Reduced Phase Space Quantization and Dirac Observables

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Abstract

In her recent work, Dittrich generalized Rovelli's idea of partial observables to construct Dirac observables for constrained systems to the general case of an arbitrary first class constraint algebra with structure functions rather than structure constants. Here we use this framework and propose how to implement explicitly a reduced phase space quantization of a given system, at least in principle, without the need to compute the gauge equivalence classes. The degree of practicality of this programme depends on the choice of the partial observables involved. The (multi-fingered) time evolution was shown to correspond to an automorphism on the set of Dirac observables so generated and interesting representations of the latter will be those for which a suitable preferred subgroup is realized unitarily. We sketch how such a programme might look like for General Relativity.

We also observe that the ideas by Dittrich can be used in order to generate constraints equivalent to those of the Hamiltonian constraints for General Relativity such that they are spatially diffeomorphism invariant. This has the important consequence that one can now quantize the new Hamiltonian constraints on the partially reduced Hilbert space of spatially diffeomorphism invariant states, just as for the recently proposed Master constraint programme.

1   Introduction

It is often stated that there are no Dirac observables known for General Relativity, except for the ten Poincaré charges at spatial infinity in situations with asymptotically flat boundary conditions. This is inconvenient for any quantization scheme because it is only the gauge invariant quantities, that is, the functions on phase space which have weakly[1] vanishing Poisson brackets with the constraints, which have physical meaning and can be measured. These are precisely the (weak) Dirac observables

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[1]We say that a relation holds weakly if it is an identity on the constraint surface of the phase space where the constraints are satisfied.

of the canonical formalism. The Dirac observables also play a prominent role for the quantization at the technical level, because the ultimate physical Hilbert space must carry a representation of their Poisson algebra, no matter whether one follows a Dirac quantization scheme (reducing after quantizing) or a reduced phase space approach (quantizing after reducing).

For General Relativity the identification of a suitably complete set of Dirac observables (that is, a set which encodes all the gauge invariant information about the system) is especially hard because the constraint algebra is not a Lie algebra: While it is a first class system, it does not close with structure constants but rather with structure functions, that is, non trivial functions on phase space. This fact has obstructed the development of a representation theory of GR's constraint algebra and hence the associated invariants. There are even obstruction theorems available in the literature [1] which state the non existence of local Dirac observables (depending on a finite number of spatial derivatives) for GR.

In [2] a proposal for how to overcome the problem of structure functions for GR for the quantum theory in the context of Loop Quantum Gravity [3] was developed. The idea is to replace the algebra by a simpler but equivalent one which closes without structure functions. The proposal was successfully applied to many test models of varying degree of complexity [4]. Besides this quantum application also a meachnism to generate strong Dirac observables by the method of ergodic averaging was given. This avoids the obstruction theorem mentioned above because indeed the resulting observables do depend on an infinite number of spatial derivatives. However, strong Dirac observables are not particularly interesting for systems with structure functions, simply because there are probably not very many of them: If $\{C_j, O\} = 0$ everywhere on phase space for all constraints C_j and $\{C_j, C_k\} = f_{jk}{}^l C_l$ then by the Jacobi identity $\{\{C_j, C_k\}, O\} = C_l\{f_{jk}{}^l, O\} = 0$ identically which due to the algebraic independence of the C_j means that strong Dirac Observables also must satisfy the additional equations $\{f_{jk}{}^l, O\} = 0$ and iterating like this it is quite possible that only the constants survive.

In [5] Dittrich addressed the problem of constructing weak Dirac observables for first class systems, possibly with structure constants. Her construction is based on the notion of partial observables which to a large part is due to Rovelli [6]. The idea is to use a relational point of view, namely to construct observables $F_{A,B}^\tau$ of the type: What is the value of a non – ivariant function A when under the gauge flow the non – invariant function B has arrived at the value τ? The functions A, B are here the partial observables and $F_{A,B}^\tau$ is called a complete observable. In [6] it was shown that the complete observable is a strong Dirac observable for the case of a single constraint. The reason is that for a *single* constraint there is always an invariant combination between *two* functions: If we study the orbits $\beta \mapsto A(\beta), B(\beta)$ of A, B under the gauge flow parameterized by a Lagrange multiplier β then we may use the value of B as the parameter, i.e. we may invert the equation $B(\beta) = \tau$ for β and insert that value into $A(\beta)$. The result is an invariant. In [5] Dittrich generalized the idea to an arbitrary number N of constraints by observing that similarly there is always a weakly invariant combination between $N + 1$ functions. This time we have to invert the $N-$parameter gauge flow for N partial observables.

In [5] not only we find the proof that the result is always a weak invariant but moreover an *explicit expression* is obtained in terms of a formal power series. To the best of our knowledge, this is the first explicit expression and concrete algorithm for how to construct Dirac observables which moreover have a concrete (relational) physical interpretation[2]. On top of that, it is possible within this framework to calculate the Poisson algebra of these Dirac observables and to implement a notion of multi – fingered time [11], along the lines of Rovelli's "evolving constants", as certain Poisson

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[2]There seems to be some overlap with [7], however, the proofs are missing in that paper.

automorphisms on that algebra. While these findings were not derived like that, in retrospect the results of [5] technically rest on the fact that, just like in [2], it is always possible to replace the constraint algebra by a simpler one.

The present paper serves three purposes:
1.
The proofs of [5] are very elegant and often use the definition of the complete obsservables in terms of the flow. In section 2 we review the parts of [5] relevant for our purposes and give sometimes alternative proofs by directly working with the local, explicit expressions and brutally working out the Poisson brackets.
2.
In section three we remark on the implications of [5] for the purposes of the quantum theory. In particular, we sketch how to perform a reduced phase space quantization of the algebra of Dirac observables by making use of a suitable choice of partial observables. Such a choice is always available and we show that then, quite surprisingly, representations of a sufficiently large subalgebra of elementary Dirac observables are easily available. The multi – fingered time evolution can be implemented unitarily provided one can quantize the corresponding Hamiltonian generators as aself – adjoint operators in those representations. While it is possible to identify those Hamiltonians, their explicit form in terms of elementary Dirac observables will be very complicated in general.

In section 4 we also sketch how such a quantization scheme might look concretely when applied to General Relativity coupled to the Standard model and in section 5 we comment on the important physical differences between the reduced phase space quantization and the Dirac contraint quantization that is currently performed in LQG.
3.
In section 6 we combine the ideas of [5] with those of [2] by showing how the Master Constraint Programme for General Relativity can be used in order to provide *spatially diffeomorphism invariant Hamiltonian constraints*. The important consequence of this is that in the constraint quantization one can implement the new Hamiltonian constraints on the spatially diffeomorphism invariant Hilbert space which is not possible for the old constraints because for those the spatial diffeomorphism subalgebra is not an ideal. As a consequence, the algebra of the new Hamiltonian constraints on the spatially dfiffeomorphism invariant Hilbert space then closes on itself (albeit with structure functions rather than structure constants in general). This might pose an attractive alternative to the previous Hamiltonian constraint quantization [10].

We conclude in section 7.

2 Review of the Classical Framework

We summarize here the work of [5] and sometimes give alternative proofs. These use only local considerations, hence no global assumptions are made. On the other hand, the results are only locally valid (e.g. the clock variables must provide a good coordinatization of the gauge orbits). This is enough from a physical perspective since physical clocks are not expected to be good coordinates everywhere on phase space.

