{diag}} = \frac{2iD}{\sqrt{\lambda}}\;\mathrm{diag}(+1,-1,0,0,0,0,\ldots) \tag{3.48}$$

and hence does not depend on σ. This leads to the following asymptotic formula for the eigenvalues of the monodromy matrix:

$$q_k = \delta_{k1}\,\frac{2\pi D}{\sqrt{\lambda}\,(x\mp 1)} + \mathcal{O}\big((x\mp 1)^0\big) \qquad \text{for } x\to\pm 1. \tag{3.49}$$

This formula allows to relate the asymptotic data at $x\to\pm 1$ with the energy of a classical state of the string and hence identify the anomalous dimension of an operator due to the AdS/CFT correspondence.

3.8 Inversion Symmetry

For a sigma model with right and left currents $j = h^{-1}dh$ and $\ell = -dh\,h^{-1}$, the families of associated flat currents $a(x)$ and $a_\ell(x)$ are related by an inversion of the spectral parameter x:

$$\begin{aligned} h\big(d+a(x)\big)h^{-1} &= h\,dh^{-1} + \frac{1}{1-x^2}\,hjh^{-1} + \frac{x}{1-x^2}\,h*jh^{-1} \\ &= d+\ell - \frac{1}{1-x^2}\,\ell - \frac{x}{1-x^2}\,*\ell \\ &= d - \frac{x^2}{1-x^2}\,\ell - \frac{x}{1-x^2}\,*\ell \\ &= d + \frac{1}{1-1/x^2}\,\ell + \frac{1/x}{1-1/x^2}\,*\ell \\ &= d + a_\ell(1/x). \end{aligned} \tag{3.50}$$

In our S^{m-1} model where $j = \ell$ this is particularly interesting since it implies the symmetry

$$h_{\mathbf{v}}\,\mathcal{L}_{\mathbf{v}}(x)\,h_{\mathbf{v}}^{-1} = \mathcal{L}_{\mathbf{v}}(1/x) \qquad \text{and} \qquad h_{\mathbf{s}}\,\mathcal{L}_{\mathbf{s}}(x)\,h_{\mathbf{s}}^{-1} = \mathcal{L}_{\mathbf{s}}(1/x) \tag{3.51}$$

and consequently

$$h_{\mathbf{v}}(2\pi)\,\Omega^{\mathbf{v}}(x)\,h_{\mathbf{v}}(0)^{-1} = \Omega^{\mathbf{v}}(1/x) \qquad \text{and} \qquad h_{\mathbf{s}}(2\pi)\,\Omega^{\mathbf{s}}(x)\,h_{\mathbf{s}}(0)^{-1} = \Omega^{\mathbf{s}}(1/x). \tag{3.52}$$

For a closed string we have $h(0) = h(2\pi)$, therefore $\Omega(x)$ and $\Omega(1/x)$ are related by a similarity transformation and thus have the same set of eigenvalues. For the vector representation it means that each of the quasi-momenta $\{q_1, -q_1, q_2, -q_2, \ldots\}$ transforms into one of $\{q_1, -q_1, q_2, -q_2, \ldots\}$ under the $x \leftrightarrow 1/x$ symmetry.

When m is even, the spinor representation is can be reduced into its chiral and antichiral components. We know that a gamma matrix $\vec{\gamma}$ interchanges both chiralities. Therefore the matrix $h_{\mathbf{s}}$ inverts chirality while $j_{\mathbf{s}} = h_{\mathbf{s}}^{-1} dh_{\mathbf{s}}$ preserves it. It follows that the inversion symmetry relates monodromy matrices of opposite chiralities

$$h_{\mathbf{s}}(2\pi)\,\Omega^{\mathbf{s}\pm}(x)\,h_{\mathbf{s}}(0)^{-1} = \Omega^{\mathbf{s}\mp}(1/x). \tag{3.53}$$

To be consistent with the singular behavior (3.49), q_1 has to transform as

$$q_1(1/x) = 4\pi n_0 - q_1(x). \tag{3.54}$$

The integer constant n_0 reflects the difference of branches of the logarithm at $x = 0$ and $x = \infty$. We need a factor of 4π, because for spinor representations (3.29) we find the exponentials $\exp(\pm\frac{i}{2}q_1)$, which must not change sign. Furthermore, for even m we require that chiral and antichiral representations are interchanged. This is achieved by an odd number of sign flips for all the q_k. A possible transformation rule for the quasi-momenta that works for all values of m is[13]

$$q_k(1/x) = (1 - 2\delta_{k1})\,q_k(x) + 4\pi n_0 \delta_{k1}, \tag{3.55}$$

i.e. q_1 is flips sign while all other q_k are invariant. We shall assume that it is the correct rule although there might be other consistent choices for $m > 4$. Note that there are no additional constant shifts for q_k, $k \neq 1$, because these would be in conflict with the even transformation rule.

[13] We cannot exclude different transformation rules at this point. It would be interesting to see whether there exist such solutions (which are not merely obtained by a relabelling of the eigenvalues).

3.9 The Sigma-Model on $\mathbb{R} \times S^3$

To test out results, we will consider the case $\mathbb{R} \times S^3$ which was extensively studied in [22]. The isometry group SO(4) of S^3 is locally isomorphic to $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$, which played a key role in the preceding discussion. Now we need not to make use of the isomorphism, nevertheless it is useful to interpret our formulation also in terms of $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$.

The spinor representations $\mathbf{2}_\mathrm{L}, \mathbf{2}_\mathrm{R}$ of SO(4) can be viewed as the fundamental representations of $\mathrm{SU}(2)_\mathrm{L}$, $\mathrm{SU}(2)_\mathrm{R}$, respectively. Monodromy matrices in these representations are then viewed as the $SU(2)$ monodromy matrices. They are diagonalized as

$$\Omega^{\mathbf{s}+} \sim \mathrm{diag}(e^{ip_\mathrm{L}}, e^{-ip_\mathrm{L}}), \quad \Omega^{\mathbf{s}-} \sim \mathrm{diag}(e^{ip_\mathrm{R}}, e^{-ip_\mathrm{R}}). \tag{3.56}$$

Now we have two independent quasi-momenta $p_\mathrm{L}, p_\mathrm{R}$, due to the reducibility of $SO(4)$. They can be related to the quasi-momenta q_1, q_2 by

$$p_\mathrm{L} = \tfrac{1}{2}q_1 + \tfrac{1}{2}q_2, \qquad p_\mathrm{R} = \tfrac{1}{2}q_1 - \tfrac{1}{2}q_2 \tag{3.57}$$

where we put constant terms for convenience.

The inversion symmetry now gives rise to

$$p(x) := p_\mathrm{R}(x) = -p_\mathrm{L}(1/x) + 2\pi n_0. \tag{3.58}$$

This means one can assemble the two quasi-momenta to a single quasi-momentum $p(x)$ without inversion symmetry. This special fact played a crucial role in the previous analysis in [22]. In fact this $p(x)$ is the quasi-momentum discussed there.

Let us deduce properties of $p(x)$ from our general results. The pole structure reads

$$p(x) = \frac{\pi D}{\sqrt{\lambda}\,(x \mp 1)} + \mathcal{O}\big((x \mp 1)^0\big). \tag{3.59}$$

It exhibits the asymptotic behavior

$$p(x) = \frac{2\pi r_\mathrm{R}}{\sqrt{\lambda}}\,\frac{1}{x} + \mathcal{O}(1/x^2) \qquad \text{for } x \to \infty \tag{3.60}$$

while the asymptotic behavior of $p_\mathrm{L}(x)$ for $x \to \infty$ is interpreted as

$$p(x) = 2\pi n_0 - \frac{2\pi r_\mathrm{L}}{\sqrt{\lambda}}\,x + \mathcal{O}(x^2) \qquad \text{for } x \to 0. \tag{3.61}$$

The Dynkin labels $r_\mathrm{L} = J_1 + J_2$ and $r_\mathrm{R} = J_1 - J_2$ specify the quantum numbers of $\mathrm{SU}(2)_\mathrm{L}$ and $\mathrm{SU}(2)_\mathrm{R}$; they equal twice the invariant SU(2) spin. Both analyticity conditions (3.33,3.34) are now simply given by

$$2\not{p}(x) = 2\pi n_a, \qquad x \in \mathcal{C}_a. \tag{3.62}$$

All of these agree exactly with the previous results [22].

4 Algebraic Curve for the Sigma-Model on $\mathbb{R} \times S^5$

In this section we will show that the generic solution to the string sigma model on $\mathbb{R} \times S^5$ is uniquely characterized by a set of mode numbers and filling fractions. These are related to certain cycles of the derivative of the quasi-momentum, $q'_k(x)$, which is an algebraic curve of degree four.

4.1 SO(6) vs. SU(4)

The isometry group SO(6) of S^5 is locally isomorphic to SU(4). This enables us to formulate the model in terms of the $\mathfrak{su}(4)$ algebra and the spinor representation which turns out to simplify the structure of the algebraic curve. Here we will translate the properties obtained in the previous section in terms of the quasi-momentum p corresponding to the spinor representation instead of q which corresponds to the vector representation.

The chiral spinor representation **4** of SO(6) is equivalent to the fundamental representation of SU(4). Therefore $\Omega^{\mathrm{s}+}$ can be regarded as the SU(4) monodromy matrix, which is diagonalized as

$$\Omega^{\mathrm{s}+} \sim \operatorname{diag}(e^{ip_1}, e^{ip_2}, e^{ip_3}, e^{ip_4}) \tag{4.1}$$

with $p_1 + p_2 + p_3 + p_4 = 0$. The quasi-momenta p_k are identified as

$$\begin{aligned} p_1 &= \tfrac{1}{2}(\ \ q_1 + q_2 - q_3), \\ p_2 &= \tfrac{1}{2}(\ \ q_1 - q_2 + q_3), \\ p_3 &= \tfrac{1}{2}(-q_1 + q_2 + q_3), \\ p_4 &= \tfrac{1}{2}(-q_1 - q_2 - q_3) \end{aligned} \tag{4.2}$$

in our general notation.

The inversion symmetry (3.55) in terms of p_k is now written as[14]

$$p_{1,2}(1/x) = 2\pi n_0 - p_{2,1}(x), \qquad p_{3,4}(1/x) = -2\pi n_0 - p_{4,3}(x). \tag{4.3}$$

This leads to the structure of branch cuts as depicted in Fig. 6. The cuts of type $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ correspond to the three simple roots of SU(4). While cuts $\mathcal{C}_1$ and $\mathcal{C}_3$ connect sheets $1,2$ and $3,4$, respectively, cuts $\mathcal{C}_2$ may connect either the sheets $2,3$ or the sheets $1,4$ due to the symmetry (4.3). The total number of cuts of either type will be denoted by A_1, A_2, A_3, mirror cuts are assumed to be explicitly included.

