Testing the Master Constraint Programme for Loop Quantum Gravity II. Finite Dimensional Systems

B. Dittrich[*], T. Thiemann[†]

Albert Einstein Institut, MPI f. Gravitationsphysik
Am Mühlenberg 1, 14476 Potsdam, Germany

and

Perimeter Institute for Theoretical Physics
31 Caroline Street North, Waterloo, ON N2L 2Y5, Canada

Preprint AEI-2004-117

Abstract

This is the second paper in our series of five in which we test the Master Constraint Programme for solving the Hamiltonian constraint in Loop Quantum Gravity. In this work we begin with the simplest examples: Finite dimensional models with a finite number of first or second class constraints, Abelean or non − Abelean, with or without structure functions.

[*]dittrich@aei.mpg.de, bdittrich@perimeterinstitute.ca
[†]thiemann@aei.mpg.de, tthiemann@perimeterinstitute.ca

Contents

1 Introduction

We continue our test of the Master Constraint Programme [1] for Loop Quantum Gravity (LQG) [6, 7, 8] which we started in the companion paper [2] and will continue in [3, 4, 5]. The Master Constraint Programme is a new idea to improve on the current situation with the Hamiltonian constraint operator for LQG [9]. In short, progress on the solution of the Hamiltonian constraint has been slow because of a technical reason: the Hamiltonian constraints themselves are not spatially diffeomorphism invariant. This means that one cannot first solve the spatial diffeomorphism constraints and then the Hamiltonian constraints because the latter do not preserve the space of solutions to the spatial diffeomorphism constraint [10]. On the other hand, the space of solutions to the spatial diffeomorphism constraint [10] is relatively easy to construct starting from the spatially diffeomorphism invariant representations on which LQG is based [11] which are therefore very natural to use and, moreover, essentially unique. Therefore one would really like to keep these structures. The Master Constraint Programme removes that technical obstacle by replacing the Hamiltonian constraints by a single Master Constraint which is a spatially diffeomorphism invariant integral of squares of the individual Hamiltonian constraints which encodes all the necessary information about the constraint surface and the associated invariants. See e.g. [1, 2] for a full discussion of these issues. Notice that the idea of squaring constraints is not new, see e.g. [12], however, our concrete implementation is new and also the Direct Integral Decomposition (DID) method for solving them, see [1, 2] for all the details.

The Master Constraint for four dimensional General Relativity will appear in [14] but before we test its semiclassical limit, e.g. using the methods of [15, 16] and try to solve it by DID methods we want to test the programme in the series of papers [2, 3, 4, 5]. We begin by studying finite dimensional systems, in particular: a finite number of Abelean constraints linear in the momenta which will also play an important role for [4], a system with second class constraints, first class constraints with structure constants at most quadratic in the momenta and first class constraints linear momenta with structure functions.

2 Finite Number of Abelean First Class Constraints Linear in the Momenta

Our first example is a system with configuration manifold $\mathbb{R}^n$, coordinatized by $x^i, i = 1, \ldots, n$ and $m < n$ commuting constraints

$$C_i = p_i \quad i = 1, \ldots, m \quad , \tag{2.1}$$

where the p_i's are the conjugated momenta to the x^i's.

All phase space functions which do not depend on x_i are Dirac observables, i.e. functions which commute with the constraints (on the constraint hypersurface). A Dirac observable which depends on p_i, $i = 1, \ldots, m$ is equivalent to the Dirac observable obtained from the first one by setting $p_i = 0$ (since these two observables will coincide on the constraint hypersurface). Therefore it is sufficient to consider observables which are independent of the first m configuration observables and of the first m conjugated momenta. A canonical choise for an observable algebra is the one generated by x^i, p_i, $i = (m + 1), \ldots, n$.

To quantize the system, we will start with an auxilary Hilbert space $L_2(\mathbb{R}^n)$ on which the operators $\hat{x}^i$ act as multiplication operators and the momenta as derivatives, i.e. we use the standard Schrödinger representation:

$$\hat{x}^i\,\psi(\mathbf{x}) = x^i\,\psi(\mathbf{x}) \qquad \hat{p}_i\psi(\mathbf{x}) = -i\hbar\,\partial_i\psi(\mathbf{x}) \quad . \tag{2.2}$$

According to the Master Constraint Programme, we have to consider the spectral resolution of the Master Constraint

$$\widehat{\mathbf{M}} = -\hbar^2 \sum_{i=1}^{m} \partial_i^2 =: -\hbar^2 \Delta_m \quad . \tag{2.3}$$

This spectral resolution can be constructed with the help of the spectral resolutions of the operators $\hat{p}_i = -i\hbar\partial_i$, which are well known to be given by the Fourier transform:

$$\psi(\mathbf{x}) = \int_{\mathbb{R}} < b_{k_i}, \psi(x^1, \ldots, x^{i-1}, \cdot, x^{i+1}, x^n) >_i b_{k_i}(x^i) dk_i \tag{2.4}$$

with generalized eigenfunctions

$$\begin{aligned} & b_{k_i}(x^i) = \tfrac{1}{\sqrt{2\pi}} \exp(ik_i x^i) \qquad \text{and} \\ & < b_{k_i}, \psi(x^1, \ldots, x^{i-1}, \cdot, x^{i+1}, x^n) >_i = \int_{\mathbb{R}} \tfrac{1}{\sqrt{2\pi}} \exp(-ik_i x^i)\psi(\mathbf{x}) dx^i \end{aligned} \tag{2.5}$$

and $\psi \in L_2(\mathbb{R}^n)$. The spectrum of $\hat{p}_i$ is therefore $\mathrm{spec}(\hat{p}_i) = \mathbb{R}$. Moreover the spectral projectors of $\hat{p}_i$ and $\hat{p}_j$ commute, so we can achieve a simultaneous diagonalization of all the C_i's (that is the Fourier transform with respect to $(x^1, \ldots, x^m)$:

$$\psi(\mathbf{x}) = \int_{\mathbb{R}^m} < \prod_{i=1}^{m} b_{k_i}(\cdot), \psi(\cdots, x_{m+1}, \ldots, x_n) >_{(m)} \left(\prod_{i=1}^{m} b_{k_i}(x^i) \right) dk_1 \cdots dk_m \tag{2.6}$$

with

$$< \prod_{i=1}^{m} b_{k_i}(\cdot), \psi(\cdots, x_{m+1}, \ldots, x_n) >_{(m)} = \int_{\mathbb{R}^m} \left(\prod_{i=1}^{m} b_{k_i}(x^i) \right) \psi(\mathbf{x}) dx^1 \cdots dx^m \quad . \tag{2.7}$$

This decomposition of a function $\psi \in L_2(\mathbb{R}^n)$ corresponds to a decomposition of the Hilbert space $L_2(\mathbb{R}^n)$ into a direct integral of Hilbert spaces

$$L_2(\mathbb{R}^n) \simeq \int_{\mathbb{R}^m} \mathcal{H}_{(k_1, \ldots, k_m)} dk_1 \cdots dk_m \quad . \tag{2.8}$$

where each $\mathcal{H}_{(k_1,\ldots,k_m)}$ is isomorphic to $L_2(\mathbb{R}^{n-m})$.

Since the Master Constraint Operator is a polynomial of the C_i's, according to spectral calculus we already achieved the spectral resolution of the Master Constraint. The spectrum of $\widehat{\mathbf{M}}$ is given by $\mathrm{spec}(\hat{M}) = \mathrm{clos}\{\sum_{i=1}^m k_i^2,\, k_i \in \mathbb{R}\} = \mathbb{R}_+$. The generalized eigenfunctions of $\hat{M}$ to the (generalized) eigenvalue 0 can be read off from (2.6) to be

$$\begin{aligned}\Psi(\mathbf{x}) &= \frac{1}{(\sqrt{2\pi})^m} \exp(i \sum_{i=1}^m k_i x^i)_{|k_i=0} \psi(x^{m+1}, \cdots, x^n) \\ &= \frac{1}{(\sqrt{2\pi})^m} \psi(x^{m+1}, \cdots, x^n) \qquad \text{with} \quad \psi \in L_2(\mathbb{R}^{n-m}) \end{aligned} \tag{2.9}$$

i.e. functions which do not depend on $(x^1, \ldots, x^m)$. So one could conclude that the physical Hilbert space is $\mathcal{H}_{(0,\ldots,0)} \simeq L_2(\mathbb{R}^{n-m})$, a space of functions of $(x^{m+1}, \ldots, x^n)$. The action of the elementary Dirac observables $\hat{x}^i$ and $\hat{p}_i$ for $i = (m+1), \ldots, n$ is well defined on this physical Hilbert space.

However we would like to go through the explicit procedure for constructing the direct integral decomposition for a seperable Hilbert space, since we will use it later on in another example. The general procedure is explained in detail in [2].

We will work in the fourier-transformed picture, i.e. the Hilbert space is $L_2(\mathbb{R}^n)$ as a function space of the momenta $(p_1, \ldots, p_n)$ and the Master Constraint Operator becomes the multiplication operator $\widehat{\mathbf{M}} = \sum_{i=1}^m p_i^2$.

To begin with we have to choose a set of orthonormal vectors $\{\Omega_i\}_i$ from which $\mathcal{H} = L_2(\mathbb{R}^n)$ can be generated through repeated applications of the Master Constraint Operator $\widehat{\mathbf{M}}$. To this end we use the fact (see [21]), that

$$\{\exp\big(-\tfrac{1}{2}\sum_{j=1}^m p_j^2\big)\big(\sum_{j=1}^m p_j^2\big)^k \psi_{s,l}\,|s,k \in \mathbb{N}, l \in \mathfrak{l}_s\} \tag{2.10}$$

is a basis for $L_2(\mathbb{R}^m)$. Here $\psi_{s,l}$ are harmonic polynomials[1] of degree s on $\mathbb{R}^m$, and the index l takes values in some finite index set $\mathfrak{l}_s$, which depends on m and s. Obviously the set (2.10) can be generated by applying repeatedly $\widehat{\mathbf{M}}$ (more correctly $\tilde{M}$ where $\widehat{\mathbf{M}} = \tilde{M} \otimes 1_{L_2(\mathbb{R}^{m-n})}$, so that $\tilde{M}$ is an operator on $L_2(\mathbb{R}^m)$) from the set

$$\{\tilde{\Omega}_{sl} := \exp\big(-\tfrac{1}{2}\sum_{j=1}^m p_j^2\big)\psi_{s,l}\,|s \in \mathbb{N}, l \in \mathfrak{l}_s\} \quad . \tag{2.11}$$

Moreover the spaces generated from different $\tilde{\Omega}_{s,l}$ are mutually orthogonal. (This can be seen if one introduces spherical coordinates. Then the restrictions of the harmonic polynomials to the unit sphere, which are given by the spherical harmonics, are orthogonal in $L_2(S^{m-1}, d\omega_{n-1})$ where $d\omega_{n-1}$ is the uniform measure on the sphere S^{m-1}.) So we conclude that the set of orthonormal vectors

$$\{\Omega_{slt} := N_{sl} \exp\big(-\tfrac{1}{2}\sum_{j=1}^m p_j^2\big)\psi_{s,l} \otimes \phi_t(p_{m+1}, \ldots, p_n)\,|s \in \mathbb{N}, l \in \mathfrak{l}_s, t \in \mathfrak{t}\} \quad . \tag{2.12}$$

where N_{sl} are normalization constants and $\{\phi_t\}_{t\in\mathfrak{t}}$ is an orthonormal basis of $L_2(\mathbb{R}^{n-m})$ fulfills all the demands required above, i.e.

$$L_2(\mathbb{R}^n) = \sum_{s,l,t}{}^{\oplus} \overline{\mathrm{span}\{p(\widehat{\mathbf{M}})\Omega_{slt}|p \quad \text{polynomial}\}} \quad . \tag{2.13}$$

[1] i.e. homogeneous polynomials annihilated by the Laplace operator on $\mathbb{R}^m$