2.1 Partial and Weak Dirac Observables

Let C_j, $j \in \mathcal{I}$ be a system of first class constraints on a phase space $\mathcal{M}$ with (strong) symplectic structure given by a Poisson bracket $\{.,.\}$ where the index set has countable cardinality. This includes the case of a field theory for which the constraints are usually given in the local form $C_\mu(x)$, $x \in \sigma$, $\mu = 1,..,n < \infty$ where σ is a spatial, $D-$dimensional manifold corresponding to the initial value formulation and μ are some tensorial and/or Lie algebra indices. This can be seen by choosing a basis b_I of the Hilbert space $L_2(\sigma, d^D x)$ consisting of smooth functions of compact support and defining $C_j := \int_\sigma d^D x b_I(x) C_\mu(x)$ with $j := (\mu, I)$. We assume the most general situation, namely that $\{C_j, C_k\} = f_{jk}{}^l C_l$ closes with structure functions, that is, $f_{jk}{}^l$ can be non – trivial functions on $\mathcal{M}$.

The partial observable Ansatz to generate Dirac observables is now as follows: Take as many functions on phase space T_j, $j \in \mathcal{I}$ as there are constraints. These functions have the purpose of providing a local (in phase space) coordinatization of the gauge orbit $[m]$ of any point m in phase space, at least in a neighbourhood of the constraint surface $\overline{\mathcal{M}} = \{m \in \mathcal{M}; C_j(m) = 0 \; \forall j \in \mathcal{I}\}$. The gauge orbit $[m]$ of m is given by $[m] := \{\alpha_{\beta_1} \circ .. \circ \alpha_{\beta_N}(m); N < \infty, \beta_k^j \in \mathbb{R}, k = 1,..,N, j \in \mathcal{I}\}$. Here α_β is the canonical transformation (automorphism of $(C^\infty(\mathcal{M}), \{.,.\})$) generated by the Hamiltonian vector field χ_β of $C_\beta := \beta^j C_j$, that is $\alpha_\beta(f) := \exp(\chi_\beta) \cdot f$. (Notice that if the system would have structure constants, then it would be sufficient to choose $N = 1$.)

In other words, we assume that it is possible to find functions T_j such that each $m \in \overline{\mathcal{M}}$ is completely specified by $[m]$ and by the $T_j(m)$. This means that if the value τ_j is in the range of T_j then the gauge fixing surface $\overline{\mathcal{M}}_\tau := \{m \in \overline{\mathcal{M}}; T_j(m) = \tau_j\}$ intersects each m in precisely one point. In practice this is usually hard to achieve globally on $\overline{\mathcal{M}}$ due to the possibility of Gribov copies but here we are only interested in local considerations. It follows that the matrix $A_{jk} := \{C_j, T_k\}$ must be locally invertible so that the condition $[\alpha_\beta(T_j)](m) = T_j(\alpha_\beta(m)) = \tau_j$ can be inverted for β (given $m' \in m$ we may write it in the form $[\alpha_\beta(m)]_{|\beta=B(m)}$ for some $B(m)$ which may depend on m).

Take now another function f on phase space. Then the weak Dirac observable $F_{f,T}^\tau$ associated to the partial observables f, T_j, $j \in \mathcal{I}$ is defined by

$$(F_{f,T}^\tau)(m) := [f(\alpha_\beta(m))]_{|\beta=B_T^\tau(m)}, \quad [T_j(\alpha_\beta(m))]_{|\beta=B_T^\tau(m)} = \tau_j \tag{2.1}$$

The physical interpretation of $F_{f,T}^\tau$ is that it is the value of f at those "times" β_j when the "clocks" T_j take the values τ_j.

In [5] a proof was given that (2.1) is indeed invariant under the flow automorphisms α_β despite the fact that the α_β do not form a group of automorphisms in the case of structure functions. This is quite astonishing given the fact that the direct proof for the case of a single constraint can be easily repeated only in the case that the constraints are mutually commuting. Then an explicit expression was derived using the system of partial differential equations (in the parameters τ_j) that the $F_{f,T}^\tau$ satisfy.

We will now derive that same explicit expression from an Ansatz for a Taylor expansion. Namely, on the gauge cut $\overline{\mathcal{M}}_\tau$ the function $F_{f,T}^\tau$ equals f since then $B_T^\tau(m) = 0$. Away from this section, $F_{f,T}^\tau$ can be expanded into a Taylor series[3]. Thus we make the Ansatz

$$F_{f,T}^\tau = \sum_{\{k_j\}_{j \in \mathcal{I}}=0}^\infty \prod_{j \in \mathcal{I}} \frac{(\tau_j - T_j)^{k_j}}{k_j!} f_{\{k_j\}_{j \in \mathcal{I}}} \tag{2.2}$$

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[3]In other words, $F_{f,T}^\tau$ is the gauge invariant extension of the restriction of f to $\overline{\mathcal{M}}_\tau$ mentioned in [8] for which however no explicit expression was given there.

with $f_{\{k_j\}=\{0\}} = f$. We assume that (2.2) converges absolutely on a open set S and is continuous there, hence is uniformly bounded on S. We may then interchange summation and differentiation on S and compute

$$
\begin{aligned}
\{C_l, F^\tau_{f,T}\} \quad = \quad & \sum_{\{k_j\}_{j\in\mathcal{I}}=0}^{\infty} \prod_{j\in\mathcal{I}} \frac{(\tau_j - T_j)^{k_j}}{k_j!} \times \\
& +[\sum_{m\in\mathcal{I}} -A_{l,m} f_{\{k'_j(m)\}_{j\in\mathcal{I}}} + \{C_l, f_{\{k_j\}_{j\in\mathcal{I}}}]
\end{aligned}
\tag{2.3}
$$

where $k'_j(m) = k_j$ for $j \neq m$ and $k'_m(m) = k_m + 1$. Setting (2.3) (weakly) to zero leads to a recursion relation with the formal solution

$$
f_{\{k_j\}_{j\in\mathcal{I}}} = \prod_{j\in\mathcal{I}}(X'_j)^k_j \cdot f, \quad X'_j \cdot f = \sum_{k\in\mathcal{I}}(A^{-1})_{jk}\{C_k, f\}
\tag{2.4}
$$

Expression (2.4) is formal because we did not specify the order of application of the vector fields X'_j. We will now show that, as a weak identity, the order in (2.4) is irrelevant. To see this, let us introduce the equivalent constraints (at least on S)

$$
C'_j := \sum_{k\in\mathcal{I}}(A^{-1})_{jk}C_k
\tag{2.5}
$$

and notice that with the Hamiltonian vector fields $X_j \cdot f = \{C'_j, f\}$ we have $X'_{j_1}..X'_{j_n} \cdot f \approx X_{j_1}..X_{j_n} \cdot f$ for any $j_1, ..j_n$ due to the first class property of the constraints. Here and what follows we write $\approx$ for a relation that becomes an identity on $\overline{\mathcal{M}}$. Then we can make the following surprising observation.

Theorem 2.1.
Let C_j be a system of first class constraints and T_j be any functions such that the matrix A with entries $A_{jk} := \{C_j, T_k\}$ is invertible on some open set S intersecting the constraint surface. Define the equivalent C'_j constraints (2.5). Then their Hamiltonian vector fields $X_j := \chi_{C'_j}$ are mutually weakly commuting.