The pole structure (3.49) reads

$$p_k(x) = \frac{\pi D}{\sqrt{\lambda}\,(x \mp 1)} + \mathcal{O}\big((x \mp 1)^0\big) \qquad \text{for } x \to \pm 1. \tag{4.4}$$

[14] This is based on the assumption (3.55). Other possibilities for $m = 6$ are $p_{1,2}(1/x) \in 2\pi\mathbb{Z} - p_{1,2}(x)$ and/or $p_{3,4}(1/x) \in 2\pi\mathbb{Z} - p_{3,4}(x)$. This will slightly change the counting of individual constraints below, but the overall number of moduli will remain the same.

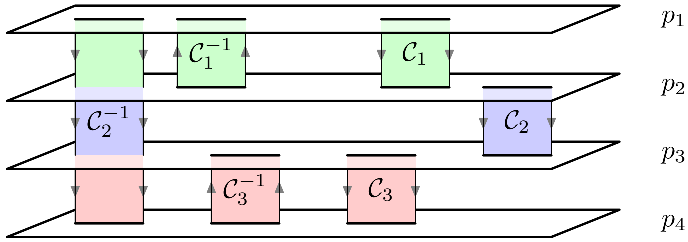

Figure 6: Structure of sheets and branch cuts in the **4** representation for the sigma model on S^5. There are three types of cuts, $\mathcal{C}_{1,2,3}$, corresponding to the simple roots of SU(4). For each cut $\mathcal{C}$, there is a mirror cut $\mathcal{C}^{-1}$. Whether or not it connects the same two sheets depends on the type of cut. The total number of cuts including the mirror images is denoted by A_1, A_2, A_3.

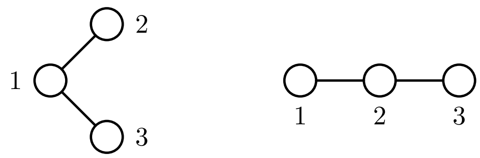

Figure 7: Dynkin diagrams of SO(6) and SU(4).

while the asymptotic behavior at $x = \infty$ (3.42) now reads

$$
\begin{aligned}
p_1(x) &= \frac{1}{x}\frac{4\pi}{\sqrt{\lambda}}\left(\ \ \tfrac{3}{4}r_1 + \tfrac{1}{2}r_2 + \tfrac{1}{4}r_3\right) + \cdots = \frac{1}{x}\frac{2\pi}{\sqrt{\lambda}}\left(\ \ J_1 + J_2 - J_3\right) + \cdots \,, \\
p_2(x) &= \frac{1}{x}\frac{4\pi}{\sqrt{\lambda}}\left(-\tfrac{1}{4}r_1 + \tfrac{1}{2}r_2 + \tfrac{1}{4}r_3\right) + \cdots = \frac{1}{x}\frac{2\pi}{\sqrt{\lambda}}\left(\ \ J_1 - J_2 + J_3\right) + \cdots \,, \\
p_3(x) &= \frac{1}{x}\frac{4\pi}{\sqrt{\lambda}}\left(-\tfrac{1}{4}r_1 - \tfrac{1}{2}r_2 + \tfrac{1}{4}r_3\right) + \cdots = \frac{1}{x}\frac{2\pi}{\sqrt{\lambda}}\left(-J_1 + J_2 + J_3\right) + \cdots \,, \\
p_4(x) &= \frac{1}{x}\frac{4\pi}{\sqrt{\lambda}}\left(-\tfrac{1}{4}r_1 - \tfrac{1}{2}r_2 - \tfrac{3}{4}r_3\right) + \cdots = \frac{1}{x}\frac{2\pi}{\sqrt{\lambda}}\left(-J_1 - J_2 - J_3\right) + \cdots \,.
\end{aligned}
\tag{4.5}
$$

Here the Dynkin labels $[r_1, r_2, r_3]$ of SU(4) are related to the Dynkin labels $[s_1; s_2, s_3]$ and charges (J_1, J_2, J_3) of SO(6) by

$$
\begin{aligned}
r_1 &= s_2 = J_2 - J_3, \\
r_2 &= s_1 = J_1 - J_2, \\
r_3 &= s_3 = J_2 + J_3.
\end{aligned}
\tag{4.6}
$$

This is due to the difference in the labelling of simple roots between the Lie algebras of SU(4) and SO(6): The labels 1 and 2 are interchanged (see Fig. 7).

4.2 Branch Cuts

The monodromy matrix $\Omega^{\mathrm{s}+}(x)$ has similar analytic properties as the one for the spin chain of Sec. 2. Therefore, as discussed in Sec. 2.5, the derivative of the quasi-momentum, $p' = (p'_1, p'_2, p'_3, p'_4)$, is again an algebraic curve of degree four.

First of all, let us define

$$y_k(x) = (x - 1/x)^2\, x\, p_k'(x). \tag{4.7}$$

This removes the poles at $x = \pm 1$ (4.4) and leads to a simple transformation rule under the symmetry (4.3). We can now write y as the solution to an algebraic equation of the same type as (2.55)

$$F(y, x) = P_4(x)\, y^4 + P_2(x)\, y^2 + P_1(x)\, y + P_0(x) = 0. \tag{4.8}$$

As explained in Sec. 2.5 we know that the branch points are given by the roots of $P_4(x)$, let us assume there are A cuts

$$P_4(x) \sim \prod_{a=1}^{A} \big(x - a_a\big)\big(x - b_a\big). \tag{4.9}$$

We need to remove all further branch points. Their positions can be obtained as roots of the discriminant R of the quartic equation

$$R = -4P_1^2 P_2^3 + 16 P_0 P_2^4 - 27 P_1^4 P_4 + 144 P_0 P_1^2 P_2 P_4 - 128 P_0^2 P_2^2 P_4 + 256 P_0^3 P_4^2. \tag{4.10}$$

This means that the discriminant must be a perfect square

$$R(x) = Q(x)^2. \tag{4.11}$$

4.3 Asymptotics

The asymptotics $p(x) \sim 1/x$ at $x = \infty$, (4.5) translate to

$$y(x) \sim x \qquad \text{at } x = \infty. \tag{4.12}$$

for y as defined in (4.7). This requires that $P_k(x) \sim x^{-k} P_0(x)$ for the highest-order terms. Similarly, the asympotics $p(x) \sim \text{const.} + x$ at $x = 0$ obtained through the symmetry (4.3) translate to

$$y(x) \sim 1/x \qquad \text{at } x = 0. \tag{4.13}$$

This requires that $P_k(x) \sim x^k P_0(x)$ for the lowest-order terms. In total this leads to polynomials of the form

$$P_k(x) = * x^k + \cdots + * x^{2A+8-k}. \tag{4.14}$$

Consequently, the discriminant (4.10) takes the form

$$R(x) = * x^8 + \cdots + * x^{10A+32}. \tag{4.15}$$

4.4 Symmetry

The symmetry of the quasi-momentum p in (4.3) translates to

$$y_{1,2}(1/x) = y_{2,1}(x), \qquad y_{3,4}(1/x) = y_{4,3}(x). \tag{4.16}$$

In order the solution to the algebraic equation (4.8) have this symmetry, the polynomials must transform according to[15]

$$P_k(1/x) = x^{-2A-8} P_k(x). \tag{4.17}$$

Similarly, the resolvent satisfies

$$R(1/x) = x^{-10A-40} R(x). \tag{4.18}$$

In other words, the coefficients of the polynomials are the same when read backwards and forwards. Note that in (4.9) we have not made the symmetry for $P_4(x)$ manifest. It requires that $a_a = 1/a_b$ and $b_a = 1/b_b$ for a pair of cuts $\mathcal{C}_{a,b}$ which interchange under the symmetry.[16]

The symmetry of $F(y, x)$, however, merely guarantees that $y_k(1/x) = y_{\pi(k)}(x)$ with some permutation $\pi(k)$. In the most general case, there can only be the trivial permutation $\pi(k) = k$. This can be seen by looking at the fixed points $x = \pm 1$ of the map $x \to 1/x$. If $y_1(\pm 1) \neq y_2(\pm 1)$ there is no chance that the permutation in (4.16) is realized. To permit (4.16) we need to make sure that

$$y_1(x) = y_2(x) \quad \text{and} \quad y_3(x) = y_4(x) \qquad \text{for} \qquad x = +1 \quad \text{and} \quad x = -1. \tag{4.19}$$

This yields four constraints on the coefficients of $F(y, x)$. At this point the trivial permutation $\pi(k) = k$ is still an option. However, now the choice between $\pi(1) = 1$ and $\pi(1) = 2$ is merely a discrete one, there are no further constraints which remove a continuous degree of freedom. In fact, as the the solution to $F(y, x) = 0$ degenerates into two pairs at $x = \pm 1$, the discriminant must have a quadruple pole at these points, i.e. we can write

$$R(x) = x^8 (x^2 - 1)^4 \bigl(*x^0 + \cdots + *x^{10A+16}\bigr). \tag{4.20}$$

4.5 Singularities

Let us now consider the poles at $x = \pm 1$. The expansion of a generic solution for y yields

$$p'_k(x) = \frac{\alpha_k^\pm}{(x \mp 1)^2} + \frac{\beta_k^\pm}{x \mp 1} + \mathcal{O}\bigl((x \mp 1)^0\bigr). \tag{4.21}$$

We have already demanded that $\alpha_1^\pm = \alpha_2^\pm$ and $\alpha_3^\pm = \alpha_4^\pm$. The symmetry furthermore requires $\beta_1^\pm = -\beta_2^\pm$ and $\beta_3^\pm = -\beta_4^\pm$. As the sum of all sheets must be zero, $p_1 + p_2 + p_3 +$

[15] We could also assume $P_k(1/x) = -x^{-2A-8} P_k(x)$, but it turns out to be too restrictive.

[16] In principle we should also allow symmetric cuts with $b_a = 1/a_a$. Apparently these do not occur for solutions which correspond to gauge theory states at weak coupling. At weak coupling one cut should grow to infinity while the other shrinks to zero. This is not compatible with symmetric cuts.

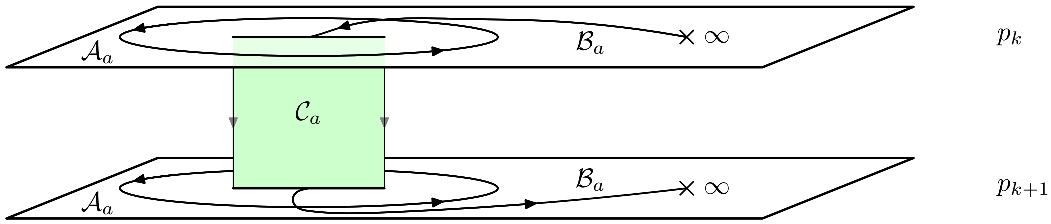

Figure 8: Branch cut $\mathcal{C}_a$ between sheets k and $k+1$ with associated $\mathcal{A}$-cycles and $\mathcal{B}$-period.