Now the vectors Ω_{slt} are tensor products of the form $\Omega_{slt} = \Omega_{sl} \otimes \phi_t$ where $\Omega_{sl} \in \mathcal{H}_1 := L_2(\mathbb{R}^m)$ and $\phi_t \in \mathcal{H}_2 := L_2(\mathbb{R}^{n-m})$ and furthermore $\widehat{\mathbf{M}} = \tilde{M} \otimes 1_{\mathcal{H}_2}$ acts only on the first factor. Hence we have for the spectral measures $\mu_{slt}(\lambda)$:

$$\begin{aligned} \mu_{slt} &:= <\Omega_{sl} \otimes \phi_t, (\Theta(\lambda - \tilde{M})\Omega_{sl}) \otimes \phi_t >_{\mathcal{H}} = <\Omega_{sl}, \Theta(\lambda - \tilde{M})\Omega_{sl} >_{\mathcal{H}_1} ||\phi_t||_{\mathcal{H}_2} \\ &= <\Omega_{sl}, \Theta(\lambda - \tilde{M})\Omega_{sl} >_{\mathcal{H}_1} =: \mu_{sl} \quad , \end{aligned} \tag{2.14}$$

so that we only need to consider the spectral measures μ_{sl} with respect to the operator $\tilde{M}$ on $\mathcal{H}_1$. Additionally, using the rotational symmetry of the Master Constraint Operator (and an idea outlined in [21]) one can simplify the calculations even more: The space of harmonic polynomials on $\mathbb{R}^m$ of degree s is an irreducible module for the rotation group $O(m)$ under the left regular representation, which acts as

$$U(R) : \psi(\vec{p}) \mapsto \psi(R^{-1} \cdot \vec{p}) \tag{2.15}$$

on the space of functions over $\mathbb{R}^m$. Here R is a rotation matrix for $\mathbb{R}^m$. This representation is unitary if considered on the Hilbert space $\mathcal{H}_1 = L_2(\mathbb{R}^m)$. The irreducibilty of the representation ensures that one can generate all vectors $\{\Omega_{sl}\}_{l \in \mathfrak{l}}$ by applying rotations $U(R)$ to just one vector, say $\Omega_s := \Omega_{s0}$. But these rotations leave the spectral projectors of $\tilde{M}$ invariant (since $\tilde{M}$ commutes with the rotations $U(R)$), so that we have

$$\begin{aligned} \mu_{sl} &= <\Omega_{sl}, \Theta(\lambda - \tilde{M})\Omega_{sl} >=< U(R)\Omega_{s0}, \Theta(\lambda - \tilde{M})U(R)\Omega_{s0} > \\ &= <\Omega_{s0}, \Theta(\lambda - \tilde{M})\Omega_{s0} >= \mu_{s0} =: \mu_s \end{aligned} \tag{2.16}$$

for an appropriate rotation R. Here one can see, that it is very helpful to know the symmetries of the Master Constraint Operator, i.e. unitary operators commuting with $\widehat{\mathbf{M}}$. These may come from (exponentiated) strong Dirac observables, that is operators which commute with all the constraints on the whole Hilbert space $\mathcal{H}$. Examples for these are operators of the type $1_{\mathcal{H}_1} \otimes U_2$, i.e. unitary operators which act only on $\mathcal{H}_2 = L_2(\mathbb{R}^{n-m})$. We used this kind of symmetry in (2.14). However the $U(R)$'s from above are not of this type: The classical counterparts of their generators (i.e. angular momentum on $\mathbb{R}^m$) Poisson-commutes with the Master Constraint but they also vanish on the constraint hypersurface $\{p_i = 0, i \leq m\}$. Nevertheless the rotation group $O(m)$ is useful in constructing the direct Hilbert space decomposition. Later we will see, that the vanishing of its (classical) generators on the constraint hypersurface corresponds to the fact, that the representation of the rotation group on the induced Hilbert space is trivial.

So we just need to select for each $s \in \mathbb{N}$ a particular homogeneous harmonic polynomial ψ_s of degree s. We choose

$$\psi_s(p_1, \ldots, p_m) = (p_1 + ip_2)^s \quad ; \ s = 0, 1, 2, \ldots \tag{2.17}$$

so that the vectors Ω_s become

$$\Omega_s = \frac{1}{\sqrt{\pi^{m/2} s!}} e^{-\frac{1}{2} r_m^2} r_2^s e^{is\varphi} \tag{2.18}$$

where we introduced the coordinates $r_m = \sum_{i=1}^m p_i^2$, $r_2^2 = p_1^2 + p_2^2$ and the angle $\varphi = \arctan(p_1/p_2)$.

With this at hand we can compute the spectral measures μ_s (we will assume that $m \geq 4$):

$$\begin{aligned}
\mu_s(\lambda) &= \quad < \Omega_s, \Theta(\lambda - \tilde{M})\Omega_s >_{\mathcal{H}_1} \\
&= \quad \frac{1}{\pi^{m/2}\, s!} \int_{\mathbb{R}^m} d^m p\, \Theta(\lambda - r_m^2) e^{-r_m^2} r_2^{2s} \\
&\underset{r_m^2 = r_2^2 + r_{m-2}^2}{=} \quad \frac{1}{\pi^{m/2}\, s!} \int_0^{2\pi} d\varphi \int_{\mathbb{R}_+} dr_2 \int_{S^{m-3}} d\omega_{m-3} \int_{\mathbb{R}_+} dr_{m-2} \Theta(\lambda - r_m^2) r_2^{2s+1} r_{m-2}^{m-3} e^{-r_m^2} \\
&\underset{\substack{r_2 = r_m \cos\phi \\ r_{m-2} = r_m \sin\phi}}{=} \quad \frac{2\pi \mathrm{Vol}(S^{m-3})}{\pi^{m/2}\, s!} \int_0^{\pi/2} d\phi \int_{\mathbb{R}_+} dr_m\, \Theta(\lambda - r_m^2) r_m^{2s+m-1} (\cos\phi)^{2s+1} (\sin\phi)^{m-3} e^{-r_m^2} \\
&\underset{y = r_m^2}{=} \quad \frac{2\pi \mathrm{Vol}(S^{m-3})}{\pi^{m/2}\, s!} \frac{s!\, \Gamma(m/2-1)}{2\,\Gamma(s+m/2)} \frac{1}{2} \int_{\mathbb{R}_+} dy\, \Theta(\lambda - y)\, y^{s+m/2-1} e^{-y} \\
&= \quad \frac{1}{\Gamma(s+m/2)} \int_{\mathbb{R}_+} dy\, \Theta(\lambda - y)\, y^{s+m/2-1} e^{-y} \qquad (2.19)
\end{aligned}$$

From the second to the third line we transformed the cartesian coordinates $(p_3, \ldots, p_m)$ to spherical coordinates on $\mathbb{R}^{m-2}$ with radial coordinate $r_{m-2}^2 = \sum_{i=3}^m p_i^2$ and angels varying over S^{m-3}. $\mathrm{Vol}(S^{m-3})$ is the euclidian volume of the $(m-3)$-dimensional unit sphere. From the third to the fourth line we again introduced spherical coordinates (r_m, ϕ) for the two radii r_2 and r_{m-2}, so that $r_m^2 = r_2^2 + r_{m-2}^2$. Since r_2, r_{m-2} are positive, ϕ varies over $[0, \pi/2]$. Then we integrated over ϕ and used a new integration variable $y = r_m^2$ for the remaining integral. In the last step we used that $\mathrm{Vol}(S^{m-3}) = 2\pi^{m/2-1}/\Gamma(m/2-1)$.

To obtain the final spectral measure $\mu(\lambda)$ we have to sum over all the measures $\mu_{slt} \equiv \mu_s$ multiplied by constants α_{slt}. We choose the constants such that $\sum_{l,t} \alpha_{slt} = 2^{-s-1}$, so that the sum over all the α_{slt}'s is one. The spectral measure becomes

$$\begin{aligned}
\mu(\lambda) &= \quad \sum_{s,l,t} \alpha_{slt}\, \mu_{slt}(\lambda) = \sum_{s=0}^{\infty} 2^{-s-1} \mu_s(\lambda) \\
&= \quad \frac{2\pi \mathrm{Vol}(S^{m-3})}{2\,\pi^{m/2}} \sum_{s=0}^{\infty} \frac{2^{-s}}{s!} \int_{\mathbb{R}_+} dr_2 \int_{\mathbb{R}_+} dr_{m-2} \Theta(\lambda - r_m^2) r_2^{2s+1} r_{m-2}^{m-3} e^{-r_m^2} \\
&= \quad \frac{\pi \mathrm{Vol}(S^{m-3})}{\pi^{m/2}} \int_{\mathbb{R}_+} dr_2 \int_{\mathbb{R}_+} dr_{m-2} \Theta(\lambda - r_m^2) r_2\, e^{\frac{1}{2} r_2^2} r_{m-2}^{m-3} e^{-r_m^2} \\
&\underset{x = r_2^2/2}{=} \quad \frac{\pi \mathrm{Vol}(S^{m-3})}{\pi^{m/2}} \int_{\mathbb{R}_+} dx \int_{\mathbb{R}_+} dr_{m-2} \Theta(\lambda - r_{m-2}^2 - 2x) e^{-x} e^{-r_{m-2}^2} r_{m-2}^{m-3} \\
&= \quad \frac{\pi \mathrm{Vol}(S^{m-3})}{\pi^{m/2}} \int_{\mathbb{R}_+} dr_{m-2} \Theta(\lambda - r_{m-2}^2) \big(1 - e^{-\frac{1}{2}\lambda + \frac{1}{2} r_{m-2}^2}\big) e^{-r_{m-2}^2} r_{m-2}^{m-3} \\
&\underset{y = r_{m-2}^2}{=} \quad \frac{1}{\Gamma(m/2-1)} \int_{\mathbb{R}_+} dy\, \Theta(\lambda - y) \big(1 - e^{-\frac{1}{2}\lambda + \frac{1}{2} y}\big) e^{-y} y^{m/2-2} \qquad (2.20)
\end{aligned}$$

Here we inserted for μ_s the third line of (2.19), exchanged integration and summation, performed a variable transformation $x = r_2^2/2$ in the fourth line, and integrated over the new variable x in the fifth line. Finally we changed to the integration variable $y = r_{m-2}^2$ in the last line and used the explicit expression $2\pi^{m/2-1}/\Gamma(m/2-1)$ for $\mathrm{Vol}(S^{m-3})$. This gives for the derivative of $\mu(\lambda)$:

$$\begin{aligned}\frac{d\mu(\lambda)}{d\lambda} &= \frac{1}{\Gamma(m/2-1)}\int_{\mathbb{R}_+} dy\,\delta(\lambda-y)\big(1-e^{-\frac{1}{2}\lambda+\frac{1}{2}y}\big)e^{-y}y^{m/2-2} \\ &\quad+\frac{1}{\Gamma(m/2-1)}\int_{\mathbb{R}_+} dy\,\Theta(\lambda-y)\big(\tfrac{1}{2}e^{-\frac{1}{2}\lambda+\frac{1}{2}y}\big)e^{-y}y^{m/2-2} \\ &= \frac{1}{2\,\Gamma(m/2-1)}\,e^{-\frac{1}{2}\lambda}\int_{\mathbb{R}_+} dy\,\Theta(\lambda-y)e^{-\frac{1}{2}y}y^{m/2-2} \quad . \end{aligned} \tag{2.21}$$

Next we have to calculate the Radon-Nikodym derivatives $\rho_s = \rho_{slt}$ of μ_s with respect to μ. Since both measures are absolutely continuous with respect to the Lebesgue measure, we can write

$$\rho_s(\lambda) = \frac{d\mu_s(\lambda)}{d\mu(\lambda)} = \frac{d\mu_s(\lambda)/d\lambda}{d\mu(\lambda)/d\lambda} = \frac{2\,\Gamma(m/2-1)}{\Gamma(m/2+s)}\lambda^{s+m/2-1}e^{-\frac{1}{2}\lambda}\left[\int_0^\lambda dy\,e^{-\frac{1}{2}y}y^{m/2-2}\right]^{-1} \tag{2.22}$$

For $0<\lambda\leq\infty$ the derivatives $\rho_s(\lambda)$ will be some strictly positive numbers for all $s\in\mathbb{N}$. But for the limit $\lambda\to 0$ we apply L'hospital's rule, resulting in

$$\lim_{\lambda\to 0}\rho_s(\lambda) = \frac{2\,\Gamma(m/2-1)}{\Gamma(m/2+s)}\lim_{\lambda\to 0}\frac{(s+m/2-1)\lambda^{s+m/2-2}-\frac{1}{2}\lambda^{s+m/2-1}}{\lambda^{m/2-2}} \tag{2.23}$$

which gives $\rho_s(0)=0$ for $s>0$ and $\rho_0(0)=2$.