Proof of theorem 2.1:
The proof consists of a straightforward computation and exploits the Jacobi identity. Abbreviating

$B_{jk} := (A^{-1})_{jk}$ we have

$$\{C'_j, \{C'_k, f\}\} - \{C'_k, \{C'_j, f\}\} \approx \sum_{m,n} B_{jm}\{C_m, [B_{kn}\{C_n, f\} + C_n\{B_{kn}, f\}]\} - j \leftrightarrow k$$

$$\approx \sum_{m,n} B_{jm}[\{C_m, B_{kn}\}\{C_n, f\} + B_{kn}\{C_m, \{C_n, f\}\}] - j \leftrightarrow k$$

$$= \sum_{m,n} B_{jm}[-\sum_{l,i} B_{kl}\{C_m, A_{li}\}B_{in}\{C_n, f\} + B_{kn}\{C_m, \{C_n, f\}\}] - j \leftrightarrow k$$

$$= \sum_{m,n} B_{jm}[-\sum_{l,i} B_{kl}B_{in}\{C_n, f\}(\{C_m, \{C_l, T_i\}\} - \{C_l, \{C_m, T_i\}\}) + B_{kn}(\{C_m, \{C_n, f\}\} - \{C_n, \{C_m, f\}\}))]$$

$$= \sum_{m,n} B_{jm}[\sum_{l,i} B_{kl}B_{in}\{C_n, f\}\{T_i, \{C_m, C_l\}\} - B_{kn}(\{f, \{C_m, C_n\}\}]$$

$$\approx \sum_{m,n} B_{jm}[-\sum_{l,i,p} B_{kl}B_{in}\{C_n, f\}f_{ml}{}^p A_{pi} + B_{kn}\sum_l f_{mn}{}^l\{C_l, f\}]$$

$$= \sum_{m,n,l} B_{jm}[-B_{kl}\{C_n, f\}f_{ml}{}^n + B_{kn}f_{mn}{}^l\{C_l, f\}]$$

$$= 0 \tag{2.6}$$

Due to

$$\{C'_j, \{C'_k, f\}\} - \{C'_k, \{C'_j, f\}\} = \{\{C'_j, C'_k\}, f\} \approx f'_{jk}{}^l\{C'_l, f\} \approx 0 \tag{2.7}$$

this means that the structure functions $f'_{jk}{}^l$ with respect to the C'_j are weakly vanishing, that is, themselves proprotional to the constraints.
□

We may therefore write the Dirac observable generated by f, T_j indeed as

$$F^\tau_{f,T} = \sum_{\{k_j\}_{j \in \mathcal{I}} = 0}^\infty \prod_{j \in \mathcal{I}} \frac{(\tau_j - T_j)^{k_j}}{k_j!} \prod_{j \in \mathcal{I}} (X_j)^{k_j} \cdot f \tag{2.8}$$

Expression (2.8) is, despite the obvious convergence issues to be checked in the concrete application, remarkably simple. Of course, especially in field theory it will not be possible to calculate it exactly and already the computation of the inverse A^{-1} may be hard, depending on the choice of the T_j. However, for points close to the gauge cut expression (2.8) is rapidly converging and one may be able to do approximate calculations.

Remark:
Let $\alpha'_\beta(f) := \exp(\sum_j \beta_j X_j) \cdot f$ be the gauge flow generated by the new constraints C'_j for real valued gauge parameters β_j. We easily calculate $\alpha'_\beta(T_j) \approx T_j + \beta_j$. The condition $\alpha'_\beta(T_j) = \tau_j$ can therefore be easily inverted to $\beta_j \approx \tau_j - T_j$. Hence the complete observable prescription with respect to the new constraints C'_j

$$F^\tau_{f,T} := [\alpha'_\beta(f)]_{|\alpha'_\beta(T) = \tau} \tag{2.9}$$

weakly coincides with (2.8).

2.2   Poisson algebra of Dirac Observables

In [8] we find the statement that the Poisson brackets among the Dirac observables obtained as the gauge invariant extension off $\overline{\mathcal{M}}_\tau$ of the respective restrictions to the gauge cut of functions f, g is

weakly given by the gauge invariant extension of their Dirac bracket with respect to the associated gauge fixing functions. However, a proof of that statement could nowhere be found by the present author. Expression (2.8) now enables us to give an explicit, local proof (modulo convergence issues). See [5] for an alternative one.

Theorem 2.2.
Let $F_{f,T}^\tau$ be defined as in (2.8) with respect to partial observables T_j. Introduce the gauge conditions $G_j := T_j - \tau_j$ and consider the system of second class constraints $C_{1j} := C_j$, $C_{2j} := G_j$ and abbreviate $\mu = (I, j)$, $I = 1, 2$. Introduce the Dirac bracket

$$\{f, f'\}^* := \{f, f'\} - \{f, C_\mu\} K^{\mu\nu} \{C_\nu, f'\} \tag{2.10}$$

where $K_{\mu\nu} = \{C_\mu, C_\nu\}$, $K^{\mu\rho} K_{\rho\nu} = \delta^\mu_\nu$. Then

$$\{F_{f,T}^\tau, F_{f',T}^\tau\} \approx F_{\{f,f'\}^*,T}^\tau \tag{2.11}$$

Proof of theorem 2.2:
Let us introduce the abbreviations

$$Y_{\{k\}} = \prod_j \frac{(\tau_j - T_j)^{k_j}}{k_j!}, \quad f_{\{k\}} = \prod_j (X_j)^{k_j} \cdot f, \quad \sum_{\{k\}} = \sum_{k_1, k_2, ..=0}^\infty \tag{2.12}$$

We have

$$\{F_{f,T}^\tau, F_{f',T}^\tau\} = \sum_{\{k\},\{l\}} \{Y_{\{k\}} f_{\{k\}}, Y_{\{l\}} f'_{\{l\}}\}$$

$$\approx \sum_{\{k\},\{l\}} Y_{\{k\}} Y_{\{l\}} [\{f_{\{k\}}, f'_{\{l\}}\} - \sum_j (X_j \cdot f)_{\{k\}} \{T_j, f'_{\{l\}}\}$$

$$+ \sum_j (X_j \cdot f')_{\{l\}} \{T_j, f_{\{k\}}\} + \sum_{j,m} (X_j \cdot f)_{\{k\}} (X_m \cdot f')_{\{l\}} \{T_j, T_m\}]$$

$$= \sum_{\{n\}} Y_{\{n\}} \sum_{\{k\};\, k_l \leq n_l} \prod_l \binom{n_l}{k_l} [\{f_{\{k\}}, f'_{\{n-k\}}\} - \sum_j (X_j \cdot f)_{\{k\}} \{T_j, f'_{\{n-k\}}\}$$

$$+ \sum_j (X_j \cdot f')_{\{n-k\}} \{T_j, f_{\{k\}}\} + \sum_{j,m} (X_j \cdot f)_{\{k\}} (X_m \cdot f')_{\{n-k\}} \{T_j, T_m\}] \tag{2.13}$$

By definition of a Hamiltonian vector field we have $X_j \{f, f'\} = \{X_j f, f'\} + \{f, X_j f'\}$. Thus, by the (multi) Leibniz rule

$$\prod_l (X_l)_l^n \{f, f'\} = \sum_{\{k\};\, k_l \leq n_l} \prod_l \binom{n_l}{k_l} [\{f_{\{k\}}, f'_{\{n-k\}}\} \tag{2.14}$$

is already the first term we need. It therefore remains to show that

$$\prod_l (X_l)_l^n [\{f, f'\}^* - \{f, f'\}] \approx \sum_{\{k\};\, k_l \leq n_l} \prod_l \binom{n_l}{k_l} [-\sum_j (X_j \cdot f)_{\{k\}} \{T_j, f'_{\{n-k\}}\}$$

$$+ \sum_j (X_j \cdot f')_{\{n-k\}} \{T_j, f_{\{k\}}\} + \sum_{j,m} (X_j \cdot f)_{\{k\}} (X_m \cdot f')_{\{n-k\}} \{T_j, T_m\}] \tag{2.15}$$

We will do this by multi induction over $N := \sum_l n_l$.