$p_4 = 0$, it moreover follows that $\alpha^{\pm}_{1,2} = -\alpha^{\pm}_{3,4}$ whereas $\beta^{\pm}_{1,2}$ and $\beta^{\pm}_{3,4}$ are independent. This means there are three independent coefficients each for the singular behavior at $x = \pm 1$. Now the residue of p at $x = \pm 1$ is proportional to the energy or dimension D. This we cannot fix as it will be the (hopefully) unique result of the calculation. However, we know that the residues at both $x = \pm 1$ are equal, (4.4), which gives one constraint on the α's. Furthermore, there is no logarithmic behavior in p, (4.4), therefore all β's must be zero which gives four constraints. In total there are five constraints from the poles at $x = \pm 1$.

4.6 A-Cycles

The eigenvalues $\exp(ip_k(x))$ of $\Omega^{\mathrm{s}+}(x)$ are holomorphic functions of x. This however does not exclude the possibility of cuts where the argument $p_k(x)$ jumps by multiples of 2π but is otherwise smooth. Such cuts originate from logarithmic or branch-cut singularities; they are required when the closed integral around the singularity does not vanish. We know that there are no logarithmic singularities, therefore we merely need to ensure that

$$\oint_{\mathcal{A}_a} dp \in 2\pi\mathbb{Z} \tag{4.22}$$

where the cycles $\mathcal{A}_a$ surrounds a cut $\mathcal{C}_a$, see Fig. 8. As was shown in [22] we can even demand that all $\mathcal{A}$-cycles are zero, which conveniently reduces the number of cuts.

Assume first $\mathcal{C}_a$ connects sheets $1, 2$ or sheets $3, 4$.[17] Then there is another cut $\mathcal{C}_b$ as the image of $\mathcal{C}_a$ under (4.3) between the same two sheets. The values of the $\mathcal{A}$-cycles are related

$$\oint_{\mathcal{A}_a} dp_1 = -\oint_{\mathcal{A}_b = 1/\mathcal{A}_a} dp_2 = \oint_{\mathcal{A}_b} dp_1. \tag{4.23}$$

The two signs flips are explained as follows: The first is related to the symmetry (4.3) and the second to changing the sheet back to 1. For a cut which connects sheets $2, 3$, the mirror image will connect sheets $1, 4$. In this case we have

$$\oint_{\mathcal{A}_a} dp_2 = -\oint_{1/\mathcal{A}_a} dp_1 = -\oint_{\mathcal{A}_b} dp_1 \tag{4.24}$$

In particular this means that there is only one constraint

$$\oint_{\mathcal{A}_a} dp = 0 \tag{4.25}$$

[17] In Fig. 6 we have illustrated how the sheets are connected by the cuts.

for each pair of cuts. Moreover, the cycle around all cuts on sheet 1 is just the negative of the corresponding one on sheet 2; equivalently for sheets 3 and 4. As there are no further single poles on any sheet, the cycle around all cuts can be contracted and must be zero. The total number of constraints from $\mathcal{A}$-cycles is thus $\frac{1}{2}A - 2$.

4.7 B-Periods

We know that the set of eigenvalues $\exp(ip_k(x))$ of $\Omega^{\mathrm{s}+}(x)$ depends analytically on x. Their labeling $k = 1, 2, 3, 4$, however, is artificial. This allows for the presence of cuts $\mathcal{C}_a$ where the p_k permute, see Sec. 3.5. In addition they can also shift by multiples of 2π without effect on $\exp(ip_k(x))$. This shift can be expressed through the integral of dp along the curve $\mathcal{B}_a$ which connects the points $x = \infty$ on the involved sheets through the cut $\mathcal{C}_a$. We know that $p(x)$ is analytic along $\mathcal{B}_a$ except at the intersection of $\mathcal{B}_a$ with $\mathcal{C}_a$. Moreover we assume that $p(\infty) = 0$ on both sheets, therefore the period

$$\int_{\mathcal{B}_a} dp \in 2\pi\mathbb{Z} \tag{4.26}$$

describes the shift in $p(x)$ at $\mathcal{C}_a$ and must be a multiple of 2π.

Note that the symmetry $x \to 1/x$ does not map $\mathcal{B}$-periods directly to $\mathcal{B}$-periods due to the explicit reference to the point $x = \infty$. First, we should therefore consider the integral

$$\int_0^\infty dp_k = p_k(\infty) - p_k(0) = -p_k(0) \tag{4.27}$$

where we have made use of our choice $p_k(\infty) = 0$. From (4.3) it follows that

$$p_{1,2}(0) = -p_{3,4}(0) = 2\pi n_0 \tag{4.28}$$

which is the momentum constraint. It reduces the number of degrees of freedom by one, because n_0 must be integer. Now consider a $\mathcal{B}$-period between sheets $1, 2$ or sheets $3, 4$. Due to the symmetry

$$\begin{aligned}
\int_{\mathcal{B}_a} dp &= \int_\infty^{x_a} dp_1 + \int_{x_a}^\infty dp_2 = -\int_0^{1/x_a} dp_2 - \int_{1/x_a}^0 dp_1 \\
&= -\int_0^\infty dp_2 - \int_\infty^{x_b} dp_2 - \int_{x_b}^\infty dp_1 - \int_\infty^0 dp_1 \\
&= p_2(0) - p_1(0) + \int_\infty^{x_b} dp_1 + \int_{x_b}^\infty dp_2 = \int_{\mathcal{B}_b} dp.
\end{aligned} \tag{4.29}$$

we see that the cycles $\mathcal{B}_a$ and $\mathcal{B}_b$ have the same value. Equivalently, for $\mathcal{B}_a$ between sheets $2, 3$ which is related to $\mathcal{B}_a$ between sheets $1, 4$

$$\begin{aligned}
\int_{\mathcal{B}_a} dp &= \int_\infty^{x_a} dp_2 + \int_{x_a}^\infty dp_3 = -\int_0^{1/x_a} dp_1 - \int_{1/x_a}^0 dp_4 \\
&= -\int_0^\infty dp_1 - \int_\infty^{x_b} dp_1 - \int_{x_b}^\infty dp_4 - \int_\infty^0 dp_4 \\
&= p_1(0) - p_4(0) - \int_\infty^{x_b} dp_1 - \int_{x_b}^\infty dp_4 = 4\pi n_0 - \int_{\mathcal{B}_b} dp.
\end{aligned} \tag{4.30}$$

Note that by demanding that both values of cycles $\mathcal{B}_a$ and $\mathcal{B}_b$ are multiples of 2π, it follows that n_0 is integer.[18] So by fixing

$$\int_{\mathcal{B}_a} dp = 2\pi n_a \tag{4.31}$$

we automatically determine the value of mirror period $\mathcal{B}_b$. Consequently, the $\mathcal{B}$-periods together with the momentum constraint fix $\frac{1}{2}A+1$ coefficients.

4.8 Filling Fractions

The polynomial $F(y,x)$ has $8A+22$ coefficients in total, see (4.14). Of them $4A+9$ are incompatible with the symmetry (4.17) and another 4 are constrained by enabling non-trivial permutations of the y_k, see (4.19). The overall normalization of $F(y,x)$ is irrelevant for the $F(y,x)=0$, this removes one degree of freedom. The discriminant R, (4.20), has $5A+8$ non-trivial pairs of roots related by the symmetry (4.18). These should all have even multiplicity, (4.11), which fixes $\frac{5}{2}A+4$ coefficients. The residues of the poles and absence of logarithmic singularities at $x=\pm 1$ leads to 5 constraints. The $\mathcal{A}$-cycles and $\mathcal{B}$-periods yield $\frac{1}{2}A-2$ and $\frac{1}{2}A+1$ constraints, respectively. In total there are $\frac{1}{2}A$ continuous degrees of freedom remaining. These can be used to assign one filling fraction to each pair of cuts. We define the filling fraction of a cut $\mathcal{C}_a$ as

$$K_a = -\frac{1}{2\pi i}\oint_{\mathcal{A}_a} dx\left(1-\frac{1}{x^2}\right)p(x) = \frac{1}{2\pi i}\oint_{\mathcal{A}_a}\left(x+\frac{1}{x}\right)dp. \tag{4.32}$$

The second form which directly relates to dp is obtained by partial integration.

4.9 Global Charges

Now let us compute the global charges at $x=\infty$, see (4.5). These are obtained as the cycles of $p_k(x)\,dx$ around $x=\infty$ which we can also write as the sum of cycles around all singularities on the same sheet (c.f. Fig. 6 for the structure of cuts)

$$\begin{aligned}
\frac{4\pi r_1}{\sqrt{\lambda}} &= \frac{1}{2\pi i}\oint_\infty dx\, p_1(x) - \frac{1}{2\pi i}\oint_\infty dx\, p_2(x) = \sum_{a=1}^{A_2/2} K_{2,a} - \sum_{a=1}^{A_1} K_{1,a},\\
\frac{4\pi r_2}{\sqrt{\lambda}} &= \frac{1}{2\pi i}\oint_\infty dx\, p_2(x) - \frac{1}{2\pi i}\oint_\infty dx\, p_3(x)\\
&= \frac{4\pi\, D}{\sqrt{\lambda}} + 2\sum_{a=1}^{A_2/2}\frac{1}{2\pi i}\oint_{\mathcal{A}_{2,a}}\frac{dx\, p_2(x)}{x^2} + \tfrac{1}{2}\sum_{a=1}^{A_1} K_{1,a} - 2\sum_{a=1}^{A_2/2} K_{2,a} + \tfrac{1}{2}\sum_{a=1}^{A_3} K_{3,a},\\
\frac{4\pi r_3}{\sqrt{\lambda}} &= \frac{1}{2\pi i}\oint_\infty dx\, p_3(x) - \frac{1}{2\pi i}\oint_\infty dx\, p_4(x) = \sum_{a=1}^{A_2/2} K_{2,a} - \sum_{a=1}^{A_3} K_{3,a}.
\end{aligned} \tag{4.33}$$

[18]This is different from spin chain for which there are states with non-integer total momentum are perfectly well-defined.