Now we can construct the induced Hilbert space $\mathcal{H}^\oplus(0)$. We notize that there is only one linearly independent harmonic polynomial of degree zero, namely $\psi_{00}\equiv 1$. Therefore $\mathcal{H}^\oplus(0)$ has an orthonormal basis $\{e_t\}_{t\in\mathfrak{t}}$ which corresponds to the set of vectors $\{\Omega_{00t}\}_{t\in\mathfrak{t}}$. Hence we can identify $\mathcal{H}^\oplus(0)$ with the Hilbert space $L_2(\mathbb{R}^{n-m})$ of functions in the variables $(p_{m+1},\ldots,p_n)$. Interestingly, because of $\rho_{slt}(\lambda)>0$ for $\lambda>0$, the induced Hilbert spaces $\mathcal{H}^\oplus(\lambda)$ for $\lambda>0$ are in some sense much bigger, since they have a basis $\{e_{slt}|s\in\mathbb{N},\,l\in\mathfrak{l},\,t\in\mathfrak{t}\}$ corresponding to all the vectors Ω_{slt}. The latter can be seen as a basis in $L_2(S^{m-1},\lambda^{\frac{m-1}{2}}d\omega_{m-1})\otimes L_2(\mathbb{R}^{n-m})$, therefore we can identify $\mathcal{H}^\oplus(\lambda)$ with $L_2(S^{m-1},\lambda^{\frac{m-1}{2}}d\omega_{m-1})\otimes L_2(\mathbb{R}^{n-m})$ for $\lambda>0$.

We would like to mention, that the space of (formal) solutions to the Master Constraint Operator is much bigger than the space of functions, which are independent of the first m coordinates. For instance, functions, harmonic in the first m coordinates, are solutions of the master constraint but they are unphysical because they do not solve the individual constraints C_i. These solutions automatically do not appear in the spectral resolution of the Master Constraint. The intuitive reason for this will be discussed in the conclusions.

3 A Second Class System

Here we will discuss a simple second class system, given by the constraints

$$C_i = x^i \qquad \text{and} \qquad B_i = p_i \qquad i=1,\ldots,m \tag{3.1}$$

on the phase space $\mathbb{R}^{2n}$. (The x^i denote configuration variables and the p_i their conjugated momenta.) The Poisson brackets are given by $\{C_i,B_j\}=\delta_{ij}$ and all other Poisson brackets vanish. Because of the Heisenberg Uncertainty relations one cannot expect to find eigenfunctions of the Master Constraint Operator corresponding to the eigenvalue zero. We will therefore alter the Master Constraint and validate whether this gives sensible results.

A complete set of observables is given by phase space functions which are independent on the first m configuration variables and momenta.

As in the last case we will quantize the system by choosing the auxilary Hilbert space $\mathfrak{L}^2(\mathbb{R}^n)$ of functions of the configuration variables $\psi(\mathbf{x})$. The operators $\hat{x}^i$ act as multiplication operators and the momentum operators p_i as derivatives.

The Master Constraint Operator is defined as the sum of the squares of the constraints but we can as well consider the following slight variation of this prescription:

$$\widehat{\mathbf{M}} = \sum_{i=1}^{m} \tfrac{1}{2}(\hat{p}_i^2 + \omega_i^2(\hat{x}^i)^2) \quad , \tag{3.2}$$

where ω_i are m positive constants. This Master Constraint coincides with the Hamiltonian for m uncoupled harmonic oscillators with different frequencies. This Hamiltonian has pure point and positive spectrum, the lowest eigenvalue being $\lambda_0 = \frac{\hbar}{2}\sum_{i=1}^m \omega_i$.

Since zero is not in the spectrum of the Master Constraint Operator we have to alter $\widehat{\mathbf{M}}$, such that its spectrum includes zero. Of course we have to check in the end, whether this procedure gives a sensible quantum theory. The simplest thing one can do, is to subtract λ_0 from the Master Constraint to obtain $\widehat{\mathbf{M}}' = \widehat{\mathbf{M}} - \lambda_0$. This is equivalent to the normal ordering of the Master Constraint Operator. This could be done also for the limit $m \to \infty$ if we multiply the individual operators in the sum of (3.2) by positive constants Q_m with $\sum_m Q_m < \infty$. This condition will naturally reappear in the free field theory examples of [4].

Now the spectrum of $\widehat{\mathbf{M}}'$ includes zero and we can construct the induced Hilbert space to the eigenvalue zero. To this end we have to find a cyclic basis, which we will choose to be a tensor basis of the following kind:

$$\Omega_{n_1,\ldots,n_m,k} = f_{n_1,\ldots,n_m}(x_1,\ldots,x_m) \otimes h_k(x_{m+1},\ldots,x_n) \tag{3.3}$$

where $f_{n_1,\ldots,n_m}$, $n_i \in \mathbb{N}$ are the (normalized) eigenfunctions of $\widehat{\mathbf{M}}$ (considered on $L_2(\mathbb{R}^m)$) and $\{h_k \,|\, k \in \mathbb{N}\}$ is an orthonormal basis in $L_2(\mathbb{R}^{n-m})$. The application of $\widehat{\mathbf{M}}$ to the vectors $\Omega_{n_1,\ldots,n_m,k}$ just multiplies them with the corresponding eigenvalue so that we need them all in our cyclic basis. The associated spectral measures are

$$\mu_{n_1,\ldots,n_m,k}(\lambda) =< \Omega_{n_1,\ldots,n_m,k},\, \Theta(\lambda - \widehat{\mathbf{M}}')\Omega_{n_1,\ldots,n_m,k} >= \Theta(\lambda - \lambda_{n_1,\ldots,n_m}) \tag{3.4}$$

where $\lambda_{n_1,\ldots,n_m} = \frac{\hbar}{2}\sum_{i=1}^m n_i\,\omega_i$. Only measures with $n_1 = n_2 = \ldots = n_m = 0$ have the point zero in their support. The final spectral measure can be defined to be

$$\mu(\lambda) = \sum_{n_1,\ldots,n_m,k\in\mathbb{N}} 2^{-(n_1+1)} \cdots 2^{-(n_m+1)} \cdot 2^{-(k+1)} \mu_{n_1,\ldots,n_m,k}(\lambda) = \Theta(\lambda) + \nu(\lambda) \tag{3.5}$$

where $\nu(\lambda) = \sum_{n_1,\ldots,n_m \geq 1, k\in\mathbb{N}} 2^{-(n_1+1)} \cdots 2^{-(n_m+1)} \cdot 2^{-(k+1)} \mu_{n_1,\ldots,n_m,k}(\lambda)$ is a pure point measure which does not have support at zero. The relevant Radon-Nikodym derivatives for the induced Hilbert space to the eigenvalue zero are therefore

$$\rho_{n_1=0,\ldots,n_m=0,k}(0) = 1 \quad , \tag{3.6}$$

hence we can interpret the set $\{\Omega_{n_1=0,\ldots,n_m=0,k} \,|\, k \in \mathbb{N}\}$ as a basis in the induced Hilbert space. So as expected for a pure point spectrum the induced Hilbert space is just the (proper) eigenspace to the eigenvalue zero. This eigenspace can be identified with the space $L_2(\mathbb{R}^{n-m})$ by mapping $\Omega_{n_1=0,\ldots,n_m=0,k}$ to the basis vector $h_k(x_{m+1},\ldots,x_n)$ in $L_2(\mathbb{R}^{n-m})$. Also the action of Dirac observables (that is quantized phase space functions, which do not depend on the first m configuration variables and momenta) coincides on the null eigenspace of $\widehat{\mathbf{M}}'$ and on $L_2(\mathbb{R}^{n-m})$. We can therefore conclude, that the physical Hilbert space is given by $L_2(\mathbb{R}^{n-m})$, which is the result one would expect beforehand.

4 Finite Number of First Class Constraints at most Quadratic in the Momenta

If the constraints generate a (semi-simple) compact Lie group it is in general straightforward to apply the Master Constraint Programme. The Master Constraint Operator coincides in this case with the Casimir of the Lie group and has a pure point spectrum. The direct integral decomposition of the kinematical Hilbert space (which in this case is truly a direct sum decomposition) is equivalent to the reduction of the given representation of the Lie group into irreducible representations and the physical Hilbert space corresponds to the isotypical component[2] of the equivalence class of the trivial representation.

4.1 $SU(2)$ Model with Compact Gauge Orbits

Here we consider the configuration space $\mathbb{R}^3$ with the three $so(3)$-generators as constraints:

$$L_i = \epsilon^k_{\ ij} x^j p_k, \qquad \{L_i, L_j\} = \epsilon^k_{\ ij} L_k \tag{4.1}$$

where $\epsilon^k_{\ ij} = \epsilon_{ijk}$ is totally antisymmetric with $\epsilon_{123} = 1$ and we summed over repeated indices. (In the following, indices will be raised and lowered with respect to the metric $g_{ik} = \mathrm{diag}(+1, +1, +1)$.)

The observable algebra of this system is generated by

$$\begin{array}{ccc} d = x^i p_i & e^+ = x^i x_i & e^- = p^i p_i \\ \{d, e^\pm\} = \mp 2 e^\pm & & \{e^+, e^-\} = 4d \quad . \end{array} \tag{4.2}$$

It constitutes an $sl(2,\mathbb{R})$-algebra. We have the identity

$$-d^2 + e^+ e^- = L_i L^i \tag{4.3}$$

between the Casimirs of the constraint and observable algebra.

4.1.1 Quantization

We start with the auxilary Hilbert space $\mathfrak{L}^2(\mathbb{R}^3)$ of square integrable functions of the coordinates. The momentum operators are $\hat{p}_j = -i(\hbar)\partial_j$ and the $\hat{x}^j$ act as multiplication operators. There arises no factor ordering ambiguity for the quantization of the constraints, but for the observable algebra to close, we have to choose:

$$\hat{d} = \tfrac{1}{2}(\hat{x}^i \hat{p}_i + \hat{p}_i \hat{x}^i) = \hat{x}^i \hat{p}_i - \tfrac{3}{2} i\hbar \qquad \hat{e}^+ = \hat{x}^i \hat{x}_i \qquad \hat{e}^- = \hat{p}^i \hat{p}_i \tag{4.4}$$

The commutators between constraints and between observables are then obtained by replacing the Poisson bracket with $\frac{1}{i\hbar}[\cdot,\cdot]$.

The identity (4.3) is altered to:

$$\hat{d}^2 - \tfrac{1}{2}(\hat{e}^+ \hat{e}^- + \hat{e}^- \hat{e}^+) - \tfrac{3}{4}\hbar^2 = -\hat{L}_i \hat{L}^i \quad . \tag{4.5}$$

For the implementation of the Master Constraint Progamme we have to construct the direct integral decomposition with respect to the Master Constraint Operator

$$\widehat{\mathsf{M}} := \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2 = -\left(\hat{d}^2 - \tfrac{1}{2}(\hat{e}^+ \hat{e}^- + \hat{e}^- \hat{e}^+) - \tfrac{3}{4}\hbar^2\right) \tag{4.6}$$

[2] Compact Lie groups are completely reducible. The isotypical component of a given equivalence class within a reducible representation is the direct sum of all its irreducible representations into which it can be decomposed and which lie in the given equialence class.

The Master Constraint Operator is the Casimir of $SO(3)$ on $\mathfrak{L}^2(\mathbb{R}^3)$. Its spectrum and its (normalized) eigenfunction are well known, the latter are given by the spherical harmonics. To discuss these we make a coordinate transformation to spherical coordinates:

$$\begin{aligned} x_1 &= r\cos(\phi)\sin(\theta) \quad \phi\in[0,2\pi) \text{ and } \theta\in[0,\pi) \\ x_2 &= r\sin(\phi)\sin(\theta) \\ x_3 &= r\cos(\theta) \end{aligned} \tag{4.7}$$

$$d^3x = r^2\sin(\theta)dr\,d\theta\,d\phi$$

$$\widehat{L^2} = \hat{L}_1^2+\hat{L}_2^2+\hat{L}_3^2 = -\hbar^2\Big[\frac{1}{\sin^2\theta}\partial_\phi^2+\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta)\Big] \tag{4.8}$$

$$\begin{aligned} \hat{p}^2 &= -\hbar^2\Delta = -\hbar^2\Big(\frac{1}{r^2}\partial_r(r^2\,\partial_r)+\frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta)+\frac{1}{r^2\sin^2\theta}\partial_\phi^2\Big) \\ &= -\hbar^2\frac{1}{r^2}\partial_r(r^2\,\partial_r)+\frac{1}{r^2}\widehat{L^2} \\ \hat{x}^2 &= r^2 \end{aligned} \tag{4.9}$$

The eigenfunctions of $\widehat{\mathbf{M}}=\widehat{L^2}$ to the eigenvalue $\hbar^2\,l(l{+}1), l\in\mathbb{N}$ are of the form $Y_{lm}(\theta,\phi)\,R(r)$ where Y_{lm}, $l\in\mathbb{N}, -l\leq m\leq l$ are the spherical harmonics on the two-dimensional sphere S^2 and $R(r)$ is an arbitrary function in $L_2(\mathbb{R}_+,r^2dr)$. To discuss the direct integral decomposition of the kinematical Hilbert space $L_2(\mathbb{R}^3)$ we have to find a cyclic basis of this Hilbert space. We will choose the eigenbasis of $\hat{H}:=\hat{e}^-+\hat{e}^+$, i.e. the three-dimensional harmonic oscillator Hamiltonian. Its (normalized) eigenfunctions are given by [22]

$$\psi_{nlm}(r,\theta,\phi) = N_{nl}\,r^l\,\exp(-\frac{r^2}{2\hbar})\,M(-n,l+\tfrac{3}{2},\tfrac{1}{\hbar}r^2)\,Y_{lm}(\theta,\phi) \tag{4.10}$$

where the index n takes values in $\mathbb{N}$, the constant N_{nl} is a normalization constant and $M(-n,l+\frac{3}{2},r^2)$ are confluent hypergeometric functions. Since the first argument of these is a whole negative number, the functions $M(-n,l{+}\frac{3}{2},r^2)$ reduce to polynomials in r^2 of order n. Therefore, for fixed indices m,l the functions ψ_{nlm} are polynomials in r with minimal degree r^l. (From this, one can see, that $L_2(\mathbb{R}^3)$ is not equivalent to a tensor product $L_2(\mathbb{R}_+,r^2dr)\otimes L_2(S^2,\sin\theta d\theta d\phi)$. If this would be the case the $\{\psi_{nlm}\,|\,n\in\mathbb{N}\}$ should span the whole Hilbert space $L_2(\mathbb{R}_+,r^2dr)$ for arbitrary fixed indices m,l. However, they span only the subspace of polynomials with minimal degree l. We will come back to this point later on.)