The case $N = 0$ reduces to the claim

$$\{f, f'\}^* - \{f, f'\} \quad \approx -\sum_j (X_j \cdot f)\{T_j, f'\}$$
$$+ \sum_j (X_j \cdot f')\{T_j, f\} + \sum_{j,m} (X_j \cdot f)(X_m \cdot f')\{T_j, T_m\} \qquad (2.16)$$

To compute the Dirac bracket explicitly we must invert the matrix $K_{Jj,Kk}$ with entries $K_{1j,1k} = \{C_j, C_k\} = f_{jk}\,{}^l C_l \approx 0$, $K_{1j,2k} = \{C_j, T_k\} = A_{jk} = -K_{2k,1j}$ and $K_{2j,2k} = \{T_j, T_k\}$. By definition $\sum_{L,l} K^{Jj,Ll} K_{Ll,Kk} = \delta_K^J \delta_k^j$ therefore $K^{1j,1k} \approx \sum_{m,n}(A^{-1})_{mj}\{T_m, T_n\}(A^{-1})_{nk}$, $K^{1j,2k} \approx -(A^{-1})_{kj} \approx -K^{2k,1j}$ and $K^{2j,2k} \approx 0$. It follows

$$-\{f, f'\}^* + \{f, f'\} = \{f, C_j\}K^{1j,1k}\{C_k, f'\} + \{f, C_j\}K^{1j,2k}\{T_k, f'\}$$
$$+\{f, T_j\}K^{2j,1k}\{C_k, f'\} + \{f, T_j\}K^{2j,2k}\{T_k, f'\}$$
$$\approx \sum_{m,n}\{f, C_j\}(A^{-1})_{mj}\{T_m, T_n\}(A^{-1})_{nk}\{C_k, f'\} - \{f, C_j\}(A^{-1})_{kj}\{T_k, f'\} + \{f, T_j\}(A^{-1})_{jk}\{C_k, f'\}$$
$$\approx -\sum_{m,n}(X_m \cdot f)\{T_m, T_n\}(X_n \cdot f') + (X_k \cdot f)\{T_k, f'\} - (X_k \cdot f')\{T_k, f\} \qquad (2.17)$$

which is precisely the negative of (2.16).

Suppose then that we have proved the claim for every configuration $\{n_l\}$ such that $\sum_l n_l \leq N$. Any configuration with $N + 1$ arises from a configuration with N by raising one of the n_l by one unit, say $n_j \to n_j + 1$. Then, by assumption

$$X_j \prod_l (X_l)_l^n[\{f, f'\}^* - \{f, f'\}]$$
$$\approx X_j \sum_{\{k\};\, k_l \leq n_l} \prod_l \binom{n_l}{k_l} [-\sum_l (X_l \cdot f)_{\{k\}}\{T_l, f'_{\{n-k\}}\}$$
$$+ \sum_l (X_l \cdot f')_{\{n-k\}}\{T_l, f_{\{k\}}\} + \sum_{l,m}(X_l \cdot f)_{\{k\}}(X_m \cdot f')_{\{n-k\}}\{T_l, T_m\}]$$
$$\approx \sum_{\{k\};\, k_l \leq n_l} \prod_l \binom{n_l}{k_l} \times$$
$$- \sum_l [(X_l \cdot f)_{\{k^j\}}\{T_l, f'_{\{n-k\}}\} + (X_l \cdot f)_{\{k\}}\{T_l, f'_{\{n^j-k\}}\} + (X_l \cdot f)_{\{k\}}\{X_j \cdot T_l, f'_{\{n-k\}}\}]$$
$$+ \sum_l [(X_l \cdot f')_{\{k^j\}}\{T_l, f_{\{n-k\}}\} + (X_l \cdot f')_{\{k\}}\{T_l, f_{\{n^j-k\}}\} + (X_l \cdot f')_{\{k\}}\{X_j \cdot T_l, f_{\{n-k\}}\}]$$
$$+ \sum_{l,m}[(X_l \cdot f)_{\{k^j\}}(X_m \cdot f')_{\{n-k\}}\{T_l, T_m\} + (X_l \cdot f)_{\{k\}}(X_m \cdot f')_{\{n^j-k\}}\{T_l, T_m\}$$
$$+(X_l \cdot f)_{\{k\}}(X_m \cdot f')_{\{n-k\}}(\{X_j T_l, T_m\} + \{T_l, X_j T_m\})]] \qquad (2.18)$$

where $\{k^j\}$ coincides with $\{k\}$ except that $k_j \to k_j + 1$ and similar for $\{n^j\}$. By the multi binomial theorem the first two terms in each of the three sums in the last equality combine precisely to what

we need. Hence it remains to show that

$$
\begin{aligned}
0 \approx & \sum_{\{k\};\, k_l \leq n_l} \prod_l \binom{n_l}{k_l} \times \\
& [-\sum_l (X_l \cdot f)_{\{k\}} \{X_j \cdot T_l, f'_{\{n-k\}}\} + \sum_l (X_l \cdot f')_{\{k\}} \{X_j \cdot T_l, f_{\{n-k\}}\} \\
& + \sum_{l,m} (X_l \cdot f)_{\{k\}} (X_m \cdot f')_{\{n-k\}} (\{X_j T_l, T_m\} + \{T_l, X_j T_m\})]
\end{aligned}
\tag{2.19}
$$

We have

$$
X_j \cdot T_l = \delta_{jl} + \sum_m C_m \{(A^{-1})_{jm}, T_l\}] =: \delta_{jl} + \sum_m C_m B_{jlm}
\tag{2.20}
$$

Hence

$$
\{X_j \cdot T_l, g\} \approx \sum_{m,n} B_{jlm} A_{mn} (X_n \cdot g) =: \sum_n D_{jln} (X_n \cdot g)
\tag{2.21}
$$

Next, using (2.20) and (2.21)

$$
\{X_j T_l, T_m\} + \{T_l, X_j T_m\} \approx \sum_n (B_{jln} A_{nm} - B_{jmn} A_{nl}) = D_{jlm} - D_{jml}
\tag{2.22}
$$

We now can simplify the right hand side of (2.19)

$$
\begin{aligned}
& \sum_{\{k\};\, k_l \leq n_l} \prod_l \binom{n_l}{k_l} \times \\
& \sum_{l,m} D_{jlm} [-(X_l \cdot f)_{\{k\}} (X_m \cdot f'_{\{n-k\}}) + (X_l \cdot f')_{\{k\}} (X_m \cdot f_{\{n-k\}}) \\
& + [D_{jlm} - D_{jml}] (X_l \cdot f)_{\{k\}} (X_m \cdot f')_{\{n-k\}}] \\
& \sum_{l,m} D_{jlm} \prod_i (X_i)^{n_i} [[-(X_l \cdot f)(X_m \cdot f') + (X_l \cdot f')(X_m \cdot f) + (X_l \cdot f)(X_m \cdot f') - (X_m \cdot f)(X_l \cdot f')] \\
= \; & 0
\end{aligned}
\tag{2.23}
$$

as claimed. Notice that by using the Jacobi identity we also have $D_{jkl} = D_{jlk}$ so the two terms in the second and third line of (2.23) even vanish separately (important for the case that $\{T_j, T_k\} = 0$).
$\square$
We can now rephrase theorem 2.2 as follows:
Consider the map

$$
F_T^\tau : \; (C^\infty(\mathcal{M}), \{.,.\}_T^*) \to (D^\infty(\mathcal{M}), \{.,.\}_T^*); \; f \mapsto F_{f,T}^\tau
\tag{2.24}
$$

where $D^\infty(\mathcal{M})$ denotes the set of smooth, weak Dirac observables and $\{.,.\}_T^*$ is the Dirac bracket with respect to the gauge fixing functions T_j. Then theorem 2.2 says that F_T^τ is a weak Poisson homomorphism (i.e. a homomorphism on the constraint surface). To see this, notice that for (weak) Dirac observables the Dirac bracket coincides weakly with the ordinary Poisson bracket. Moreover,

the map F_T^τ is linear and trivially

$$
\begin{aligned}
F_{f,T}^\tau\, F_{f',T}^\tau &= \sum_{\{k\},\{l\}} Y_{\{k\}} Y_{\{l\}} f_{\{k\}} f'_{\{l\}} \\
&= \sum_{\{n\}} Y_{\{n\}} \sum_{\{k\};\, k_l \le n_l} \prod_l \binom{n_l}{k_l} f_{\{k\}} f'_{\{n-k\}} \\
&\approx \sum_{\{n\}} Y_{\{n\}} \prod_l (X_l)^{n_l} (f\, f') = F_{ff',T}^\tau
\end{aligned}
\tag{2.25}
$$