Here we have made use of the symmetry to write our findings in terms of the filling fractions $K_{k,a}$. Note that the sums $\sum_{a=1}^{A_2/2}$ extend only over those cuts which connect sheets $2, 3$, but not sheets $1, 4$, while $\sum_{a=1}^{A_1}, \sum_{a=1}^{A_3}$ extend over all cuts between sheets $1, 2$ and $3, 4$, respectively. The first and last equation directly relate the total filling fractions for cuts between sheets $1, 2$ and sheets $3, 4$ with two of the Dynkin labels r_1, r_3 of the state. The middle equation serves a similar purpose for r_2, but it also involves the scaling dimension. The following identity might be useful in connection with the dimension

$$-p_1'(0)-p_2'(0) = \frac{4\pi}{\sqrt{\lambda}}(\tfrac{1}{2}r_1+r_2+\tfrac{1}{2}r_3) = \frac{4\pi\, D}{\sqrt{\lambda}} + \sum_{a=1}^{A_2/2} \frac{1}{2\pi i}\oint_{\mathcal{A}_{2,a}} \left(1+\frac{1}{x^2}\right) dx\, p_2(x). \quad (4.34)$$

4.10 Comparison to Gauge Theory

Here we will show that the algebraic curve of the SYM theory in $\mathfrak{so}(6)$ sector coincides with the algebraic curve of the string sigma model on $\mathbb{R}\times S^5$ in two loops, in accordance with the proposal of [48].

Let us compare the analytical data defining the curves the weak coupling limit $\lambda \to 0$. We will define for the convenience through the Sec. 4 a rescaled variable $u = (\sqrt{\lambda}/4\pi D)x$, which makes it similar to the spectral parameter u of Sec. 2. In this limit, for each pair of mutually symmetric branch points in u variable, $(a_k, 16\pi^2 D^2/\lambda a_k)$, the second one goes to infinity, and we are left with a half of the cuts (a_k, b_k) having no symmetry with respect to the inversion of u, as in the case of the SYM curve. It is easy to see that the equation (4.8) becomes (2.55) in this limit, for the similar definitions of $y(u)$.

Both curves enjoy the following common properties:

1. Four sheets for the quasi-momentum (which we call $p(u)$ even after rescaling) in the u-projection, connected by finite cuts (no branch points at ∞ of each sheet), as discussed in Sec. 2 and in Sec. 4.

2. The same simple poles for $p(u)$ at $u \to \infty$ for each sheet, given through the similar SU(4) charges by the formulas (2.50) and (4.5), if we redefine in (4.5) $r_k \to D r_k$.

3. The same set of equations (2.40) and (5.29) defining the symmetric part of $p(u)$ on the cuts and hence, the same $\mathcal{B}$-periods (2.61) of the cuts.

4. The same condition of zero $\mathcal{A}$-cycles.

5. The same conditions at $u \to 0$ $p_1(u)+p_2(u) = \frac{1}{u}+2\pi n_0+(u^0)$, which for the SYM means the cyclicity of the trace $U = 1$, or (2.43,2.44), and for the sigma model it follows from (4.3). The extra pole at zero $D/u \sim 1/u$ comes here from the poles at $u = \pm\sqrt{\lambda}/4\pi$ at $\lambda \to 0$ (at weak coupling the anomalous dimension behaves as $D = 1 + O(\lambda)$, see below).

These properties define the one loop algebraic curves and its relation to the physical data unambiguously, and consequently, they coincide.

The anomalous dimensions extracted from those two curves also coincide, as will be shown in Sec. 5.6.

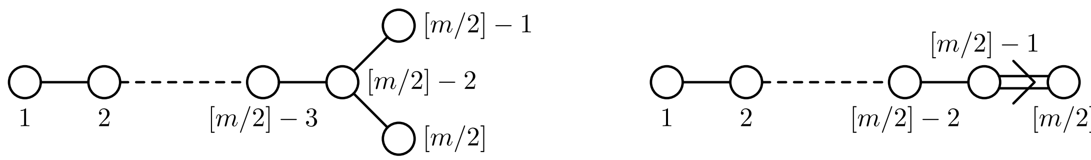

Figure 9: Dynkin diagram of SO(m) for even and odd m.

At two loops the full proof of the equivalence of two curves was only for the SU(2) sector [22], the only one where the two loop dilatation operator is actually calculated [31]. But we can borrow the idea of [48] where closeness of the SO(6) sector was demonstrated in all loops in the classical limit and the Bethe equation in the second loop was guessed. We can do the comparison of the curves at two loops along the same guidelines. In the next section it will be done using the Bethe equations.

As we also know [25], at three loops the curves do not match already in the SU(2) sector, most probably, due to yet unidentified non-perturbative corrections arising on the way from the weak to strong coupling.

5 Bethe Ansatz for the Sigma-Model on $\mathbb{R} \times S^{m-1}$

Having constructed the algebraic curve for the classical string on $\mathbb{R} \times S^5$ and having convinced ourselves that we have identified all relevant parameters, we proceed by constructing an integral representation of the curve (for all $\mathbb{R} \times S^{m-1}$). The obtained equations are similar to the Bethe equations for integrable spin chains in the thermodynamic limit which in fact form a Riemann-Hilbert problem. We finally compare the obtained equations to the one derived for gauge theory and find agreement up to two loops.

5.1 Simple Roots

To reveal the group theory structure of the equations, we will now introduce singular resolvents $\tilde{H}_k(x)$ which can be associated to the simple roots of $\mathfrak{so}(m)$. See Fig. 9 for the Dynkin diagram of the algebra and the labelling of the simple roots. They are related to the quasi-momenta $q_k(x)$ by

$$\tilde{H}_k = \sum_{j=1}^{k} q_j. \tag{5.1}$$

with the exceptions for the simple roots associated to spinors for even m

$$\tilde{H}_{[m/2]-1} = \sum_{j=1}^{[m/2]-1} \tfrac{1}{2} q_j - \tfrac{1}{2} q_{[m/2]}, \qquad \tilde{H}_{[m/2]} = \sum_{j=1}^{[m/2]-1} \tfrac{1}{2} q_j + \tfrac{1}{2} q_{[m/2]} \tag{5.2}$$

and for odd m

$$\tilde{H}_{[m/2]} = \sum_{j=1}^{[m/2]} \tfrac{1}{2} q_j. \tag{5.3}$$

Let us now collect the facts about the analytic properties of $\tilde{H}_k(x)$. First of all we know that the expansion at $x=\infty$ is related to the representation $[s_1, s_2, \ldots]$ of the state. Using the Cartan matrix M_{kj} of SO(m) (see App. E) it can be summarized as

$$\tilde{H}_k(x) = \frac{1}{x}\sum_{j=1}^{[m/2]} M^{-1}_{kj}\frac{4\pi s_j}{\sqrt{\lambda}} + \mathcal{O}(1/x^2). \tag{5.4}$$

Secondly, we have derived the singular behavior at $x = \pm 1$ [19]

$$\tilde{H}_k(x) = \frac{1}{x \mp 1}\,\frac{2\pi M^{-1}_{k1} D}{\sqrt{\lambda}} + \mathcal{O}\big(1/(x\mp 1)^0\big). \tag{5.5}$$

Finally, we will use the assumption (3.55) for the symmetry of $\tilde{H}_k(x)$ under the map $x \mapsto 1/x$ [20]

$$\tilde{H}_k(1/x) = \tilde{H}_k(x) - 2M^{-1}_{k1}\tilde{H}_1(x) + 4\pi n_0 M^{-1}_{k1}. \tag{5.6}$$

5.2 Integral Representation

We make properties (5.5) and (5.6) manifest by defining

$$\begin{aligned}\tilde{H}_k(x) &= H_k(x) + H_k(1/x) - 2M^{-1}_{k1}H_1(1/x)\\ &\quad + \frac{1}{x-1/x}\,\frac{4\pi D}{\sqrt{\lambda}}\,M^{-1}_{k1} + c_k - c_1 M^{-1}_{k1} + 2\pi n_0 M^{-1}_{k1},\end{aligned} \tag{5.7}$$

where the c_k's are a set of constants. The resolvents $H_k(x)$ are assumed to be analytic except at a collection of branch cuts $\mathcal{C}_k$ and approach zero at $x=\infty$. Note that this representation of $\tilde{H}_k(x)$ is *ambiguous*. We can add to H_k an antisymmetric function

$$H_k(x) \to H_k(x) + f_k(x) - f_k(1/x) + 2M^{-1}_{k1}f_1(1/x), \tag{5.8}$$

this has no effect on the physical function $\tilde{H}_k(x)$.

Let us introduce the density $\rho_k(x)$ which describes the discontinuity across a cut

$$\rho_k(x) = \frac{1-1/x^2}{2\pi i}\big(H_k(x-i\epsilon) - H_k(x+i\epsilon)\big) \qquad \text{for } x \in \mathcal{C}_k. \tag{5.9}$$

The factor of $1-1/x^2$ was introduced for later convenience and will allow the interpretation of $\rho_k(x)$ as a density. The apparent pole at $x=0$ is irrelevant as long as the cuts do not cross this point. We could also demand positivity of the density, $dx\,\rho_k(x) > 0$. This would fix the position of the cuts $\mathcal{C}_k$ in the complex plane, but will not be essential for the treatment of the classical sigma model. From $\rho_k(x)$ we can reconstruct the function $H_k(x)$

$$H_k(x) = \int_{\mathcal{C}_k} \frac{dy\,\rho_k(y)}{1-1/y^2}\,\frac{1}{y-x}\,. \tag{5.10}$$

[19] We could also write M^{-1}_{k1} as $\sum_j M^{-1}_{kj}V^{\mathrm{v}}_j$, where $V^{\mathrm{v}}_j = (1,0,0,\ldots)$ are the Dynkin labels of the vector representation.

[20] Again, $\tilde{H}_1 = \sum_j \tilde{H}_j V^{\mathrm{v}}_j$ could be written in a more 'covariant' way.