Applying $\widehat{\mathbf{M}}$ to the functions ψ_{nlm} just multiplies them with $\hbar^2l(l+1)$, hence we need them all as cyclic vectors Ω_{nlm}. The associated spectral measures can easily be calculated to be

$$\mu_{nlm}(\lambda) =<\psi_{nlm},\Theta(\lambda-\widehat{\mathbf{M}})\psi_{nlm}> =<\psi_{nlm},\Theta(\lambda-\hbar^2l(l+1))\psi_{nlm}> = \Theta(\lambda-\hbar^2l(l+1)) \quad . \tag{4.11}$$

The spectral measures are pure point, they are homogeneous in n and m and have only the point $\hbar^2l(l+1)$ as support. Hence the kinematical Hilbert space decomposes into eigenspaces of $\widehat{\mathbf{M}}$ in the following way

$$L_2(\mathbb{R}^3) = \sum_{l=0}^{\infty}\mathrm{clos}(\mathrm{span}\{\psi_{nlm}\,|\,n\in\mathbb{N},-l\leq m\leq l\}) \quad . \tag{4.12}$$

The induced Hilbert space to the eigenvalue $\lambda=0$ is given by $\mathrm{clos}(\mathrm{span}\{\psi_{n00}\,|\,n\in\mathbb{N}\})$. These functions are constant in θ,ϕ and are polynomials in r^2 of order n (weighted with a gaussian factor). The above set can be taken as an orthonormal basis in $L_2(\mathbb{R}_+,r^2dr)$, therefore we can identify the physical Hilbert space with $L_2(\mathbb{R}_+,r^2dr)$.

On this (physical) Hilbert space the observable algebra is given by

$$\hat{e}^+ = r^2 \qquad \hat{e}^- = -\hbar^2 \frac{1}{r^2}\partial_r(r^2\,\partial_r) \qquad \hat{d} = -i\hbar\,(r\partial_r + \tfrac{3}{2}) \tag{4.13}$$

and because of (4.5) we have

$$\hat{d}^2 - \tfrac{1}{2}(\hat{e}^+\hat{e}^- + \hat{e}^-\hat{e}^+) = \tfrac{3}{4}\hbar^2 \quad . \tag{4.14}$$

4.2 Model with Structure Functions rather than Structure Constants Linear in the Momenta

As an example for a model with structure functions we will discuss a sort of a deformed $SO(3)$ constraint algebra

$$C_1 = x_2 p_3^m - x_3^n p_2 \qquad C_2 = x_3^n p_1 - x_1 p_3^m \qquad C_3 = x_1 p_2 - x_2 p_1 \tag{4.15}$$

$$\{C_1, C_2\} = m\,n\,p_3^{m-1} x_3^{n-1}\,C_3 \qquad \{C_2, C_3\} = C_1 \qquad \{C_3, C_1\} = C_2 \tag{4.16}$$

where n and m are positive natural numbers. For $n = m = 1$ we recover the $SO(3)$-algebra. One can do a similar deformation for $SO(p,q)$-algebras.

The Dirac observables (see [18]) for this system generate an $sl(2,\mathbb{R})$ algebra:

$$\begin{aligned} e^+ &= x_1^2 + x_2^2 + \frac{2}{m+n} p_3^{1-m} x_3^{n+1} \qquad\qquad e^- = p_1^2 + p_2^2 + \frac{2}{m+n} x_3^{1-n} p_3^{m+1} \\ d &= x_1 p_1 + x_2 p_2 + \frac{2}{m+n} x_3 p_3 \end{aligned} \tag{4.17}$$

$$\{d, e^\pm\} \;=\; \mp 2 e^\pm \qquad \{e^+, e^-\} = 4d \quad . \tag{4.18}$$

For general m, n these observables commute only on the constraint hypersurface with the constraints.

We define the Master Constraint as

$$\mathsf{M} = C_1^2 + C_2^2 + C_3^2 \quad . \tag{4.19}$$

The constraints C_1 and C_2 do not strongly Poisson-commute with M (for $n \neq 1$ or $m \neq 1$), but C_3 does. For the rest of this chapter, we will consider the case $m = 1$ and n odd. For these parameter values the phase space function e^+ is non-negative and commutes on the whole phase space $\mathbb{R}^3 \times \mathbb{R}^3$ with the Master Constraint. This will be helpful for the spectral analysis of the quantized Master Constraint. The case $n = 1$ and m odd can be treated in the same way, using Fourier transformation. For $m > 1$ the observable e^+ contains negative powers of p_3, therefore the analysis gets more complicated.

4.2.1 Quantization

We will quantize this system in the usual way by assigning multiplication operators $\hat{x}_i$ to the configuration variables x_i and differential operators $\hat{p}_j = -i\hbar\partial_j$ to the momenta p_j. The kinematical Hilbert space, we are starting with, is $L_2(\mathbb{R}^3)$.

There arises no factor ordering ambiguity for the quantization of the constraints, and since the structure function $f = n(\hat{x}_3)^{n-1}$ commutes with C_3 (see (4.16)), it is possible to quantize all the constraints as symmetric operators as we will explain in the conclusion section and to have $\hat{C}_3$ standing to the right of the structure function:

$$\big[\hat{C}_1, \hat{C}_2\big] = i\hbar\, n(\hat{x}_3)^n\, \hat{C}_3 \qquad \big[\hat{C}_2, \hat{C}_3\big] = i\hbar\, \hat{C}_1 \qquad \big[\hat{C}_3, \hat{C}_1\big] = i\hbar\, \hat{C}_2 \qquad . \tag{4.20}$$

We have to analyze the Master Constraint Operator

$$\widehat{\mathbf{M}} = \hat{C}_1^2 + \hat{C}_2^2 + \hat{C}_3^2 \quad , \tag{4.21}$$

which is a second order partial differential operator. It can be densely defined and is symmetric on the linear span of the Hermite functions.

To solve the eigenvalue equation for the Master Constraint Operator we will introduce new coordinates (t, θ, ϕ) analogous to the coordinates (r, θ, ϕ) in the $SO(3)$ case. However, for $n \neq 1$ we have to consider the regions $x_3 \geq 0$ and $x_3 \leq 0$ seperately: For $x_3 \geq 0$ we define

$$\begin{aligned} x_1 &= t\,\cos\phi\,\sin\theta \qquad t \geq 0 \qquad \phi \in \big[0, 2\pi\big) \qquad \theta \in \big[0, \tfrac{\pi}{2}\big] \\ x_2 &= t\,\sin\phi \sin\theta \\ x_3 &= (\frac{n+1}{2})^{\frac{1}{n+1}} (t\,\cos\theta)^{\frac{2}{n+1}} \end{aligned} \tag{4.22}$$

and for $x_3 \leq 0$

$$x_3 = -(\frac{n+1}{2})^{\frac{1}{n+1}} (-t\,\cos\theta)^{\frac{2}{n+1}} \qquad \theta \in \big[\tfrac{\pi}{2}, \pi\big] \quad . \tag{4.23}$$

The measure is transformed to $dx_1 dx_2 dx_3 = \sin\theta\, (\frac{2}{n+1})^{\frac{n}{n+1}} t^{\frac{n+3}{n+1}} |\cos\theta|^{\frac{-n+1}{n+1}} dt d\theta d\phi$. With these coordinates we have

$$x_1^2 + x_2^2 + \frac{2}{n+1} x_3^{n+1} = t^2 = e^+ \tag{4.24}$$

and $\widehat{\mathbf{M}}$ does not include derivatives with respect to t. Indeed, for $\cos\theta \geq 0$,

$$\widehat{\mathbf{M}} = -\hbar^2 \Big(\frac{n+1}{2}\Big)^{\frac{2n}{n+1}} \Bigg[(t\cos\theta)^{\frac{2n-2}{n+1}} \partial_\theta^2 + \Bigg(\frac{(t\cos\theta)^{\frac{3n-1}{n+1}}}{t\sin\theta} - t^{\frac{2n-2}{n+1}} (\cos\theta)^{\frac{n-3}{n+1}} \Big(\frac{n-1}{n+1}\Big) \sin\theta \Bigg) \partial_\theta + \Bigg(\frac{(t\cos\theta)^{\frac{4n}{n+1}}}{t^2 \sin^2\theta} + \Big(\frac{2}{n+1}\Big)^{\frac{2n}{n+1}} \Bigg) \partial_\phi^2 \Bigg] \tag{4.25}$$

and $\widehat{\mathbf{M}}$ for $\cos\theta \leq 0$ is obtained from the above formula by replacing t with $-t$. The operator $\widehat{\mathbf{M}}$ simplifies considerably, if we introduce the coordinate

$$\begin{aligned} u &= (\cos\theta)^{\frac{2}{n+1}} \quad \text{for} \quad \cos\theta \geq 0 \\ u &= -(-\cos\theta)^{\frac{2}{n+1}} \quad \text{for} \quad \cos\theta \leq 0 \qquad u \in [-1, 1] \quad . \end{aligned} \tag{4.26}$$

The coordinate u is proportional to the coordinate x_3 and therefore one gets the same operator for x_3 positive and for x_3 negative:

$$\widehat{\mathbf{M}} = -\hbar^2 \Big(\frac{n+1}{2}\Big)^{\frac{2n}{n+1}} \Bigg[\Bigg(\frac{t^{\frac{2n-2}{n+1}}\, u^{2n}}{1 - u^{n+1}} + \Big(\frac{2}{n+1}\Big)^{\frac{2n}{n+1}} \Bigg) \partial_\phi^2 + \frac{2t^{\frac{2n-2}{n+1}}}{n+1} \partial_u \Big((1 - u^{n+1}) \partial_u \Big) \Bigg] \tag{4.27}$$

The measure is now $dx_1 dx_2 dx_3 = (\frac{n+1}{2})^{\frac{1}{n+1}} t^{\frac{n+3}{n+1}} dt du d\phi$.

Since the operators $\hat{C}_3 = -i\hbar\partial_\phi$ and $\widehat{\mathbf{M}}$ commute, we can diagonalize them simultaneously. Choosing periodic boundary conditions in ϕ one obtains $\text{spec}(\hat{C}_3) = \{\hbar k | k \in \mathbb{Z}\}$ and the null eigenspace consists of constant functions in ϕ. We are interested in the spectrum of $\widehat{\mathbf{M}}$ near zero, so it suffices to consider $\widehat{\mathbf{M}}$ restricted to the null eigenspace of $\hat{C}_3$, as it has elsewhere spectrum bounded from below by $\hbar^2$ (since $\widehat{\mathbf{M}} = \hat{C}_1^2 + \hat{C}_2^2 + \hat{C}_3^2$). The restriction of $\widehat{\mathbf{M}}$ to this space is

$$\widehat{\mathbf{M}}_{|\hat{C}_3=0} = -\hbar^2 \Big(\frac{n+1}{2}\Big)^{\frac{2n}{n+1}} \frac{2t^{\frac{2n-2}{n+1}}}{n+1} \partial_u \Big((1-u^{n+1})\partial_u\Big) \quad , \tag{4.28}$$

and can be seen as a product of two commuting operators B_1 and B_2:

$$B_1 := \hbar^2 \Big(\frac{n+1}{2}\Big)^{\frac{2n}{n+1}} \frac{2t^{\frac{2n-2}{n+1}}}{n+1} \quad , \qquad B_2 := -\partial_u \Big((1-u^{n+1})\partial_u\Big) \quad . \tag{4.29}$$

In the following, we will consider the operator B_2 on the Hilbert space $L_2((-1,1), du)$ and show that its spectrum is purely discrete. Afterwards we will come back to the product of the operators $B_1 \cdot B_2$. (The product $B_1 \cdot B_2$ cannot be seen as a tensor product of B_1 and B_2 since $L_2(\mathbb{R}^3, dx^3)$ is not equivalent to $L_2(\mathbb{R}_+, t^{\frac{n+3}{n+1}} dt) \otimes L_2((-1,1), du) \otimes L_2((0,2\pi), d\phi)$. Moreover one has to be careful, since it is not guaranteed that B_2 has a self-adjoint extension as an operator on $L_2(\mathbb{R}_3)$. This might be the case, because B_2 does not preserve the space of Hermite functions in the three variables (x_1, x_2, x_3), only the combination $B_1 \cdot B_2$ does preserve this space. Nevertheless one can consider the operator B_2 defined on the Hilbert space $L_2((-1,1), du)$.)