(We can make the homomorphism exact by dividing both $C^\infty(\mathcal{M})$ and $D^\infty(\mathcal{M})$ by the ideal (under pointwise addition and multiplication) of smooth functions vanishing on the constraint surface, see [5]). We will use this important fact, to the best of our knowledge first observed in [5], for a new proposal towards quantization. Notice, that F_T^τ is onto because $F_{f,T}^\tau \approx f$ if f is already a weak Dirac observable.

2.3 Evolving Constants

The whole concept of partial observables was invented in order to remove the following conceptual puzzle:
In a time reparameterization invariant system such as General Relativity the formalism asks us to find the time reparameterization invariant functions on phase space. However, then "nothing happens" in the theory, there is no time evolution, in obvious contradiction to what we observe. This puzzle is removed by using the partial observables by taking the relational point of view: The partial observables f, T_j can be measured but not predicted. However, we can predict $F_{f,T}^\tau$, it has the physical interpretation of giving the value of f when the T_j assume the values τ_j. In constrained field theories we thus arrive at the multi fingered time picture, there is no preferred time but there are infinitely many. Accordingly, we define a multi – fingered time evolution on the image of the maps F_T^τ by

$$
\alpha^\tau:\; F_T^{\tau^0}(C^\infty(\mathcal{M})) \to F_T^{\tau + \tau^0}(C^\infty(\mathcal{M}));\; F_{f,T}^{\tau^0} \mapsto F_{f,T}^{\tau + \tau^0}
\tag{2.26}
$$

As defined, α^τ forms an Abelean group. However, it has even more interesting properties:

$$
\begin{aligned}
F_{f,T}^{\tau + \tau^0} &= \sum_{\{n\}} \prod_j \frac{(\tau_j + \tau_j^0 - T_j)^{n_j}}{n_j!} \prod_j X_j^{n_j} \cdot f \\
&\approx \sum_{\{n\}} \sum_{\{k\};\, k_l \le n_l} \prod_l \frac{1}{n_l!} \binom{n_l}{k_l} \prod_j (\tau_j^0 - T_j)^{k_j} \tau_j^{n_j - k_j} \prod_j X_j^{k_j} X_j^{n_j - k_j} \cdot f \\
&\approx \sum_{\{k\}} \prod_j \frac{(\tau_j^0 - T_j)^{k_j}}{k_j!} \prod_j X_j^{k_j} \cdot \Big[\sum_{\{l\}} \frac{\tau_j^{l_j}}{l_j!} \prod_j X_j^{l_j} \Big] \cdot f \\
&= F_{\alpha_\tau'(f),T}^{\tau^0}
\end{aligned}
\tag{2.27}
$$

where $\alpha'_\tau(f)$ is the automorphism on $C^\infty(\mathcal{M})$ generated by the Hamiltonian vector field of $\sum_j \tau_j C'_j$ with the equivalent constraints $C'_j = \sum_k (A^{-1})_{jk} C_k$. This is due to the multi – nomial theorem

$$
\begin{aligned}
\alpha'_\tau(f) &= \sum_{n=0}^\infty \frac{1}{n!} (\sum_j \tau_j X_j)^n \cdot f \\
&= \sum_{n=0}^\infty \frac{1}{n!} \sum_{j_1,..,j_n} \prod_{k=1}^n \tau_{j_k} X_{j_k} \cdot f \\
&= \sum_{n=0}^\infty \frac{1}{n!} \sum_{\{k\}; \sum_j k_j = n} \frac{n!}{\prod_j (k_j)!} \prod_j \tau_j^{k_j} \prod_j X_j^{k_j} \cdot f \\
&= \sum_{\{k\}} \prod_j \frac{\tau_j^{k_j}}{k_j!} \prod_j X_j^{k_j} \cdot f
\end{aligned}
\tag{2.28}
$$

Thus, our time evolution on the observables is induced by a gauge transformation on the partial observables. From this observation it follows, together with the weak homomorphism property, that

$$
\begin{aligned}
\{\alpha^\tau(F_{f,T}^{\tau_0}), \alpha^\tau(F_{f',T}^{\tau_0})\} &= \{F_{f,T}^{\tau_0+\tau}, F_{f',T}^{\tau_0+\tau}\} \\
&\approx F_{\{f,f'\}^*,T}^{\tau_0+\tau} = \alpha^\tau(F_{\{f,f'\}^*,T}^{\tau_0}) \\
&\approx \alpha^\tau(\{F_{f,T}^{\tau_0}, F_{f',T}^{\tau_0}\})
\end{aligned}
\tag{2.29}
$$

In other words, $\tau \mapsto \alpha^\tau$ is a weak, Abelean, multi – parameter group of automorphisms on the image of each map $F_{f,T}^{\tau_0}$. This is in strong analogy to the properties of the one parameter group of automorphisms on phase space generated by a true Hamiltonian. Also this observation, in our opinion due to [5], will be used for a new proposal towards quantization.

3   Reduced Phase Space Quantization of the Algebra of Dirac Observables and Unitary Implementation of the Multi – Fingered Time Evolution

We will now describe our proposal. We assume that it is possible to to choose the functions T_j as canonical coordinates. In other words, we choose a canonical coordinate system consisting of canonical pairs (q^a, p_a) and (T_j, P^j) where the first system of coordinates has vanishing Poisson brackets with the second so that the only non vanishing brackets are $\{p_a, q^b\} = \delta_a^b$, $\{P^j, T_k\} = \delta_k^j$. (In field theory the label set of the $a, b, ..$ will be indefinite corresponding to certain smeared quantities of the canonical fields). The virtue of this assumption is that the Dirac bracket reduces to the ordinary Poisson bracket on functions which depend only on q^a, p_a. We will shortly see why this is important. We define with $F_T := F_T^0$ the weak Dirac observables at multi fingered time $\tau = 0$ (or anny other fixed allowed value of τ).

$$
Q^a := F_T(q^a), \ P_a := F_T(p_a)
\tag{3.1}
$$

Notice that $F_{T_j,T}^\tau \approx \tau_j$, so the Dirac observable corresponding to T_j is just a constant and thus not very interesting (but evolves precisely as a clock). Likewise $F\tau_{C_j,T} \approx 0$ is not very interesting. Since at least locally we can solve the constraints C_j for the momenta P^j, that is $P^j \approx E_j(q^a, p_a, T_k)$ and F_T is a homomorphism with respect to pointwise operations we have

$$
F_T(P_j) \approx E_j(F_T(q^a), F_T(p_a), F_T(T_k)) \approx E_j(Q^a, P_a, \tau_k)
\tag{3.2}
$$

and thus also does not give rise to a Dirac observable which we could not already construct from Q^a, P_a. The importance of our assumption is now that due to the homomorphism property

$$\{P_a, Q^b\} \approx F^0_{\{p_a, q^b\}^*, T} = F^0_{\delta^b_a, T} = \delta^b_a, \quad \{Q^a, Q^b\} \approx \{P_a, P_b\} \approx 0 \tag{3.3}$$

In other words, even though the functions P_a, Q^a are very complicated expressions in terms of q^a, p_a, T_j they have nevertheless canonical brackets at least on the constraint surface. If we would have had to use the Dirac bracket then this would not be the case and the algebra among the Q^a, P_a would be too complicated and no hope would exist towards its quantization. However, under our assumption there is now a chance.