5.3 Asymptotic Behavior

We should now relate the SO(m) representation of a state to the cuts and densities. For that purpose, we note the expansion of the resolvents $H_k(x)$ at $x = \infty$

$$H_k(x) = -\frac{1}{x}\big(K_k + H_k'(0)\big) + \mathcal{O}(1/x^2) \tag{5.11}$$

and $x = 0$

$$H_k(x) = H_k(0) + x\,H_k'(0) + \mathcal{O}(x^2) \tag{5.12}$$

where we have defined the normalizations or filling fractions of the densities

$$K_k = \int_{\mathcal{C}_k} dy\,\rho_k(y). \tag{5.13}$$

For $\tilde{H}_k(x)$ we find the asymptotic behavior at $x \to \infty$

$$\begin{aligned}\tilde{H}_k(x) = {}& c_k - c_1 M_{k1}^{-1} + 2\pi n_0 M_{k1}^{-1} + H_k(0) - 2M_{k1}^{-1} H_1(0) \\ & + \frac{1}{x}\left(\frac{4\pi D}{\sqrt{\lambda}}\,M_{k1}^{-1} - 2H_1'(0)M_{k1}^{-1} - K_k\right) + \mathcal{O}(1/x^2)\,,\end{aligned} \tag{5.14}$$

We compare this to (5.4) and find the relation between fillings and Dynkin labels

$$K_k = M_{k1}^{-1}\left(\frac{4\pi D}{\sqrt{\lambda}} - 2H_1'(0)\right) - \sum_{j=1}^{[m/2]} M_{kj}^{-1}\frac{4\pi s_j}{\sqrt{\lambda}}. \tag{5.15}$$

as well as the constants

$$c_k = c_1 M_{k1}^{-1} - 2\pi n_0 M_{k1}^{-1} - H_k(0) + 2M_{k1}^{-1} H_1(0). \tag{5.16}$$

In fact, this equation for $k = 1$ cannot be solved for c_1, it drops out. This leads to an additional condition for the resolvents, the momentum constraint

$$H_1(0) = 2\pi n_0. \tag{5.17}$$

whereas c_1 is not fixed. When substituting the constants into (5.7) we obtain

$$\begin{aligned}\tilde{H}_k(x) = {}& H_k(x) + H_k(1/x) - H_k(0) \\ & + M_{k1}^{-1}\left(-2H_1(1/x) + 2H_1(0) + \frac{1}{x - 1/x}\,\frac{4\pi D}{\sqrt{\lambda}}\right).\end{aligned} \tag{5.18}$$

Now the expansion of the functions $\tilde{H}_k$ is fixed at the points $x = \infty$ and $x = 0$ and it turns out that the the ambiguity (5.8) must not modify $H_k(x)$ at $x = \infty$.[21]

[21] $H_k(x)$ was assumed to be zero at $x = \infty$ anyway.

5.4 Bethe Equations

The singular resolvents $\tilde{H}_k$ now satisfy the desired symmetries and expansions at specific points. They however have branch cuts along the curves $\mathcal{C}_k$. These must not be seen in the transfer matrices, which are analytic except at the special points. This leads us to the Bethe equations, which are manifestations of the analyticity conditions for the monodromy matrix, see Sec. 3.5

$$\sum_{j=1}^{[m/2]} M_{kj} \not{\tilde{H}}_j(x) = 2\pi n_{k,a}, \qquad \text{for } x \in \mathcal{C}_{k,a}. \tag{5.19}$$

As explained in Sec. 2.4,3.5, they ensure that across a cut only the labelling of sheets and the branch of the logarithm changes. The slash through a resolvent implies a principal part prescription,

$$\not{H}_k(x) = \tfrac{1}{2} H_k(x + i\epsilon) + \tfrac{1}{2} H_k(x - i\epsilon). \tag{5.20}$$

Here we have split up the curves $\mathcal{C}_k$ into their connected components $\mathcal{C}_{k,a}$. For each connected curve we have introduced a mode number $n_{k,a}$ due to the allowed shift by multiples of $2\pi i$ in the exponent. When we substitute (5.18) the Bethe equations read

$$\begin{aligned} 2\pi n_{k,a} = {} & \sum_{j=1}^{[m/2]} M_{kj} \big(\not{H}_j(x) + H_j(1/x) - H_j(0) \big) \\ & + \delta_{k1} \left(-2H_1(1/x) + 2H_1(0) + \frac{1}{x - 1/x} \frac{4\pi D}{\sqrt{\lambda}} \right), \quad \text{for } x \in \mathcal{C}_{k,a}. \end{aligned} \tag{5.21}$$

Note the explicit appearance of the dimension/energy D which constitutes the physical quantity of main interest. For a given set of mode numbers $n_{k,a}$ and filling fractions

$$K_{k,a} = \int_{\mathcal{C}_{k,a}} dy \, \rho_k(y), \tag{5.22}$$

the equations (5.21) should only have a solution if D has the appropriate value. The Bethe equations of the spin chain in Sec. 2.4 are qualitatively different: They should always be soluble and the dimension is subsequently read off from (2.43,2.45).

It is useful to go to the u-plane which is related to the x-plane by [26]

$$x(u) = \tfrac{1}{2} u + \tfrac{1}{2} \sqrt{u^2 - 4}\,, \qquad u(x) = x + 1/x. \tag{5.23}$$

We can introduce a resolvent in the u-plane by

$$\bar{H}_k(u) = \int \frac{dy \, \rho_k(y)}{y + 1/y - u} = \int \frac{dv \, \rho_k(v)}{v - u}\,. \tag{5.24}$$

Note that $\rho_k(x)$ transforms as a density, i.e. $dx \, \rho_k(x) = du \, \rho_k(u)$. It is related to a symmetric combination of the resolvents in the x-space

$$H_k(x) + H_k(1/x) = \bar{H}_k(x + 1/x) + H_k(0). \tag{5.25}$$

This allows us to write the Bethe equations in the u-plane

$$\sum_{j=1}^{[m/2]} M_{kj} \not{\bar{H}}_j(u) + \delta_{k1} F_{\text{string}}(u) = 2\pi n_{k,a} \qquad \text{for } u \in \bar{\mathcal{C}}_{k,a} \tag{5.26}$$

with

$$F_{\text{string}}(u) = \frac{1}{\sqrt{u^2-4}} \frac{4\pi D}{\sqrt{\lambda}} + 2H_1(0) - 2H_1\big(1/x(u)\big). \tag{5.27}$$

5.5 The Sigma-Model on $\mathbb{R} \times S^5$

Let us now apply our results to the case $m = 6$, i.e. the sigma model on $\mathbb{R} \times S^5$. Here we shall adopt a SU(4) notation instead of the one for SO(6). The benefit of SU(4) is that it is manifestly a subgroup of SU(2, 2|4), the full supergroup of the superstring on $AdS_5 \times S^5$. The change merely amounts to swapping the labels of the first two simple roots, c.f. Fig. 7. We introduce the singular resolvents $\tilde{G}_k(x)$ corresponding to the simple roots of SU(4) by

$$\begin{aligned} \tilde{G}_1(x) &= \tilde{H}_2(x), \\ \tilde{G}_2(x) &= \tilde{H}_1(x), \\ \tilde{G}_3(x) &= \tilde{H}_3(x). \end{aligned} \tag{5.28}$$

We also interchange labels 1, 2 for the densities ρ_k and fillings K_k.

The Bethe equations are written as

$$\begin{aligned} 2\not{\tilde{G}}_1(x) - \tilde{G}_2(x) &= \not{p}_1(x) - \not{p}_2(x) = 2\pi n_{1,a}, \qquad x \in \mathcal{C}_{1,a}, \\ 2\not{\tilde{G}}_2(x) - \tilde{G}_1(x) - \tilde{G}_3(x) &= \not{p}_2(x) - \not{p}_3(x) = 2\pi n_{2,a}, \qquad x \in \mathcal{C}_{2,a}, \\ 2\not{\tilde{G}}_3(x) - \tilde{G}_2(x) &= \not{p}_3(x) - \not{p}_4(x) = 2\pi n_{3,a}, \qquad x \in \mathcal{C}_{3,a}. \end{aligned} \tag{5.29}$$

Now one can draw the Riemann sheets picture as in Section 2.4. The Riemann surface consists of four sheets each of which corresponds to p_k while resolvents $\tilde{G}_k$ describe how to connect the sheets with cuts. The main difference is that due to the inversion symmetry all the cuts appear in pairs as depicted in Fig. 6.

5.6 Comparison to Gauge Theory

Let us now consider the limit where the Dynkin labels r_k and the dimension D are large with respect to $\sqrt{\lambda}$. For this purpose we rescale x, u, K_k according to

$$\{x, u, K_k\} \to \frac{4\pi L}{\sqrt{\lambda}} \{x, u, K_k\}, \qquad \{D, r_k\} \to L\{D, r_k\} \tag{5.30}$$

while keeping $\rho_k(x), G_k(x)$ fixed. Here L is defined to be the limiting value of D (before rescaling) at $\lambda = 0$ corresponding to the classical dimension in gauge theory.[22] For

[22]The 'length' L was conjectured to be an action variable in [62]. If true, it would be interesting to identify it in the framework of the monodromy matrix. See also [21] on the definition of 'length' in the coherent approach.

convenience, we define the effective coupling constant g as

$$g^2 = \frac{\lambda}{8\pi^2 L^2} = \frac{g_{\mathrm{YM}}^2 N}{8\pi^2 L^2}\,. \tag{5.31}$$

The Bethe equations (5.26) are left invariant, but the function $F_{\mathrm{string}}(u)$ changes to

$$F_{\mathrm{string}}(u) = \frac{D}{\sqrt{u^2-2g^2}} + 2G_2(0) - 2G_2\big(g^2/2x(u)\big) \tag{5.32}$$

where now $x(u) = \frac{1}{2}u + \frac{1}{2}\sqrt{u^2-2g^2}$. Due to the inversion symmetry $x \to g^2/2x$, the resolvents $G_k(x)$ contain unphysical information which is projected out in $\tilde{G}_k(x)$, see the discussion around (5.8). By expanding $\tilde{G}_k(x)$ up to first order around $x=0$ we find that $G_2'(0)$ is not physical on its own and we can set it to an arbitrary value. We fix a gauge by setting

$$D = 1 + g^2 G_2'(0) = 1 + g^2 \int_{\mathcal{C}_2} \frac{dy\, \rho_2(y)}{1-g^2/2y^2}\,\frac{1}{y^2}\,. \tag{5.33}$$

This has two favorable effects: Firstly, (5.33) resembles the gauge theory relation (2.1) where the length is assumed to be rescaled to $L=1$ in the thermodynamic limit. Secondly, the relation of the charges and filling fractions (5.15) also reduces to the gauge theory analog (2.21)

$$K_k = \sum_{j=1}^{3} M_{kj}^{-1}(\delta_{j2} - r_j). \tag{5.34}$$

When we substitute and expand (5.32) we obtain[23]

$$\begin{aligned} F_{\mathrm{string}}(u) &= \frac{1}{\sqrt{u^2-2g^2}} + \left(2G_2(0) + \frac{g^2 G_2'(0)}{\sqrt{u^2-2g^2}} - 2G_2\big(g^2/2x(u)\big)\right) \\ &= F_{\mathrm{gauge}}(u) + \mathcal{O}(g^4). \end{aligned} \tag{5.35}$$

This means that the functions $F_{\mathrm{string}}(u)$ and $F_{\mathrm{gauge}}(u)$ agree up to and including order g^2 corresponding to two loops for the scaling dimension D in the gauge fixed by (5.33). We have thus demonstrated the generic two-loop matching of scaling dimensions in gauge theory and energies of spinning strings in the SO(6) sector.[24] This complies with the one-loop results of [21] in the coherent state approach to spinning strings [20] and also with the matching of integrable charges in special cases [63].