The operator B_2 is symmetric and positive on the dense domain

$$\begin{aligned}\mathfrak{D}(B_2) = \{f \in L_2((-1,1), du) | f \in AC(-1,1), (1-u^{n+1})f' \in AC(-1,1), \\ B_2 \cdot f \in L_2((-1,1), du), \lim_{u\to\pm 1} (1-u^{n+1}) f'(u) = 0\}\end{aligned} \tag{4.30}$$

where AC denotes the class of absolutely continuous functions. The positivity can be seen by using integration by parts

$$< f, B_2 \cdot f >_{L_2(u)} = \int_{-1}^{1} \overline{\partial_u f(u)} (1-u^{n+1}) \partial_u f(u)\, du \ \geq 0 \tag{4.31}$$

for functions $f \in \mathfrak{D}(B_2)$. A positive and symmetric operator has always selfadjoint extensions (see [19]).

The operator B_2 is an ordinary differential operator, more specifically it is a Sturm-Liouville operator on the interval $I := (-1,1)$, so we can utilize the theory of Sturm-Liouville operators, see for example [20]. Sturm-Liouville operators can be classified according to the behaviour of eigensolutions at the endpoints of the interval I: A Sturm-Liouville operator A is limit circle at the endpoint a if for one $\lambda \in \mathbb{C}$ all solutions to $(A-\lambda)f = 0$ are square integrable near a. One can prove, that if this is the case for one λ, then it holds for all $\lambda \in \mathbb{C}$. It therefore suffices to consider the solutions to $B_2 \cdot f = \lambda f$ for one $\lambda \in \mathbb{C}$, in particular $\lambda = 0$.

One solution for $\lambda = 0$ is obviously given by $f_{01}(u) \equiv 1$, the other solution is given by

$$f_{02}(u) = \int_0^u \frac{1}{(1-u'^{n+1})(f_{01}(u'))^2} du' = \int_0^u \frac{1}{(1-u'^{n+1})} du' \quad . \tag{4.32}$$

Both solutions are square integrable on $(-1,1)$. The function f_{02} is integrable because for $n > 1$ and n odd, the function $|f_{02}(n>1)|$ is less than $|f_{02}(n=1)|$ on the interval $(-1,1)$. But $f_{02}(n=1) = \operatorname{artanh}(u)$ is square integrable on $(-1,1)$. We are therefore in the limit circle case for both boundaries $u_\pm = \pm 1$.

If both boundary points are limit circle for an operator A, the following holds (see [20]): The spectrum of any self adjoint extension is purely discrete, the eigenfunctions are simple (multiplicity one) and form an orthonormal basis and the resolvent of A is a Hilbert – Schmidt operator. Notice that Hilbert – Schmidt operators are in particular compact, hence the spectrum of the resolvent $R_z(A)$ for $z \in \mathbb{R}$ not in the spectrum of A has an accumulation point at most at zero. Thus A does not have any accumulation point and since $\widehat{\mathbf{M}}$ is unbounded we know that the eigenvalues actually diverge (when ordered according to size).

Hence B_2 has a discrete simple spectrum, and since $f_{01}(u) \equiv 1$ is a square integrable solution for $\lambda = 0$ (which fulfills contrary to f_{02} the boundary conditions), zero is included in the spectrum. This holds for any selfadjoint extension of B_2. Now B_2 with the domain (4.30) is positive. We choose in the following any selfadjoint extension such that it is still positive and such that $B_1 \cdot B_2$ as an operator in $L_2(\mathbb{R}^3)$ corresponds to the positive self-adjoint extension of $\widehat{\mathbf{M}}$, we have chosen before (e.g. the Friedrich extension)[3]. The eigenvalues of B_2 are therefore positive (that is $\lambda_j \geq 0$ and $\lambda_j = 0$ if and oly if $j = 0$).

Since the operator B_2 has a null eigenspace consisting of functions constant in u and therefore θ, one would expect that the physical Hilbert space consists of functions functionally independent of θ (and ϕ as was argued above). However we are dealing not just with the operator B_2 but with the product $B_1 \cdot B_2$. The operator B_1 is a multiplication operator and has continuous spectrum on the whole positive axis including zero.

Therefore we have to discuss the decomposition of the Hilbert space into a direct integral Hilbert space and to calculate the spectral measures. To this end we will consider the restriction of the Master Constraint Operator to the subspace $\hat{C}_3 = 0$ as it has elsewhere a spectrum, which is bounded from below by $\hbar^2$. Since this subspace coincides with functions in $L_2(\mathbb{R}^3)$, which are constant in the angle variable ϕ, we can identify this subspace with $\mathcal{H}' := L_2(\mathbb{R}_+ \times \mathbb{R}, \rho\, d\rho dx_3)$ where the variable ρ is defined by $\rho^2 = x_1^2 + x_2^2$.

A dense system of vectors in this Hilbert space is generated by the set

$$\{f_{pq} := \rho^{2q} x_3^p \exp(-\tfrac{1}{2}t^2) \quad | \quad q, p \in \mathbb{N}\} \quad . \tag{4.33}$$

We need just the even polynomials in ρ since the variable ρ has range only in the positive half axis[4] $\mathbb{R}_+$. Now one has to find a cyclic system of vectors with respect to $\widehat{\mathbf{M}}$. We will begin with functions that are polynomials in $t^2 = (\rho^2 + \frac{2}{n+1}x_3^{n+1})$ weighted with $\exp(-t^2)$. All these functions are annihilated by $\widehat{\mathbf{M}}$, hence to get a cyclic system, we need all the even powers of t. (Again, we just need the even powers, since t extends over the positive half axis.) If one performs the Gram-Schmidt procedure for this system $\{t^{2k}\exp(-t^2) \,|\, k \in \mathbb{N}\}$ one will get an ortho-normal set of vectors $\{v_{0k} \,|\, k \in \mathbb{N}\}$ which can be identified as a basis in the Hilbert space $L_2(\mathbb{R}_+, t^{\frac{n+3}{n+1}}dt)$. The associated spectral measures are given by

$$\mu_{0k}(\lambda) =< v_{0k}, \Theta(\lambda - B_1 \cdot B_2)v_{0k} >=< v_{0k}, \Theta(\lambda - B_1 \cdot 0)v_{0k} >= \Theta(\lambda) \quad , \tag{4.34}$$

so that the set of vectors $\{v_{0k}\}$ is associated with a pure point spectral measure. We will call the subspace spanned by these vectors $\mathcal{H}_{pp}$ (anticipating that all vectors orthogonal to this subspace are associated to spectral measures, which are absolutely continuous).

Now we have to find the orthogonal complement to $\mathcal{H}_{pp}$. If one rewrites the functions f_{pq} in terms of the coordinates (t, u) one gets

$$f_{pq} = \rho^{2q} x_3^p = C(p,q)(1 - u^{n+1})^q u^p t^{2q + \frac{2p}{n+1}} e^{-\frac{1}{2}t^2} \tag{4.35}$$

where $C(p,q)$ is a p, q dependent constant. It is important to note here that a power of u^a is alway accompanied by a power of at least $t^{\frac{2a}{n+1}}$. The underlying reason for this is that $u \sim x_3 t^{-\frac{2}{n+1}}$. Using this fact, it is straightforward to see, that also

$$g_{pq} = u^p t^{\frac{2p}{n+1} + 2q} e^{-\frac{1}{2}t^2} \quad p, q \in \mathbb{N} \tag{4.36}$$

[3]The operator B_1 is self-adjoint with the domain $\mathfrak{D}_{SA}(B_1) = \{f \in L_2(\mathbb{R}^3) | B_1 \cdot f \in L_2(\mathbb{R}^3)\}$.

[4]Indeed, polynomials of any positive power of t^r could be used in order to construct a basis of the Hilbert space by the Gram – Schmidt procedure because any square integrable function $t \mapsto f(t)$ can be written as $t \mapsto f_r(t^r)$ where $f_r(s) := f(s^{1/r})$. Notice that we do not transform the measure to the variable t^r here.

or

$$h_{pq} = P_p(u) t^{\frac{2p}{n+1}+2q} e^{-\frac{1}{2}t^2} \tag{4.37}$$

generate a dense subspace in $\mathcal{H}'$. Here $P_p(u)$ are the Legendre polynomials of order p, that is $P_p(u)$ is a polynomial with degree p. To see this, notice that (4.35) is a polynomial in terms of t^2 and $x := ut^{2/(n+1)}$ which explains the statement for g_{pq}. Next notice that P_p is a polynomial of order p in u which can be written as a polynomial in x of the same order and $t^{2(p-k)/(n+1)}$, $k = 0, .., p$. Now any positive power of t can be expanded in terms of the v_{0k} and thus in terms of the t^{2q} again.

Since $P_0(u)$ is the constant function, the space spanned by $\{h_{0q} \,|q \in \mathbb{N}\}$ coincides with the space spanned by $\{v_{0k} \,|k \in \mathbb{N}\}$. Moreover $P_p, p > 0$ is orthogonal to P_0 (in $L_2((-1,1), du)$), so that the orthogonal complement to $\mathcal{H}_{pp}$ is spanned by $\{h_{pq} \,|\mathbb{N} \ni p > 0, q \in \mathbb{N}\}$. We will now show, that the spectral measures associated to vectors from this subspace are absolutely continuous. In order to do this we have to expand the functions $P_p(u) \in L_2((-1,1), du), p > 0$ into eigenfunctions of B_2:

$$P_p(u) = \sum_{k=1}^{\infty} a^p_k \psi_k(u) \tag{4.38}$$

where ψ_k is the normalized $k-th$ eigenfunction of B_2 (and we assume that the corresponding eigenvalues λ_k are ordered and that ψ_0 is the constant function). The constant function (i.e. ψ_0) does not appear in this decomposition since P_p is orthogonal (in $L_2((-1,1), du)$) to the constant function for $p > 0$.

Using this decomposition and abbreviating $c_n = \hbar^2(\frac{n+1}{2})^{\frac{2n}{n+1}}\frac{2}{n+1}$ we can write

$$\begin{aligned}
\mu_{p'q'pq}(\lambda) &:= < h_{p'q'}, \theta(\lambda - \widehat{\mathbf{M}}) h_{pq} > \\
&= \left(\frac{n+1}{2}\right)^{\frac{1}{n+1}} \int_{\mathbb{R}_+}\int_{[-1,1]} dt du \, \exp(-t^2)\, t^{\frac{2(p+p')}{n+1}+2(q+q')+\frac{n+3}{n+1}} \left(\sum_{k'=1}^{\infty} \overline{a}^{p'}_{k'} \overline{\psi}_{k'}(u)\right) \\
&\qquad \theta(\lambda - c_n t^{\frac{2n-2}{n+1}} B_2) \left(\sum_{k=1}^{\infty} a^p_k \psi_k(u)\right) \\
&= \left(\frac{n+1}{2}\right)^{\frac{1}{n+1}} \sum_{k',k=1}^{\infty} \int_{\mathbb{R}_+}\int_{[-1,1]} dt du \, \exp(-t^2)\, t^{\frac{2(p+p')}{n+1}+2(q+q')+\frac{n+3}{n+1}} \\
&\qquad \theta(\lambda - c_n t^{\frac{2n-2}{n+1}} \lambda_k)\, \overline{a}^{p'}_{k'} a^p_k \, \overline{\psi}_{k'}(u)\psi_k(u) \\
&= \left(\frac{n+1}{2}\right)^{\frac{1}{n+1}} \sum_{k=1}^{\infty} \int_{\mathbb{R}_+} dt \, \exp(-t^2)\, t^{\frac{2(p+p')}{n+1}+2(q+q')+\frac{n+3}{n+1}} \\
&\qquad \theta(\lambda - c_n t^{\frac{2n-2}{n+1}} \lambda_k)\, \overline{a}^{p'}_k a^p_k
\end{aligned} \tag{4.39}$$

where in the last line we used the ortho-normality of ψ_k in $L_2((-1,1), du)$.