Now reduced phase space quantization consists in quantizing the subalgebra of $\mathcal{D}$, spanned by our preferred Dirac observables Q^a, P_a evaluated on the constraint surface. As we have just seen, the algebra $\mathcal{D}$ itself is given by the Poisson algebra of the functions of the Q^a, P_a evaluated on the constraint surface. Hence all the weak equalities that we have derived now become exact. We are therefore looking for a representation $\pi : \mathcal{D} \rightarrow \mathcal{L}(\mathcal{H})$ of that subalgebra of $\mathcal{D}$ as self – adjoint, linear operators on a Hilbert space such that $[\pi(P_a), \pi(Q^b)] = i\hbar \delta^b_a$.

At this point it looks as if we have completely trivialized the reduced phase space quantization problem of our constrained Hamiltonian system because there is no Hamiltonian to be considered and so it seems that we can just choose any of the standard kinematical representations for quantizing the phase space coordinatized by the q^a, p_a and simply use it for Q^a, P_a because the respective Poisson algebras are (weakly) isomorphic. However, this is not the case. In addition to satisfying the canonical commutation relations we want that the multi parameter group of automorphisms α^τ on $\mathcal{D}$ be represented unitarily on $\mathcal{H}$ (or at least a suitable, preferred one parameter group thereof). In other words, we want that there exists a multi parameter group of unitary operators $U(\tau)$ on $\mathcal{H}$ such that $\pi(\alpha^\tau(Q^a)) = U(\tau)\pi(Q^a)U(\tau)^{-1}$ and similarly for P_a.

Notice that due to the relation (which is exact on the constraint surface)

$$\alpha^\tau(Q^a) = F_{\alpha'_\tau(q^a), T} = \sum_{\{k\}} \prod_j \frac{\tau_j^{k_j}}{k_j!} F_{\prod_j X_j^{k_j} \cdot q^a, T} \tag{3.4}$$

and where on the right hand side we may replace any occurence of P_j, T_j by functions of Q^a, P_a according to the above rules. Hence the automorphism α^τ preserves the algebra of functions of the Q^a, P_a, although it is a very complicated map in general and in quantum theory will suffer from ordering ambiguities. On the other hand, for short time periods (3.4) gives rise to a quickly converging perturbative expansion. Hence we see that the representation problem of $\mathcal{D}$ will be severely constrained by our additional requirement to implement the multi time evolution unitarily, if at all possible. Whether or not this is feasible will strongly depend on the choice of the T_j.

A possible way to implement the multi – fingered time evolution unitarily is by quantizing the Hamiltonians H_j that generate the Hamiltonian flows $\tau_j \mapsto \alpha^\tau$ where $\tau_k = \delta_{jk}\tau_j$. This can be done as follows: The original constraints C_j can be solved for the momenta P^j conjugate to T_j and we get equivalent constraints $\tilde{C}_j = P^j + E_j(q^a, p_a, T_k)$. These constraints have a strongly Abelean constraint algebra[4]. We may write $C'_j = K_{jk}\tilde{C}_k$ for some regular matrix K. Since $\{C'_j, T_k\} \approx \delta_{jk} = \{\tilde{C}_j, T_k\}$ it follows that $K_{jk} \approx \delta_{jk}$. In other words $C'_j = \tilde{C}_j + O(C^2)$ where the notation $O(C^2)$ means that the two constraints set differ by terms quadratic in the constraints. It follows that the

---

[4]Proof: We must have $\{\tilde{C}_j, \tilde{C}_k\} = \tilde{f}_{jk}{}^l \tilde{C}_l$ for some new structure functions $\tilde{f}$ by the first class property. The left hand side is independent of the functions P^j, thus must be the right hand side, which may therefore be evaluated at any value of P^j. Set $P^j = -E_j$. □.

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Hamiltonian vector fields X_j, $\tilde{X}_j$ of $C'_j, \tilde{C}_j$ are weakly commuting. We now set $H_j(Q^a, P_a) := F^0_{E_j,T} \approx E_j(F^0_{q^a,T}, F^0_{p_a,T}, F^0_{T_k,T}) \approx E_j(Q^a, P_a, 0)$. Let now f be any function which depends only q^a, p_a. Then we have

$$
\begin{aligned}
\{H_j, F^0_{f,T}\} &\approx F^0_{\{E_j,f\}^*,T} = F^0_{\{E_j,f\},T} = F^0_{\{\tilde{C}_j,f\},T} \\
&= \sum_{\{k\}} \prod_l \frac{(\tau_l - T_l)^{k_l}}{k_l!} \prod_l X_l^{k_l} \cdot \tilde{X}_j \cdot f \\
&\approx \sum_{\{k\}} \prod_l \frac{(\tau_l - T_l)^{k_l}}{k_l!} \tilde{X}_j \cdot \prod_l X_l^{k_l} \cdot f \\
&\approx \tilde{X}_j \cdot F^0_{f,T} - \sum_{\{k\}} (\tilde{X}_j \cdot \prod_l \frac{(\tau_l - T_l)^{k_l}}{k_l!}) \prod_l X_l^{k_l} \cdot f \\
&\approx +\sum_{\{k\}} \prod_l \frac{(\tau_l - T_l)^{k_l}}{k_l!} X_j \cdot \prod_l X_l^{k_l} \cdot f \\
&= (\frac{\partial}{\partial \tau_j})_{\tau=0} \alpha^\tau(F_T(f))
\end{aligned}
\tag{3.5}
$$

where we have used in the second step that $\{T_j, E_k\} = \{T_j, f\} = 0$, in the third we have used that $\{P_j, f\} = 0$, in the fifth we have used that the $X_j, \tilde{X}_k$ are weakly commuting, in the seventh we have used that $F^0_{f,T}$ is a weak observable, and in the last the definition of the flow. We conclude that the Dirac observables H_j generate the multifingered flow on the space of functions of the Q^a, P_a when restricted to the constraint surface. The algebra of the H_j is weakly Abelian because the flow α^τ is a weakly Abelian group of automorphisms.

Thus, the problem of implementing the flow unitarily can be reduced to finding a self adjoint quantization of the functions H_j. Preferred one parameter subgroups will be those for which the corresponding Hamiltonian generator is bounded from below. Notice, however, that in (3.5) we have computed the infinitesimal flow at $\tau = 0$ only. For an arbitrary value of τ the infinitesimal generator $H_j(Q^a, P_a, \tau)$ defined by

$$
\{H_j(\tau), F^\tau_{f,T}\} := \frac{\partial}{\partial \tau_j} \alpha^\tau(F_T(f))
\tag{3.6}
$$

may not coincide with $F^0_{E_j,T}$ since the Hamiltonian could be explicitly time τ dependent. In particular, the calculation (3.5) does not obviously hold any more even by setting $H_j(\tau) := F^\tau_{E_j,T}$ because even if f depends on q^a, p_a only, $\alpha'_\tau(f), \alpha'_\tau(E_j)$ may depend on P_j as well.