[23] This equation along with the generic form of the Bethe equations (5.26) was proposed independently by M. Staudacher [52]. He also showed that the solutions discussed in Sec. 2.6 for this deformation of the equations yield precisely the energies computed from the string equations of motion [14, 10, 48, 21]. We would like to thank him for insightful discussions.

[24] In both theories we have focussed on the low-lying modes of the spectrum. States which have no expansion in the effective coupling g are disregarded.

6 Discussion

In this work we continued the investigation of integrability of the multicolor $\mathcal{N} = 4$ SYM theory and its close relation (and hopefully equivalence) to the $AdS_5 \times S^5$ string sigma model. The general solutions of the one-loop SYM theory and of the classical sigma model, constructed here in the $\mathbb{R} \times S^5$ or $\mathfrak{so}(6)$ sector, give, as expected, the same result in the weakly coupled region of the classical, BMN limit. Elsewhere, the algebraic curves of the two models appear to have a very similar structure and differ only in the details. We hope that a quantized version of the sigma model will reproduce the known SYM perturbative data precisely and give in addition the complete non-perturbative information on the gauge theory, compatible with these perturbative data.

We see the most natural way to prove this complete AdS/CFT duality in construction of the full algebraic curve of the model, with the following quantization based on this curve. Often the quantization means an appropriate discretization of the model and the curve gives a hint on the right procedure. For example, the matrix model with a finite size N of a matrix, is often completely defined by its large N algebraic curve and can be considered as its quantum counterpart. The other example is the discrete Bethe ansatz equations, as those considered here, which provide the right quantization of the classical algebraic curve. This procedure of quantization is carried out in one-loop here. There were recently some interesting attempts to find the quantum version of the $AdS_5 \times S^5$ sigma model [43], though it is too early to claim that we are close to the whole resolution of this formidable problem.

As a next important step in this program we would consider the generalization of the present construction, to the algebraic curve of the one-loop SYM theory for the full dilatation Hamiltonian of [28], on the one hand and of the full $AdS_5 \times S^5$ classical sigma model, on the other hand. The present paper, together with [23, 22], provides most of the necessary technique for the completion of this task.

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A Vector Spin Chains of $\mathfrak{so}(m)$

In this appendix we present the generalizations of several expression of Sec. 2 to the case of $\mathfrak{su}(m)$. The R-matrix of two vectors ($\mathbf{V}$) [53] or of a vector and a spinor ($\mathbf{S}$) are given by

$$\begin{aligned}\mathcal{R}^{\mathbf{V},\mathbf{V}}(u) &= \mathcal{P}^{\mathbf{T}} + \frac{u-i}{u+i}\,\mathcal{P}^{\mathbf{A}} + \frac{(u-i)(u-\frac{i}{2}m+i)}{(u+i)(u+\frac{i}{2}m-i)}\,\mathcal{P}^{\mathbf{1}} \\ &= \frac{i}{u+i}\,\mathcal{S} + \frac{u}{u+i}\,\mathcal{I} - \frac{iu}{(u+i)(u+\frac{i}{2}m-i)}\,\mathcal{K}^{\mathbf{V},\mathbf{V}} \end{aligned} \tag{A.1}$$

and

$$\mathcal{R}^{\mathbf{S},\mathbf{V}}(u) = \mathcal{P}^{\mathbf{VS}} + \frac{u-\frac{i}{4}m}{u+\frac{i}{4}m}\,\mathcal{P}^{\mathbf{S}} = \mathcal{I} + \frac{u-\frac{i}{4}m}{\frac{i}{2}}\,\mathcal{K}^{\mathbf{S},\mathbf{V}}. \tag{A.2}$$

The operators $\mathcal{P}^{\mathbf{T}}, \mathcal{P}^{\mathbf{A}}, \mathcal{P}^{\mathbf{1}}$ project to the symmetric-traceless ($\mathbf{T}$), antisymmetric/adjoint ($\mathbf{A}$) and singlet ($\mathbf{1}$) modules which appear in the tensor product of two vectors, while $\mathcal{P}^{\mathbf{VS}}, \mathcal{P}^{\mathbf{S}}$ project to the traceless vector-spinor ($\mathbf{VS}$) and the spinor ($\mathbf{S}$) in the product of a vector and a spinor. These can be written using the operators $\mathcal{I}, \mathcal{S}, \mathcal{K}^{\mathbf{V},\mathbf{V}}, \mathcal{K}^{\mathbf{V},\mathbf{S}}$ which are the identity, permutation and trace operators, respectively

$$\mathcal{P}^{\mathbf{T}} = \tfrac{1}{2}\mathcal{I} + \tfrac{1}{2}\mathcal{S} - \tfrac{1}{m}\,\mathcal{K}^{\mathbf{V},\mathbf{V}}, \qquad \mathcal{P}^{\mathbf{A}} = \tfrac{1}{2}\mathcal{I} - \tfrac{1}{2}\mathcal{S}, \qquad \mathcal{P}^{\mathbf{1}} = \tfrac{1}{m}\,\mathcal{K}^{\mathbf{V},\mathbf{V}} \tag{A.3}$$

and

$$\mathcal{P}^{\mathbf{VS}} = \mathcal{I} - \tfrac{1}{m}\,\mathcal{K}^{\mathbf{S},\mathbf{V}}, \qquad \mathcal{P}^{\mathbf{S}} = \tfrac{1}{m}\,\mathcal{K}^{\mathbf{S},\mathbf{V}} \tag{A.4}$$

with $(\mathcal{K}^{\mathbf{V},\mathbf{V}})^{ij}{}_{kl} = \delta^{ij}\delta_{kl}$ and $(\mathcal{K}^{\mathbf{S},\mathbf{V}})^{\beta j}{}_{\alpha i} = (\gamma^j\gamma_i)^{\beta}{}_{\alpha}$.

For the monodromy and transfer matrices it is convenient to define

$$\begin{aligned}\mathbf{\Omega}^{\mathbf{V}}_a(u) &= \frac{(u-\frac{i}{4}m+\frac{3i}{2})^L(u+\frac{i}{4}m-\frac{i}{2})^L}{u^{2L}}\,\mathcal{R}^{\mathbf{V},\mathbf{V}}_{a1}(u-\tfrac{i}{4}m+\tfrac{i}{2})\cdots\mathcal{R}^{\mathbf{V},\mathbf{V}}_{aL}(u-\tfrac{i}{4}m+\tfrac{i}{2}), \\ \mathbf{\Omega}^{\mathbf{S}}_a(u) &= \frac{(u+\frac{i}{2})^L}{u^L}\,\mathcal{R}^{\mathbf{S},\mathbf{V}}_{a1}(u-\tfrac{i}{4}m+\tfrac{i}{2})\mathcal{R}^{\mathbf{S},\mathbf{V}}_{a2}(u-\tfrac{i}{4}m+\tfrac{i}{2})\cdots\mathcal{R}^{\mathbf{S},\mathbf{V}}_{aL}(u-\tfrac{i}{4}m+\tfrac{i}{2}). \end{aligned} \tag{A.5}$$

Then the transfer matrices for the Bethe ansatz have a rather symmetric form.

B R-Matrices

In this appendix we present the R-matrices between two spinors of $\mathfrak{so}(6)$. Together with the R-matrices in Sec. 2.1 this is a complete set for the representations $\mathbf{6}, \mathbf{4}, \bar{\mathbf{4}}$. For instance one can now explicitly prove the YBE (2.5) for all combinations of these representations. The R-matrices of two fundamentals are

$$\begin{aligned}\mathcal{R}^{\mathbf{4},\mathbf{4}}(u) &= \mathcal{P}^{\mathbf{10}} + \frac{u-\frac{i}{2}}{u+\frac{i}{2}}\,\mathcal{P}^{\mathbf{6}} = \frac{\frac{i}{2}}{u+\frac{i}{2}}\,\mathcal{S} + \frac{u}{u+\frac{i}{2}}\,\mathcal{I} = \frac{u+\frac{i}{4}}{u+\frac{i}{2}}\,\mathcal{I} + \frac{i}{u+\frac{i}{2}}\,\mathcal{J}^{\mathbf{4},\mathbf{4}}, \\ \mathcal{R}^{\bar{\mathbf{4}},\bar{\mathbf{4}}}(u) &= \mathcal{P}^{\overline{\mathbf{10}}} + \frac{u-\frac{i}{2}}{u+\frac{i}{2}}\,\mathcal{P}^{\mathbf{6}} = \frac{\frac{i}{2}}{u+\frac{i}{2}}\,\mathcal{S} + \frac{u}{u+\frac{i}{2}}\,\mathcal{I} = \frac{u+\frac{i}{4}}{u+\frac{i}{2}}\,\mathcal{I} + \frac{i}{u+\frac{i}{2}}\,\mathcal{J}^{\bar{\mathbf{4}},\bar{\mathbf{4}}}, \\ \mathcal{R}^{\mathbf{4},\bar{\mathbf{4}}}(u) &= \mathcal{P}^{\mathbf{15}} + \frac{u-2i}{u+2i}\,\mathcal{P}^{\mathbf{1}} = \mathcal{I} - \frac{i}{u+2i}\,\mathcal{K}^{\mathbf{4},\bar{\mathbf{4}}} = \frac{u+\frac{7i}{4}}{u+2i}\mathcal{I} + \frac{i}{u+2i}\mathcal{J}^{\mathbf{4},\bar{\mathbf{4}}}. \end{aligned} \tag{B.1}$$

The projectors for $\mathbf{4}\times\mathbf{4}=\mathbf{10}+\mathbf{6}$ are given by

$$\mathcal{P}^{\mathbf{10}}=\tfrac{1}{2}\mathcal{I}+\tfrac{1}{2}\mathcal{S},\qquad \mathcal{P}^{\mathbf{6}}=\tfrac{1}{2}\mathcal{I}-\tfrac{1}{2}\mathcal{S} \tag{B.2}$$

while for $\bar{\mathbf{4}}\times\bar{\mathbf{4}}=\overline{\mathbf{10}}+\mathbf{6}$ one finds essentially the same expressions

$$\mathcal{P}^{\bar{\mathbf{10}}}=\tfrac{1}{2}\mathcal{I}+\tfrac{1}{2}\mathcal{S},\qquad \mathcal{P}^{\mathbf{6}}=\tfrac{1}{2}\mathcal{I}-\tfrac{1}{2}\mathcal{S}. \tag{B.3}$$