The sum in the last line converges absolutely, since the integral over t can be bounded (k-independently) from above by ignoring the θ-function. Then we are left with the sum over the absolute values of $\overline{a}^{p'}_k a^p_k$, which can be estimated using the Cauchy-Schwarz inequality and the fact that the a^p_k are the expansion coefficients of $P_k \in L_2((-1,1), du)$.

This shows that all the measures $\mu_{p'q'pq}$ are absolutely continuous with respect to the

Lebesgue measure on $\mathbb{R}_+$. Indeed we have

$$\begin{aligned}\frac{d\mu_{p'q'pq}(\lambda)}{d\lambda} &= \Big(\frac{n+1}{2}\Big)^{\frac{1}{n+1}}\sum_{k=1}^{\infty}\overline{a}_k^{p'}a_k^p\int_{\mathbb{R}_+}dt\,\exp(-t^2)\,t^{\frac{2(p+p')}{n+1}+2(q+q')+\frac{n+3}{n+1}}\delta(\lambda-c_n t^{\frac{2n-2}{n+1}}\lambda_k)\\ &= \Big(\frac{n+1}{2}\Big)^{\frac{1}{n+1}}\sum_{k=1}^{\infty}\overline{a}_k^{p'}a_k^p\,\big(\frac{n+1}{2c_n\lambda_k(n-1)}\big)\exp\big(-(\frac{\lambda}{c_n\lambda_k})^{\frac{n+1}{n-1}}\big)\,\big(\frac{\lambda}{c_n\lambda_k}\big)^{\frac{3+p+p'+(q+q')(n+1)}{n-1}}\\ &=: \lambda^{\frac{3+p+p'+(q+q')(n+1)}{n-1}}f_{p'q'pq}(\lambda) \end{aligned}\tag{4.40}$$

where in the second line we performed a coordinate transformation $s_k = c_n\lambda_k t^{\frac{2n-2}{n+1}}$ in order to solve the delta-function. The sum in the second line converges absolutely (uniformly in λ) since the exponential factor can be estimated by 1 and the λ_k have to be bigger than 1 for some finite k because the spectrum of B_2 does not have an accumation point as we showed above. Hence $f_{p'q'pq}$ is well defined, in particular $f_{p'q'pq}(0) > 0$ for $p' = p$ (since all terms in the sum are then positive).

To summarize, our Hilbert space $\mathcal{H}'$ can be decomposed into two subspaces $\mathcal{H}_{pp}$ and $\mathcal{H}_{ac}$ where $\widehat{\mathbf{M}}$ restricted to $\mathcal{H}_{pp}$ has a pure point spectral measure and $\widehat{\mathbf{M}}$ restricted to $\mathcal{H}_{ac}$ has an absolutely continuous spectral measure. We therefore have to discuss the direct integral decomposition of $\mathcal{H}_{pp}$ and $\mathcal{H}_{ac}$ separately. The 'direct integral decomposition' for $\mathcal{H}_{pp}$ is straightforward: Since $\mathcal{H}_{pp}$ is a proper eigenspace to to the null eigenvalue of $\widehat{\mathbf{M}}$, the physical Hilbert space corresponding to the pure point spectrum coincides with $\mathcal{H}_{pp}$ (which can be identified with $L_2(\mathbb{R}_+, t^{\frac{n+3}{n+1}}dt)$).

To perform the direct integral Hilbert space decomposition of $\mathcal{H}_{ac}$ spanned by $\{h_{pq}\,|\mathbb{N}\ni p > 0, q\in\mathbb{N}\}$ we choose an ortho-normal cyclic system $\{\Omega_m\,|\mathbb{N}\ni m>0\}$ in that space, such that $\Omega_1 = N_1 h_{10}$, where N_1 is a normalization constant. Since $\{h_{pq}\,|\mathbb{N}\ni p>0, q\in\mathbb{N}\}$ generates $\mathcal{H}_{ac}$ we can always find coefficients A^m_{pq} such that

$$\Omega_m = \sum_{p=1,q=0}^{\infty} A^m_{pq}h_{pq} \quad . \tag{4.41}$$

Consider the spectral measures $\mu_m, m>1$ associated to this cyclic system

$$\begin{aligned}\mu_m(\lambda) &:= <\Omega_m, \theta(\lambda-\widehat{\mathbf{M}})\Omega_m>\\ &= <\Omega_m,\,\theta(\lambda-\widehat{\mathbf{M}})\sum_{p=1,q=0}^{\infty}A^m_{pq}h_{pq}>\\ &= <\Omega_m,\,\theta(\lambda-\widehat{\mathbf{M}})\sum_{\substack{p=1,q=0\\(p,q)\neq(1,0)}}^{\infty}A^m_{pq}h_{pq}> \end{aligned}\tag{4.42}$$

since $<\Omega_m, \theta(\lambda-\widehat{\mathbf{M}})\Omega_1> = N_1 <\Omega_m,\theta(\lambda-\widehat{\mathbf{M}})h_{10}> = 0$ due to the defining property of a cyclic system. Hence

$$\begin{aligned}\mu_m(\lambda) &= \sum_{p'=1,q'=0}^{\infty}\sum_{\substack{p=1,q=0\\(p,q)\neq(1,0)}}^{\infty}\overline{A}^m_{p'q'}A^m_{pq}<h_{p'q'},\theta(\lambda-\widehat{\mathbf{M}})h_{pq}>\\ &= \sum_{p'=1,q'=0}^{\infty}\sum_{\substack{p=1,q=0\\(p,q)\neq(1,0)}}^{\infty}\overline{A}^m_{p'q'}A^m_{pq}\;\mu_{p'q'pq}(\lambda) \quad . \end{aligned}\tag{4.43}$$

Differentiating under the sums[5]

$$\begin{aligned}\frac{d\mu_m(\lambda)}{d\lambda} &= \sum_{p'=1,q'=0}^{\infty}\sum_{\substack{p=1,q=0\\(p,q)\neq(1,0)}}^{\infty}\overline{A}^m_{p'q'}A^m_{pq}\;\frac{d\mu_{p'q'pq}(\lambda)}{d\lambda}\\ &= \lambda^{\frac{6}{n-1}}\sum_{p'=1,q'=0}^{\infty}\sum_{\substack{p=1,q=0\\(p,q)\neq(1,0)}}^{\infty}\overline{A}^m_{p'q'}A^m_{pq}\;\lambda^{\frac{p+p'+(q+q')(n+1)-3}{n-1}}f_{p'q'pq}(\lambda)\\ &=: \lambda^{\frac{6}{n-1}}g_m(\lambda) \end{aligned}\tag{4.44}$$

where $g_m(\lambda)$ is a non-singular function of λ near zero (that is $g_m(0)$ is either zero or some finite value). This holds for $m>1$; for $m=1$ we get

$$\frac{d\mu_1(\lambda)}{d\lambda} = \lambda^{\frac{5}{n-1}}(N_1)^2 f_{1010}(\lambda)\tag{4.45}$$

where $f_{1010}(0) = c > 0$.

The total spectral measure $\mu_{ac}(\lambda)$ on $\mathcal{H}_{ac}$ is given by (if the maximal multiplicity is less than infinity, choose appropriate normalization constants different from 2^{-m})

$$\begin{aligned}\frac{d\mu_{ac}(\lambda)}{d\lambda} &:= \frac{1}{2}\frac{d\mu_1(\lambda)}{d\lambda} + \sum_{m=2}^{\infty}2^{-m}\frac{d\mu_m(\lambda)}{d\lambda}\\ &= \frac{1}{2}\lambda^{\frac{5}{n-1}}(N_1)^2 f_{1010}(\lambda) + \lambda^{\frac{6}{n-1}}\sum_{m=2}2^{-m}g_m(\lambda)\quad . \end{aligned}\tag{4.46}$$

Now we can consider the Radon-Nikodym derivatives in the limit $\lambda\to 0$. Assume that $m>1$:

$$\lim_{\lambda\to 0}\rho_m(\lambda) = \lim_{\lambda\to 0}\frac{d\mu_m(\lambda)/d\lambda}{d\mu_{ac}(\lambda)/d\lambda} = \lim_{\lambda\to 0}\frac{\lambda^{\frac{6}{n-1}}g_m(\lambda)}{\frac{1}{2}\lambda^{\frac{5}{n-1}}(N_1)^2 f_{1010}(\lambda) + \lambda^{\frac{6}{n-1}}\sum_{m=2}2^{-m}g_m(\lambda)} = 0\tag{4.47}$$

Hence all the $\rho_m, m>1$ vanish at $\lambda = 0$. But the same calculation for $m=0$ gives $\rho_1(0) = 2$. Therefore the induced Hilbert space for $\lambda = 0$ from $\mathcal{H}_{ac}$ is one-dimensional, that is unitarily equivalent to $\mathbb{C}$. Putting the contributions from $\mathcal{H}_{pp}$ and $\mathcal{H}_{ac}$ together we get $\mathcal{H}_{phys}\simeq L_2(\mathbb{R}_+, t^{\frac{n+3}{n+1}})\oplus\mathbb{C}$ (where the inner product in $\mathbb{C}$ can be rescaled arbitrarily). The first term corresponds to the (proper) null subspace with respect to $\widehat{\mathbf{M}}$ the second to the continious spectrum of the multiplication operator in t.

Remark:
By the standard theory for Sturm – Liouville operators such as B_2 we know that the eigenfunction ψ_k has k zeroes and that the eigenvalues asymptote to $\lambda_k \propto k^2$. By the Weierstrass theorem, ψ_k can be approximated in the sup – norm arbitrarily well on the compact set $[-1,1]$ by polynomials of degree at least k (in order to have k real roots). Thus, the ψ_k are actually not too different from the standard Legendre polynomials P_k and if they would be really polynomials of degree k we could just use the functions $\Omega_k = \psi_k t^{2k/(n+1)}$ as a cyclic system. Then the above calculations would become entirely trivial because the Radon – Nikodym derivative at $\lambda = 0$ would be obviously non – vanishing for the lowest order $k=1$. Unfortunately the eigenvectors ψ_k are not polynomials unless $n=1$ and so we had to go through this very elaborous analysis.

[5]Strictly speaking one must verify that we may interchange summation and differentiation. This could be done for instance by verifying that the series for $\mu_m(\lambda)$ converges absolutely and uniformly in λ at least close to zero (the absolute and uniform convergence of the series for $\mu(\lambda)$ then follows). We have not checked that this is the case but given the fact that the $\mu_{p'q'pq}$ converge rapidly to zero in the vicinity of $\lambda = 0$ this should be true. With the tools given here, one could check this explicitly by a tedious but straightforward calculation.

5 Pure Point and Absolutely Continuous Spectra

Here we will discuss an example, where, similarly to the previous one, the Master Constraint Operator has pure point and absolutely continuous spectrum at zero. The example is simpler without structure constants and serves the purpose to illustrate an important point when choosing the cyclic system.

We will start with a kinematical Hilbert space $L_2(\mathbb{R}^2)$ and a Master Constraint Operator

$$\widehat{\mathbf{M}} = (\hat{x}_1^2 + \hat{x}_2^2)(\hat{x}_1\hat{p}_2 - \hat{x}_2\hat{p}_1)^2 \tag{5.1}$$

where the $\hat{x}_i$ are multiplication operators and the $\hat{p}_i := -i\hbar\partial_i$ act by differentiation. Apriori one would expect that the physical Hilbert space includes the space of functions with zero angular momentum and a one-dimensional part which corresponds to the generalized eigenfunction $\delta(x_1)\cdot\delta(x_2)$.