4 Reduced Phase Space Quantization of Geometry and Matter

In what follows we sketch a possible application of these ideas to field theory coupled to gravity. Details will appear elsewhere.

One would like to apply this formalism to the theory which presently describes most accurately what we observe, namely General Relativity for geometry coupled to the Standard Model for matter. In its canonical formulation coupled to gravity, see e.g. [9], we encounter the following set of constraints: The electroweak $U(1) \times SU(2)_L$ Gauss constraints (before symmetry breaking), the QCD $SU(3)$ Gauss constraint, the $SU(2)$ Gauss constraint for geometry, the spatial diffeomorphism constraint

and the Hamiltonian constraint. The various Gauss constraints are rather easy to cope with and we will focus on the latter four constraints which are, roughly speaking, the Hamiltonian incarnation of the generators of the four dimensional diffeomorphism group under which the theory is invariant. The point of view is that we carry out a reduced phase space quantization with respect to these constraints and solve the remaining Gauss constraints after that. This is possible because the Gauss constraints Poisson commute with the four other constraints.

The most natural choice for clock variables are scalar fields and remarkably there are precisely four real scalars in the standard model: The real and imaginary part of the two components of the Higgs dublett. (Notice however that the Higgs field still awaits its observation). The Higgs field consists of two complex scalar fields ϕ_I, $I = 1, 2$ and given a complete orthonormal basis b_j of $L_2(\sigma, d^3 x)$ where the spacetime manifold M is assumed to be diffeomorphic to $\mathbb{R} \times \sigma$, we can form the 4 functions

$$T_j^{1I} = \Re(< b_j, \phi_I >), \; T_j^{2I} = \Im(< b_j, \phi_I >) \tag{4.1}$$

where the inner product denoted is that on $L_2(\sigma, d^3 x)$. The four sets of constraints $C_{\mu j} :=< b_j, C_\mu >$ where $C\mu, \mu = 1, 2, 3$ stands for the spatial diffeomorphism constraint and $C_0 = C$ for the Hamiltonian constrainet for the combined matter and geometry system, are algebraic expressions in the momenta conjugate to the $T_j^{\alpha I}$ and thus can be solved in terms of them. Interestingly, for the appropriate choice of sign, the corresponding Hamiltonians E_j are *positive* functions (which however does not imply that the corresponding $H_j(\tau)$ remains positive for all τ). Moreover, the Higgs field and its conjugate momentum have vanishing Poisson brackets with the remaining gauge fields (including gravity) and the fermions (leptons and quarks).

This is precisely the situation pictured in the previous section and we can now start applying the formalism. In particular, we would need to look for representations of these remaining fields which have a unitary implementation of the "Higgs Time evolution". Notice that we could use the standard kinematical representations for these fields while having already accounted for the Hamiltonian constraint which is very difficult to solve in the Dirac procedure of solving the constraints at the quantum level. The Gauss constraints mentioned above can be solved within these representations because the coummute with the constraints $C_{\mu j}$.

Notice that in this picture the Higgs field drops out from quantization which looks bad, see the discussion in the next section[5]. The clock times τ_j to be inserted in the formalism are the measured values of what one usually calls the Higgs expectation value and that one uses in the spontaneous symmetry breaking picture and which find its value into the masses and couplings of the standard model. Notice also that for every function on phase space f which is gauge invariant under the electroweak gauge group fails to do so after we apply the map F_T which effectively replaces the dependence of the Higgs field in f by its nondynamical "expectation value" τ_j. Thus, the map F_T also accomplishes for the spontaneous symmetry breaking. In order that the Higgs field serves as a good clock not only must the corresponding matrix A_{jk} be invertible (which is actually the case, it is proportional to its corresponding conjugate momentum which we assume to be non vanishing) but also that the Higgs field values τ_j should really evolve in nature. Thus, one expects that masses of leptons and quarks are evolving, if only very slowly, if the Higgs field is to be a good clock.

---

[5]One might speculate that this could explain why it has not been directly observed yet, however, it is hard to imagine how the already successful calculations within the electroweak theory and which use the Higgs interactions in an essential way could be accounted for otherwise.

5 Dirac Quantization of Dirac Observables

Reduced Phase Space quantization is not what one usually does, mostly one quantizes before reducing. This is done because the usual belief is that the algebra of Dirac observables, if they can be found at all classically, is too complicated in order that one can control its representation theory. Therfore one starts with a kinematical representation of the full unconstrained phase space which supports the constraints as (densely defined and closable) operators, determines the joint (generalized) kernel of the quantum constraints, computes the induced inner product on that kernel which then becomes the physical Hilbert space and finally represents the (weak) Dirac observables as self adjoint operators on the physical Hilbert space. Here weak Dirac observables are operators which preserve, in an appropriate (generalized) sense, the kernel of the quantum constraints.

On the other hand, we have seen in section 3 that the machinery of [5] allows one to find easily representations of at least a subclass of Dirac observables if the convergence issues mentioned can be resolved. In particular, in the reduced phase space quantization of the algebra of the Q^a, P_a sketched in section 3, the representation that one chooses is already the physical Hilbert space and the Dirac observables are represented by self – adjoint operators there. Thus it seems that the reduced phase space quantization is preferred as it circumvents to work with the representation problem of unphysical quantities altogether. One could object that while the construction of [5] works in principle it is rather involved and $F_{f,T}^\tau$ can hardly be computed explicitly. However, as we must deal with the $F_{f,T}^\tau$ also in the Dirac quantization picture in order to arrive at physical predictions, we are confronted with the same problem also in the constraint quantization programme.

There is, however, a difference between the constraint quantization programme and the reduced phase space quantization programme which leads to physically different predictions: As we have seen, in the reduced phase space programme the clock variables T_j are replaced by real numbers τ_j and their conjugate momenta by the functions $E_j(q^a, p_a, \tau_j)$. Thus T_j is not quantized. On the other hand, in the Dirac quantization programme also the clock variables T_j are quantized as well as the conjugate momenta P^j which are not replaced, via the constraints, in terms of q^a, p_a, T_j. Hence, the representations of the Dirac observables that come from constraint quantization know about the quantum fluctuations of T_j while those of the reduced phase space quantization do not. In particular, different choices of clocks will lead to representations which suppress fluctuations of different clocks. In that sense, the constraint quantization is universal because it treats all variables on the same (quantum) footing. For instance, using the Higgs field as a clock along the lines of section 4 and using a constraint quantization procedure would allow the Higgs field to fluctuate which should be the case as one uses Feynman diagrammes involving Higgs vertices quite successfully in order to compute electroweak processes that are actually measured at CERN. We conclude that the reduced phase space quantization of the Dirac observables can be useful only in a regime where the clocks T_j can be assumed to behave classically. This is of course not the case with respect to any choice of clocks in extreme situations that we would like to access in quantum gravity such as at the big bang. There we necessarily need a constraint quantization of the system.

Notice, however, that the additional representation problem of implementing the multi – fingered time evolution unitarily in the reduced phase space picture is also present in the Dirac quantization picture when we try to simply find an ordering of the quantities Q^a, P_a and $\alpha^t(Q^a), \alpha^t(P_a)$. This problem can be resolved in the same manner, namely by trying to find a self adjoint quantization of the Dirac observables corresponding to the Hamiltonians H_j defined in section 3.