For the mixed product $\mathbf{4}\times\bar{\mathbf{4}}=\mathbf{15}+\mathbf{1}$ we get

$$\mathcal{P}^{\mathbf{15}}=\mathcal{I}-\tfrac{1}{4}\mathcal{K}^{\mathbf{4},\bar{\mathbf{4}}}\mathcal{P}^{\mathbf{1}}=\tfrac{1}{4}\mathcal{K}^{\mathbf{4},\bar{\mathbf{4}}} \tag{B.4}$$

with $(\mathcal{K}^{\mathbf{4},\bar{\mathbf{4}}})^{\beta\dot{\alpha}}{}_{\delta\dot{\gamma}}=\delta^{\beta\dot{\alpha}}\delta_{\delta\dot{\gamma}}$. The rotation generators can be written as follows

$$\mathcal{J}^{\mathbf{4},\mathbf{4}}=\mathcal{S}-\tfrac{1}{4}\mathcal{I},\qquad \mathcal{J}^{\bar{\mathbf{4}},\bar{\mathbf{4}}}=\mathcal{S}-\tfrac{1}{4}\mathcal{I},\qquad \mathcal{J}^{\mathbf{4},\bar{\mathbf{4}}}=\tfrac{1}{4}\mathcal{I}-\mathcal{K}^{\mathbf{4},\bar{\mathbf{4}}}. \tag{B.5}$$

C Antisymmetric Transfer Matrices of $\mathfrak{su}(4)$

There is a nice formula to obtain expressions for the transfer matrices for the Bethe ansatz in all totally antisymmetric products of the fundamental representation of $\mathfrak{su}(m)$, see [50]. In this appendix we shall present it for our case of interest, $\mathfrak{su}(4)$. It allows us to obtain the transfer matrices in $\mathbf{4},\mathbf{6},\bar{\mathbf{4}}$. It is based on a differential operator $\mathit{\Psi}_u^{\mathbf{4}}$

$$\begin{aligned}\mathit{\Psi}_u^{\mathbf{4}}=&\left(\exp(i\partial_u)-\frac{R_1(u)}{R_1(u+i)}\,\frac{V(u+2i)}{V(u+\frac{3i}{2})}\right)\\ &\cdot\left(\exp(i\partial_u)-\frac{R_2(u-\frac{i}{2})}{R_2(u+\frac{i}{2})}\,\frac{R_1(u+i)}{R_1(u)}\,\frac{V(u+i)}{V(u+\frac{i}{2})}\right)\\ &\cdot\left(\exp(i\partial_u)-\frac{R_3(u-i)}{R_3(u)}\,\frac{R_2(u+\frac{i}{2})}{R_2(u-\frac{i}{2})}\,\frac{V(u-i)}{V(u-\frac{i}{2})}\right)\\ &\cdot\left(\exp(i\partial_u)-\frac{R_3(u)}{R_3(u-i)}\,\frac{V(u-2i)}{V(u-\frac{3i}{2})}\right).\end{aligned} \tag{C.1}$$

The operator is slightly modified from [50] to accommodate for a non-fundamental spin representation and a different normalization of rapidities. When this operator is expanded in powers of $\exp(i\partial_u)$, which shifts u by i, it should yield

$$\begin{aligned}\mathit{\Psi}_u^{\mathbf{4}}=&\frac{V(u+2i)V(u+i)V(u-i)V(u-2i)}{V(u+\frac{3i}{2})V(u+\frac{i}{2})V(u-\frac{i}{2})V(u-\frac{3i}{2})}\\ &-\exp(\tfrac{i}{2}\partial_u)\,\frac{V(u+\frac{3i}{2})}{V(u+i)}\,\frac{V(u-\frac{3i}{2})}{V(u-i)}\,T_{\bar{\mathbf{4}}}(u)\,\exp(\tfrac{i}{2}\partial_u)\\ &+\exp(i\partial_u)\,\frac{V(u)}{V(u+\frac{i}{2})}\,\frac{V(u)}{V(u-\frac{i}{2})}\,T_{\mathbf{6}}(u)\,\exp(i\partial_u)\\ &-\exp(\tfrac{3i}{2}\partial_u)\,T_{\mathbf{4}}(u)\,\exp(\tfrac{3i}{2}\partial_u)\\ &+\exp(4i\partial_u).\end{aligned} \tag{C.2}$$

Here we can read off the expressions for $\mathbf{T_R}(u)$, they agree with (2.22,2.24,2.25). Alternatively one can use the conjugate operator

$$\begin{aligned}
\Psi_u^{\bar{\mathbf{4}}} = &\left(\exp(i\partial_u) - \frac{R_3(u)}{R_3(u+i)}\,\frac{V(u+2i)}{V(u+\frac{3i}{2})}\right)\\
&\cdot\left(\exp(i\partial_u) - \frac{R_2(u-\frac{i}{2})}{R_2(u+\frac{i}{2})}\,\frac{R_3(u+i)}{R_3(u)}\,\frac{V(u+i)}{V(u+\frac{i}{2})}\right)\\
&\cdot\left(\exp(i\partial_u) - \frac{R_1(u-i)}{R_1(u)}\,\frac{R_2(u+\frac{i}{2})}{R_2(u-\frac{i}{2})}\,\frac{V(u-i)}{V(u-\frac{i}{2})}\right)\\
&\cdot\left(\exp(i\partial_u) - \frac{R_1(u)}{R_1(u-i)}\,\frac{V(u-2i)}{V(u-\frac{3i}{2})}\right)
\end{aligned}\tag{C.3}$$

which expands as follows

$$\begin{aligned}
\Psi_u^{\bar{\mathbf{4}}} = &\frac{V(u+2i)V(u+i)V(u-i)V(u-2i)}{V(u+\frac{3i}{2})V(u+\frac{i}{2})V(u-\frac{i}{2})V(u-\frac{3i}{2})}\\
&- \exp(\tfrac{i}{2}\partial_u)\,\frac{V(u+\frac{3i}{2})}{V(u+i)}\,\frac{V(u-\frac{3i}{2})}{V(u-i)}\,T_{\mathbf{4}}(u)\,\exp(\tfrac{i}{2}\partial_u)\\
&+ \exp(i\partial_u)\,\frac{V(u)}{V(u+\frac{i}{2})}\,\frac{V(u)}{V(u-\frac{i}{2})}\,T_{\mathbf{6}}(u)\,\exp(i\partial_u)\\
&- \exp(\tfrac{3i}{2}\partial_u)\,T_{\bar{\mathbf{4}}}(u)\,\exp(\tfrac{3i}{2}\partial_u) + \exp(4i\partial_u)
\end{aligned}\tag{C.4}$$

In the thermodynamic limit, see Sec. 2.3, we find the following limits for the operators

$$\begin{aligned}
\Psi_u^{\mathbf{4}} \to &\big(\exp(i\partial_u/L) - \exp(ip_1)\big)\\
&\cdot\big(\exp(i\partial_u/L) - \exp(ip_2)\big)\\
&\cdot\big(\exp(i\partial_u/L) - \exp(ip_3)\big)\\
&\cdot\big(\exp(i\partial_u/L) - \exp(ip_4)\big)
\end{aligned}\tag{C.5}$$

and the conjugate one

$$\begin{aligned}
\Psi_u^{\bar{\mathbf{4}}} \to &\big(\exp(i\partial_u/L) - \exp(-ip_4)\big)\\
&\cdot\big(\exp(i\partial_u/L) - \exp(-ip_3)\big)\\
&\cdot\big(\exp(i\partial_u/L) - \exp(-ip_2)\big)\\
&\cdot\big(\exp(i\partial_u/L) - \exp(-ip_1)\big).
\end{aligned}\tag{C.6}$$

D Higher Charges of the Sigma Model

Here we shall continue the expansion of q_k at the singular points $x = \pm 1$ to higher orders. Note, first of all, that the straight perturbative diagonalization fails if there are degenerate eigenvalues. In the case at hand the eigenvalue zero is indeed degenerate. However, the zero subspace can be decoupled completely from the non-zero eigenvectors in perturbation theory. This is not a problem, because the zero subspace does not display singular behavior by definition; there the monodromy matrix behaves as for any other point x and we cannot expect to be able to find a simple expression for $q_k(x)$, $k \neq 1$ at $x = \pm 1$. For $q_1(x)$ the situation is different, it starts with a pole whose residue is non-degenerate. We would now like to find a solution of the equation

$$\mathcal{L}(x,\sigma)\vec{V}(x,\sigma) = i f(x,\sigma)\vec{V}(x,\sigma) \tag{D.1}$$

such that the eigenvalue $f(x,\sigma)$ is singular at $x = +1$. As explained around (3.45), the leading-order eigenvector $\vec{V}(x,\sigma)$ is an eigenvector of $j_+(\sigma)$. It therefore must be a linear combination of $\vec{X}$ and $\vec{X}_+ := (\sqrt{\lambda}/D)\partial_+\vec{X}$, we find $\vec{X} - \frac{i}{2}\vec{X}_+$ with eigenvalue $f(x,\sigma) = D/\sqrt{\lambda}(x-1) + \cdots$. When we substitute this in the above equation for $\mathcal{L}$ we can solve for the subleading terms. In the first few orders we find the eigenstate

$$\begin{aligned}\vec{V}(x) = {} & \big(\vec{X} - i\vec{X}_+\big) + \big(-\tfrac{i}{2}\vec{X}_+ - \tfrac{1}{2}\vec{X}_{++}\big)(x-1) \\ & + \big(+\tfrac{i}{8}\vec{X}_+ + \tfrac{i}{4}\vec{X}_{3+} - \tfrac{1}{8}(\vec{X}\cdot\vec{X}_{4+})\vec{X} + \tfrac{i}{4}(\vec{X}\cdot\vec{X}_{4+})\vec{X}_+\big)(x-1)^2 \\ & + \big(-\tfrac{i}{16}\vec{X}_+ - \tfrac{i}{8}\vec{X}_{3+} + \tfrac{1}{8}\vec{X}_{4+} + \tfrac{3}{16}(\vec{X}\cdot\vec{X}_{4+})\vec{X}_{++} + \tfrac{1}{20}(\vec{X}\cdot\vec{X}_{5+})\vec{X}_+\big)(x-1)^3 \\ & + \mathcal{O}\big((x-1)^4\big)\end{aligned} \tag{D.2}$$

where $\vec{X}_{n+}$ is defined as $\vec{X}_{n+} := (\sqrt{\lambda}/D)^n\partial_+^n\vec{X}$. The value $q_1(x)$ is now given as the integral of $f(x,\sigma)$ over the closed string

$$\begin{aligned} q_1(x) &= \int_0^{2\pi} d\sigma\, f(x,\sigma) \\ &= \frac{D}{\sqrt{\lambda}}\int_0^{2\pi} d\sigma \bigg[\frac{1}{x-1} + \tfrac{1}{2} + (x-1)\big(-\tfrac{1}{8} + \tfrac{1}{4}\vec{X}\cdot\vec{X}_{+-} + \tfrac{1}{8}\vec{X}\cdot\vec{X}_{4+}\big) \\ &\qquad + (x-1)^2\big(\tfrac{1}{16} - \tfrac{1}{8}\vec{X}\cdot\vec{X}_{+-} - \tfrac{1}{16}\vec{X}\cdot\vec{X}_{4+}\big) + \mathcal{O}\big((x-1)^3\big)\bigg].\end{aligned} \tag{D.3}$$