We will now construct the direct integral decomposition of the kinematical Hilbert space with respect to $\widehat{\mathbf{M}}$. First of all, we have to find a cyclic basis for $L_2(\mathbb{R}^2)$. To this end we note, that $L_2(\mathbb{R}^2)$ is spanned by polynomials in x_1, x_2 weighted by a gaussian factor. These can be generated by the set

$$\{x_1^{n_1} x_2^{n_2} \exp(-\tfrac{1}{2}(x_1^2 + x_2^2) \,|\, n_1, n_2 \in \mathbb{N}\} \tag{5.2}$$

and also by the set

$$\{v_{mn} = x_+^m x_-^n \; \exp(-\tfrac{1}{2}x_+x_-) \,|\, n, m \in \mathbb{N}\} \tag{5.3}$$

where we defined $x_\pm := x_1 \pm ix_2$. In spherical coordinates defined by

$$\begin{aligned} x_1 &= r\cos\phi \qquad r \in \mathbb{R}_+, \phi \in [0, 2\pi) \\ x_2 &= r\sin\phi \end{aligned} \tag{5.4}$$

the vectors v_{nm} can be expressed as

$$v_{nm} = r^{n+m} \exp(i\phi(m-n)) \; e^{-r^2/2} \quad . \tag{5.5}$$

By introducing new indices $N := (n-m) \in \mathbb{Z}$ and $k := \frac{1}{2}(m + n - |n-m|) \in \mathbb{N}$ the set (5.3) can also be written as

$$\{v_{Nk} = r^{|N|+2k} \exp(iN\phi) \exp(\tfrac{1}{2}r^2) \,|\, N \in \mathbb{Z}, k \in \mathbb{N}\} \quad . \tag{5.6}$$

Applying repeatedly $\widehat{\mathbf{M}}$ to a vector $v_{N0}, N \neq 0$ generates all the other vectors $v_{Nk}, k \in \mathbb{N}$ with the same index N. We therefore need just the v_{N0} as cyclic vectors. But for $N = 0$ applying $\widehat{\mathbf{M}}$ to v_{0k} gives zero, so that we also need the whole set $\{v_{0k} \,|\, k \in \mathbb{N}\}$. However, notice that this set is not orthonormal, one has therefore to perform a Gram-Schmidt procedure. This will result in an orthonormal set of the form $\{\Omega_{0k} := f_k(r^2)\exp(-r^2/2)\}$ where the f_k are polynomials of order k in r^2. This set can be taken as a basis for $L_2(\mathbb{R}_+, rdr)$.

Remark:
At this point it is important to draw attention to the following subtlety: Recall the definition of the tensor product of two Hilbert spaces $\mathcal{H}_j$: This is the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ consisting of pairs $\psi_1 \otimes \psi_2 := (\psi_1, \psi_2)$ with $\psi_j \in \mathcal{H}_j$ equipped with the inner product $< \psi_1 \otimes \psi_2, \psi_1' \otimes \psi_2' >_{\mathcal{H}_1 \otimes \mathcal{H}_2} := \prod_{j=1}^2 < \psi_j, \psi_j' >_{\mathcal{H}_j}$. One can show that if $b_k^{(j)}$ is a basis for $\mathcal{H}_j$ then $b_k^{(1)} \otimes b_n^{(2)}$ is a basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$. If $(X_j, \mathcal{B}_j, \mu_j)$ are measure spaces and $\mathcal{H}_j := L_2(X_j, d\mu_j)$ are separable Hilbert spaces then one can show by using Fubinis theorem that $\mathcal{H}_1 \otimes \mathcal{H}_2$ is isometrically isomorphic to $L_2(M_1 \times M_2, d\mu_1 \otimes d\mu_2)$ via $\psi_1 \otimes \psi_2 \mapsto \psi_1(x_1)\psi_2(x_2)$ where $\mu_1 \otimes \mu_2$ is the measure on $M_1 \times M_2$

based on the smallest $\sigma-$algebra $\mathcal{B}_1 \otimes \mathcal{B}_2$ containing the "rectangles" $B_1 \times B_2$ where $B_j \in \mathcal{B}_j$ and by definition $\mu_1 \otimes \mu_2(B_1 \times B_2) := \mu_1(B_1)\mu_2(B_2)$.

Now consider the problem at hand: In polar coordinates we have $L_2(\mathbb{R}^2, d^2x) = L_2(\mathbb{R}^2, rdrd\phi)$ with $\mathbb{R}^2 = [(\mathbb{R}_+ - \{0\}) \times S^1] \cup \{(0,0)\}$. One could now think that since the one point sets $\{(0,0)\}$ and $\{0\}$ respectively have d^2x and dx Lebesgue measure zero respectively that

$$\begin{aligned} L_2(\mathbb{R}^2, d^2x) &= L_2(\mathbb{R}^2 - \{(0,0)\}, d^2x) = L_2((\mathbb{R}_+ - \{0\}) \times S^1, d^2x) \\ &= L_2((\mathbb{R}_+ - \{0\}, rdr) \otimes L_2(S^1, d\phi) = L_2((\mathbb{R}_+, rdr) \otimes L_2(S^1, d\phi) \end{aligned} \tag{5.7}$$

If (5.7) would be true then, given some ONB $b(1)_k(r)$ for $L_2(\mathbb{R}_+, rdr)$ consisting of polynomials in r times $e^{-r^2/2}$ obtained via the Gram – Schmidt procedure and an ONB $b_n^{(2)}(\phi) = e^{in\phi}/\sqrt{2\pi}$ for $L_2(S^1, d\phi)$ we would obtain a basis $b_k^{(1)} \otimes b_n^{(2)}$ for $L_2(\mathbb{R}^2, d^2x)$. However, in the tensor product we then obtain a dense set of vectors of the form $r^k e^{in\phi} e^{-r^2/2}$ where the pair $(k, n) \in \mathbb{N}_0 \times \mathbb{Z}$ is unrestricted. On the other hand the Hermite polynomial basis for $L_2(\mathbb{R}^2, d^2x)$ provide a dense set consisting of vectors of the form $r^{|n|+2l} e^{in\phi} e^{r^2/2}$ as we have just seen. We conclude that the basis $b_k^{(1)} \otimes b_l^{(2)}$ for $L_2((\mathbb{R}_+, rdr) \otimes L_2(S^1, d\phi)$ is overcomplete for $L_2(\mathbb{R}^2, d^2x)$ and hence these Hilbert spaces are not isometrically isomorphic. In other words, the functions $r^k e^{in\phi} e^{-r^2/2}$ such that $k - |n|$ is not a non – negative and even integer could be expanded in terms of Hermite polynomials because they are obviously square integrable.

What has gone wrong in (5.7) is that on $\mathbb{R}^2$ the coordinates (r, ϕ) are singular at $r = 0$. The coordinates x_1, x_2 are globally defined which results in the fact that there is the restriction $k - |n| = 2l,\ l = 0, 1, 2, ..$ in order that the functions $r^k e^{in\phi}$ are are regular at $r = 0$ when expressed in terms of x_1, x_2. This topological subtlety that $\mathbb{R}^2$ is a plane while $\mathbb{R}_+ \times S^1$ is a half – infinite cylinder with different orthonormal bases is of outmost importance for the Direct Integral Decomposition (DID) because neglecting this difference would result in a much bigger physical Hilbert space as we will see shortly.

The normalization of the vectors v_{N0} gives

$$\Omega_{N0} = (\pi\, |N!|)^{-1/2} r^{|N|} \exp(iN\phi) \exp(-r^2/2) \quad . \tag{5.8}$$

Their associated spectral measure are

$$\begin{aligned} \mu_{N0} = < \Omega_{N0}, \Theta(\lambda - \widehat{\mathbf{M}})\Omega_{N0} > &= \frac{2}{|N|!} \int_0^\infty r^{2|N|} \exp(-r^2)\, \Theta(\lambda - r^2 N^2)\, rdr \\ &= \frac{1}{|N|!} \int_0^\infty x^{|N|} \exp(-x)\, \Theta(\lambda - N^2 x)\, dx \end{aligned} \tag{5.9}$$

which shows, that these measures are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$.

The spectral measures associated to the Ω_{0k} can be calculated to

$$\mu_{0k} = < \Omega_{0k}, \Theta(\lambda - \widehat{\mathbf{M}})\Omega_{0k} > = \Theta(\lambda) < \Omega_{0k}, \Omega_{0k} > = \Theta(\lambda) \tag{5.10}$$

which shows, that these measures are of pure point type.

Hence the kinematical Hilbert space $L_2(\mathbb{R}^2)$ decomposes into a direct sum of two Hilbert spaces $\mathcal{H}_{ac}$ and $\mathcal{H}_{pp}$, on which $\widehat{\mathbf{M}}$ has either absolutely continuous or pure point spectrum. $\mathcal{H}_{ac}$ is defined to be the closure of the span of $\{\widehat{\mathbf{M}}^k\, \Omega_{N0} \,|\, N \neq 0, N \in \mathbb{Z}, k \in \mathbb{N}\}$ and $\mathcal{H}_{pp}$ is the closure of the span of $\{\Omega_{0k} \,|\, k \in \mathbb{N}\}$. We will now discuss the direct integral decomposition on each of these two spaces seperately.

The direct integral decomposition of $\mathcal{H}_{pp}$ (which can be identified with $L_2(\mathbb{R}_+, rdr)$ is easy to obtain, since all the measures μ_{0k} are the same and have just the point zero in their support.

Hence $\mathcal{H}_{pp}$ is alredy decomposed with respect to $\widehat{\mathbf{M}}$ – the null eigenspace of $\widehat{\mathbf{M}}$ on $\mathcal{H}_{pp}$ coincides with $\mathcal{H}_{pp}$. The contribution from the pure point part of the spectrum to the physical Hilbert space is therefore given by $\mathcal{H}_{pp}$.

For the direct integral decomposition of $\mathcal{H}_{ac}$ we have to calculate first the total spectral measure

$$\begin{aligned} \mu_{ac}(\lambda) &= \tfrac{1}{2}\sum_{N=1}^{\infty} 2^{-N}\left(\mu_{N0}(\lambda)+\mu_{-N0}(\lambda)\right) \\ &= \int_0^{\infty}\left(\sum_{N=1}^{\infty}\frac{(x/2)^N}{N!}\,\Theta(\lambda-N^2x)\right)\exp(-x)\,dx \quad . \end{aligned} \tag{5.11}$$

This gives for the Radon-Nikodym derivative with respect to the Lebesgue measure

$$\frac{d\mu_{ac}}{d\lambda}=\sum_{N=1}^{\infty}\frac{1}{N^2\,N!}\left(\frac{\lambda}{2N^2}\right)^N\exp(-\lambda/N^2) \quad . \tag{5.12}$$

The Radon-Nikodym derivative of the measures μ_{N0} with respect to the Lebesgue measure is

$$\frac{d\mu_{N0}}{d\lambda}=\frac{\lambda^{|N|}}{N^2\,|N|!}\exp(-\lambda/N^2) \tag{5.13}$$

so that the Radon-Nikodym derivatives of μ_{N0} with respect to μ_{ac} can be calculated to

$$\rho_{N0}(\lambda)=\frac{d\mu_{N0}/d\lambda}{d\mu_{ac}/d\lambda}=\lambda^{|N|}\frac{(N^2\,|N|!)^{-1}\exp(-\lambda/N^2)}{\sum_{M=1}^{\infty}\frac{1}{M^2\,M!}\left(\frac{\lambda}{2M^2}\right)^M\exp(-\lambda/M^2)} \quad . \tag{5.14}$$

In the limit $\lambda\to 0$ most of the ρ_{N0} vanish:

$$\lim_{\lambda\to 0}\rho_{N0}(\lambda)=\begin{cases}2 & \text{for } |N|=1,\\ 0 & \text{for } |N|>1.\end{cases} \tag{5.15}$$

Hence the contribution from the absolutely continuous part of the spectrum to the physical Hilbert space consists of just two vectors. This contribution corresponds to the fact that the delta function $\delta(x_1)\delta(x_2)$ is a generalized eigenvector of $\widehat{\mathbf{M}}$.

The total physical Hilbert space is the sum of the contributions from $\mathcal{H}_{pp}$ and $\mathcal{H}_{ac}$, i.e the sum of $\mathcal{H}_{pp}$ which can be identified with $L_2(\mathbb{R}_+, r dr)$ and two vectors $\{e_{10}, e_{-10}\}$. Notice that if we had made the wrong identification of $L_2(\mathbb{R}^2, d^2x)$ with $L_2(\mathbb{R}_+, r dr)\otimes L_2(S^1, d\phi)$ discussed above then we would have had to use the vectors $\Omega'_{N0}\propto e^{in\phi}e^{-r^2}$, $n\neq 0$ to get a cyclic system. The spectral measures of these vectors all coincide and therefore the corresponding Radon – Nikodym derivatives would all be non vanishing at $\lambda=0$ and the contribution to the physcal Hilbert space from the continuous spectrum would be infinite dimensional which would be physically wrong because the constraint $x_1^2+x_2^2=0$ corresponds to only one point in phase space and the correponding physical Hilbert space should be finite dimensional.