6 Reducing in Steps and Master Constraint

In [5] it was shown that, remarkably, if one is given a constraint algebra of the form[6]

$$\{C_J, C_K\} = f_{JK}{}^L C_L, \quad \{C_J, C_k\} = f_{Jk}{}^l C_l, \quad \{C_j, C_k\} = f_{jk}{}^L C_L \tag{6.1}$$

one can construct weak Dirac observables of the form $F_{f,T}^\tau$ if one has functions f, T_j which are strong Dirac observables with respect to the C_I only. The obvious application is General Relativity where the C_I play the role of the spatial diffeomorphism constraints while the C_j play the role of the Hamiltonian constraint. The former close with structure constants so that it makes sense to compute strong Dirac observables rather than weak ones with respect to the spatial Diffeomorphism constraint.

So the statement is that we need only to choose clock variables T_j with respect to the second set of constraints C_j if f, T_j are already invariant with respect to the first set of the C_I. This is quite remarkable because the $X_{j_1}..X_{j_n} f$ are not obviously invariant under the C_I.

In this section we report on a related observation which results from the considerations in [2] which is to overcome the dificult problem of quantizing constraint algebras of the form (6.1) with structure functions $f_{jk}{}^l$ rather than structure functions and such that the C_I do not form an ideal. The idea of [9] is to construct the (partial) Master constraint

$$M := \frac{1}{2} \sum_{j,k} Q_{jk} C_j C_k \tag{6.2}$$

where Q_{jk} is a positive, symmetric matrix valued function on phase space. The Master Constraint $M = 0$ imposes all the individual constraints $C_j = 0$ simultaneously and the condition $\{F, \{F, M\}\}_{M=0}$ is equivalent to the condition that $\{F, C_j\} \approx 0 \; \forall j$ is a weak Dirac observable with respect to the C_j. In application to General Relativity it is important that one uses a matrix Q_{jk} such that the Master Constraint is invariant with respect to the C_I so that one can first solve the C_I constraints in quantum theory and then the Master constraint (the Master constraint should leave the kernel with respect to the C_I, that is, the associated spatially diffeomorphism invariant Hilbert space, invariant).

It turns out that the Master constraint is very useful in the quantum theory [4] but so far in the classical theory it has not yet been possible to construct weak Dirac observables directly using the condition $\{F, \{F, M\}\}_{M=0}$. One might think that one should simply construct $C' := M/\{M, T\}$ for a single Master clock function T and then apply the machinery of [5] however, the corresponding functions are ill – defined on the constraint surface in general. Thus one must use a different method for instance the one developed in [5].

However, one can fruitfully combine the methods of [5] and [2] as follows: Consider C_I−invariant functions T_j and a C_I−invariant Master constraint M. We may now define new constraints

$$\tilde{C}_j := \{M, T_j\} \approx \sum_{kl} Q_{kl} C_k A_{lj} \tag{6.3}$$

which are equivalent to the old ones since the matrices Q, A are invertible by assumption. Interestingly, the new constraints are C_I−invariant, $\{C_I, \tilde{C}_j\} = 0$ and the constraint algebra (6.1) can be simplified to

$$\{C_J, C_K\} = f_{JK}{}^L C_L, \quad \{C_J, \tilde{C}_k\} = 0, \quad \{\tilde{C}_j, \tilde{C}_k\} = \tilde{f}_{jk}{}^L C_L + \tilde{f}_{jk}{}^l C_l \tag{6.4}$$

---

[6]In [5] the specific form of (6.1) was not assumed, it is enough that the C_I form a subalgebra. However, here we are interested in the application to GR only.

This means that, using the Master constraint, in *General Relativity we may generate new Hamiltonian constraints which are spatially diffeomorphism invariant.* Therefore now the spatial diffeomorphism constraints form an ideal and it is possible to first solve the spatial diffeomorphism constraints and then to impose the Hamiltonian constraints on the spatially diffeomorphism invariant Hilbert space where they then close among themselves (with structure functions). In other words, using the above procedure one can use the philosophy of [2] to work on the spatially diffeomorphism invariant Hilbert space while still using an infinite number of constraints rather than a Master constraint. However, unless the T_j and Q_{jk} can be chosen in such a way that $\tilde{f}_{jk}{}^l = 0$ the algebra still contains structure functions in contrast to the Master constraint proposal of [2] which contains only structure constants which might make the quantization of that algebra difficult.

Nevertheless, at the classical level, using the $\tilde{C}_j$ it is possible to choose C_I-invariant clock variables T_j just with respect to the C_j and still $F_{f,T}^\tau$ is a weak Dirac observable with respect to *all* constraints. The proof is simpler than the one in [5] and goes as follows: Cconsider the yet different but equivalent set of constraints $\tilde{C}_j' = \sum_k (\tilde{A}^{-1})_{jk} \tilde{C}_k$, where $\tilde{A}_{jk} = \{\tilde{C}_j, T_k\}$ is a non - degenerate (symmetric, if $\{T_j, T_k\} = 0$) matrix which is now C_I-invariant as well. It is clear that the corresponding functions $F_{f,T}^\tau$ are exactly C_I-invariant since the $X_{j_1}'..X_{j_n}' \cdot f$ are. We thus just have to show that the Hamiltonian vector fields $\tilde{X}_j'$ of the $\tilde{C}_j'$ are weakly Abelean when applied to C_I-invariant functions f. This follows from a calculation similar to (2.6): Abbreviating $\tilde{B}_{jk} := (\tilde{A}^{-1})_{jk}$ we now have, following exactly the same steps

$$
\begin{aligned}
&\{\tilde{C}_j', \{\tilde{C}_k', f\}\} - \{\tilde{C}_k', \{\tilde{C}_j', f\}\} \\
=& \sum_{m,n} \tilde{B}_{jm} [\sum_{l,i} \tilde{B}_{kl} \tilde{B}_{in} \{\tilde{C}_n, f\} \{T_i, \{\tilde{C}_m, \tilde{C}_l\}\} - \tilde{B}_{kn}(\{f, \{\tilde{C}_m, \tilde{C}_n\}\})] \\
\approx& \sum_{m,n} \tilde{B}_{jm} [-\sum_{l,i,p} \tilde{B}_{kl} \tilde{B}_{in} \{\tilde{C}_n, f\} \tilde{f}_{ml}{}^p \tilde{A}_{pi} + \tilde{B}_{kn} \sum_l \tilde{f}_{mn}{}^l \{\tilde{C}_l, f\}] \\
=& \sum_{m,n,l} \tilde{B}_{jm} [-\tilde{B}_{kl} \{\tilde{C}_n, f\} \tilde{f}_{ml}{}^n + B_{kn} \tilde{f}_{mn}{}^l \{C_l, f\}] = 0
\end{aligned} \tag{6.5}
$$

where the terms proportional to $\{C_I, T_j\}$, $\{C_I, f\}$, which appeared at an intermediate stage in the third step, drop out exactly due to the invariance of T_j, f under the C_I.

7   Conclusions

The proposal of [5] shows that the issue of the construction of Dirac observables for General Relativity is not as hopeless as it seems. While there are many open issues even in the classical theory such as convergence and differentiability of the formal power series constructed, we now have analytical expressions available and these can be used in order to make the framework rigorous in principle. Physical insight will be necessary in order to identify the mathematically most convenient and physically most relevant clocks especially for field theories such as General Relativity.

In the quantum theory, either in the reduced phase space picture or the Dirac constraint quantization picture, the challenge will be to construct the corresponding self – adjoint operators on the physical Hilbert space as well as the generators of the multi – fingered time evolution. This will be very hard but in a dynamical system as complicated as General Relativity this is to be expected. One hopes, of course, that the power series provided in [5] will help to develop a systematic approximation scheme or perturbation theory close to the gauge cut $T = \tau$.

In the present paper we have mainly reported some first observations and ideas. We hope to fill the many gaps in future publications.

Acknowledgements

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