We make use of the following two identities

$$\operatorname{Tr} j_{\mathbf{v},+} j_{\mathbf{v},-} = \frac{8D^2}{\lambda}\,\vec{X}\cdot\vec{X}_{+-}, \qquad \operatorname{Tr}\partial_\pm j_{\mathbf{v},\pm}\,\partial_\pm j_{\mathbf{v},\pm} = \frac{8D^4}{\lambda^2}\big(1 - \vec{X}\cdot\vec{X}_{4\pm}\big) \tag{D.4}$$

to express the expansion of $q_1(x)$ in terms of the currents $j_{\mathbf{v}}$

$$q_1(x) = \frac{2\pi D}{\sqrt{\lambda}\,(x\mp 1)} \pm \frac{\pi D}{\sqrt{\lambda}} + (x\mp 1)\frac{\lambda^{3/2}\,Q_\pm}{64D^3} \mp (x\mp 1)^2\frac{\lambda^{3/2}\,Q_\pm}{128D^3} + \mathcal{O}\big((x\mp 1)^3\big). \tag{D.5}$$

with

$$Q_\pm = \int_0^{2\pi} d\sigma\,\left(\frac{2D^2}{\lambda}\operatorname{Tr} j_{\mathbf{v},+} j_{\mathbf{v},-} - \operatorname{Tr}\partial_\pm j_{\mathbf{v},\pm}\,\partial_\pm j_{\mathbf{v},\pm}\right). \tag{D.6}$$

Here we have also included the expansion around $x = -1$. The charges $Q_\pm$ are the first two non-trivial local commuting charges of the sigma model.

E Cartan Matrices

The Cartan metric for $\mathfrak{so}(m)$ are given by The metric M for the space of simple roots is given by

$$M_{jk} = \begin{pmatrix} +2 & -1 & & & & \\ -1 & \ddots & \ddots & & & \\ & \ddots & \ddots & -1 & & \\ & & -1 & +2 & -1 & -1 \\ & & & -1 & +2 & \\ & & & -1 & & +2 \end{pmatrix}, \qquad \text{for } m \text{ even} \tag{E.1}$$

and by

$$M_{jk} = \begin{pmatrix} +2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & -1 & +2 & -2 \\ & & & -2 & +4 \end{pmatrix}, \qquad \text{for } m \text{ odd.} \tag{E.2}$$

The inverse metric is given by

$$M^{-1}_{jk} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 2 & 2 & \cdots & 2 & 1 & 1 \\ 1 & 2 & 3 & \cdots & 3 & \frac{3}{2} & \frac{3}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & \frac{1}{2}(m-4) & \frac{1}{4}(m-4) & \frac{1}{4}(m-4) \\ \frac{1}{2} & 1 & \frac{3}{2} & \cdots & \frac{1}{4}(m-4) & \frac{1}{8}(m-0) & \frac{1}{8}(m-4) \\ \frac{1}{2} & 1 & \frac{3}{2} & \cdots & \frac{1}{4}(m-4) & \frac{1}{8}(m-4) & \frac{1}{8}(m-0) \end{pmatrix}, \qquad \text{for } m \text{ even} \tag{E.3}$$

and by

$$M^{-1}_{jk} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & \frac{1}{2} \\ 1 & 2 & 2 & \cdots & 2 & 1 \\ 1 & 2 & 3 & \cdots & 3 & \frac{3}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & \frac{1}{2}(m-3) & \frac{1}{4}(m-3) \\ \frac{1}{2} & 1 & \frac{3}{2} & \cdots & \frac{1}{4}(m-3) & \frac{1}{8}(m-1) \end{pmatrix}, \qquad \text{for } m \text{ odd.} \tag{E.4}$$

For $\mathfrak{so}(6)$ this reduces to

$$M_{jk} = \begin{pmatrix} +2 & -1 & -1 \\ -1 & +2 & \\ -1 & & +2 \end{pmatrix}, \qquad M^{-1}_{jk} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{3}{4}, \end{pmatrix} \tag{E.5}$$

while for the $\mathfrak{su}(4)$ notation we need to permute the first two rows and columns

$$M_{jk} = \begin{pmatrix} +2 & -1 & \\ -1 & +2 & -1 \\ & -1 & +2 \end{pmatrix}, \qquad M^{-1}_{jk} = \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix}. \tag{E.6}$$

F The Sigma-Model on $\mathbb{R} \times S^2$

In this section we apply the general results obtained in section Sec. 3 to the case of the Sigma-Model on $\mathbb{R} \times S^2$.

F.1 Properties of the Monodromy

The isometry group SO(3) of S^2 is locally isomorphic to SU(2). The spinor representation **2** of SO(3) can be viewed as the fundamental representation of SU(2). Correspondingly the monodromy matrix Ω^{s} can be regarded as the SU(2) monodromy matrix which is diagonalized as

$$\Omega^{\mathrm{s}} \simeq \mathrm{diag}(e^{ip}, e^{-ip}). \tag{F.1}$$

This quasi-momentum p is identified as

$$p = \tfrac{1}{2} q_1. \tag{F.2}$$

It exhibits the inversion symmetry

$$p(1/x) = -p(x) + 2\pi n_0, \tag{F.3}$$

has the pole structure

$$p(x) = \frac{\pi D}{\sqrt{\lambda}\,(x \mp 1)} + \mathcal{O}\big((x \mp 1)^0\big) \qquad \text{for } x \to \pm 1 \tag{F.4}$$

as well as the asymptotic behavior

$$p(x) = \frac{1}{x} \frac{2\pi J}{\sqrt{\lambda}} + \mathcal{O}(1/x^2). \tag{F.5}$$

where $r = 2J$ is the Dynkin label of SU(2). The analyticity condition reads

$$2\not{p}(x) = 2\pi n_a, \qquad x \in \mathcal{C}_a. \tag{F.6}$$

F.2 Algebraic Curve

Here we repeat the counting of moduli of the algebraic curve for the case at hand. We make the most general ansatz

$$p'(x) = \frac{Q(x + 1/x)}{x(x - 1/x)^2 \sqrt{\prod_{b=1}^{A} (x - x_b^*)(1/x - x_b^*)}}\,, \tag{F.7}$$

where $Q(u)$ is a polynomial of degree $\frac{1}{2}A+1$. This corresponds to an inversion-symmetric two-sheeted algebraic curve with $p_1' + p_2' = 0$ and A cuts and the correct singular and aymptotic behavior described in the previous section. Note that A must be even to satisfy the general behavior.

There are in total $\frac{3}{2}A+2$ free parameters which we shall now constrain. Firstly, the residues of double poles at $x=\pm 1$ need to be equated (F.4) giving one condition. The absolute value represents the dimension which we shall not fix directly. Single poles, which would give rise to undesired logarithmic behavior in $p(x)$ are automatically absent due to the symmetry (F.3). Vanishing of $\mathcal{A}$-cycles yields $\frac{1}{2}A$ conditions (c.f. Fig. 8 for an illustration of the cycles corresponding to a branch cut $\mathcal{C}_a$). Note that the sum of $\mathcal{A}$-cycles around a symmetric pair of cuts vanishes due to the symmetry (F.3)

$$0=\oint_{\mathcal{A}_a} dp = -\oint_{1/\mathcal{A}_a} dp \tag{F.8}$$

Therefore there sum of all $\mathcal{A}$-cycles is automatically zero and the number of conditions is not reduced from the known behavior at $x=\infty$. Finally, integrality of $\mathcal{B}$-periods gives $\frac{1}{2}A+1$ constraints, because each symmetric pair is related

$$2\pi n_a = \int_{\mathcal{B}_a} dp = 4\pi n_0 - \int_{1/\mathcal{B}_a} dp, \qquad \int_0^\infty dp = -2\pi n_0. \tag{F.9}$$

up to the total momentum $2\pi n_0$. In total we have constrained $1+\frac{1}{2}A+(\frac{1}{2}A+1)$ coefficients and end up with $\frac{1}{2}A$ moduli. These correspond to one filling fraction for each pair of cuts.

F.3 Example

In the simplest case, the curve is described by two cuts $A=2$.

$$p'(x) = -\frac{2\pi}{\sqrt{\lambda}}\,\frac{J\sqrt{ab}\,(x-1/x)^2 + D(1+ab)(x+1/x) - 2D(a+b)}{x(x-1/x)^2\sqrt{(x-a)(1/x-a)(x-b)(1/x-b)}}\,. \tag{F.10}$$

where we fixed the all parameters of $Q(u)$ in (F.7) but two branch points a,b, from (F.4,F.5). We can fix the branch points and the dimension D from the integrals

$$\oint_{\mathcal{A}_1} dp = 0, \qquad \oint_{\mathcal{B}_1} dp = 2\pi n_1, \qquad \oint_{\mathcal{B}_2} dp = 2\pi n_2. \tag{F.11}$$

Note that the cycle $\mathcal{A}_2$ is equivalent to $\mathcal{A}_1$ under the symmetry and the mode numbers are related to the total momentum by $n_1+n_2=2n_0$. The filling fraction is directly related to the charge J and does not require a further condition. The most general solution to these equations corresponds to the solution of the Neumann-Rosochatius system, see [64], restricted to a single spin. Three particularly simple solutions have $n_1=\pm n_2$ or $n_2=0$ corresponding to two folded and one circular string. They are obtained by setting $a=-b$. These were investigated in [26], Appendices C.1.1 and C.2.1. There the sigma model on $\mathbb{R}\times S^3$ as used which effectively does not have the $x\to 1/x$ symmetry, see Sec. 3.9. Instead, the four branch points were assumed to be at $\pm a, \pm b$. When one spin is sent to zero to reduce to the $\mathbb{R}\times S^2$ model ($\alpha=0,1$)[25]

[25]The folded solution with $\alpha=1$ corresponds to the one found in [12].

the solution recovers the inversion symmetry: One finds that a and b are related by $b = \pm 1/a$. This is a nice confirmation of the symmetry property. Unfortunately all these solutions have a singular/fractional/trivial weak-coupling expansion as proposed by Frolov and Tseytlin and can therefore not be compared directly to gauge theory.

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