6 Conclusions

The main lesson learnt in this article is that the Master Constraint Programme can deal essentially with all situations that one usually encounters when quantizing finite dimensional systems. It is even possible to deal with second class constraints as already emphasized by Klauder in his Affine Quantization Programme for gravity [12]. A situation that we have not dealt with are constraints which are quadratic in the momenta with structure constants but such that the corresponding Lie group is not compact. We will deal with this difficult case in a seperate paper [3].

In all the examples studied we recover the usual results. This might seem surprising because one would think, e.g. that the space of solutions to the single quadratic constraint $\widehat{\mathbf{M}} = \hat{p}_1^2 + \hat{p}_2^2 = 0$ is larger than the space of solutions to the individual linear constraints $\hat{p}_1 = \hat{p}_2 = 0$. Indeed, the general solution of the former is of the form $f_1(z) + f_2(\bar{z})$ where $z = x_1 + ix_2$ and f_1, f_2 are smooth functions while the general solution to the latter are the constants where we have used the representation $\hat{p}_j = i\hbar\partial/\partial x_j$ on $\mathcal{H} = L_2(\mathbb{R}^2, d^2x)$. However, solutions of the first type do not appear in the spectral resolution of the Master Constraint. Intuitively, this comes about because a physical Hilbert space based on square integrable linear combinations of holomorphic or antiholomorphic functions must be of the form $L_2(\mathbb{C}, dz d\bar{z}\rho(|z|))$ with a damping factor $\rho(|z|)$. However, this Hilbert space is not a representation space for a self adjoint representation of the Dirac obsrvables with respect to $\widehat{\mathbf{M}}$: Indeed, the induced action of the Dirac observables $\hat{p}_j$ is not self-adjoint in this representation. Thus, this representation is not viable unless we restrict ourselves to the constant functions. These are automatically selected by the spectral analysis of the Master Constraint which in turn is induced by the spectral analysis of the individual constraints $\hat{C}_j = \hat{p}_j$.

Another point worth mentioning is the following: Assume a simple situation, a finite number of first class constraints C_j which may close with structure functions only. In the case of non – trivial structure functions, the constraints $\hat{C}_j$ *generically must not be self – adjoint* for the following reason: Let $\{C_j, C_k\} = f_{jk}\ ^l C_l$ with non – trivial, real valued structure functions $f_{jk}\ ^l$. Suppose that the $\hat{C}_j$ are self –adjoint then in quantum theory we expect a relation of the form $[\hat{C}_j, \hat{C}_k] = i\hbar(\hat{f}_{jk}\ ^l \hat{C}_l + \hat{C}_l(\hat{f}_{jk}\ ^l)^\dagger)/2$ where the symmetric ordering is forced on us due to the antisymmetry of the commutator. Since $f_{jk}\ ^l$ is real valued $\hat{f}_{jk}\ ^l - (\hat{f}_{jk}\ ^l)^\dagger$ should vanish as $\hbar \to 0$. Now suppose that Ψ is a generalized solution[6] of all constraints, $\hat{C}_j\Psi = 0$ for all j. Applying the commutator we find $0 = i\hbar[\hat{C}_l, (\hat{f}^l_{jk})^\dagger]\Psi$ for all j, k. Now as $\hbar \to 0$, the expression $i\hbar[\hat{C}_l, (\hat{f}^l_{jk})^\dagger]$ becomes the Poisson bracket $\hbar^2\{C_l, f_{jk}\ ^l\}$ to lowest order in $\hbar$. Unless this classical quantity vanishes, the solution Ψ not only satisfies $\hat{C}_j\Psi = 0$ but the additional constraints $[\hat{C}_l, (\hat{f}^l_{jk})^\dagger]\Psi = 0$ which have no classical counterpart. Iterating like this it could happen, in the worst case, that Ψ satisfies an indefinite tower of new constraints, leaving us with the only solution $\Psi = 0$. In general the solution space will be too small as to capture the physics of the classical reduced phase space, displying a quantum anomaly. In the example with structure functions studied in this paper it was actually the case that $\{C_l, f_{jk}\ ^l\} = 0$ which is why we succeeded in using self adjoint constraint operators there[7]. This argument just given is not new, it was presented first in [24], but it is often forgotten.

Finally, let us assume that zero lies in the point spectrum of all the $\hat{C}_j$ for a given Hilbert space representation and set $\widehat{\mathbf{M}} := \sum_j \hat{C}_j^\dagger \hat{C}_j$. Now the eigenvalue equations $\hat{C}_j\Psi = 0$ and $\widehat{\mathbf{M}}\,\Psi = 0$ make sense for elements $\Psi \in \mathcal{H}$. Then we claim that that $\hat{C}_j\Psi = 0$ for all j if and only if $\widehat{\mathbf{M}}\,\Psi = 0$. The first implication is clear. For the converse we compute $0 =< \Psi, \widehat{\mathbf{M}}\,\Psi >= \sum_j ||\hat{C}_j\Psi||^2$. Thus we see that in this case the Master Constraint Operator captures exactly the same physics as the individual constraints, in particular it will suffer under potentially the same quantum anomalies. Hence, nothing is swept under the rug[8]. In case that zero also lies in the continuous spectrum, the DID procedure of [2] generalizes the argument just made.

[6]More precisely we should look for solutions in a space of distributions Φ^* dual to a dense and invariant (under the constraints) subspace $\Phi \subset \mathcal{H}$ in the form $\Psi(\hat{C}_j^\dagger f) = 0$ for all $f \in \Phi$ and all j.

[7]We should study another example in the future where this is not the case.

[8]However, it is interesting to notice that by subtracting the minimum of the spectrum from the Master Constraint we may also be able to deal with anomalous situations!

Acknowledgements

BD thanks the German National Merit Foundation for financial support. This research project was supported in part by a grant from NSERC of Canada to the Perimeter Institute for Theoretical Physics.

References

[1] T. Thiemann, "The Phoenix Project: Master Constraint Programme for Loop Quantum Gravity", gr-qc/0305080

[2] B. Dittrich, T. Thiemann, "Testing the Master Constraint Programme for Loop Quantum Gravity I. General Framework", gr-qc/0411138

[3] B. Dittrich, T. Thiemann, "Testing the Master Constraint Programme for Loop Quantum Gravity III. SL(2,R) Models", gr-qc/0411140

[4] B. Dittrich, T. Thiemann, "Testing the Master Constraint Programme for Loop Quantum Gravity IV. Free Field Theories", gr-qc/0411141

[5] B. Dittrich, T. Thiemann, "Testing the Master Constraint Programme for Loop Quantum Gravity V. Interacting Field Theories", gr-qc/0411142

[6] C. Rovelli, "Loop Quantum Gravity", Living Rev. Rel. **1** (1998) 1, gr-qc/9710008
T. Thiemann,"Lectures on Loop Quantum Gravity", Lecture Notes in Physics, **631** (2003) 41 – 135, gr-qc/0210094
A. Ashtekar, J. Lewandowski, "Background Independent Quantum Gravity: A Status Report", Class. Quant. Grav. **21** (2004) R53; [gr-qc/0404018]
L. Smolin, "An Invitation to Loop Quantum Gravity", hep-th/0408048

[7] C. Rovelli, "Quantum Gravity", Cambridge University Press, Cambridge, 2004

[8] T. Thiemann, "Modern Canonical Quantum General Relativity", Cambridge University Press, Cambridge, 2005, gr-qc/0110034

[9] T. Thiemann, "Anomaly-free Formulation of non-perturbative, four-dimensional Lorentzian Quantum Gravity", Physics Letters **B380** (1996) 257-264, [gr-qc/9606088]
T. Thiemann, "Quantum Spin Dynamics (QSD)", Class. Quantum Grav. **15** (1998) 839-73, [gr-qc/9606089]; "II. The Kernel of the Wheeler-DeWitt Constraint Operator", Class. Quantum Grav. **15** (1998) 875-905, [gr-qc/9606090]; "III. Quantum Constraint Algebra and Physical Scalar Product in Quantum General Relativity", Class. Quantum Grav. **15** (1998) 1207-1247, [gr-qc/9705017]; "IV. 2+1 Euclidean Quantum Gravity as a model to test 3+1 Lorentzian Quantum Gravity", Class. Quantum Grav. **15** (1998) 1249-1280, [gr-qc/9705018]; "V. Quantum Gravity as the Natural Regulator of the Hamiltonian Constraint of Matter Quantum Field Theories", Class. Quantum Grav. **15** (1998) 1281-1314, [gr-qc/9705019]; "VI. Quantum Poincaré Algebra and a Quantum Positivity of Energy Theorem for Canonical Quantum Gravity", Class. Quantum Grav. **15** (1998) 1463-1485, [gr-qc/9705020]; "Kinematical Hilbert Spaces for Fermionic and Higgs Quantum Field Theories", Class. Quantum Grav. **15** (1998) 1487-1512, [gr-qc/9705021]

[10] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão, T. Thiemann, "Quantization for diffeomorphism invariant theories of connections with local degrees of freedom", Journ. Math. Phys. **36** (1995) 6456-6493, [gr-qc/9504018]

[11] H. Sahlmann, "When do Measures on the Space of Connections Support the Triad Operators of Loop Quantum Gravity?", gr-qc/0207112; "Some Comments on the Representation Theory of the Algebra Underlying Loop Quantum Gravity", gr-qc/0207111
H. Sahlmann, T. Thiemann, "On the Superselection Theory of the Weyl Algebra for Diffeomorphism Invariant Quantum Gauge Theories", gr-qc/0302090; "Irreducibility of the

Ashtekar-Isham-Lewandowski Representation", gr-qc/0303074
A. Okolow, J. Lewandowski, "Diffeomorphism Covariant Representations of the Holonomy Flux Algebra", gr-qc/0302059

[12] J. Klauder, "Universal Procedure for Enforcing Quantum Constraints", Nucl.Phys.B547:397-412,1999, [hep-th/9901010]; "Quantization of Constrined Systems", Lect. Notes Phys. **572** (2001) 143-182, [hep-th/0003297]
A. Kempf, J. Klauder, "On the Implementation of Constraints through Projection Operators", J. Phys. **A34** (2001) 1019-1036, [quant-ph/0009072]

[13] J.E. Marsden, P.R. Chernoff, "Properties of Infinite Dimensional Hamiltonian Systems", Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1974

[14] T. Thiemann, "Quantum Spin Dynamics (QSD): VIII. The Master Constraint", in preparation

[15] H. Sahlmann, T. Thiemann, "Towards the QFT on Curved Spacetime Limit of QGR. 1. A General Scheme", [gr-qc/0207030]; "2. A Concrete Implementation", [gr-qc/0207031]

[16] T. Thiemann, "Quantum Spin Dynamics (QSD): VII. Symplectic Structures and Continuum Lattice Formulations of Gauge Field Theories", Class.Quant.Grav.18:3293-3338,2001, [hep-th/0005232]; "Gauge Field Theory Coherent States (GCS): I. General Properties", Class.Quant.Grav.18:2025-2064,2001, [hep-th/0005233]; "Complexifier Coherent States for Canonical Quantum General Relativity", gr-qc/0206037
T. Thiemann, O. Winkler, "Gauge Field Theory Coherent States (GCS): II. Peakedness Properties", Class.Quant.Grav.18:2561-2636,2001, [hep-th/0005237]; "III. Ehrenfest Theorems", Class. Quantum Grav. **18** (2001) 4629-4681, [hep-th/0005234]; "IV. Infinite Tensor Product and Thermodynamic Limit", Class. Quantum Grav. **18** (2001) 4997-5033, [hep-th/0005235]
H. Sahlmann, T. Thiemann, O. Winkler, "Coherent States for Canonical Quantum General Relativity and the Infinite Tensor Product Extension", Nucl.Phys.B606:401-440,2001 [gr-qc/0102038]

[17] M. Reed, B. Simon, "Methods of Modern Mathematical Physics I: Functional Analysis", Academic Press, New York, 1980

[18] B. Dittrich, "Partial and Complete Observables for Hamiltonian Constrained Systems", gr-qc/0411013

[19] M. Reed, B. Simon, "Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness", Academic Press, New York, 1975

[20] G. Teschl, "Schrödinger Operators", lecture notes for a cours at the University Vienna 1999 and 2002, available at http://www.mat.univie.ac.at/~gerald/ftp/book~schroe/

[21] R. Howe,E. C. Tan, "Non-Abelian Harmonic Analysis – Applications of SL(2,R)", Springer 1992, New-York

[22] A. Galindo, P. Pascual, "Quantum Mechanics I", (Springer, Berlin) 1990

[23] R. Courant, D. Hilbert, "Methods of Mathematical Physics", vol. 1, Interscience Publishers, New York, 1953

[24] P. Hajíček, K. Kuchař, Phys. Rev. **D41** (1990) 1091, Journ. Math. Phys. **31** (1990) 1723